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An explicit basis for the rational higher Chow groups of abelian number fields

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We review and simplify A. Beĭlinson's construction of a basis for the motivic cohomology of a point over a cyclotomic field, then promote the basis elements to higher Chow cycles and evaluate the KLM regulator map on them.

1. Introduction

Let $\zeta_N \in \mathbb{C}^*$ be a primitive N-th root of 1 ($N \geq 2$). The seminal article [Beĭlinson 1984] concludes with a construction of elements Ξ_b (for $b \in (\mathbb{Z}/N\mathbb{Z})^*$) in motivic cohomology

$$H^1_{\mathcal{M}}(\mathrm{Spec}(\mathbb{Q}(\zeta_N)), \mathbb{Q}(n)) \cong K^{(n)}_{2n-1}(\mathbb{Q}(\zeta_N)) \otimes \mathbb{Q}$$

mapping to $\mathrm{Li}_n(\zeta_N^b) = \sum_{k\geq 1} \zeta_N^{kb}/k^n \in \mathbb{C}/(2\pi \boldsymbol{i})^n\mathbb{R}$ under his regulator. Since by Borel's theorem [1974], we have $\mathrm{rk}\, K^{(n)}_{2n-1}(\mathbb{Q}(\zeta_N))_{\mathbb{Q}} = \frac{1}{2}\phi(N)$ (for $N \geq 3$), an immediate consequence is that the $\{\Xi_b\}$ span $K^{(n)}_{2n-1}(\mathbb{Q}(\zeta_N))_{\mathbb{Q}}$; indeed, Beĭlinson's results anticipated the eventual proofs [Rapoport 1988; Burgos Gil 2002] of the equality (for number fields) of his regulator with that of Borel [1977]. An expanded account of his construction was written up by Neukirch (with Rapoport and Schneider) in [Neukirch 1988], up to a "crucial lemma" [op. cit., Part II, Lemma 2.4] required for the regulator computation, which was subsequently proved by Esnault [1989].

The intervening years have seen some improvements in technology, especially Bloch's introduction of higher Chow groups [Bloch 1986], which yield an *integral* definition of motivic cohomology for smooth schemes X. In particular, we have[1]

$$H^1_{\mathcal{M}}(\mathrm{Spec}(\mathbb{Q}(\zeta_N)), \mathbb{Z}(n)) \cong CH^n(\mathbb{Q}(\zeta_N), 2n-1)$$
$$:= H_{2n-1}\{Z^n(\mathbb{Q}(\zeta_N), \bullet), \partial\},$$

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[1] We use the shorthand $CH^*(F, *)$ ($Z^*(F, *)$, etc.) for $CH^*(\mathrm{Spec}(F), *)$ (F a field).

and can ask for explicit cycles in $\ker(\partial) \subset Z^n(\mathbb{Q}(\zeta_N), 2n-1)$ representing (multiples of) Beĭlinson's elements Ξ_b. Another relevant development was the explicit realization of Beĭlinson's regulator in [Kerr et al. 2006; Kerr and Lewis 2007] as a morphism $\widetilde{\mathrm{AJ}}$ of complexes, from a *rationally* quasi-isomorphic subcomplex $Z^n_{\mathbb{R}}(X, \bullet)$ of $Z^n(X, \bullet)$ to a complex computing the absolute Hodge cohomology of X. Here this "KLM morphism" yields an Abel–Jacobi mapping

$$\mathrm{AJ} : CH^n(\mathbb{Q}(\zeta_N), 2n-1) \otimes \mathbb{Q} \to \mathbb{C}/(2\pi \boldsymbol{i})^n \mathbb{Q}, \tag{1.1}$$

and in the present note we shall construct (for all n) higher Chow cycles

$$\hat{\mathscr{Z}}_b \in \ker(\partial) \subset Z^n_{\mathbb{R}}(\mathbb{Q}(\zeta_N), 2n-1) \otimes \mathbb{Q}$$

satisfying

$$(n-3)N^{n-1}\hat{\mathscr{Z}}_b \in Z^n_{\mathbb{R}}(\mathbb{Q}(\zeta_N), 2n-1)) \quad \text{and} \quad \mathrm{AJ}(\hat{\mathscr{Z}}_b) = \mathrm{Li}_n(\zeta_N^b).$$

(See Theorems 3.3, 3.8, and 4.2, with $\hat{\mathscr{Z}} = (-1)^n \widetilde{\mathscr{Z}}/N^{n-1}$.) This is entirely more explicit than the constructions in [Beĭlinson 1984; Neukirch 1988], and yields a brief and transparent evaluation of the regulator, which moreover allows us to dispense with some of the hypotheses of [Neukirch 1988, Part II, Lemma 2.4] or [Esnault 1989, Theorem 3.9] and thus avoid the more complicated construction of [Neukirch 1988, Part II, Lemma 3.1]. Furthermore, in concert with the anticipated extension of $\widetilde{\mathrm{AJ}}$ to the entire complex $Z^n(X, \bullet)$ (making (1.1) integral), we expect that our cycles will be useful for studying the torsion in $CH^n(\mathbb{Q}(\zeta_N), 2n-1)$, as begun in [Petras 2008; 2009]; see Remark 4.1 and Section 4E.

2. Beĭlinson's construction

In this section we show that (the graph of) the n-tuple of functions

$$\left\{1-\zeta_N z_1 \cdots z_{n-1}, \left(\frac{z_1}{z_1-1}\right)^N, \dots, \left(\frac{z_{n-1}}{z_{n-1}-1}\right)^N\right\}$$

completes to a relative motivic cohomology class on $(\square^{n-1}, \partial\square^{n-1})$. Most of the work that follows is to show that its image under a residue map vanishes; see (2.12). It also serves to establish notation for Section 3, where we recast this class as a higher Chow cycle and compute its regulator.

2A. ***Notation.*** Let $N \geq 2$, and $\zeta \in \mathbb{C}$ be a primitive N-th root of unity; i.e., $\zeta = e^{2\pi \boldsymbol{i} a/N}$, where a is coprime to N. Denoting by $\Phi_N(x)$ the N-th cyclotomic polynomial, each such a yields an embedding σ of $\mathbb{F} := \mathbb{Q}[\omega]/(\Phi_N(\omega))$ into $\mathbb{C}$ (by sending $\omega \mapsto \zeta$). (If $N = 2$, then $\mathbb{F} = \mathbb{Q}$ and $\omega = \zeta = -1$.)

Working over any subfield of $\mathbb{C}$ containing ζ, write

$$\square^n := (\mathbb{P}^1 \backslash \{1\})^n \supset (\mathbb{P}^1 \backslash \{0, 1\})^n =: \mathbb{T}^n,$$

with coordinates $(z_1, \dots, z_n)$. We have isomorphisms from $\mathbb{T}^n$ to $\mathbb{G}_m^n$ (with coordinates $(t_1, \dots, t_n)$), given by $t_i := z_i/(z_i - 1)$. Define a function $f_n(\underline{z}) := 1 - \zeta^b t_1 \cdots t_n$ on $\mathbb{T}^n$ (with b coprime to N), and normal crossing subschemes

$$S^n := \{\underline{z} \in \mathbb{T}^n \mid \text{some } z_i = \infty\} \subset S^n \cup |(f_n)_0| =: \widetilde{S}^n \subset \mathbb{T}^n.$$

(Alternatively, we may view these schemes as defined over $\mathbb{F}$ by replacing ζ^b with ω^b.)

Now consider the morphism

$$\imath_n : \mathbb{T}^{n-1} \to \mathbb{T}^n, \qquad (t_1, \dots, t_{n-1}) \mapsto (t_1, \dots, t_{n-1}, (\zeta^b t_1 \cdots t_{n-1})^{-1}).$$

Lemma 2.1. *The morphism $\imath_n$ sends $\mathbb{T}^{n-1}$ isomorphically onto $|(f_n)_0|$, with*

$$\imath_n(\widetilde{S}^{n-1}) = |(f_n)_0| \cap S^n.$$

We also remark that the Zariski closure of $\imath_n(\mathbb{T}^{n-1})$ in $\square^n$ is just $\imath_n(\mathbb{T}^{n-1})$.

2B. ***Results for Betti cohomology.*** The construction just described has quite pleasant cohomological properties, as we shall now see.

Lemma 2.2. *As a $\mathbb{Q}$-MHS,*

$$H^q(\mathbb{T}^n, S^n) \cong \begin{cases} \mathbb{Q}(-n), & q = n, \\ 0, & q \neq n. \end{cases}$$

Proof. Apply the Künneth formula to $(\mathbb{T}^n, S^n) \cong (\mathbb{G}_m, \{1\})^n$. □

Lemma 2.3. *As a $\mathbb{Q}$-MHS,*

$$H^q(\mathbb{T}^n, \widetilde{S}^n) \cong \begin{cases} \mathbb{Q}(0) \oplus \mathbb{Q}(-1) \oplus \cdots \oplus \mathbb{Q}(-n), & q = n, \\ 0, & q \neq n. \end{cases}$$

Proof. This is clear for $(\mathbb{T}^1, \widetilde{S}^1) \cong (\mathbb{G}_m, \{1, \bar{\zeta}\})$. Now consider the exact sequence

$$H^{*-1}(\mathbb{T}^n, S^n) \xrightarrow{\imath_n^*} \underline{H^{*-1}(\mathbb{T}^{n-1}, \widetilde{S}^{n-1})} \xrightarrow{\delta} H^*(\mathbb{T}^n, \widetilde{S}^n) \to \underline{H^*(\mathbb{T}^n, S^n)} \xrightarrow{\imath_n^*} H^*(\mathbb{T}^{n-1}, \widetilde{S}^{n-1})$$

of $\mathbb{Q}$-MHS, associated to the inclusion $(\mathbb{T}^{n-1}, \widetilde{S}^{n-1}) \overset{\imath_n}{\hookrightarrow} (\mathbb{T}^n, S^n)$. (This is just the relative cohomology sequence, once one notes that $((\mathbb{T}^n, S^n), \imath_n(\mathbb{T}^{n-1}, \widetilde{S}^{n-1})) = (\mathbb{T}^n, S^n \cup \imath_n(\mathbb{T}^{n-1})) = (\mathbb{T}^n, \widetilde{S}^n)$ by Lemma 2.1.) If $* \neq n$, then the underlined terms are 0 via Lemma 2.2 and induction. If $* = n$, then the *end* terms are 0 via Lemma 2.2 and induction, and

$$0 \to H^{n-1}(\mathbb{T}^{n-1}, \widetilde{S}^{n-1}) \xrightarrow{\delta} H^n(\mathbb{T}^n, \widetilde{S}^n) \to H^n(\mathbb{T}^n, S^n) \to 0 \tag{2.4}$$

is a short-exact sequence.

