Algebraic & Geometric Topology 19 (2019) 2837–2854

Nonorientable Lagrangian surfaces in rational 4–manifolds

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We show that for any *nonzero* class A in $H_2(X;\mathbb{Z}_2)$ in a rational 4–manifold X, A is represented by a nonorientable embedded Lagrangian surface L (for some symplectic structure) if and only if $\mathcal{P}(A)\equiv\chi(L) \pmod 4$, where $\mathcal{P}(A)$ denotes the mod 4 valued Pontryagin square of A.

53D12, 57Q35

1 Introduction

A smooth immersion or embedding $f\colon L\to X$ from a smooth manifold L into a symplectic manifold (X,ω) is called Lagrangian if $\dim L=\frac{1}{2}\dim X$ and $f^*\omega=0$. The existence of Lagrangian submanifolds is an important problem in symplectic topology and is studied by many people; see Audin [1], Audin, Lalonde and Polterovich [2], Biran [3], Lalonde and Sikorav [7], Polterovich [13] and others. When X is a uniruled 4–manifold, it is well known that the only Lagrangian embedding of closed orientable surfaces to X are spheres and tori, and Lagrangian tori are homologically trivial in X. The existence question for Lagrangian spheres in a uniruled 4–manifold was completely answered by Li and Wu [9].

The focus of this paper will be on the existence of embedded nonorientable Lagrangian surfaces for a given mod 2 homology class. First of all, we have the following simple observation by the Lagrangian immersion h–principle and the Lagrangian surgery construction:

Proposition 1.1 *Let (X,ω) be a symplectic 4–manifold. For any mod 2 homology class A, there exists an embedded nonorientable Lagrangian surface representing this class.*

In light of the general existence, we ask the following question:

Published: 20 October 2019 DOI: 10.2140/agt.2019.19.2837

Question 1.2 *Given a mod 2 class A, what are the possible topological type of nonorientable Lagrangian surfaces in the class A? In particular, what is the maximal Euler number (equivalently, the minimal genus)?*

We will study this question for rational 4–manifolds. Here, a smooth 4−manifold is called rational if it is $S^2 \times S^2$ or $\mathbb{CP}^2 \# k\overline{\mathbb{CP}^2}$ for some $k \in \mathbb{Z}_{\geq 0}$.

Recall that any nonorientable surface is diffeomorphic to $N_k = k\mathbb{RP}^2 = \mathbb{RP}^2 \# \cdots \# \mathbb{RP}^2$ for some $k \in \mathbb{N}$. The Euler number of N_k is $2-k$, and the genus of N_k is defined to be k. Audin's congruence theorem [1] states that, for a mod 2 class A in an arbitrary symplectic 4–manifold (X, ω), if A is represented by an embedded nonorientable Lagrangian surface L, then the Pontryagin square of A is congruent to the Euler number of L modulo 4.

The Pontryagin square referred here is a certain cohomology operation

$$\mathcal{P}\colon H^2(X;\mathbb{Z}_2) \to H^4(X;\mathbb{Z}_4),$$

which is a lift of the mod 2 cup product (see eg Thomas [17]). Furthermore, if A is the reduction of an integral class $\bar{A}$, then $\mathcal{P}(A)$ is the mod 4 reduction of $\bar{A}^2$. In particular, if X is a rational manifold, then $H_2(X;\mathbb{Z})$ has no torsions, in particular 2–torsions, thus every mod 2 class A has an integral lift $\bar{A}$, and $\mathcal{P}(A) \equiv \bar{A}^2 \pmod 4$.

For the zero class, it follows from Givental's construction in $\mathbb{R}^4$ [4] that there are nonorientable Lagrangian surfaces with Euler number divisible by 4, except possibly the Klein bottle. Remarkably, it was shown by Shevchishin [15] and Nemirovskiĭ [12] that the mod 2 class of a Lagrangian Klein bottle in a uniruled manifold must be nonzero. Together with Audin's congruence theorem, the problem for the zero class is thus completely understood for a uniruled manifold.

Now we assume that A is a nonzero class. The first step is to consider the situation where the symplectic form ω is not fixed. We apply the classification of Lagrangian spheres in [9] together with the Lagrangian blowup construction of Rieser [14] and Givental's local construction in [4] to show that Audin's congruence is also sufficient when A is a nonzero class in a rational manifold.

Theorem 1.3 *Let X be a rational 4–manifold and A be a nonzero class in $H_2(X;\mathbb{Z}_2)$. Let $\mathcal{P}(A)$ denote the Pontryagin square of A. Then A is represented by an embedded nonorientable Lagrangian surface of Euler number χ for some symplectic structure if and only if*

$$\mathcal{P}(A) \equiv \chi \pmod 4.$$

Remark 1.4 If we let $|\mathcal{P}(A)|$ be the normalized $\mathcal{P}(A)$ with values in $\{-2,-1,0,1\}$, then the minimal genus of embedded nonorientable Lagrangian surfaces in a class A is

$$2-|\mathcal{P}(A)| \in \{1,2,3,4\}.$$

Notice that for Lagrangian immersions, Proposition 1.1 holds for *any symplectic structure*. The next step is to fix the symplectic form, or, equivalently, classify for which symplectic forms there exist an embedded Lagrangian surface representing A. A distinct feature is that, unlike for a Lagrangian sphere which only exists for a codimension one locus of the symplectic cone due to the null symplectic area condition, this seems to be an open condition. We will deal with this problem in future work.

The structure of this paper is as following. In Section 2, we introduce several general approaches in constructing Lagrangian submanifolds and use them to prove Proposition 1.1. In Section 3, we construct embedded Lagrangian surfaces with desired genus and prove Theorem 1.3.

Acknowledgements Li would like to thank Banghe Li for useful discussions on Proposition 1.1. The research of Dai is partially supported by NSFC 11771232 and 11431001. The research of Ho is partially supported by MOST 105-2115-M-017-005-MY2. The research of Li is partially supported by the NSF.

2 Constructing nonorientable Lagrangian surfaces

2.1 Existence of immersed Lagrangian surfaces

In this subsection we establish the existence of immersed Lagrangian surfaces in an arbitrary symplectic 4–manifold.

Let us recall Gromov and Lee's h–principle [6; 8]. Let L be a closed n–manifold and (W,ω) be a symplectic $2n$–manifold.

A smooth map $f\colon L\to (W,\omega)$ is called an *almost* (or *formal*) *Lagrangian immersion* if the following two conditions are satisfied:

(1) $f^*[\omega]=0$ in $H^2(L;\mathbb{R})$.

(2) There is an injective bundle map $F\colon TL\to f^*TW$ over L such that $F(T_pL)\subset (f^*TW|_p, f^*\omega|_p)$ is a Lagrangian subspace for any $p\in L$.

