Algebraic & Geometric Topology 18 (2018) 2897–2918

Kakimizu complexes of Seifert fibered spaces

JENNIFER SCHULTENS

Kakimizu complexes of Seifert fibered spaces can be described as either horizontal or vertical, depending on what type of surfaces represent their vertices. Horizontal Kakimizu complexes are shown to be trivial. Each vertical Kakimizu complex is shown to be isomorphic to a Kakimizu complex of the base orbifold minus its singular points.

57M10; 57M50

1 Introduction

For nearly a century, it has been known that for every knot K there is a compact orientable surface whose boundary is K. Seifert surfaces, named after Herbert Seifert, who proved the existence of such surfaces, are used extensively for both topological and quantitative investigations into knots and 3–manifolds. There can be infinitely many distinct Seifert surfaces for a given knot, obtained, for instance, by adding trivial handles to a given Seifert surface, or, more interestingly, by "spinning" it around a decomposing annulus or torus. This was first proved by J R Eisner [5]. The abundance of Seifert surfaces led Osamu Kakimizu, in the 1990s, to define a complex, later named after him, whose vertices are isotopy classes of Seifert surfaces of a given knot and whose n–simplices are $(n+1)$–tuples of vertices that admit pairwise disjoint representatives.

The Kakimizu complex of a knot has been described by several authors, most notably Makoto Sakuma and Kenneth Shackleton, who exhibited diameter bounds in terms of the genus of a knot (see [17]); Jessica Banks, who described in full detail how and when the Kakimizu complex of a knot fails to be locally finite (see [2]) and also how to compute the Kakimizu complex of a composite knot from the Kakimizu complexes of its summands (see [3]); Piotr Przytycki and the author, who established that the Kakimizu complexes of knots and certain more general 3–manifolds are contractible (see [16]); and Johnson, Pelayo and Wilson, who proved that the Kakimizu complex of a knot is quasi-Euclidean (see [14]).

Published: 22 August 2018 DOI: 10.2140/agt.2018.18.2897

In [18], the author generalized the definition of Kakimizu complex to the context of (codimension 1) submanifolds of n–manifolds. The argument used in [16] still applies and shows the Kakimizu complex to be contractible in this larger context.

This paper grew out of a desire to study concrete examples of Kakimizu complexes of 3–manifolds other than knot complements. As a case study, driven by personal experience rather than the innate poetry, we consider Seifert fibered spaces. Seifert fibered spaces, first studied by Herbert Seifert [20], are 3–dimensional manifolds that are foliated by circles. We give a brief overview in Section 3. Seifert fibered spaces provide an arena in which much is known about incompressible surfaces. This knowledge proves sufficient to characterize Kakimizu complexes for this class of 3–manifolds. Some of these Kakimizu complexes are easily proved to be trivial; see Theorem 17. Others prove less tractable — see Theorem 37 — but can be expressed in terms of Kakimizu complexes of the base orbifold. We prove the following:

Theorem 17 *Every horizontal Kakimizu complex of an orientable Seifert fibered space with a given fibration is trivial.*

Theorem 37 *Every vertical Kakimizu complex of an orientable Seifert fibered space with a given fibration is isomorphic to the corresponding Kakimizu complex of the surface obtained from the base orbifold by removing neighborhoods of the singular points.*

Theorem 37 is proved for closed orientable Seifert fibered spaces to avoid technicalities arising from case discussions. Most notably, to prove an analogous theorem for closed orientable Seifert fibered spaces with nonempty boundary it would be necessary to prove a more general version of Proposition 25. Whereas the universal cover of a good aspherical orbifold without boundary is usually the hyperbolic plane, the universal cover of good aspherical orbifolds with boundary is more complicated. In particular, its compactification has boundary that is partitioned into segments alternately consisting of limit points and lifts of boundary points.

I wish to thank the referee for many insightful comments and Misha Kapovich for numerous helpful conversations.