Now observe that:

- $H^n(\mathbb{T}^n, S^n; \mathbb{C}) = F^n H^n(\mathbb{T}^n, S^n; \mathbb{C})$ is generated by the holomorphic form
$$\eta := \frac{1}{(2\pi \boldsymbol{i})^n} \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_n}{t_n};$$
- $H_{n-1}(\mathbb{T}^{n-1}, \widetilde{S}^{n-1}; \mathbb{Q})$ is generated by images $\underline{e}(U_i)$ of the cells
$$\bigcup_{i=0}^{n} U_i = [0,1]^n \setminus \bigcup_{\ell=1}^{n} \Bigl\{ \sum x_i = \ell - \frac{a}{N} \Bigr\},$$
where $\underline{e} : [0,1]^n \to \mathbb{T}^n$ is defined by $(x_1, \dots, x_n) \mapsto (e^{2\pi \boldsymbol{i} x_1}, \dots, e^{2\pi \boldsymbol{i} x_n}) = (t_1, \dots, t_n)$;
- $\displaystyle\int_{\underline{e}(U_i)} \eta = \int_{U_i} dx_1 \wedge \cdots \wedge dx_n \in \mathbb{Q}.$

(Writing $\mathscr{S}^1$ for the unit circle, $((\mathscr{S}^1)^n, (\mathscr{S}^1)^n \cap \widetilde{S}^n)$ is a deformation retract of $(\mathbb{T}^n, \widetilde{S}^n)$. The $\underline{e}(U_i)$ visibly yield all the relative cycles in the former, justifying the second observation.) Together these immediately imply that (2.4) is split, completing the proof. □

2C. ***Results for Deligne cohomology.*** Recall that Beĭlinson's absolute Hodge cohomology [1986] of an analytic scheme Y over $\mathbb{C}$ sits in an exact sequence

$$0 \to \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{r-1}(\mathrm{Y}, \mathbb{A}(p))) \to H^r_{\mathcal{D}}(\mathrm{Y}, \mathbb{A}(p)) \\ \to \mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^r(\mathrm{Y}, \mathbb{A}(p))) \to 0.$$

(Here we use a subscript "$\mathcal{D}$" since the construction after all is a "weight-corrected" version of Deligne cohomology; the subscript "MHS" of course means "$\mathbb{A}$-MHS".) We shall not have any use for details of its construction here, and refer the reader to [Kerr and Lewis 2007, §2].

Lemma 2.5. *The map* $\iota_n^* : H^n_{\mathcal{D}}(\mathbb{T}^n, S^n; \mathbb{A}(n)) \to H^n_{\mathcal{D}}(\mathbb{T}^{n-1}, \widetilde{S}^{n-1}; \mathbb{A}(n))$ *is zero* ($\mathbb{A} = \mathbb{Q}$ *or* $\mathbb{R}$).

Proof. Consider the exact sequence

$$\cdots \to H^n_{\mathcal{D}}(\mathbb{T}^n, S^n; \mathbb{Q}(n)) \xrightarrow{\iota_n^*} H^n_{\mathcal{D}}(\mathbb{T}^{n-1}, \widetilde{S}^{n-1}; \mathbb{Q}(n)) \xrightarrow{\delta_{\mathcal{D}}} H^{n+1}_{\mathcal{D}}(\mathbb{T}^n, \widetilde{S}^n; \mathbb{Q}(n)) \to \cdots.$$

It suffices to show that $\delta_{\mathcal{D}}$ is injective. Now

$$\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^n(\mathbb{T}^{n-1}, \widetilde{S}^{n-1}; \mathbb{Q}(n))) = \{0\},$$
$$\mathrm{Hom}_{\mathrm{MHS}}(\mathbb{Q}(0), H^{n+1}(\mathbb{T}^n, \widetilde{S}^n; \mathbb{Q}(n))) = \{0\},$$

by Lemma 2.3, and so $\delta_{\mathcal{D}}$ is given by

$$\mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^{n-1}(\mathbb{T}^{n-1}, \widetilde{S}^{n-1}; \mathbb{Q}(n))) \xrightarrow{\delta_{\mathcal{D}}} \mathrm{Ext}^1_{\mathrm{MHS}}(\mathbb{Q}(0), H^n(\mathbb{T}^n, \widetilde{S}^n; \mathbb{Q}(n))).$$

Since (2.4) is split, the corresponding sequence of Ext^1-groups is exact, and $\delta_{\mathcal{D}}$ is injective. □

2D. ***Results for motivic cohomology.*** Let X be any smooth simplicial scheme (of finite type), defined over a subfield of $\mathbb{C}$. We have Deligne class maps

$$c_{\mathcal{D},\mathbb{A}} : H^r_{\mathcal{M}}(X, \mathbb{Q}(p)) \to H^r_{\mathcal{D}}(X^{\mathrm{an}}_{\mathbb{C}}, \mathbb{A}(p))$$

(for $\mathbb{A} = \mathbb{Q}$ or $\mathbb{R}$). The case of particular interest here is when $r = 1$, X is a point, and

$$c_{\mathcal{D},\mathbb{A}}(\mathrm{Z}) = \frac{1}{(2\pi \boldsymbol{i})^{p-1}} \int_{\mathrm{Z}^{\mathrm{an}}_{\mathbb{C}}} R_{2p-1} \in \mathbb{C}/\mathbb{A}(p), \tag{2.6}$$

where, interpreting $\log(z)$ as the 0-current with branch cut along $T_z := z^{-1}(\mathbb{R}_-)$,

$$\begin{aligned} R_{2p-1} &:= \sum_{k=1}^{2p-1} (2\pi \boldsymbol{i})^{k-1} R^{(k)}_{2p-1} \\ &:= \sum_{k=1}^{2p-1} (2\pi \boldsymbol{i})^{k-1} \log(z_k) \frac{dz_{k+1}}{z_{k+1}} \wedge \cdots \wedge \frac{dz_{2p-1}}{z_{2p-1}} \cdot \delta_{T_{z_1} \cap \cdots \cap T_{z_{k-1}}} \end{aligned} \tag{2.7}$$

is the regulator current of [Kerr et al. 2006; Kerr and Lewis 2007] belonging to $D^{2p-2}((\mathbb{P}^1)^{\times(2p-1)})$. Here it is essential that the representative higher Chow cycle Z belong to the quasi-isomorphic subcomplex $Z^p_{\mathbb{R}}(\mathrm{pt.}, \bullet)_{\mathbb{Q}} \subset Z^p(\mathrm{pt.}, \bullet)_{\mathbb{Q}}$ comprising cycles in good position with respect to certain real analytic chains; see [Kerr and Lewis 2007, §8] or Remark 3.4 below.

Now take a number field K, $[K : \mathbb{Q}] = d = r_1 + 2r_2$, and set

$$d_m = d_m(K) := \begin{cases} r_1 + r_2 - 1, & m = 1, \\ r_1 + r_2, & m > 1 \text{ odd}, \\ r_2, & m > 0 \text{ even}. \end{cases}$$

For X defined over K, write $\widetilde{X^{\mathrm{an}}_{\mathbb{C}}} := \coprod_{\sigma \in \mathrm{Hom}(K,\mathbb{C})} ({}^{\sigma}X)^{\mathrm{an}}_{\mathbb{C}}$ and

$$\begin{array}{ccc} H^r_{\mathcal{M}}(X, \mathbb{Q}(p)) & \xrightarrow{\quad \tilde{c}_{\mathcal{D},\mathbb{R}} \quad} & H^r(\widetilde{X^{\mathrm{an}}_{\mathbb{C}}}, \mathbb{R}(p)) \\ & {\scriptstyle \tilde{c}^{+}_{\mathcal{D},\mathbb{R}}} \searrow \qquad \nearrow & \\ & H^r_{\mathcal{D}}(\widetilde{X^{\mathrm{an}}_{\mathbb{C}}}, \mathbb{R}(p))^+ & \end{array}$$

for the map $\mathrm{Z} \mapsto (c_{\mathcal{D},\mathbb{R}}({}^{\sigma}\mathrm{Z}))_{\sigma}$, which factors through the invariants under de Rham conjugation. If $X = \mathrm{Spec}(K)$, then we have $H^1_{\mathcal{D}}(\widetilde{X^{\mathrm{an}}_{\mathbb{C}}}, \mathbb{R}(p)) \cong \mathbb{R}(p-1)^{\oplus d}$ and $H^1_{\mathcal{D}}(\widetilde{X^{\mathrm{an}}_{\mathbb{C}}}, \mathbb{R}(p))^+ \cong \mathbb{R}(p-1)^{\oplus d_p}$. Write $H^r_{\mathcal{M}}(X, \mathbb{R}(p)) = H^r_{\mathcal{M}}(X, \mathbb{Q}(p)) \otimes_{\mathbb{Q}} \mathbb{R}$.