Theorem 2.1 (Gromov, Lee) *Every almost Lagrangian immersion is homotopic through almost Lagrangian immersions to a Lagrangian immersion.*

To apply this h–principle, here is a useful observation. If we take an ω–compatible almost complex structure on W, then symplectic vector bundles and complex vector bundles are closely related and Lagrangian subbundles correspond to real subbundles. Hence condition (2) can be replaced by an isomorphism as complex vector bundles

$$TL \otimes \mathbb{C} \cong f^* TW. \tag{2-1}$$

In our situation, we have:

Lemma 2.2 *Rank two complex vector bundles over a nonorientable surface are classified by w_2.*

Proof Let Σ be a nonorientable surface and E a rank two complex vector bundle over Σ. In particular, there is an almost complex structure J on E. For dimensional reasons, there is a nowhere-zero section $\tau \in \Gamma(E)$. Then $\tau \oplus J\tau$ forms a trivial complex line bundle and induces a splitting $\mathbb{C} \oplus \xi$. The complex line bundle ξ is classified by $c_1 \in H^2(\Sigma;\mathbb{Z}) = \mathbb{Z}_2$. Notice that $c_1 \equiv w_2$ under the mod 2 reduction homomorphism $H^2(\Sigma;\mathbb{Z}) \to H^2(\Sigma;\mathbb{Z}_2)$, which is an isomorphism in this case. □

Proposition 2.3 *Let (X,ω) be a symplectic 4–manifold and A a mod 2 homology class. Suppose Σ is a nonorientable surface and $f\colon \Sigma \to X$ is a smooth map such that $f_*([\Sigma]) = A$ and $\chi(\Sigma) \equiv \langle w_2(X), A\rangle \pmod 2$. Then there is a Lagrangian immersion from Σ to (X,ω) which is homotopic to f.*

Proof By Theorem 2.1 it suffices to show that f is an almost Lagrangian immersion. Since Σ is a nonorientable surface, $H^2(\Sigma;\mathbb{R}) = 0$ and hence $f^*[\omega]$ is automatically zero.

Let us now analyze the bundle f^*TX as a real vector bundle by calculating the Stiefel–Whitney classes. Firstly, $w_1(f^*TX) = f^*w_1(TX) = 0$ since X is orientable. Since Σ is nonorientable, we have the pairing $\langle w_2(f^*TX), [\Sigma]\rangle$, and

$$\langle w_2(f^*TX), [\Sigma]\rangle = \langle w_2(TX), A\rangle.$$

For the bundle $T\Sigma \otimes \mathbb{C}$, as a real bundle,

$$T\Sigma \otimes \mathbb{C} = T\Sigma \oplus T\Sigma$$

has $w_1 = 0$ and, by Wu's formula (see Theorem 11.14 in [11]),

$$w_2 = 2w_2(T\Sigma) + w_1(T\Sigma)\cdot w_1(T\Sigma) = w_1(T\Sigma)\cdot w_1(T\Sigma) = w_2(T\Sigma).$$

Since $w_2(T\Sigma)$ is the mod 2 reduction of the Euler class of $T\Sigma \to \Sigma$ we have

$$\langle w_2(T\Sigma\otimes\mathbb{C}), [\Sigma]\rangle \equiv \chi(\Sigma) \pmod 2.$$

It follows from Lemma 2.2 that (2-1) holds if and only if

$$\chi(\Sigma) \equiv \langle w_2(TX), A\rangle \pmod 2. \qquad \square$$

Remark 2.4 Since $\langle w_2(W), A\rangle \equiv A\cdot A \pmod 2$, the condition $\chi(\Sigma) \equiv \langle w_2(TX), A\rangle \pmod 2$ is equivalent to $\chi(\Sigma) \equiv A\cdot A \pmod 2$.

2.2 Existence of embedded nonorientable Lagrangian surfaces

In this subsection, we review the Lagrangian surgery and Givental's beautiful local constructions. Then we use these tools to construct new embedded Lagrangian surfaces from old or immersed ones in a given symplectic 4–manifold.

2.2.1 Lagrangian surgery [13] A Lagrangian surgery is a desingularization of a transversal Lagrangian intersection point. Let $(\mathbb{R}^{2n}, \omega_0)$ be the standard symplectic vector space and l_1 and l_2 Lagrangian subspaces of $\mathbb{R}^{2n}$ which intersect transversally at the origin. Let $J\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ be an ω_0–compatible almost complex structure with $l_2 = J(l_1)$. Let $W = \{\xi \in l_1 \mid \omega_0(\xi, J\xi) = 1\} \cong S^{n-1}$. Define a map $F\colon W\times\mathbb{R} \to \mathbb{R}^{2n}$, $(\xi, t) \mapsto e^{-t}\xi + e^t J\xi$. Then F is a Lagrangian embedding, and F is asymptotic to l_1 as $t\to-\infty$, and asymptotic to l_2 as $t\to+\infty$. One can smooth F outside a large ball to obtain a Lagrangian embedding $F'\colon W\times\mathbb{R}\to\mathbb{R}^{2n}$ such that $F'(W\times(-\infty,-c]) \subset l_1$ and $F'(W\times[c,+\infty)) \subset l_2$ for some $c>0$. The image of F', $\Gamma(l_1,l_2)$, is called a Lagrangian handle joining l_1 and l_2.

In general, if x is a transversal intersection point of two Lagrangian submanifolds L_1 and L_2 or a transversal self-intersection point of a Lagrangian submanifold L, by the Weinstein neighborhood theorem, we can choose a neighborhood U of x such that $(L_1\cup L_2)\cap U$ or $L\cap U$ is the union of two Lagrangian disks. The Lagrangian surgery is cutting out U and gluing back a portion of the Lagrangian handle carefully to construct a new Lagrangian submanifold.

Notice that the Lagrangian isotopy class of a Lagrangian handle is independent of the choice of almost complex structures and smoothing. It turns out that the topological type of the resulting manifold is also completely determined. It will be much easier to describe it if we introduce the signs or orientations for the manifolds and the intersections. Assume the orientations of l_1 and l_2 are given. The Lagrangian handle $\Gamma(l_1, l_2)$ is called positive or has sign 1 if the orientations of l_1 and l_2 coincide in the image of F'. Otherwise, it is called negative or has sign -1. There is a natural orientation for the positive Lagrangian handle induced by the orientation of l_1 and l_2.

We know that:

Proposition 2.5 [13]

(1) *The sign of the Lagrangian handle* $\Gamma(l_1, l_2)$ *is* $(-1)^{\frac{n(n-1)}{2}+1}\,\mathrm{ind}(l_1, l_2)$.

(2) *Let* $P^n = S^{n-1} \times S^1$ *and* $Q^n = S^{n-1} \times [-1,1]/{\sim}$, *where* $(x,1) \sim (\tau(x), -1)$, *and* $\tau\colon S^{n-1} \to S^{n-1}$ *is an orientation-reversing involution. Suppose* L *is a connected immersed Lagrangian manifold with a self-intersection point* x *and* N *is the resulting manifold of a Lagrangian surgery at* x. *If* L *is orientable, then* $N \cong L \,\#\, P^n$ *when the surgery is positive and* $N \cong L \,\#\, Q^n$ *when the surgery is negative. If* L *is nonorientable, then* $N \cong L \,\#\, P^n \cong L \,\#\, Q^n$.

