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CORRIGENDUM TO "THE GROMOV NORM OF THE KAEHLER CLASS AND THE MASLOV INDEX", ASIAN J. MATH., VOL. 7, NO. 2, 269–296, 2003*

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The proof of Theorem 3.1, as presented in the article needs more explanation and does require a little more work.

Before coming to the reformulation of the proof, we need some auxiliary results. Recall first the properties of the *Poisson-Szegö kernel* $P(z,\sigma)$ for the domain $\mathcal{D}$ (see [F-K], XIII, p. 270 for tube-type domains, but the results are valid more generally). It is a smooth positive real-valued function on $\mathcal{D}\times S$, which reproduces the functions which are continuous on $\overline{\mathcal{D}}$ and holomorphic in $\mathcal{D}$. If f is such a function, then

$$\forall z\in\mathcal{D}\qquad f(z)=\int_S P(z,\sigma)f(\sigma)d\sigma \quad , \tag{1}$$

where $d\sigma$ is the probability measure on S invariant under the action of U. In particular,

$$\forall z\in\mathcal{D}\qquad \int_S P(z,\sigma)d\sigma = 1 \ .$$

The following result is a refined version of the maximum principle. The proof of Theorem 3.1 will use a variation of this result (see below).

LEMMA 1. *Let f be a holomorphic on $\mathcal{D}$, and assume that $\Re f$ is bounded on $\mathcal{D}$. Assume further that there is a mesurable function $\phi : S\to\mathbb{R}$ with $|\phi(\sigma)|\le m$ almost everywhere in S and such that*

$$\Re f(r\sigma)\longrightarrow \phi(\sigma)$$

as $r\longrightarrow 1$, for all σ in a set of full measure in S. Then

$$\forall z\in\mathcal{D}\qquad |\Re f(z)|\le m \ .$$

Proof. For any $r, 0<r<1$, the function $z\longrightarrow f(rz)$ is continuous on $\overline{\mathcal{D}}$ and holomorphic in $\mathcal{D}$. Hence, using (1), we get for all $z\in\mathcal{D}$

$$f(rz)=\int_S P(z,\sigma)f(r\sigma)d\sigma \ .$$

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As the Poisson-Szegö kernel is real-valued, we may take the real part of both sides to get, for all $z \in \mathcal{D}$

$$\Re f(rz) = \int_S P(z,\sigma)\ \Re f(r\sigma) d\sigma\ .$$

By assumption, there exists a constant M such that $|\Re f(rz)| \leq M$ for all $z \in \mathcal{D}$ and $r, 0 < r < 1$. Moreover,

$$\lim_{r\to 1} \Re f(r\sigma) = \phi(\sigma) \quad a.e.$$

Hence we may apply Lebesgue's dominated convergence theorem, to conclude that for all $z \in \mathcal{D}$,

$$\Re f(z) = \int_S P(z,\sigma)\phi(\sigma) d\sigma,$$

and hence

$$|\Re f(z)| \leq\ m \int_S |P(z,\sigma)| d\sigma = m\ .$$

The next ingredient is an *a priori* uniform bound (but not the sharp one) for the symplectic area of a geodesic triangle in a tube-type domain $\mathcal{D}$. We now use the tube realization of $\mathcal{D}$. Let $T_\Omega = J + i\Omega \subset \mathbb{J}$. It is holomorphically equivalent under the Cayley transform to the domain $\mathcal{D}$. The (normalized) Bergman kernel is (up to a positive constant)

$$k(z,w) = \big(\det(\frac{z-\overline{w}}{2i})\big)^{-2}$$

(confer [F-K] prop. X.1.3), and a formula for the symplectic area of geodesic triangles in T_Ω also holds.

Lemma 2. *Let $\Delta = (w_1, w_2, w_3)$ be an (oriented) geodesic triangle in T_Ω. Then*

$$\int_\Delta \omega = -\big(\arg k(w_1,w_2) + \arg k(w_2,w_3) + \arg k(w_3,w_1)\big) \tag{2}$$

where $\arg k(z,w)$ *is the unique continous determination of the argument such that* $\arg k(z,z) = 0$ *for any* $z \in T_\Omega$.