Lemma 2.8. *For* $X = \mathrm{Spec}(K)$, $\mathbb{G}_{m,K}^{\times n}$, $(\mathbb{T}_K^n, S_K^n)$, *or* $(\mathbb{T}_K^n, \widetilde{S}_K^n)$,

$$\tilde{c}_{\mathcal{D},\mathbb{R}}^{+} \otimes \mathbb{R} : H_{\mathcal{M}}^r(X, \mathbb{R}(p)) \to H_{\mathcal{D}}^r(\widetilde{X_{\mathbb{C}}^{\mathrm{an}}}, \mathbb{R}(p))^{+}$$

is an isomorphism ($\forall r, p$).

Proof. By [Burgos Gil 2002], the composition

$$K_{2p-1}(\mathcal{O}_K) \otimes \mathbb{Q} \xrightarrow{\cong} H_{\mathcal{M}}^1(\mathrm{Spec}(K), \mathbb{Q}(p)) \xrightarrow{\tilde{c}_{\mathcal{D},\mathbb{R}}^{+}} \mathbb{R}(p-1)^{\oplus d_p} \xrightarrow{\cdot 2/(2\pi \boldsymbol{i})^{p-1}} \mathbb{R}^{d_p}$$

is exactly the Borel regulator (and the groups are zero for $r \neq 1$). The lemma follows for $X = \mathrm{Spec}(K)$.

Let Y be a smooth quasiprojective variety, defined over K, and pick $p \in \mathbb{G}_m(K)$. Write $Y \overset{\imath}{\hookrightarrow} \mathbb{G}_{m,Y} \overset{\jmath}{\hookrightarrow} \mathbb{A}_Y^1 \overset{\kappa}{\hookleftarrow} Y$ for the Cartesian products with Y of the morphisms

$$\mathrm{Spec}(K) \overset{\imath_p}{\hookrightarrow} \mathbb{G}_{m,K} \overset{\jmath}{\hookrightarrow} \mathbb{A}_K^1 \overset{\imath_0}{\hookleftarrow} \mathrm{Spec}(K).$$

Then by the homotopy property,

$$\imath^* : H_{\mathcal{K}}^r(\mathbb{G}_{m,Y}, \mathbb{R}(p)) \to H_{\mathcal{K}}^r(Y, \mathbb{R}(p)) \cong H_{\mathcal{K}}^r(\mathbb{A}_Y^1, \mathbb{R}(p))$$

splits the localization sequence

$$\cdots \xrightarrow{\kappa_*} H_{\mathcal{K}}^r(\mathbb{A}_Y^1, \mathbb{R}(p)) \xrightarrow{\jmath^*} H_{\mathcal{K}}^r(\mathbb{G}_{m,Y}, \mathbb{R}(p)) \xrightarrow{\mathrm{Res}} H_{\mathcal{K}}^{r-1}(Y, \mathbb{R}(p-1)) \xrightarrow{\kappa_*} \cdots$$

for $\mathcal{K} = \mathcal{M}, \mathcal{D}$ (in particular, $\kappa_* = 0$). It follows that

$$H_{\mathcal{K}}^r(\mathbb{G}_{m,Y}, \mathbb{R}(p)) \cong H_{\mathcal{K}}^r(Y, \mathbb{R}(p)) \oplus H_{\mathcal{K}}^{r-1}(Y, \mathbb{R}(p-1)),$$

compatibly with $c_{\mathcal{D},\mathbb{R}}$; applying this iteratively gives the lemma for $\mathbb{G}_{m,K}^{\times n}$.

Finally, both $(\mathbb{T}^n, S_K^n)$ and $(\mathbb{T}_K^n, \widetilde{S}_K^n)$ may be regarded as (co)simplicial normal crossing schemes $X^\bullet$. (That is, writing $\widetilde{S}_K^n = \bigcup Y_i$, we take $X^0 = \mathbb{T}_K^n$, $X^1 = \coprod_i Y_i$, $X^2 = \coprod_{i<j} Y_i \cap Y_j$, etc.) We have spectral sequences

$$E_1^{i,j} = H_{\mathcal{K}}^{2p+j}(X^i, \mathbb{R}(p)) \Longrightarrow H_{\mathcal{K}}^{2p+i+j}(X^\bullet, \mathbb{R}(p)),$$

compatible with $c_{\mathcal{D},\mathbb{R}}$, and all X^i are disjoint unions of powers of $\mathbb{G}_{m,K}$. The lemma is proved. □

Lemma 2.9. *The map* $\imath_n^* : H_{\mathcal{M}}^n(\mathbb{T}^n, S^n; \mathbb{A}(n)) \to H_{\mathcal{M}}^n(\mathbb{T}^{n-1}, \widetilde{S}^{n-1}; \mathbb{A}(n))$ *is zero* (*for* $\mathbb{A} = \mathbb{Q}$ *or* $\mathbb{R}$).

Proof. Form the obvious commutative square and use the results of Lemmas 2.5 and 2.8. □

2E. ***The Beĭlinson elements.*** To each $I \subset \{1, \dots, n\}$ and $\epsilon : I \to \{0, \infty\}$ we associate a face map $\rho_I^\epsilon : \square^{n-|I|} \hookrightarrow \square^n$, with $z_i = \epsilon(i)$ (for all $i \in I$) on the image, and degeneracy maps $\delta_i : \square^n \twoheadrightarrow \square^{n-1}$ killing the i-th coordinate. For any smooth quasiprojective variety X (say, over a field $K \supseteq \mathbb{Q}$), let $c^p(X, n)$ denote the free abelian group on subvarieties (of codimension p) of $X \times \square^n$ meeting all faces $X \times \rho_I^\epsilon(\square^{n-|I|})$ properly, and $d^p(X, n) = \sum \operatorname{im}(\operatorname{id}_X \times \delta_i^*) \subset c^p(X, n)$. Then $Z^p(X, \bullet) := c^p(X, \bullet)/d^p(X, \bullet)$ defines a complex with differential

$$\partial = \sum_{i=1}^{n} (-1)^{i-1} \big((\operatorname{id}_X \times \rho_i^0)^* - (\operatorname{id}_X \times \rho_i^\infty)^* \big),$$

whose r-th homology defines Bloch's higher Chow group

$$CH^p(X, r) \cong H_{\mathcal{M}}^{2p-r}(X, \mathbb{Z}(p)). \tag{2.10}$$

This isomorphism does not apply for singular varieties (e.g., our simplicial schemes above), and for our purposes in this paper it is the right-hand side of (2.10) that provides the correct generalization. In particular, we have

$$H_{\mathcal{M}}^r(X \times (\square^a, \partial\square^a), \mathbb{Q}(p)) \cong H_{\mathcal{M}}^{r-a}(X, \mathbb{Q}(p)),$$

where $\partial\square^a := \bigcup_{i \in \{1,\dots,a\},\, \varepsilon \in \{0,\infty\}} \rho_i^\varepsilon(\square^{a-1})$. We note here that the (rational) motivic cohomology of a cosimplicial normal-crossing scheme $X^\bullet$ can be computed via (the simple complex associated to) a double complex:

$$E_0^{a,b} := Z^p(X^a, -b)_{\mathbb{Q}}^{\#} \Longrightarrow H_{\mathcal{M}}^{2p+a+b}(X^\bullet, \mathbb{Q}(p)), \tag{2.11}$$

where # denotes cycles meeting all components of all $X^{q>a} \times \partial_I^\epsilon \square^{-b}$ properly.[2]

Continuing to write t_i for $z_i/(z_i - 1)$, we now consider

$$f(\underline{z}) = f_{n-1}(z_1, \dots, z_{n-1}) := 1 - \omega^b t_1 \cdots t_{n-1}$$

as a regular function on $\square_{\mathbb{F}}^{n-1}$, and

$$\mathcal{Z} := \{(\underline{z};\, f(\underline{z}), t_1^N, \dots, t_{n-1}^N) \mid \underline{z} \in \square^{n-1} \backslash |(f)_0|\}$$

as an element of

$$\ker\Big\{ Z^n(\square^{n-1} \backslash |(f)_0|, n)_{\mathbb{Q}}^{\#} \xrightarrow{\partial \oplus \sum (\rho_i^\varepsilon)^*} Z^n(\square^{n-1} \backslash |(f)_0|, n-1) \oplus \bigoplus\nolimits_{i,\varepsilon} Z^n\big(\square^{n-2} \backslash \big|(f|_{z_i=\varepsilon})_0\big|, n\big)_{\mathbb{Q}} \Big\},$$

and hence of

$$H_{\mathcal{M}}^n(\square_{\mathbb{F}}^{n-1} \backslash |(f)_0|, \partial\square^{n-1} \backslash \partial|(f)_0|; \mathbb{Q}(n))$$

[2]See [Levine 1994, §3] and [Kerr and Lewis 2007, §8.2] for the relevant moving lemmas (and for detailed discussion of differentials, etc.).

(where $\partial|(f)_0| := \partial\square^{n-1} \cap |(f)_0| = \bigcup_{i,\varepsilon} |(f|_{z_i=\varepsilon})_0|$, and # indicates cycles meeting faces of $\partial\square^{n-1}\backslash\partial|(f)_0|$ properly). The powers t_i^N are unnecessary at this stage but will be crucial later. For simplicity, we write the class of Z in this group as a symbol $\{f_{n-1}, t_1^N, \ldots, t_{n-1}^N\}$.