(3) (a) L *and* N *are homologous* (*mod* 2).

(b) *If* L *is oriented and the surgery is positive*, *then* L *and* N *are homologous.*

When $n = 2$, the sign of a Lagrangian handle coincides with $\mathrm{ind}(l_1, l_2)$. N is diffeomorphic to $L \,\#\, T^2$ if L is orientable and the index of self-intersection point is 1. Otherwise, N is diffeomorphic to $L \,\#\, \mathrm{KB}$, where KB denotes a Klein bottle.

2.2.2 Wavefront construction Let $\boldsymbol{x} = (x_1, \dots, x_n)$ be a coordinate system of $\mathbb{R}^n$ and $\boldsymbol{y} = (y_1, \dots, y_n)$ the coordinates for fibers of $T^*\mathbb{R}^n$ corresponding to the basis $dx_1, \dots, dx_n$. Then $\lambda_{\mathrm{can}} = \sum_i y_i dx_i$ is the Liouville form on $T^*\mathbb{R}^n$, and $\omega_0 = -d\lambda_{\mathrm{can}}$, the standard symplectic structure.

A smooth section $f\colon L(\subset \mathbb{R}^n) \to (T^*\mathbb{R}^n, \omega_0)$ is Lagrangian if $f^*\omega_0 = 0$. So $f^*d\lambda_{\mathrm{can}} = df^*\lambda_{\mathrm{can}} = 0$ and $f^*\lambda_{\mathrm{can}}$ is closed. We call f exact if $f^*\lambda_{\mathrm{can}} = dh$ for some function h on L. We call h a generating function of f. The graph of h in $L \times \mathbb{R}$ is called a wavefront of f. Note that h is unique up to an addition by a constant.

Conversely, let h be a smooth function on $\mathbb{R}^n$. Consider the gradient function $\nabla h\colon \mathbb{R}^n \to \mathbb{R}^n$. The graph of ∇h is a Lagrangian section of $\mathbb{R}^{2n} \cong T^*\mathbb{R}^n$ with a generating function h.

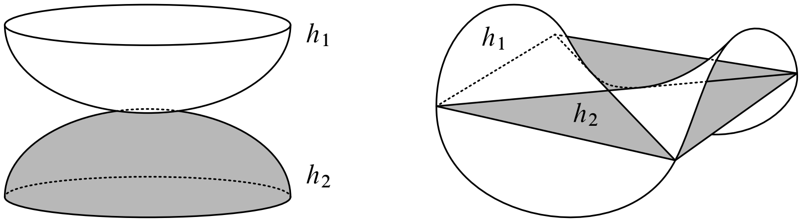

Figure 1

In general, if L is a Lagrangian submanifold of a $2n$–dimensional symplectic manifold (X,ω) and $x \in L$, we can choose a neighborhood U of x and a local chart $\phi\colon U \to T^*\mathbb{R}^n$ carefully so that $\phi(L \cap U)$ is a Lagrangian section of $T^*\mathbb{R}^n$ with a generating function h. If f_1 and f_2 are two local sections with generating functions h_1 and h_2, respectively, then f_1 and f_2 coincide at $x \in \mathbb{R}^n$ if and only if $\nabla h_1(x) = \nabla h_2(x)$. Geometrically, it is equivalent to the condition that the wavefronts of f_1 and f_2 are parallel at x. This approach is extremely useful for $n = 2$ since we can explicitly draw the wavefronts, ie the graphs of h_1 and h_2 in $\mathbb{R}^3$.

Definition 2.6 Let L_1 and L_2 be two Lagrangian sections of $T^*\mathbb{R}^n$ which are given by $f_k\colon \mathbb{R}^n \to T^*\mathbb{R}^n$, ie $L_k = f_k(\mathbb{R}^n)$, and h_1 and h_2 be their generating functions. Assume further that $\nabla h_1(x) = \nabla h_2(x)$ and $p = (x, h_1(x))$. We define $\operatorname{sgn}(p) = 1$ if $\det(\operatorname{Hess}(h_2 - h_1)) > 0$ and $\operatorname{sgn}(p) = -1$ if $\det(\operatorname{Hess}(h_2 - h_1)) < 0$ at p.

When $n = 2$, $\operatorname{sgn}(p)$ can be visualized from the wavefronts easily. We can assume further that $h_1(x) = h_2(x)$ by adding constants. If $\det(\operatorname{Hess}(h_2 - h_1)) > 0$, then in a small neighborhood of x, one wavefront is completely in the top of the other one except x (see Figure 1, left). When $\det(\operatorname{Hess}(h_2 - h_1)) < 0$, these two wavefronts always intersect at some point near but not equal to x (see Figure 1, right).

Definition 2.7 Let L_1 and L_2 be two oriented Lagrangian sections of $T^*\mathbb{R}^n$ which are given by $f_k\colon \mathbb{R}^n \to T^*\mathbb{R}^n$ and h_1 and h_2 be their generating functions. Let $\mathbb{R}^n$ be oriented naturally by the ordered basis $\partial/\partial x_1, \dots, \partial/\partial x_n$. We define:

(1) $s(L_k) = 1$ if f_k is orientation-preserving; $s(L_k) = -1$ if f_k is orientation-reversing.

(2) L_1 and L_2 have the same orientation or $s(L_1, L_2) = 1$ if $f_2 \circ f_1^{-1}\colon L_1 \to L_2$ is orientation-preserving; L_1 and L_2 have opposite orientation or $s(L_1, L_2) = -1$ if $f_2 \circ f_1^{-1}$ is orientation-reversing.

It is clear that $s(L_1, L_2) = s(L_1)s(L_2)$.

Lemma 2.8 *Let L_1 and L_2 be two oriented Lagrangian sections of $T^*\mathbb{R}^n$ intersecting transversally at $p=(x,y)$. Let h_1 and h_2 be their generating functions. Let $\operatorname{ind}_p(L_1,L_2)$ denote the intersection index of L_1 and L_2 at p. Then*

$$\operatorname{ind}_p(L_1,L_2)=(-1)^{\frac{n(n-1)}{2}}s(L_1,L_2)\operatorname{sgn}(p).$$

Proof The tangent plane $T_{(x,f_k(x))}L_k$ has a basis

$$e_i^k=\frac{\partial}{\partial x_i}+\sum_j\frac{\partial^2 h_k}{\partial x_i\partial x_j}(x)\frac{\partial}{\partial y_j}$$

for $1\le i\le n$.