2 The Kakimizu complex

In the following we will always assume:

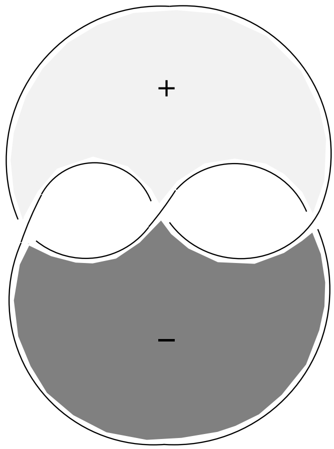

Figure 1: A Seifert surface for the trefoil

(1) M is a compact (possibly closed) connected oriented 3–manifold.

(2) α is an element of $H_2(M, \partial M, \mathbb{Z})$.

In the context of knots, a Seifert surface is an orientable spanning surface of a knot. This wording obscures the essential features of interest in the present investigation. Here we are interested primarily in three features of Seifert surfaces of knots:

(1) They represent a generator of the second relative homology $H_2(\mathbb{S}^3 \backslash K, K)$ (we will be interested in surfaces that represent relative second homology classes).

(2) They have connected complements (we will make an analogous assumption).

(3) They can be "projected" onto each other in the sense described in [16].

Definition 1 A *Seifert surface* for (M, α) is a pair (w, S), where S is a multisurface, ie a union, $S_1 \sqcup \cdots \sqcup S_n$, of pairwise disjoint oriented properly embedded 2–sided surfaces in M and w is an n–tuple of natural numbers $(w^1, \dots, w^n)$ such that the homology class $w^1[\![S_1]\!] + \cdots + w^n[\![S_n]\!]$ equals α. Moreover, we require that S have connected complement. We call S the *underlying surface* of (w, S).

The existence of Seifert surfaces of knots was first established by Seifert. This existence result has been generalized in several ways. A proof of the existence of a hypersurface in an n–manifold realizing a given $(n-1)$–dimensional homology class can be found in Bruno Martelli's book [15]. We include a brief discussion in the appendix (see Proposition 41).

Our definition of Seifert surface disallows null-homologous subsets. Indeed, a null-homologous subset would bound a component of $M \setminus S$ and would hence be separating. In fact, S contains no bounding subsets. Conversely, if S' contains no bounding subsets, then $M \setminus S'$ is connected.

Lemma 2 *If (w, S) represents α, then w is determined by the underlying surface S.*

Proof Suppose that (w, S) and (w', S) represent α, where $w = (w^1, \dots, w^n)$ and $w' = ((w')^1, \dots, (w')^n)$. Then

$$w^1[\![S_1]\!] + \dots + w^n[\![S_n]\!] = \alpha = (w')^1[\![S_1]\!] + \dots + (w')^n[\![S_n]\!],$$

hence

$$(w^1 - (w')^1)[\![S_1]\!] + \dots + (w^n - (w')^n)[\![S_n]\!] = 0.$$

Since S has no null-homologous subsets, this ensures that

$$w^1 - (w')^1 = 0, \quad \dots, \quad w^n - (w')^n = 0.$$

Thus,

$$w^1 = (w')^1, \quad \dots, \quad w^n = (w')^n. \qquad \square$$

Since the underlying multisurface S of a Seifert surface (w, S) determines w, we will often speak of a Seifert surface S, when w does not feature in our discussion.

Definition 3 For each pair (M, α), the isomorphism between $H_1(M, \partial M)$ and $H^1(M)$ identifies an element a^* of $H^1(M)$ corresponding to α that lifts to a homomorphism $h_a\colon \pi_1(M) \to \mathbb{Z}$. We denote the covering space corresponding to $N_\alpha = \ker(h_a)$ by $(p_\alpha, \widehat{S}_\alpha, S)$, or simply $(p, \widehat{S}, S)$, and call it the *infinite cyclic covering space associated with α*.