Using Lemma 2.1, we have a (vertical) localization exact sequence

$$\begin{array}{ccc} \downarrow & & \\ H^n_{\mathcal{M}}(\square^{n-1}, \partial\square^{n-1}; \mathbb{Q}(n)) & \xleftarrow{\;\cong\;} & CH^n(\mathbb{F}, 2n-1)_{\mathbb{Q}} \\ \downarrow & & \\ H^n_{\mathcal{M}}(\square^{n-1}\backslash|(f)_0|, \partial\square^{n-1}\backslash|(f)_0|; \mathbb{Q}(n)) & & \\ \downarrow{\scriptstyle \mathrm{Res}_{|(f)_0|}} & & \\ H^{n-1}_{\mathcal{M}}(\mathbb{T}^{n-2}, \widetilde{S}^{n-2}; \mathbb{Q}(n-1)) & \xleftarrow{\;\iota_{n-1}^*\;} & H^{n-1}_{\mathcal{M}}(\mathbb{T}^{n-1}, S^{n-1}; \mathbb{Q}(n-1)) \\ \downarrow & & \end{array} \tag{2.12}$$

in which evidently

$$\mathrm{Res}_{|(f)_0|}\{f_{n-1}, t_1^N, \ldots, t_{n-1}^N\} = \iota_{n-1}^*\{t_1^N, \ldots, t_{n-1}^N\}.$$

Proposition 2.13. *$\mathcal{Z}$ lifts to a class $\widetilde{\Xi} \in CH^n(\mathbb{F}, 2n-1)_{\mathbb{Q}}$.*

Proof. Apply (2.12) and Lemma 2.9. □

This is essentially Beĭlinson's construction; we normalize the class by

$$\Xi := \frac{(-1)^n}{N^{n-1}} \widetilde{\Xi}.$$

3. The higher Chow cycles

3A. ***Representing Beĭlinson's elements.*** We first describe (2.11) more explicitly in the relevant cases. As above, write $\partial : Z^n(\square^r, s)^{\#}_{\mathbb{Q}} \to Z^n(\square^r, s-1)^{\#}_{\mathbb{Q}}$ for the higher Chow differential, and

$$\delta : Z^n(\square^r, s)^{\#}_{\mathbb{Q}} \to \bigoplus_{i,\varepsilon} Z^n(\square^{r-1}, s)^{\#}_{\mathbb{Q}}$$

for the cosimplicial differential $\sum_{i=1}^r (-1)^{i-1}((\rho_i^0 \times \mathrm{id}_{\square^s})^* - (\rho_i^\infty \times \mathrm{id}_{\square^s})^*)$. A complex of cocycles for the top motivic cohomology group in (2.12) is given by

$$\mathfrak{Z}^n_{\square}(k) := Z^n_{\mathcal{M}}((\square^{n-1}_{\mathbb{F}}, \partial\square^{n-1}_{\mathbb{F}}), k)_{\mathbb{Q}} := \bigoplus_{a=0}^{n-1} \bigoplus_{(I,\epsilon),\,|I|=a} Z^n(\square^{n-a-1}_{\mathbb{F}}, a+k)^{\#}_{\mathbb{Q}} \tag{3.1}$$

with differential $\mathbb{D} := \partial + (-1)^{n-a-1}\delta$. These are, of course, the simple complex and total differential associated to the natural double complex

$$E_0^{a,b} = \bigoplus_{(I,\epsilon),\,|I|=a} Z^n(\square_{\mathbb{F}}^{n-a-1}, -b)^{\#}_{\mathbb{Q}}.$$

Analogously, one defines

$$\mathfrak{Z}^n_{\square\backslash f}(k) := Z^n_{\mathcal{M}}\big((\square_{\mathbb{F}}^{n-1}\backslash|(f)_0|, \partial\square_{\mathbb{F}}^{n-1}\backslash\partial|(f)_0|), k\big)_{\mathbb{Q}},$$
$$\mathfrak{Z}^{n-1}_f(k) := Z^{n-1}_{\mathcal{M}}((\mathbb{T}^{n-2}, \widetilde{\mathbb{S}}^{n-2}), k)_{\mathbb{Q}},$$

so that $\mathfrak{Z}^{n-1}_f(\bullet) \xrightarrow{\imath_*} \mathfrak{Z}^n_{\square}(\bullet) \to \mathfrak{Z}^n_{\square\backslash f}(\bullet)$ are morphisms of (homological) complexes.

Now define

$$\theta : \mathfrak{Z}^n_{\square}(k) \to Z^n(\mathbb{F}, n+k-1)_{\mathbb{Q}}$$

by simply adding up the cycles (with no signs) on the right-hand side of (3.1). (Use the natural maps $\square^{n-a-1} \times \square^{a+k} \to \square^{n+k-1}$ obtained by concatenating coordinates.) Then we have:

Lemma 3.2. *The map θ is a quasi-isomorphism of complexes.*

Proof. Checking that θ is a morphism of complexes is easy and left to the reader. The $a = n-1$, $(I, \epsilon) = (\{1, \dots, n-1\}, \underline{0})$ term of (3.1) is a copy of $Z^n(\mathbb{F}, n+k-1)$ in $\mathfrak{Z}^n_{\square}(k)$, which leads to a morphism $\psi : Z^n(\mathbb{F}, n+\bullet-1) \to \mathfrak{Z}^n_{\square}(\bullet)$ with $\theta \circ \psi = \mathrm{id}$. Moreover, it is elementary that ψ is a quasi-isomorphism: taking $d_0 = \partial$ gives

$$E_1^{a,b} = \bigoplus_{(I,\epsilon),\,|I|=a} CH^n(\square_{\mathbb{F}}^{n-a-1}, -b)_{\mathbb{Q}} \cong CH^n(\mathbb{F}, -b)^{\oplus 2^a\binom{n-1}{a}},$$

so $E_2^{a,b} = 0$ except for $E_2^{n-1,b} \cong CH^n(\mathbb{F}, -b)$, which is exactly the image of $\psi(\ker\partial)$.[3] □

In particular, we may view θ as yielding the isomorphism in the top row of (2.10).

By the moving lemmas of Bloch [1994] and Levine [1994], we have another quasi-isomorphism

$$\frac{\mathfrak{Z}^n_{\square}(\bullet)}{\imath_*\mathfrak{Z}^{n-1}_f(\bullet)} \xrightarrow{\simeq} \mathfrak{Z}^n_{\square\backslash f}(\bullet),$$

which enables us to replace any $\mathcal{Y}_{\square\backslash f} \in \ker(\mathbb{D}) \subset \mathfrak{Z}^n_{\square\backslash f}(n)$ by a homologous $\mathcal{Y}'_{\square\backslash f}$ arising as the restriction of some $\mathcal{Y}'_{\square} \in \mathfrak{Z}^n_{\square}(n)$ with $\mathbb{D}\mathcal{Y}'_{\square} = \imath_*(\mathcal{Y}''_f)$ and $\mathcal{Y}''_f \in \ker(\mathbb{D}) \in \mathfrak{Z}^{n-1}_f(n-1)$. This gives an "explicit" prescription for computing $\mathrm{Res}_{|(f)_0|}$ in (2.10).

Now we come to our central point: the cycle $\mathcal{Z} = \{f_{n-1}, t_1^N, \dots, t_{n-1}^N\}$ of Section 2E already belongs to $(Z^n(\square_{\mathbb{F}}^{n-1}, n)^{\#}_{\mathbb{Q}} \subseteq)\mathfrak{Z}^n_{\square}(n)$, without "moving" it by a boundary. Its restriction to $\mathfrak{Z}^n_{\square\backslash f}(n)$ is clearly $\mathbb{D}$-closed, and $\mathbb{D}\mathcal{Z} = \imath_*\{t_1^N, \dots, t_{n-1}^N\} =: \imath_*\mathcal{T}$.

---

[3] This is true for any field, but specifically for our $\mathbb{F} = \mathbb{Q}(\omega)$, the only nonzero term is $E_2^{n-1,n}$.

By Proposition 2.13, the class of $\mathcal{T}$ in homology of $\mathfrak{Z}_f^{n-1}(\bullet)$ is trivial, and so there exists $\mathcal{T}' \in \mathfrak{Z}_f^{n-1}(n)$ with $\mathbb{D}\mathcal{T}' = -\mathcal{T}$. Defining

$$\mathcal{W} := \imath_* \mathcal{T}', \qquad \widetilde{\mathcal{Z}} := \mathcal{Z} + \mathcal{W},$$

we now have $\mathbb{D}\widetilde{\mathcal{Z}} = 0$. This allows us to make a rather precise statement about the lift in Proposition 2.13. Denote the projection $(z_1, \ldots, z_{2n-1}) \mapsto (z_1, \ldots, z_{n-i})$ by $\mathsf{p}_i : \square^{2n-1} \twoheadrightarrow \square^{n-i}$.

Theorem 3.3. $\widetilde{\Xi}$ *has a representative in* $Z^n(\mathbb{F}, 2n-1)_{\mathbb{Q}}$ *of the form*

$$\widetilde{\mathscr{Z}} = \mathscr{Z} + \mathscr{W} = \mathscr{Z} + \mathscr{W}_1 + \mathscr{W}_2 + \cdots + \mathscr{W}_{n-1},$$

where $\mathscr{Z} = \theta(\mathcal{Z})$ *(i.e.,* $\mathcal{Z}$ *interpreted as an element of* $Z^n(\mathbb{F}, 2n-1)_{\mathbb{Q}}$*) and* $\mathscr{W}_i$ *is supported on* $\mathsf{p}_i^{-1}|(f_{n-i})_0|$.