Since L_1 and L_2 intersect transversally at $p=(x,y)$, then $y=f_1(x)=\nabla h_1(x)=\nabla h_2(x)=f_2(x)$ and $e_1^1,\dots,e_n^1,e_1^2,\dots,e_n^2$ form a basis of $T_p(T^*\mathbb{R}^n)$.

The coefficient matrix of the ordered basis $e_1^1,\dots,e_n^1,e_1^2,\dots,e_n^2$ with respect to the ordered basis $\partial/\partial x_1,\dots,\partial/\partial x_n,\partial/\partial y_1,\dots,\partial/\partial y_n$ is

$$C=\begin{pmatrix}\mathrm{I} & \mathrm{Hess}(h_1)\\ \mathrm{I} & \mathrm{Hess}(h_2)\end{pmatrix}.$$

We can show that $\det(C)=\det(\mathrm{Hess}(h_2-h_1))$. The sign between the basis $\partial/\partial x_1,\dots,\partial/\partial x_n,\partial/\partial y_1,\dots,\partial/\partial y_n$ and the standard basis $\partial/\partial x_1,\partial/\partial y_1,\dots,\partial/\partial x_n,\partial/\partial y_n$ is $(-1)^{\frac{n(n-1)}{2}}$. Hence

$$\begin{aligned}\operatorname{ind}_p(L_1,L_2)&=s(L_1)s(L_2)\operatorname{sgn}(\det(C))\cdot(-1)^{\frac{n(n-1)}{2}}\\&=(-1)^{\frac{n(n-1)}{2}}s(L_1,L_2)\operatorname{sgn}(p).\end{aligned}$$ □

Examples 2.9 (1) Let L be a constant section of $T^*\mathbb{R}^2$ given by $y_1=a$ and $y_2=b$; L is Lagrangian and a wavefront of L is a plane $h(x_1,x_2)=ax_1+bx_2+c$ in $\mathbb{R}^3$. The wavefronts of two different constant sections are two nonparallel planes.

(2) **Whitney sphere** Let $S^n=\{(\boldsymbol{x},y)\in\mathbb{R}^n\times\mathbb{R}\mid \boldsymbol{x}=(x_1,\dots,x_n),\ \|\boldsymbol{x}\|^2+y^2=1\}$ and $w\colon S^n\to\mathbb{R}^{2n}$, $w(x_1,\dots,x_n,y)=(x_1,\dots,x_n,x_1y,\dots,x_ny)$.

Then w can be viewed as the union of two sections of $T^*\mathbb{R}^n\cong\mathbb{R}^{2n}$ over the unit disk, $w_\pm(\boldsymbol{x})=(\boldsymbol{x},t_\pm(\boldsymbol{x})\boldsymbol{x})$, where $t_\pm(\boldsymbol{x})=\pm\sqrt{1-\|\boldsymbol{x}\|^2}$. Let $L_\pm$ denote the graph of $w_\pm$.

The $w_\pm$ have generating functions $h_\pm=\mp\frac{1}{3}(1-\|\boldsymbol{x}\|^2)^{\frac{3}{2}}$, and $\nabla h_+=\nabla h_-$ when $\|\boldsymbol{x}\|=1$ or $\boldsymbol{x}$ is the origin. $L_\pm$ can be sewed smoothly along $\|\boldsymbol{x}\|=1$. So w is an immersed Lagrangian n–sphere with a double point at the origin.

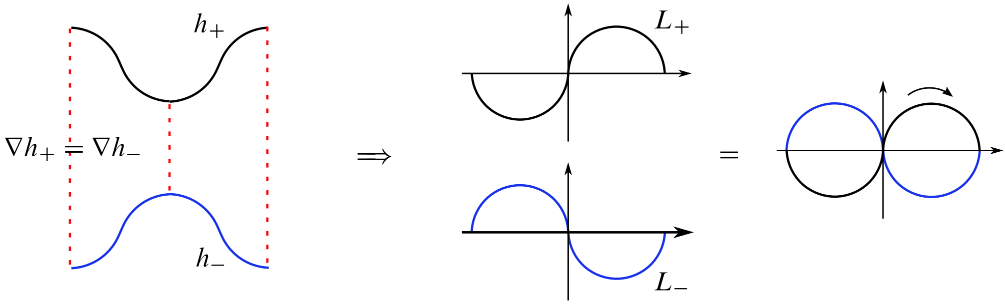

Figure 2

When $n = 1$, h_+ and h_- have same slopes at $x = 0, 1, -1$. Moreover, if an orientation of $w(S^1)$ is given, it induces orientations for $L_\pm$. For instance, if it is oriented as shown in Figure 2, then L_+ is oriented in the x_1 direction and L_- is oriented in the $-x_1$ direction. So $s(L_+, L_-) = -1$.

When $n = 2$, h_+ and h_- have same slopes on the circle $\|x\| = 1$ and at the origin $(0, 0)$. Assume an orientation of $w(S^2)$ is given by a frame near the common boundary of $L_\pm$ as shown in Figure 3. When the frame is moved away from the boundary, we can keep the first vector in the x_2 direction, then the second one is in the $-x_1$ direction for L_+ but in the x_1 direction for L_-. So the induced orientations for L_+ and L_- are different (compare with the natural orientation of $\mathbb{R}^2$) and $s(L_+, L_-) = -1$.