We describe the *Kakimizu complex* of (M, α): The *vertices* are Seifert surfaces (w, S) of (M, α) that minimize Thurston norm, considered up to isotopy of underlying surfaces. We write $v = [(w, S)]$, where $[(w, S)]$ is the isotopy class of S and v denotes the corresponding vertex. Let v and v' be a pair of vertices with representatives (w, S) and (w', S'). Here $M \setminus S$ and $M \setminus S'$ are connected. It follows that lifts of $M \setminus S$ and $M \setminus S'$ to the covering space associated with α are simply components of $p^{-1}(M \setminus S)$ and $p^{-1}(M \setminus S')$. We construct a graph $\Gamma(M, \alpha)$ by declaring the vertices v and v' to span an edge $e = (v, v')$ if and only if representatives (w, S) and (w', S') of v and v' can be chosen so that a lift of $M \setminus S$ to the covering space associated with α intersects exactly two lifts of $M \backslash S'$. (This condition implies that S and S' are disjoint, but not vice versa.)

Definition 4 Let X be a simplicial complex. If, whenever the 1–skeleton of a simplex σ is in X, the simplex σ is also in X, then X is said to be a *flag* complex.

Definition 5 The *Kakimizu complex of* (M,α), denoted by $\mathrm{Kak}(M,\alpha)$, is the flag complex with $\Gamma(M,\alpha)$ as its 1–skeleton.

Theorem 6 *For every* 3*–manifold* M *and every* $\alpha \in H_2(M,\partial M)$, $\mathrm{Kak}(M,\alpha)$ *is contractible.*

Proof See Theorem 1.1 in [16] and the proof of Theorem 5 in [18]. □

Theorem 7 *For every* 3*–manifold* M *and every* $\alpha \in H_2(M,\partial M)$, $\mathrm{Kak}(M,\alpha)$ *is connected.*

Proof This is a corollary of Theorem 6. □

Analogously, we define the Kakimizu complex for a surface S and a relative first homology class β of S:

Definition 8 A *Seifert curve* for (S,β) is a pair (w,c), where c is a multicurve, ie a union, $c_1 \sqcup \cdots \sqcup c_n$, of pairwise disjoint oriented properly embedded 2–sided arcs and curves in S, and w is an n–tuple of natural numbers $(w^1,\ldots,w^n)$ such that the homology class $w^1[\![c_1]\!] + \cdots + w^n[\![c_n]\!]$ equals β. Moreover, we require that c have connected complement. We call c the *underlying curve* of (w,c).

The *Kakimizu complex* of (S,α) is defined analogously to the Kakimizu complex of a 3–manifold. The *vertices* are Seifert curves (w,c) of (S,α), considered up to isotopy of underlying curves. We write $v = [(w,c)]$, where $[(w,c)]$ is the isotopy class of c and v is the corresponding vertex. Let v and v' be a pair of vertices with representatives (w,c) and (w',c'). Here $S \backslash c$ and $S \backslash c'$ are connected, hence path-connected. It follows that lifts of $S \backslash c$ and $S \backslash c'$ to the covering space associated with α are simply path components of $p^{-1}(S \backslash c)$ and $p^{-1}(S \backslash c')$. We construct a graph $\Gamma(S,\alpha)$ by declaring the vertices v and v' to span an edge $e = (v,v')$ if and only if representatives (w,c) and (w',c') of v and v' can be chosen so that a lift of $S \backslash c$ to the covering space associated with α intersects exactly two lifts of $S \backslash c'$. (This condition implies that c and c' are disjoint, but not vice versa.)