It is easily deduced from the formula for geodesic triangles in $\mathcal{D}$ (Theorem 2.1) after performing the Cayley transform.

Lemma 3. *Let Δ be an (oriented) geodesic triangle in T_Ω. Then*

$$\left| \ \int_\Delta \omega\ \right| \leq\ 3\ r\pi\ . \tag{3}$$

Proof. Observe that for $z, w \in T_\Omega$, the element $\dfrac{z-\overline{w}}{2i}$ belongs to the right half-plane $\Omega + iJ$ (denoted by R in section 4). From Lemma 4.9 (the proof of which is independant of any prior result in the paper), we conclude that

$$\left|\arg\det\ (\frac{z-\overline{w}}{2i})\right| \leq r\,\frac{\pi}{2}\ .$$

The formula (2) shows that the symplectic area of a triangle is a sum of three terms of this type, and hence the estimate (3) follows.

Needless to say the same bound yields for the symplectic area of the geodesic triangles in $\mathcal{D}$.

With these results, we may now give the proof of Theorem 3.1, which we reset for convenience.

THEOREM 3.1. *For any geodesic triangle Δ in $\mathcal{D}$*

$$\mid \int_\Delta \omega \mid < r\pi \ .$$

Step 1 (reduction to tube-type case) remains as stated originally, and we introduce as in step 2 the function φ and φ_t for $0 < t < 1$. If we freeze two of the variables (say z_1 and z_2), then, as a function of z_3, $\varphi_t(z_1, z_2, .)$ is the real part of a holomorphic function in $\mathcal{D}$. Hence, we get for all $z_1, z_2, z_3 \in \mathcal{D}^3$

$$\varphi_t(z_1, z_2, z_3) = \int_S P(z_3, \sigma_3)\, \varphi_t(z_1, z_2, \sigma_3)\ d\sigma_3\ .$$

Repeting twice this argument, we get for all $z_1, z_2, z_3 \in \mathcal{D}^3$

$$\varphi_t(z_1, z_2, z_3) = \int_S \int_S \int_S\ P(z_1, \sigma_1)P(z_2, \sigma_2)P(z_3, \sigma_3)\varphi_t(\sigma_1, \sigma_2, \sigma_3)\ d\sigma_1\, d\sigma_2\, d\sigma_3 \tag{4}$$

Lemma 3 shows that φ is a bounded function in $\mathcal{D} \times \mathcal{D} \times \mathcal{D}$. Now the main result of [C-Ø] says that for $(\sigma_1, \sigma_2, \sigma_3) \in S^3_\top$

$$\frac{1}{\pi}\varphi_t(\sigma_1, \sigma_2, \sigma_3) \longrightarrow \iota(\sigma_1, \sigma_2, \sigma_3)$$

where $\iota(\sigma_1, \sigma_2, \sigma_3)$ is the *Maslov index* of the triple $(\sigma_1, \sigma_2, \sigma_3)$. As the transversality condition is given by a polynomial condition, $S^3_\top$ is of full measure in S^3.

Now, in the spirit of Lemma 1, we let t tend to 1 in (4), using, as we may, Lebesgue dominated convergence theorem to get for all $z_1, z_2, z_3 \in \mathcal{D}^3$

$$\varphi(z_1, z_2, z_3) = \pi \int_{S^3_\top} P(z_1, \sigma_1)P(z_2, \sigma_2)P(z_3, \sigma_3)\, \iota(\sigma_1, \sigma_2, \sigma_3)\, d\sigma_1\, d\sigma_2\, d\sigma_3\ ,$$

and then estimate this integral, using the fact that the Maslov index belongs to the interval $[-r, r]$. This gives the estimate of Theorem 3.1, except for the strict inequality, for which the original argument is still valid.