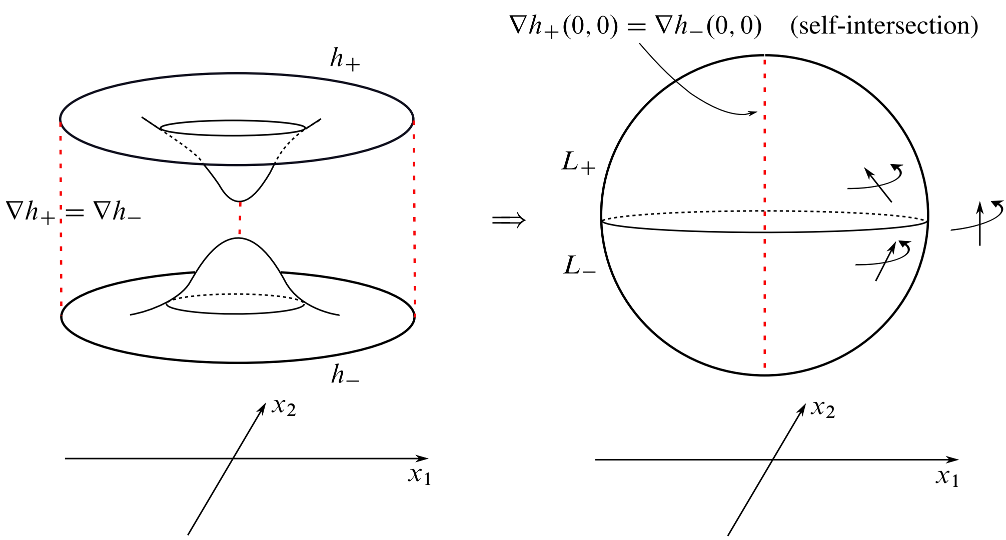

Figure 3

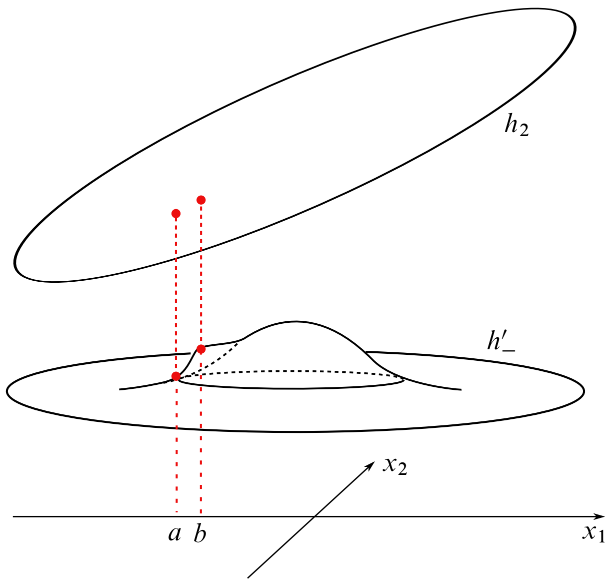

Figure 4

Moreover, at the intersection point $p=(0,0,0,0)$, we know that $\mathrm{Hess}(h_+ - h_-) = 2\mathrm{I}$, $\det(\mathrm{Hess}(h_+ - h_-)) = 4 > 0$ and $\mathrm{sgn}(p) = 1$. This also can be observed from Figure 1, left.

(3) Let $L_1 = L_+ \cup L_-$ be the 2–dimensional Whitney sphere in (2) and L_2 be a nonzero constant section close to but not intersecting with L_1. Without loss of generality, we may assume L_2 has a generating function $h_2(x_1, x_2) = x_1$. Let h'_- be a generating function deformed from h_- as shown in Figure 4 and L'_- be its Lagrangian section. $L'_1 = L_+ \cup L'_-$ is another Lagrangian sphere in $T^*\mathbb{R}^2$ with one double point $p = (0,0,0,0)$.

Along the line $x_2 = 0$, $\nabla h'_- = \nabla h_2 = (1,0)$ at two points $(a,0)$ and $(b,0)$ with $a < b < 0$. Moreover, $\partial h'_-/\partial x_2 < 0$ when $x_2 > 0$ and $\partial h'_-/\partial x_2 > 0$ when $x_2 < 0$ in the deformed area. So $\nabla h'_-$ is never parallel to ∇h_2 when $x_2 \neq 0$ and L'_- and L_2 intersect transversally at two points $p_1 = (a,0,1,0)$ and $p_2 = (b,0,1,0)$. Moreover, $\det(\mathrm{Hess}(h'_- - h_2)) < 0$ at $(a,0)$ and $\det(\mathrm{Hess}(h'_- - h_2)) > 0$ at $(b,0)$. So $\mathrm{sgn}(p_1) = -1$ and $\mathrm{sgn}(p_2) = 1$.

2.2.3 Surgeries on wavefronts When we consider Lagrangian sections on the cotangent bundle $T^*\mathbb{R}^n$, the sign of Lagrangian handles can be easily read out from their generating functions.

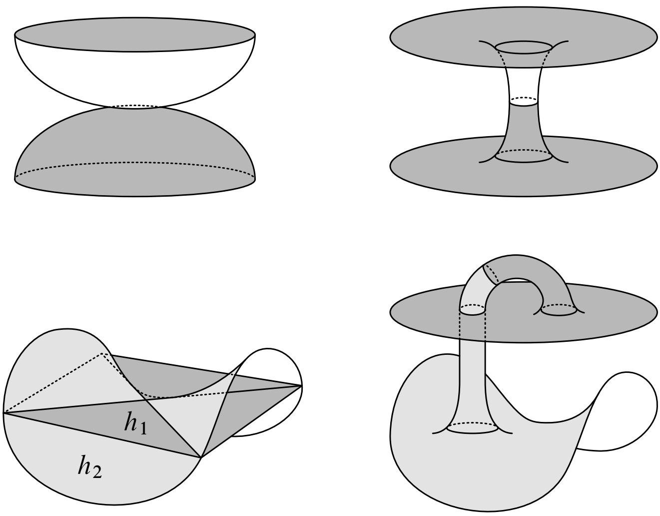

Figure 5

Lemma 2.10 *Suppose two Lagrangian sections L_1 and L_2 are oriented and intersect transversally at $p = (x, y)$. Then the sign of the Lagrangian handle at p is*

$$-s(L_1, L_2)\,\mathrm{sgn}(p).$$

Proof By Proposition 2.5(1) and Lemma 2.8, the sign of the Lagrangian handle at p is

$$(-1)^{\frac{n(n-1)}{2}+1} \cdot (-1)^{\frac{n(n-1)}{2}} s(L_1, L_2)\,\mathrm{sgn}(p) = -s(L_1, L_2)\,\mathrm{sgn}(p). \qquad \square$$

As we mentioned in the paragraph before Proposition 2.5, the topological feature of the resulting manifold after Lagrangian surgery is independent of the choice of the Lagrangian sections. When $n = 2$, the effect of the surgery is completely determined by $\mathrm{sgn}(p)$ and can be visualized from their wavefronts easily and can be visualized as shown in Figure 5. The graphs on the left are the wavefronts and the ones on the right are the Lagrangian surfaces after surgery.

Examples 2.11 (1) The 2–dimensional Whitney sphere in Example 2.9(2) is an immersed sphere in $T^*\mathbb{R}^n$ with a self-intersection point at p. If it is oriented, we have shown that $s(L_+, L_-) = -1$ and $\mathrm{sgn}(p) = 1$. By Lemma 2.10, the Lagrangian handle at p is positive and the resulting manifold is an embedded Lagrangian torus in $T^*\mathbb{R}^2$.

(2) Consider the Lagrangian surfaces L_1' and L_2 in Example 2.9(3). There are three intersection or self-intersection points, p, p_1 and p_2. Assume L_1' and

L_2 are oriented so that $s(L'_-, L_2) = 1$. By Lemma 2.10, the Lagrangian handle is positive at p and p_1 and negative at p_2. If we apply Lagrangian surgery at p_1, the resulting manifold L_3 is diffeomorphic to $L'_1 \# L_2$ with natural orientation and two double points p and p_2. Applying the surgery to L_3 at p, by Proposition 2.5, the resulting manifold L_4 also has natural orientation and is diffeomorphic to $L'_1 \# L_2 \# T^2$ with one double point p_2. Finally, we can apply the surgery to L_4 at p_2 and get an embedded Lagrangian surface which is diffeomorphic to $\mathrm{KB} \# L_2 \# T^2$ or $L_2 \# 4\mathbb{RP}^2$.