Proof. Viewing $(|(f_{n-1})_0|, \partial|(f_{n-1})_0|) \cong (\mathbb{T}^{n-2}, \widetilde{S}^{n-2})$ as a simplicial subscheme $\mathfrak{X}^\bullet$ of $(\square^{n-1}, \partial\square^{n-1}) =: X^\bullet$, the subscheme $\mathfrak{X}^{i-1} \subset X^{i-1}$ comprises $2^{i-1}\binom{n-1}{i-1}$ copies of $|(f_{n-i})_0| \subset \square^{n-1}$. We may decompose

$$\mathcal{W} \in \bigoplus_{i=1}^{n} \bigoplus_{(I,\epsilon),\, |I|=i-1} \imath_* Z^{n-1}(|(f_{n-i})_0|, n+i-1)_{\mathbb{Q}}^{\#} \subset \bigoplus_{i=1}^{n-1} E_0^{i-1,-n-i+1}$$

into its constituent pieces $\mathcal{W}_i \in E_0^{i-1,-n-i+1}$, and define $\mathscr{W}_i := \theta(\mathcal{W}_i)$ and $\mathscr{W} := \theta(\mathcal{W})$. Clearly $\mathrm{supp}(\mathscr{W}_i) \subset \mathsf{p}_i^{-1}|(f_{n-i})_0|$, and $\widetilde{\mathscr{Z}} := \theta(\widetilde{\mathcal{Z}})$ is ∂-closed, giving the desired representation. □

Remark 3.4. In fact, $\sigma(\mathscr{Z}) \in Z^n_{\mathbb{R}}(\mathrm{Spec}(\mathbb{C}), 2n-1)_{\mathbb{Q}}$ for any $\sigma \in \mathrm{Hom}(\mathbb{F}, \mathbb{C})$: the intersections $T_{z_1} \cap \cdots \cap T_{z_k} \cap (\rho_I^\epsilon)^* \sigma(\mathscr{Z})$ are empty excepting $T_{z_1} \cap \cdots \cap T_{z_k} \cap \sigma(\mathscr{Z})$ for $k \leq n-1$ and $T_{z_1} \cap \cdots \cap T_{z_k} \cap (\rho_n^0)^* \sigma(\mathscr{Z})$ for $k \leq n-2$, which are both of the expected real codimension. A trivial modification of the above argument then shows that the $\mathscr{W}_i$ may be chosen so that the $\sigma(\mathscr{W}_i)$ (and hence $\sigma(\widetilde{\mathscr{Z}})$) are in $Z^n_{\mathbb{R}}(\mathrm{Spec}(\mathbb{C}), 2n-1)_{\mathbb{Q}}$ as well. We shall henceforth assume that this has been done.

3B. ***Computing the KLM map.*** We begin by simplifying the formula (2.6) for the regulator map.

Lemma 3.5. *Let* $K \subset \mathbb{C}$ *and suppose* $Z \in \ker(\partial) \subset Z^n_{\mathbb{R}}(\mathrm{Spec}(K), 2n-1)_{\mathbb{Q}}$ *satisfies*

$$T_{z_1} \cap \cdots \cap T_{z_n} \cap Z^{\mathrm{an}}_{\mathbb{C}} = \varnothing. \tag{3.6}$$

Then

$$c_{\mathcal{D},\mathbb{Q}}(Z) = \int_{Z^{\mathrm{an}}_{\mathbb{C}} \cap T_{z_1} \cap \cdots \cap T_{z_{n-1}}} \log(z_n) \frac{dz_{n+1}}{z_{n+1}} \wedge \cdots \wedge \frac{dz_{2n-1}}{z_{2n-1}}$$

in $\mathbb{C}/\mathbb{Q}(n)$.

Proof. We have

$$c_{\mathcal{D},\mathbb{Q}}(Z) = \sum_{k=1}^{n-1} (2\pi \boldsymbol{i})^{k-n} \int_{Z_{\mathbb{C}}^{\mathrm{an}}} R_{2n-1}^{(k)} + \int_{Z_{\mathbb{C}}^{\mathrm{an}}} R_{2n-1}^{(n)} + \sum_{k=1}^{n-1} (2\pi \boldsymbol{i})^{k} \int_{Z_{\mathbb{C}}^{\mathrm{an}}} R_{2n-1}^{(n+k)}.$$

The terms $\int_{Z_{\mathbb{C}}^{\mathrm{an}}} R_{2n-1}^{(k)}$ are zero by type, since $\dim_{\mathbb{C}} Z_{\mathbb{C}} = n-1$, and the $\int_{Z_{\mathbb{C}}^{\mathrm{an}}} R_{2n-1}^{(n+k)}$ are integrals over $Z_{\mathbb{C}}^{\mathrm{an}} \cap T_{z_1} \cap \cdots \cap T_{z_{n+k-1}} = \varnothing$. So only the middle term remains. □

Lemma 3.7. *For any* $\sigma \in \mathrm{Hom}(\mathbb{F}, \mathbb{C})$, $T_{z_1} \cap \cdots \cap T_{z_n} \cap \sigma(\widetilde{\mathscr{Z}}) = \varnothing$.

Proof. From Theorem 3.3, $\sigma(\mathscr{W}_i)$ is supported over $\mathsf{p}_i^{-1}(|(f_{n-i})_0|)$; that is, on $\sigma(\mathscr{W}_i)$ we have $z_1 \cdots z_{n-i} = \bar{\zeta}^b$, and so $T_{z_1} \cap \cdots \cap T_{z_{n-i}} \cap \sigma(\mathscr{W}_i) = \varnothing$, since $\bar{\zeta}^b \notin (-1)^{n-i}\mathbb{R}_+$. On $\sigma(\mathscr{Z})$, $z_n = f_{n-1}(z_1, \ldots, z_{n-1}) = 1 - \zeta^b t_1 \cdots t_{n-1}$ (where $t_i = z_i/(z_i - 1)$), and on T_{z_i}, $t_i \in [0, 1]$. It follows that on $T_{z_1} \cap \cdots \cap T_{z_n} \cap \sigma(\mathscr{Z})$, z_n belongs to $\mathbb{R}_- \cap (1 - \zeta^b[0, 1])$, which is empty. □

We may now compute the regulator on the cycle of Theorem 3.3, independently of the choice of the $\mathscr{W}_i$.

Theorem 3.8. $$c_{\mathcal{D},\mathbb{Q}}(\sigma(\Xi)) = \mathrm{Li}_n(\zeta^b) \in \mathbb{C}/\mathbb{Q}(n).$$

Proof. By Lemmas 3.5 and 3.7, we obtain

$$c_{\mathcal{D},\mathbb{Q}}(\sigma(\widetilde{\mathscr{Z}})) = \int_{\sigma(\mathscr{Z})_{\mathbb{C}}^{\mathrm{an}} \cap T_{z_1} \cap \cdots \cap T_{z_{n-1}}} \log(z_n) \frac{dz_{n+1}}{z_{n+1}} \wedge \cdots \wedge \frac{dz_{2n-1}}{z_{2n-1}} + \sum_{i=1}^{n-1} \int_{\sigma(\mathscr{W}_i)_{\mathbb{C}}^{\mathrm{an}} \cap T_{z_1} \cap \cdots \cap T_{z_{n-1}}} \log(z_n) \frac{dz_{n+1}}{z_{n+1}} \wedge \cdots \wedge \frac{dz_{2n-1}}{z_{2n-1}},$$

in which (by the proof of Lemma 3.7) $\sigma(\mathscr{W}_i)_{\mathbb{C}}^{\mathrm{an}} \cap T_{z_1} \cap \cdots \cap T_{z_{n-1}} = \varnothing$ for all i. The remaining (first) term becomes

$$\begin{aligned} &\int_{\underline{z} \in \mathbb{R}_-^{\times(n-1)}} \log(f_{n-1}(\underline{z})) \frac{dt_1^N}{t_1^N} \wedge \cdots \wedge \frac{dt_{n-1}^N}{t_{n-1}^N} \\ &\qquad = (-N)^{n-1} \int_{\underline{t} \in [0,1]^{\times(n-1)}} \log(1 - \zeta^b t_1 \cdots t_{n-1}) \frac{dt_1}{t_1} \wedge \cdots \wedge \frac{dt_{n-1}}{t_{n-1}} \\ &\qquad = (-N)^{n-1} \int_0^{\zeta^b} \int_0^{u_{n-1}} \cdots \int_0^{u_2} \log(1 - u_1) \frac{du_1}{u_1} \wedge \cdots \wedge \frac{du_{n-1}}{u_{n-1}} \\ &\qquad = (-1)^n N^{n-1} \mathrm{Li}_n(\zeta^b), \end{aligned}$$

where $u_{n-1} = \zeta^b t_{n-1},\ u_{n-2} = \zeta^b t_{n-2} t_{n-1},\ \ldots,\ u_1 = \zeta^b t_1 \cdots t_{n-1}$. □

To write the image of our cycles under the Borel regulator, we refine notation by writing σ_a (for $\sigma : \omega \mapsto e^{2\pi \boldsymbol{i} a/N}$), $f_{n-1,b} = 1 - \omega^b t_1 \cdots t_{n-1}$, Ξ_b, $\widetilde{\mathscr{Z}}_b$, $\mathscr{Z}_b$, etc. So Theorem 3.8 reads $c_{\mathcal{D},\mathbb{Q}}(\sigma_a(\Xi_b)) = \mathrm{Li}_n(e^{2\pi \boldsymbol{i} ab/N})$, and one has the following corollary.