2.2.4 Existence of embedded nonorientable Lagrangian surfaces Now we can prove the existence of embedded nonorientable Lagrangian surfaces.

Proof of Proposition 1.1 By a classical result of Thom [16], any mod 2 homology class is represented by a smooth map $f\colon \Sigma \to M$, for some surface Σ. If necessary, by taking connected sum with an appropriate number of locally null-homologous $\mathbb{RP}^2$, we can assume that Σ is nonorientable and $\chi(\Sigma) \equiv \langle w_2(X), A\rangle \pmod 2$. Then there is a Lagrangian immersion $\Sigma \to (X, \omega)$ by Proposition 2.3. Perform Lagrangian surgeries on the double points to obtain an embedded nonorientable Lagrangian surface in the class A. □

We can further construct more Lagrangian surfaces with arbitrary small Euler numbers.

Proposition 2.12 *If A is represented by a Lagrangian surface L, then it is represented by a Lagrangian surface of type $L \# (4l)\mathbb{RP}^2$ for any positive integer l.*

Proof Let $f\colon L \to X$ be a Lagrangian submanifold. By a local version of Weinstein's Lagrangian neighborhood theorem, for any $x \in L$, there exists a neighborhood U and a symplectic embedding $\phi\colon (U, \omega) \to (T^*\mathbb{R}^n, \omega_0)$ such that $\phi(L \cap U)$ is the intersection of $\phi(U)$ with the constant section L_2 in Example 2.9(3). As shown in Example 2.11(3), we can take a null-homologous immersed Lagrangian sphere with a self-intersection point, and intersecting L at two other points. Apply Lagrangian surgeries to construct a Lagrangian submanifold L', which is diffeomorphic to $L\#4\mathbb{RP}^2$. By Proposition 2.5(3), L' is in the same mod 2 homology class of L.

We can repeat this process and get $L \# (4l)\mathbb{RP}^2$ for any $l \in \mathbb{N}$. □

Remark 2.13 It is more natural to consider the Clifford torus to increase the genus. Here we choose a deformation of the Whitney sphere because it is easier to give a clear explanation for the whole process via wavefront in this case. Actually, the construction in Example 2.11(2) is generic and should work in more general situations.

2.3 Lagrangian blowup

In this section, we describe the procedure of Lagrangian blowup in [14] and give an important example. Let $\widetilde{B}=\{(z,l)\in\mathbb{C}^n\times\mathbb{CP}^{n-1}\mid z\in l\}$, $\widetilde{B}_\varepsilon=\{(z,l)\in\widetilde{B}\mid |z|\le\varepsilon\}$ and $B_r=\{z\in\mathbb{C}^n\mid |z|\le r\}$. There are two natural projections, $p_1\colon \widetilde{B}\to\mathbb{C}^n$ and $p_2\colon \widetilde{B}\to\mathbb{CP}^{n-1}$. The projection p_1 implies that $\widetilde{B}$ is the blowup of $\mathbb{C}^n$ at the origin, and p_2 implies that $\widetilde{B}$ is also the universal line bundle over $\mathbb{CP}^{n-1}$. For any $\lambda>0$, let $\omega_\lambda=p_1^*\omega_0+\lambda^2 p_2^*\omega_{\mathrm{FS}}$ be the induced symplectic form on $\widetilde{B}$.

There is a symplectomorphism $\alpha\colon(\widetilde{B}_\varepsilon-\widetilde{B}_0,\omega_\lambda)\cong(B_{\sqrt{\lambda^2+\varepsilon^2}}-B_\lambda,\omega_0)$ (see Lemma 7.11 of [10]). In particular, this preserves the real parts of $\widetilde{B}$ and $\mathbb{C}^n$.

Suppose X is a symplectic manifold and $x\in X$. By Darboux's theorem, there is a symplectic embedding $\psi\colon(B_\delta,\omega_0)\to(X,\omega)$ for $\delta\ge\sqrt{\lambda^2+\varepsilon^2}$ when λ and ε are sufficiently small, and $\psi(0)=x$. Let $\widetilde{X}=(X-\psi(B_\lambda))\amalg\widetilde{B}_\varepsilon/\sim$, with $\psi(\alpha(y))\sim y$ for any $y\in\widetilde{B}_\varepsilon-\widetilde{B}_0$. The closed 2–form $\widetilde{\omega}$ induced by ω and ω_0 is a symplectic structure on $\widetilde{X}$.

If X is a symplectic 4–manifold, L is a Lagrangian surface in X and $x\in L$, then there exists a neighborhood U of x and a symplectic embedding $\phi\colon(U,\omega)\to(\mathbb{C}^2,\omega_0)$ such that $\phi(L\cap U)$ is the real part. By Theorem 1.21 in [14], after blowing up at x, L is lifted to a Lagrangian surface $\widetilde{L}\subset\widetilde{X}$ with $\widetilde{L}\cong L\,\#\,\mathbb{RP}^2$. Moreover, the mod 2 class $[\widetilde{L}]_2$ represented by $\widetilde{L}$ satisfies

$$[\widetilde{L}]_2=[L]_2+E,$$

where $[L]_2$ is the mod 2 class represented by L and E is the mod 2 reduction of the exceptional divisor, and we use the natural decomposition $H_2(\widetilde{X},\mathbb{Z}_2)=H_2(X,\mathbb{Z}_2)\oplus\mathbb{Z}_2E$.

An important example is the holomorphic blowup of $\mathbb{CP}^2$,

$$\widetilde{X}=\{([W_1:W_2],[Z_0:Z_1:Z_2])\in\mathbb{CP}^1\times\mathbb{CP}^2\mid Z_1W_2=Z_2W_1\}.$$

There are natural projections $p_i\colon\widetilde{X}\to\mathbb{CP}^i$, $i=1,2$. The projection $p_1\colon\widetilde{X}\to\mathbb{CP}^1$ is a nontrivial $\mathbb{CP}^1$–bundle over $\mathbb{CP}^1$. The preimage $p_2^{-1}([1{:}0{:}0])=\mathbb{CP}^1\times\{[1{:}0{:}0]\}$ is the exceptional curve, and $p_2\colon\widetilde{X}-p_2^{-1}([1{:}0{:}0])\to\mathbb{CP}^2-\{[1{:}0{:}0]\}$ is a diffeomorphism.

There is a family of Kähler forms on $\mathbb{CP}^1\times\mathbb{CP}^2$, given by $\omega=p_1^*\tau_1+\lambda^2p_2^*\tau_2$, where $\lambda>0$, and τ_i are the Fubini–Study forms on $\mathbb{CP}^i$. Then $\widetilde{X}$ inherits a family of Kähler forms.

Let $\overline{H} \in H_2(\widetilde{X};\mathbb{Z})$ be the hyperplane class of $\mathbb{CP}^2$, and $\overline{E}$ the exceptional class. The real locus of $\widetilde{X}$ is diffeomorphic to $\mathbb{RP}^2\#\mathbb{RP}^2 \cong \mathrm{KB}$ representing $H+E \in H_2(\widetilde{X},\mathbb{Z}_2)$. It is an embedded Lagrangian surface with respect to that family of Kähler forms, and also has the structure of a nontrivial $\mathbb{RP}^1$–bundle over $\mathbb{RP}^1$. Note that a fiber of nontrivial S^2–bundle over S^2 represents the integral class $\overline{H}-\overline{E}$. Hence the Lagrangian KB and a fiber of the nontrivial S^2–bundle represent the same mod 2 class.

3 Minimal genus Lagrangian surfaces in rational 4–manifolds

We will prove the following result:

Proposition 3.1 *Let X be a rational $4-$manifold and $A \in H_2(X;\mathbb{Z}_2)$ a nonzero class. A is represented by a Lagrangian $l\mathbb{RP}^2$ with $0 \le l \le 3$ (for some symplectic structure) if and only if $\mathcal{P}(A) \equiv 2-l \pmod 4$. Here, we use the convention $S^2 = 0\mathbb{RP}^2$.*

Proof The conditions are necessary by [1]. So we just need to show that they are sufficient.

First we consider the case $X = \mathbb{CP}^2\#k\overline{\mathbb{CP}^2}$ with $k \in \mathbb{N}$. Assume $\overline{H}$ is the generator of $H_2(\mathbb{CP}^2,\mathbb{Z})$, the $\overline{E}_i$ are the exceptional classes of $H_2(X,\mathbb{Z})$ and $H, E_i \in H_2(X,\mathbb{Z}_2)$ are the reductions of $\overline{H}$ and $\overline{E}_i$, respectively. Any class in $H_2(X,\mathbb{Z}_2)$ is of the form

$$A = aH + b_1E_1 + \cdots + b_kE_k,$$

where a and the b_i are either 0 or 1. Note that $\mathcal{P}(E_1+\cdots+E_k) = -k \pmod 4$ and $\mathcal{P}(H+E_1+\cdots+E_k) = 1-k \pmod 4$. All our conclusions and results are valid under permutation of exceptional divisors. For convenience, we will choose one class in discussion, which also works for any other classes of the same type. For example, $H+E_1$ can also represent $H+E_2$, $H+E_3$, etc.

Let $K_X = -3\overline{H} + \sum_i \overline{E}_i$ be the standard canonical class. For any $t \in \mathbb{N}$, let

$$\overline{Z}_t = t\overline{H} - \overline{E}_1 - \cdots - \overline{E}_{2t+1} - (t-1)\overline{E}_{(2t+2)}.$$

We explain that $\overline{Z}_t$ is represented by a smooth sphere. Let D be a configuration of t degree-one algebraic curves $D_1,\dots,D_t$ in $\mathbb{CP}^2$ such that $D_1,\dots,D_{t-1}$ pass through a common point x, while D_t misses x. Blowing up at x and the other $2t+1$ points in D_t which are away from the intersection points $D_1\cap D_t,\dots,D_{t-1}\cap D_t$, the lift $\widetilde{D}$ of D in $\mathbb{CP}^2 \# (2t+2)\overline{\mathbb{CP}^2}$ is in the class $\bar{Z}_t$ if we arrange the indices of exceptional divisors appropriately. A sphere is given by resolving the intersecting points $D_1 \cap D_t, \dots, D_{t-1} \cap D_t$. Hence $\bar{Z}_t$ is represented by a smooth sphere.

By [9, Proposition 5.6], an integral class Z is represented by a Lagrangian sphere (with respect to some symplectic structure ω with canonical class K_X) if and only if Z is represented by a smooth sphere, $Z \cdot K_X = 0$ and $Z^2 = -2$. It can be shown straightforwardly that $\bar{Z}_t$ satisfies $\bar{Z}_t \cdot K_X = 0$ and $\bar{Z}_t^2 = -2$. Hence $\bar{Z}_t$ (and its reduction Z_t) is represented by a Lagrangian sphere.

The reduction of $\bar{Z}_t$ is $E_1 + \cdots + E_{4l+2}$ when $t = 2l$ and is $H + E_1 + \cdots + E_{4l+3}$ when $t = 2l+1$. Therefore, the mod 2 classes in the two sequences

$$\{E_1 + \cdots + E_{4l+2} \mid l \ge 0\} \quad \text{and} \quad \{H + E_1 + \cdots + E_{4l+3} \mid l \ge 0\}$$

are represented by Lagrangian spheres.

We perform the Lagrangian blowup construction (see Section 2.3) at one point of a Lagrangian sphere in the mod 2 class $E_1 + \cdots + E_{4l+2}$ to obtain a Lagrangian $\mathbb{RP}^2$ in the mod 2 class

$$E_1 + \cdots + E_{4l+3}$$

for any $l \ge 0$. And, by repeating this process, for any $l \ge 0$ we obtain a Lagrangian $2\mathbb{RP}^2$ in the mod 2 class

$$E_1 + \cdots + E_{4l+4}$$

and a Lagrangian $3\mathbb{RP}^2$ in the mod 2 class

$$E_1 + \cdots + E_{4l+5}.$$

Similarly, for any $l \ge 0$, by blowing up at 1, 2 or 3 points of a Lagrangian sphere in the mod 2 class $H + E_1 + \cdots + E_{4l+3}$, we can construct a Lagrangian $\mathbb{RP}^2$, $2\mathbb{RP}^2 = \mathrm{KB}$ or $3\mathbb{RP}^2$ in the mod 2 classes

$$H + E_1 + \cdots + E_{4l+4}, \quad H + E_1 + \cdots + E_{4l+5}, \quad H + E_1 + \cdots + E_{4l+6},$$

respectively.

We are only left with the mod 2 classes

$$0, \quad H, \quad E_1, \quad H + E_1, \quad H + E_1 + E_2$$

to consider. The real part of $\mathbb{CP}^2$ is a Lagrangian $\mathbb{RP}^2$ of class H. A Lagrangian KB of $H + E_1$ and a Lagrangian $3\mathbb{RP}^3$ of $H + E_1 + E_2$ can be constructed by blowing up one or two points on this $\mathbb{RP}^2$. Blowing up at a point of the Clifford torus, we get a Lagrangian $3\mathbb{RP}^2$ representing E_1.

Since $(S^2 \times S^2) \# k\overline{\mathbb{CP}^2}$ is diffeomorphic to $\mathbb{CP}^2 \# (k+1)\overline{\mathbb{CP}^2}$ for any $k \geq 1$, the remaining rational 4–manifold is $S^2 \times S^2$. $H_2(S^2 \times S^2, \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}$ has two generators, the base class (or section class) $\bar{B} = [S^2 \times \mathrm{pt}]$ and the fiber class $\bar{F} = [\mathrm{pt} \times S^2]$. Let $B, F \in H_2(S^2 \times S^2, \mathbb{Z}_2)$ denote their reductions.