Corollary 3.9. *Let $N \geq 3$ and set*

$$A := \{a \in \mathbb{N} \mid (a, N) = 1 \text{ and } 1 \leq a \leq \lfloor \tfrac{N}{2} \rfloor\};$$

then for any $b \in A$,

$$\tilde{c}^{+}_{\mathcal{D},\mathbb{R}}(\Xi_b) = (\pi_n(\mathrm{Li}_n(e^{2\pi \boldsymbol{i} ab/N})))_{a\in A} \in \mathbb{R}(n-1)^{\oplus \frac{1}{2}\phi(N)},$$

where $\pi_n : \mathbb{C} \to \mathbb{R}(n-1)$ is $\boldsymbol{i}\mathrm{Im}$ for n even, and Re for n odd. If $N = 2$, then $\tilde{c}^{+}_{\mathcal{D},\mathbb{R}} = 0$ for n even and $\tilde{c}^{+}_{\mathcal{D},\mathbb{R}}(\Xi_1) = \zeta(n) \in \mathbb{R}(n-1)$ for n odd.

As an immediate consequence, we get a (rational) basis for the higher Chow cycles on a point over any abelian extension of $\mathbb{Q}$.

Corollary 3.10. *The $\{\Xi_b\}_{b\in A}$ span $CH^n(\mathbb{F}, 2n-1)_{\mathbb{Q}}$. Moreover, for any subfield $\mathbb{E} \subset \mathbb{F}$, with $\Gamma = \mathrm{Gal}(\mathbb{F}/\mathbb{E})$, there exists a subset $B \subset A$ (with $|B| = d_n(\mathbb{E})$) such that the $\{\sum_{\gamma\in\Gamma} {}^{\gamma}\Xi_b\}_{b\in B}$ span $CH^n(\mathbb{E}, 2n-1)_{\mathbb{Q}}$.*

Proof. In view of Lemma 2.8, for the first statement we need only check the linear independence of the vectors $\underline{v}^{(b)}$ in Corollary 3.9. Let χ be one of the $\frac{1}{2}\phi(N)$ Dirichlet characters modulo N with $\chi(-1) = (-1)^{n-1}$; and let $\rho_\alpha : \mathbb{C}^{|A|} \to \mathbb{C}^{|A|}$ be the permutation operator defined by $\mu(\underline{v})_j = v_{\alpha\cdot j}$, where $\alpha \in (\mathbb{Z}/N\mathbb{Z})^*$ is a generator. Then the linear combinations

$$\underline{v}^{\chi} := \sum_{b\in A} \chi(b)\underline{v}^{(b)} = \left(\frac{1}{2}\sum_{b=1}^{N} \chi(b)\pi_n(\mathrm{Li}_n(e^{2\pi \boldsymbol{i} ab/N}))\right)_{a\in A}$$

are independent (over $\mathbb{C}$) provided they are nonzero, since their eigenvalues $\overline{\chi(\alpha)}$ under ρ_α are distinct. By the computation in [Zagier 1991, pp. 420–422], if χ is induced from a primitive character χ_0 modulo $N_0 = N/M$, then (with μ being the Möbius function and $\tau(\,\cdot\,)$ the Gauss sum)

$$v_1^{\chi} = \frac{1}{2M^{n-1}}\left\{\sum_{d\mid M} \mu(d)\chi_0(d)d^{n-1}\right\}\tau(\chi_0)L(\overline{\chi_0}, n),$$

the last two factors of which are nonzero by primitivity of χ_0; the bracketed term is $\prod_{p>1 \text{ prime},\, p\mid M}(1-\chi_0(p)p^{n-1})$, hence also nonzero.

The second statement follows at once, since the composition of $\sum_{\gamma\in\Gamma}$ with $CH^n(\mathbb{E}, 2n-1)_{\mathbb{Q}} \hookrightarrow CH^n(\mathbb{F}, 2n-1)_{\mathbb{Q}}$ is a multiple of the identity. □

4. Explicit representatives

We finally turn to the construction of the cycles described by Theorem 3.3. Here the benefit of using t_i^N (at least, if one is happy to work rationally) comes to the fore: it allows us to obtain uniform formulas for all N, and to use as few terms as possible. In fact, it turns out that *for all n* it is possible to take $\mathscr{W}_3 = \cdots =$

$\mathscr{W}_{n-1} = 0$. (While it is easy to argue abstractly that $\mathscr{W}_{n-1}$ can always be taken to be zero, this stronger statement surprised us.) For brevity, we shall use the notation $(f_1(\underline{t}, u, v), \dots, f_m(\underline{t}, u, v))$ for

$$\{(f_1(\underline{t}, u, v), \dots, f_m(\underline{t}, u, v)) \mid t_i, u, v \in \mathbb{P}^1\} \cap \square^m;$$

all precycles are defined over $\mathbb{F} = \mathbb{Q}(\omega)$, and we write $\xi := \omega^b$.

4A. K_3 *case* ($n = 2$). Let $\mathscr{Z} = (t/(t-1), 1-\xi t, t^N)$, as dictated by Theorem 3.3; then all $\partial_i^{\varepsilon} \mathscr{Z} = 0$. In particular,

$$\partial_1^0 \mathscr{Z} = (1-\xi t, t^N)|_{t/(t-1)=0} = (1, 0) = 0$$

and

$$\partial_2^0 \mathscr{Z} = \left(\frac{\xi^{-1}}{\xi^{-1}-1}, \xi^{-N}\right) = \left(\frac{1}{1-\xi}, 1\right) = 0.$$

So we may take $\mathscr{W} = 0$ and $\widetilde{\mathscr{Z}} = \mathscr{Z}$.

In contrast, if we took $\mathscr{Z} = (t/(t-1), 1-\xi t, t)$, then $\partial_2^0 \mathscr{Z} = (1/(1-\xi), \xi^{-1})$ and a nonzero $\mathscr{W}$-term is required.

4B. K_5 *case* ($n = 3$). Of course $\mathscr{Z} = (t_1/(t_1-1), t_2/(t_2-1), 1-\xi t_1 t_2, t_1^N, t_2^N)$. Taking

$$\mathscr{W}_1 = \frac{1}{2}\left(\frac{t_1}{t_1-1}, \frac{1}{1-\xi t_1}, \frac{(u-t_1^N)(u-t_1^{-N})}{(u-1)^2}, t_1^N u, \frac{u}{t_1^N}\right),$$

we note that $z_2 = 1/(1-\xi t_1)$ implies $t_2 = (1-\xi t_1)^{-1}/((1-\xi t_1)^{-1}-1) = 1/\xi t_1$, which in turn implies $f_2(t_1, t_2) = 0$. Now we have

$$\partial \mathscr{Z} = \partial_3^0 \mathscr{Z} = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, t_1^N, t_2^N\right)\Bigg|_{1-\xi t_1 t_2 = 0} = \left(\frac{t_1}{t_1-1}, \frac{1}{1-\xi t_1}, t_1^N, \frac{1}{t_1^N}\right)$$

and

$$\partial \mathscr{W}_1 = -\partial_3^{\infty} \mathscr{W}_1 = -2 \cdot \frac{1}{2}\left(\frac{t_1}{t_1-1}, \frac{1}{1-\xi t_1}, t_1^N, \frac{1}{t_1^N}\right) = -\partial \mathscr{Z}.$$

Therefore $\widetilde{\mathscr{Z}} = \mathscr{Z} + \mathscr{W}_1$ is closed.

Remark 4.1. See [Petras 2008, §3.1] for a detailed discussion of the properties of these cycles, especially the (integral!) distribution relations of [loc. cit., Proposition 3.1.26].

In particular, we can specialize to $N = 2$ to obtain

$$2\widetilde{\mathscr{Z}} = 2\left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, 1+t_1 t_2, t_1^2, t_2^2\right) + \left(\frac{t_1}{t_1-1}, \frac{1}{1+t_1}, \frac{(u-t_1^2)(u-t_1^{-2})}{(u-1)^2}, t_1^2 u, \frac{u}{t_1^2}\right)$$

in $Z^3_{\mathbb{R}}(\mathbb{Q}, 5)$, spanning $CH^3(\mathbb{Q}, 5)_{\mathbb{Q}} \cong K_5(\mathbb{Q})_{\mathbb{Q}}$, with

$$c_{\mathcal{D},\mathbb{Q}}(2\widetilde{\mathscr{Z}}) = -8\,\mathrm{Li}_3(-1) = 6\zeta(3) \in \mathbb{C}/\mathbb{Q}(3).$$

4C. K_7 case ($n = 4$). Set

$$\begin{aligned}
\mathscr{Z} &= \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{t_3}{t_3-1}, 1-\xi t_1t_2t_3, t_1^N, t_2^N, t_3^N\right),\\
\mathscr{W}_1 &= \frac{1}{2}(\mathscr{W}_1^{(1)}+\mathscr{W}_1^{(2)}),\\
\mathscr{W}_1^{(1)} &= \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1t_2}, \frac{(u-t_1^N)(u-t_2^N)}{(u-1)(u-t_1^Nt_2^N)}, \frac{u}{t_1^N}, \frac{u}{t_2^N}, \frac{1}{u}\right),\\
\mathscr{W}_1^{(2)} &= \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1t_2}, \frac{(u-t_1^N)(u-t_2^N)}{(u-1)(u-t_1^Nt_2^N)}, \frac{t_1^N}{u}, \frac{t_2^N}{u}, \frac{u}{t_1^Nt_2^N}\right),\\
\mathscr{W}_2 &= -\frac{1}{2}\left(\frac{t_1}{t_1-1}, \frac{1}{1-\xi t_1}, \frac{(v-t_1^Nu)(v-ut_1^{-N})}{(v-u^2)(v-1)},\right.\\
&\qquad\qquad \left.\frac{(u-t_1^N)(u-vt_1^{-N})}{(u-v)^2}, \frac{vt_1^N}{u}, \frac{v}{t_1^Nu}, \frac{u}{v}\right).
\end{aligned}$$