Let $\phi\colon S^2 \to S^2$ be the antipodal map, and ω a symplectic form on S^2 such that $\phi^*\omega = -\omega$. Note that the standard symplectic form obeys this condition. Equip $S^2 \times S^2$ with the product symplectic form $\omega \oplus \omega$. It is easy to see that the graph of the antipodal map $x \mapsto (x, \phi(x))$ is an embedded Lagrangian sphere representing the integral class $\bar{B} - \bar{F}$, hence the mod 2 class $B + F$.

Since the mod 2 classes B and F are symmetric, it suffices to construct Lagrangian surfaces for the class F. We will construct such a Lagrangian representative by the real part of $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$, and embed this Lagrangian surface into the symplectic fiber sum of two copies of $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$.

We recall the operation of symplectic fiber sum briefly (see [5; 10]). Let (X_1, ω_1) and (X_2, ω_2) be symplectic manifolds of the same dimension $2n$, and (Q, τ) be a compact symplectic manifold of dimension $2n-2$. Suppose that

$$\iota_i\colon Q \to X_i$$

are symplectic embeddings such that their images $\iota_i(Q)$ have trivial normal bundles. By the symplectic neighborhood theorem, there are symplectic embeddings

$$f_i\colon Q \times B^2(\epsilon) \to X_i, \quad f_i^*\omega_i = \tau \oplus dx \wedge dy$$

such that $f_i(q, 0) = \iota_i(q)$ for $q \in Q$ and $i = 1, 2$.

Let $A(\underline{\epsilon}, \epsilon)$ be the annulus on $B^2(\epsilon)$ with radius $\underline{\epsilon} < r < \epsilon$, and $\phi\colon A(\underline{\epsilon}, \epsilon) \to A(\underline{\epsilon}, \epsilon)$ be an area- and orientation-preserving diffeomorphism which swaps the two boundary components. Then the symplectic fiber sum is defined by

$$X_1 \#_Q X_2 = (X_1 - f_1(Q \times B^2(\underline{\epsilon})) \cup (X_2 - f_2(Q \times B^2(\underline{\epsilon}))/\sim,$$

where

$$f_2(q, z) \sim f_1(q, \phi(z)) \quad \text{for all } (q, z) \in Q \times A(\underline{\epsilon}, \epsilon).$$

There is a natural symplectic structure on $X_1 \#_Q X_2$ induced by ω_1 and ω_2.

Let us take two copies of $\mathbb{CP}^2 \# \overline{\mathbb{CP}^2}$ as in the last part of Section 2, and denote them by X_1 and X_2. Let L_1 be the real locus of X_1, which is a Lagrangian Klein bottle. Regard X_1 and X_2 as nontrivial S^2–bundles over S^2, and form the symplectic fiber sum $X_1 \#_{S^2} X_2$ so that the gluing region in the base of X_1 is away from the real locus $\mathbb{RP}^1 \subset \mathbb{CP}^1$. Denote the resulting manifold and symplectic form by $(\widehat{X}, \widehat{\omega})$. Then $\widehat{X}$ is a trivial S^2–bundle over S^2, since $\pi_1(\mathrm{Diff}^+(S^2)) \cong \pi_1(SO(3)) \cong \mathbb{Z}_2$. It is easy to see that L_1 is embedded in $\widehat{X}$ as a Lagrangian Klein bottle, representing the fiber class in $H_2(\widehat{X}, \mathbb{Z}_2)$. □

Let us complete the proof of Theorem 1.3.

Proof of Theorem 1.3 It is given by Propositions 3.1 and 2.12. □

References

[1] **M Audin**, *Quelques remarques sur les surfaces lagrangiennes de Givental*, J. Geom. Phys. 7 (1990) 583–598 MR

[2] **M Audin**, **F Lalonde**, **L Polterovich**, *Symplectic rigidity: Lagrangian submanifolds*, from "Holomorphic curves in symplectic geometry" (M Audin, J Lafontaine, editors), Progr. Math. 117, Birkhäuser, Basel (1994) 271–321 MR

[3] **P Biran**, *Geometry of symplectic intersections*, from "Proceedings of the International Congress of Mathematicians" (T Li, editor), volume II, Higher Ed. Press, Beijing (2002) 241–255 MR

[4] **A B Givental**, *Lagrangian imbeddings of surfaces and the open Whitney umbrella*, Funktsional. Anal. i Prilozhen. 20 (1986) 35–41 MR In Russian; translated in Funct. Anal. Appl. 20 (1986) 197–203

[5] **R E Gompf**, *A new construction of symplectic manifolds*, Ann. of Math. 142 (1995) 527–595 MR

[6] **M Gromov**, *Partial differential relations*, Ergeb. Math. Grenzgeb. 9, Springer (1986) MR

[7] **F Lalonde**, **J-C Sikorav**, *Sous-variétés lagrangiennes et lagrangiennes exactes des fibrés cotangents*, Comment. Math. Helv. 66 (1991) 18–33 MR

[8] **J A Lees**, *On the classification of Lagrange immersions*, Duke Math. J. 43 (1976) 217–224 MR

[9] **T-J Li**, **W Wu**, *Lagrangian spheres, symplectic surfaces and the symplectic mapping class group*, Geom. Topol. 16 (2012) 1121–1169 MR

[10] **D McDuff**, **D Salamon**, *Introduction to symplectic topology*, 2nd edition, Clarendon, New York (1998) MR

[11] **J W Milnor**, **J D Stasheff**, *Characteristic classes*, Annals of Mathematics Studies 76, Princeton Univ. Press (1974) MR

[12] **S Y Nemirovskiĭ**, *The homology class of a Lagrangian Klein bottle*, Izv. Ross. Akad. Nauk Ser. Mat. 73 (2009) 37–48 MR In Russian; translated in Izv. Math. 73 (2009) 689–698

[13] **L Polterovich**, *The surgery of Lagrange submanifolds*, Geom. Funct. Anal. 1 (1991) 198–210 MR

[14] **A Rieser**, *Lagrangian blow-ups, blow-downs, and applications to real packing*, J. Symplectic Geom. 12 (2014) 725–789 MR

[15] **V V Shevchishin**, *Lagrangian embeddings of the Klein bottle and the combinatorial properties of mapping class groups*, Izv. Ross. Akad. Nauk Ser. Mat. 73 (2009) 153–224 MR In Russian; translated in Izv. Math. 73 (2009) 797–859

[16] **R Thom**, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. 28 (1954) 17–86 MR

[17] **E Thomas**, *A generalization of the Pontrjagin square cohomology operation*, Proc. Nat. Acad. Sci. U.S.A. 42 (1956) 266–269 MR

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Received: 25 August 2017 Revised: 16 December 2018