Direct computation shows

$$\begin{aligned}
\partial\mathscr{Z} &= -\partial_4^0\mathscr{Z} = -\partial_4^\infty\mathscr{W}_1^{(1)} = -\partial_4^\infty\mathscr{W}_1^{(2)},\\
\partial\mathscr{W}_1 &= -\tfrac{1}{2}\partial_3^\infty\mathscr{W}_1^{(1)} + \tfrac{1}{2}\partial_4^\infty\mathscr{W}_1^{(1)} - \tfrac{1}{2}\partial_3^\infty\mathscr{W}_1^{(2)} + \tfrac{1}{2}\partial_4^\infty\mathscr{W}_1^{(2)},\\
\partial\mathscr{W}_2 &= -\partial_3^\infty\mathscr{W}_2 = \tfrac{1}{2}\partial_3^\infty\mathscr{W}_1^{(1)} + \tfrac{1}{2}\partial_3^\infty\mathscr{W}_1^{(2)},
\end{aligned}$$

which sum to zero.

Alternately, we can take

$$\begin{aligned}
\mathscr{W}_1 &= \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1t_2}, \frac{(u-t_1^N)(u-t_2^N)}{(u-1)(u-t_1^Nt_2^N)}, \frac{t_1^N}{u}, \frac{t_2^N}{u}, \frac{u}{t_1^Nt_2^N}\right),\\
\mathscr{W}_2 &= \left(\frac{t_1}{t_1-1}, \frac{1}{1-\xi t_1}, \frac{(u-vt_1^N)(u-vt_1^{-N})}{(u-v)^2}, \frac{vt_1^N}{u}, \frac{v}{t_1^Nu}, \frac{u}{v}, v-1\right).
\end{aligned}$$

Writing

$$\begin{aligned}
\mathscr{V}_1 &= \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1t_2}, t_1^N, t_2^N, \frac{1}{t_1^Nt_2^N}\right),\\
\mathscr{V}_2 &= \left(\frac{t_1}{t_1-1}, \frac{1}{1-\xi t_1}, \frac{(u-t_1^N)(u-t_1^{-N})}{(u-1)^2}, \frac{t_1^N}{u}, \frac{1}{t_1^Nu}, u\right),
\end{aligned}$$

one has $\partial\mathscr{Z} = -\mathscr{V}_1$, $\partial\mathscr{W}_1 = -\mathscr{V}_2 + \mathscr{V}_1$, $\partial\mathscr{W}_2 = \mathscr{V}_2$; so again $\widetilde{\mathscr{Z}}$ is a closed cycle.

We present the general n construction next, but include the $n = 5$ case as an appendix (as the authors only saw the pattern after working out this case).

4D. *General n construction* ($n \geq 4$). To state the final result, we define

$$\mathscr{Z} := \left(\frac{t_1}{t_1-1}, \dots, \frac{t_{n-1}}{t_{n-1}-1}, 1-\xi t_1\cdots t_{n-1}, t_1^N, \dots, t_{n-1}^N\right),$$

$$\mathscr{W}_1 := \frac{1}{n-3}\widetilde{\mathscr{W}}_1$$

$$:= \frac{(-1)^{n-1}}{n-3}\left(\frac{t_1}{t_1-1}, \dots, \frac{t_{n-2}}{t_{n-2}-1}, \frac{1}{1-\xi t_1\cdots t_{n-2}}, \frac{(u-t_1^N)\cdots(u-t_{n-2}^N)}{(u-t_1^N\cdots t_{n-2}^N)(u-1)^{n-3}}, \frac{t_1^N}{u}, \dots, \frac{t_{n-2}^N}{u}, \frac{u}{t_1^N\cdots t_{n-2}^N}\right),$$

$$\mathscr{W}_2 := \frac{1}{n-3}\sum_{i=1}^{n-1}(-1)^{i-1}\mathscr{W}_2^{(i)},$$

where for $1 \leq i \leq n-2$,

$$\mathscr{W}_2^{(i)} := \left(\frac{t_1}{t_1-1}, \dots, \frac{t_{n-3}}{t_{n-3}-1}, \frac{1}{1-\xi t_1\cdots t_{n-3}}, \frac{(u-t_1^N v)\cdots(u-t_{n-3}^N v)}{(u-t_1^N\cdots t_{n-3}^N v)(u-v)^{n-4}}, \frac{vt_1^N}{u}, \dots, \frac{v}{u}, \dots, \frac{vt_{n-3}^N}{u}, \frac{u}{vt_1^N\cdots t_{n-3}^N}, v-1\right),$$

(with v/u occurring in the $(n+i-1)$-st entry[4]) and

$$\mathscr{W}_2^{(n-1)} := \left(\frac{t_1}{t_1-1}, \dots, \frac{t_{n-3}}{t_{n-3}-1}, \frac{1}{1-\xi t_1\cdots t_{n-3}}, \frac{(u-t_1^N v)\cdots(u-t_{n-3}^N v)}{(u-t_1^{-N}\cdots t_{n-3}^{-N} v)^{-1}(u-v)^{n-2}}, \frac{vt_1^N}{u}, \dots, \frac{vt_{n-3}^N}{u}, \frac{v}{ut_1^N\cdots t_{n-3}^N}, \frac{u}{v}, v-1\right).$$

Theorem 4.2. *$\widetilde{\mathscr{Z}} = \mathscr{Z} + \mathscr{W}_1 + \mathscr{W}_2$ yields a closed cycle, with the properties described in Theorem 3.3. (In particular, this recovers the second K_7 construction and the K_9 construction above, for $n = 4$ and 5.)*

Proof. Writing

$$\mathscr{Y}_0 := \partial_n^0 \mathscr{Z} = \left(\frac{t_1}{t_1-1}, \dots, \frac{t_{n-2}}{t_{n-2}-1}, \frac{1}{1-\xi t_1\cdots t_{n-2}}, t_1^N, \dots, t_{n-2}^N, \frac{1}{t_1^N\cdots t_{n-2}^N}\right),$$

$\mathscr{Y}_i := \partial_{2n-1}^0 \mathscr{W}_2^{(i)}$ ($i = 1, \dots, n-1$), and $\mathscr{X}_{i,j} := \partial_j^\infty \mathscr{W}_2^{(i)}$ ($j = 1, \dots, n-2$), one computes that $\partial\mathscr{Z} = (-1)^{n-1}\mathscr{Y}_0$,

[4]That is, either before ($i = 1$), after ($i = n-2$), or in the middle of the sequence $vt_1^N/u, vt_2^N/u, \dots, vt_{n-3}^N/u$.

$$\partial\widetilde{\mathscr{W}}_1 = (-1)^n \partial_n^\infty \widetilde{\mathscr{W}}_1 + \sum_{i=1}^{n-1} (-1)^i \partial_i^\infty \widetilde{\mathscr{W}}_1 = (-1)^n (n-3)\mathscr{Y}_0 + \sum_{i=1}^{n-1} (-1)^i \mathscr{Y}_i,$$

and $\partial\mathscr{W}_2^{(i)} = \mathscr{Y}_i + \sum_{j=1}^{n-2} (-1)^j \mathscr{X}_{i,j}$. We have, therefore,

$$\partial\widetilde{\mathscr{Z}} = \frac{1}{n-3} \sum_{i=1}^{n-1} \sum_{j=1}^{n-2} (-1)^{i+j-1} \mathscr{X}_{i,j}, \tag{4.3}$$

and for each $i > j$, the reader may verify that $\mathscr{X}_{i,j} = \mathscr{X}_{j,i-1}$, so that the terms on the right-hand side of (4.3) cancel in pairs. □

4E. ***Expected implications for torsion.*** One of the anticipated applications of the explicit AJ maps of [Kerr et al. 2006; Kerr and Lewis 2007] has been the detection of torsion in higher Chow groups. While they provide an explicit map of complexes from $Z^p_{\mathbb{R}}(X, \bullet)$ to the *integral* Deligne cohomology complex, the fact that $Z^p_{\mathbb{R}}(X, \bullet) \subset Z^p(X, \bullet)$ is only a *rational* quasi-isomorphism leaves open the possibility that a given cycle with (nontrivial) torsion KLM-image is bounded by a precycle in the larger complex. So far, therefore, any conclusions we can try to draw about torsion are speculative, as they depend on the (so far) conjectural extension of the KLM map to an *integrally* quasi-isomorphic subcomplex.

Let us describe what the existence of such an extension, together with the cycles just constructed, would yield. Let $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}$ be a function which is zero off $(\mathbb{Z}/N\mathbb{Z})^*$, with $f(-b) = (-1)^n f(b)$, and write

$$\varepsilon_n := \begin{cases} 1, & n = 2, \\ 2, & n = 3, \\ n-3, & n \geq 4. \end{cases}$$

Then (fixing $\sigma(\omega) = \zeta_N = e^{2\pi \boldsymbol{i}/N}$) the cycle

$$Z^n_f(N) := \varepsilon_n \sum_{b=0}^{N-1} f(b)\sigma(\widetilde{\mathscr{Z}}_b) \in Z^n_{\mathbb{R}}(\mathbb{Q}(\zeta_N), 2n-1)$$

is integral. Working up to sign, we compute (in $\mathbb{C}/\mathbb{Z}$) by Theorem 3.8

$$\tau^n_f(N) := \frac{\pm 1}{(2\pi\boldsymbol{i})^n} c_{\mathcal{D}}(Z^n_f(N)) = \frac{\pm\varepsilon_n N^{n-1}}{(2\pi\boldsymbol{i})^n} \sum_{b=0}^{N-1} f(b) \sum_{k\geq 1} \frac{\zeta_N^{kb}}{k^n}$$

$$= \frac{\pm\varepsilon_n N^{n-1}}{2(2\pi\boldsymbol{i})^n} \sum_{b=0}^{N-1} f(b) \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{\zeta_N^{kb}}{k^n} = \frac{\pm\varepsilon_n N^{n-1}}{2\cdot n!} \sum_{b=0}^{N-1} f(b) B_n(\frac{b}{N}),$$

which is evidently a rational number.[5] This (nonconjecturally) establishes that $Z_f^n(N)$ is torsion. Under our working (conjectural!) hypothesis, if $\tau_f^n(N) = \pm A_f^n(N)/C_f^n(N)$ in lowest form, we may additionally conclude that the order of $Z_f^n(N)$ is a multiple of $C_f^n(N)$.

For example, taking $N = 5$, $n = 2$, and $f(1) = f(4) = 1$, $f(2) = f(3) = 0$, we obtain $Z_f^2(5) \in Z_{\mathbb{R}}^2(\mathbb{Q}(\sqrt{5}), 3)$ with $\tau_f^2(5) = \frac{\pm 1}{120}$. This checks out with what is known (cf. Proposition 6.9 and Remark 6.10 of [Petras 2009]), and would make $Z_f^2(5)$ a generator of $CH^2(\mathbb{Q}(\sqrt{5}), 3)$.

For $N = 2$, $f(1) = 1$, and $n = 2m$ (i.e., $CH^{2m}(\mathbb{Q}, 4m-1)$), the above computation simplifies to

$$|\tau_f^{2m}(2)| = \frac{\pm\varepsilon_{2m} 2^{2m-2}}{(2m)!} B_{2m}\left(\frac{1}{2}\right)$$
$$= \frac{\pm(2m-3)(2^{2m-1}-1)}{2(2m)!} B_{2m},$$

which yields $\frac{1}{24}$, $\frac{7}{1440}$, $\frac{31}{20160}$, $\frac{635}{483840}$ for $m = 1, 2, 3, 4$, respectively. It is known that $CH^2(\mathbb{Q}, 3) \cong \mathbb{Z}/24\mathbb{Z}$ [Petras 2009], but the other orders seem unexpectedly large and should warrant further investigation.

Appendix: K_9 case ($n = 5$)

Begin by writing

$$\mathscr{Z} = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{t_3}{t_3-1}, \frac{t_4}{t_4-1}, 1 - \xi t_1 t_2 t_3 t_4, t_1^N, t_2^N, t_3^N, t_4^N \right),$$

$$\mathscr{W}_1 = \frac{1}{2}\left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{t_3}{t_3-1}, \frac{1}{1-\xi t_1 t_2 t_3}, \frac{(u-t_1^l)(u-t_2^l)(u-t_3^l)}{(u-1)^2(u-t_1^l t_2^l t_3^l)}, \frac{t_1^N}{u}, \frac{t_2^N}{u}, \frac{t_3^N}{u}, \frac{u}{t_1^N t_2^N t_3^N} \right),$$

$$\mathscr{W}_2^{(1)} = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1 t_2}, \frac{(u-t_1^N v)(u-t_2^N v)}{(u-t_1^N t_2^N v)(u-v)}, \frac{v}{u}, \frac{t_1^N v}{u}, \frac{t_2^N v}{u}, \frac{u}{v t_1^N t_2^N}, v-1 \right),$$

$$\mathscr{W}_2^{(2)} = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1 t_2}, \frac{(u-t_1^N v)(u-t_2^N v)}{(u-t_1^N t_2^N v)(u-v)}, \frac{v t_1^N}{u}, \frac{v}{u}, \frac{t_2^N v}{u}, \frac{u}{v t_1^N t_2^N}, v-1 \right),$$

$$\mathscr{W}_2^{(3)} = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1 t_2}, \frac{(u-t_1^N v)(u-t_2^N v)}{(u-t_1^N t_2^N v)(u-v)}, \frac{v t_1^N}{u}, \frac{v t_2^N}{u}, \frac{v}{u}, \frac{u}{v t_1^N t_2^N}, v-1 \right),$$

[5] $B_n(x) = \sum_{j=0}^n \binom{n}{j} B_j x^{n-j}$ is the n-th Bernoulli polynomial (and $\{B_j\}$ the Bernoulli numbers).

$$\mathscr{W}_2^{(4)} = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1 t_2}, \right.$$
$$\left. \frac{(u-t_1^N v)(u-t_2^N v)}{(u-vt_1^{-N}t_2^{-N})^{-1}(u-v)^3}, \frac{vt_1^N}{u}, \frac{vt_2^N}{u}, \frac{v}{ut_1^N t_2^N}, \frac{u}{v}, v-1\right),$$

$$\mathscr{W}_2 = \frac{1}{2}(\mathscr{W}_2^{(1)} - \mathscr{W}_2^{(2)} + \mathscr{W}_2^{(3)} - \mathscr{W}_2^{(4)}).$$

To compute the boundaries, introduce

$$\mathscr{U}_1 = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{t_3}{t_3-1}, \frac{1}{1-\xi t_1 t_2 t_3}, t_1^N, t_2^N, t_3^N, \frac{1}{t_1^N t_2^N t_3^N}\right),$$

$$\mathscr{U}_2 = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1 t_2}, \frac{(u-t_1^N)(u-t_2^N)}{(u-t_1^N t_2^N)(u-1)}, \frac{1}{u}, \frac{t_1^N}{u}, \frac{t_2^N}{u}, \frac{u}{t_1^N t_2^N}\right),$$

$$\mathscr{U}_3 = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1 t_2}, \frac{(u-t_1^N)(u-t_2^N)}{(u-t_1^N t_2^N)(u-1)}, \frac{t_1^N}{u}, \frac{1}{u}, \frac{t_3^N}{u}, \frac{u}{t_1^N t_2^N}\right),$$

$$\mathscr{U}_4 = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1 t_2}, \frac{(u-t_1^N)(u-t_2^N)}{(u-t_1^N t_2^N)(u-1)}, \frac{t_1^N}{u}, \frac{t_2^N}{u}, \frac{1}{u}, \frac{u}{t_1^N t_2^N}\right),$$

$$\mathscr{U}_5 = \left(\frac{t_1}{t_1-1}, \frac{t_2}{t_2-1}, \frac{1}{1-\xi t_1 t_2}, \frac{(u-t_1^N)(u-t_2^N)(u-t_1^{-N}t_2^{-N})}{(u-1)^3}, \frac{t_1^N}{u}, \frac{t_2^N}{u}, \frac{1}{ut_1^N t_2^N}, u\right),$$

and

$$\mathscr{V}_1 = \left(\frac{t_1}{t_1-1}, \frac{1}{1-\xi t_1}, \frac{(u-t_1^N v)(u-t_1^{-N} v)}{(u-v)^2}, \frac{v}{u}, \frac{t_1^N v}{u}, \frac{v}{ut_1^N}, \frac{u}{v}, v-1\right),$$

$$\mathscr{V}_2 = \left(\frac{t_1}{t_1-1}, \frac{1}{1-\xi t_1}, \frac{(u-t_1^N v)(u-t_1^{-N} v)}{(u-v)^2}, \frac{vt_1^N}{u}, \frac{v}{u}, \frac{v}{t_1^N u}, \frac{u}{v}, v-1\right),$$

$$\mathscr{V}_3 = \left(\frac{t_1}{t_1-1}, \frac{1}{1-\xi t_1}, \frac{(u-t_1^N v)(u-t_1^{-N} v)}{(u-v)^2}, \frac{vt_1^N}{u}, \frac{v}{ut_1^N}, \frac{v}{u}, \frac{u}{v}, v-1\right).$$

Then $\partial\mathscr{Z} = \mathscr{U}_1$, $\partial\mathscr{W}_1 = -\mathscr{U}_1 + \frac{1}{2}(-\mathscr{U}_2 + \mathscr{U}_3 - \mathscr{U}_4 + \mathscr{U}_5)$, $\partial\mathscr{W}_2^{(1)} = -\mathscr{V}_1 + \mathscr{U}_2$, $\partial\mathscr{W}_2^{(2)} = -\mathscr{V}_2 + \mathscr{U}_3$, $\partial\mathscr{W}_2^{(3)} = -\mathscr{V}_3 + \mathscr{U}_4$, and $\partial\mathscr{W}_2^{(4)} = \mathscr{U}_5 - \mathscr{V}_1 + \mathscr{V}_2 - \mathscr{V}_3$; and so $\widetilde{\mathscr{Z}}$ is closed.

As for $n = 3$, we obtain a generator for $CH^5(\mathbb{Q}, 9)_{\mathbb{Q}} \cong K_9(\mathbb{Q})_{\mathbb{Q}}$ by setting $N = 2$ and $\xi = -1$; the integral cycle $2\widetilde{\mathscr{Z}}$ has $c_{\mathcal{D},\mathbb{Q}}(2\widetilde{\mathscr{Z}}) = 15\zeta(5)$.

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