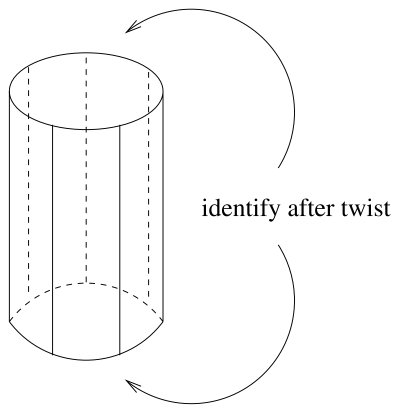

Figure 2: A fibered solid torus

Definition 9 The *Kakimizu complex of* (S, α), denoted by $\mathrm{Kak}(S, \alpha)$, is the flag complex with $\Gamma(S, \alpha)$ as its 1–skeleton.

3 Seifert fibered spaces

Several excellent sources describe Seifert fibered spaces in great detail. We recommend Seifert's original paper on the subject (see [20]) and W Heil's translation (see [21]).

Definition 10 A *fibered solid torus* is a solid torus obtained from $\mathbb{D}^2 \times [0,1]$ by gluing $\mathbb{D} \times \{0\}$ to $\mathbb{D} \times \{1\}$ after a rotation by a rational multiple of 2π. For the rotation by $\frac{2\pi\nu}{\mu}$, where $\mu, \nu \in \mathbb{Z}$ and $\gcd(\mu, \nu) = 1$, we denote the resulting solid torus by $V(\nu, \mu)$. For notational convenience, we require that $0 < \nu < \mu$. The simple closed curves resulting from identifying intervals of the form $\{y\} \times [0,1]$ for $y \in D^2$ are called *fibers*.

A fiber other than the one resulting by identifying $\{(0,0)\} \times [0,1]$ is called a *regular fiber*. The fiber resulting by identifying $\{(0,0)\} \times [0,1]$ is called an *exceptional fiber* if regular fibers intersect a meridian disk more than once and a regular fiber otherwise.

We are interested in orientable Seifert fibered spaces and hence need only consider fibered solid tori. To understand nonorientable Seifert fibered spaces one needs also to understand the fibered solid Klein bottle and allow it to take the place of fibered solid tori in the definition below.

Definition 11 A *Seifert fibered space* M is a compact connected 3–manifold that can be decomposed into a union of disjoint circles each of which has a neighborhood that is homeomorphic to a fibered solid torus via a homeomorphism that takes circles to fibers. The circles into which M decomposes are also called *fibers* of M. Fibers of M are *regular* or *exceptional* in accordance with their images under the relevant homeomorphisms. A particular decomposition of M into fibers is called a *fibration* of M.

Definition 12 Given a Seifert fibered space M together with a fibration, we form the quotient space B by identifying each fiber to a point. We denote the quotient map by $p: M \to B$. Topologically, the quotient space is a surface. However, we add information by recording the points corresponding to exceptional fibers. Specifically, for a fiber e, if nearby regular fibers wrap around e exactly μ times, then we declare $p(e)$ to be a *singular point* of B of *index* μ. We thereby endow B with an orbifold structure and call it the *base orbifold*.

See [19] for the basics of orbifolds and their relation to Seifert fibered spaces. Note that even for an orientable Seifert fibered space, the base orbifold can be orientable or nonorientable. Consider, for instance, the twisted circle bundle over the Möbius band. It is homeomorphic to a twisted I–bundle over the Klein bottle. This is an orientable 3–manifold and, via the former description, a Seifert fibered space. Its double is of interest, because it admits two distinct, though homeomorphic, Seifert fibrations as twisted circle bundles over the Klein bottle.

Seifert fibered spaces are completely determined by a set of invariants computed from the base orbifold and the μ's and ν's of their exceptional fibers. Their fundamental groups can be computed from this set of invariants. Relevant to the investigation here is that for a Seifert fibered space M there is a short exact sequence

$$1 \to C \to \pi_1(M) \to \pi_1(B) \to 1,$$

where C is a normal cyclic subgroup of $\pi_1(M)$ generated by a regular fiber, B is the base orbifold and $\pi_1(B)$ is the orbifold-fundamental group of B (see [19]).

4 Incompressible surfaces in Seifert fibered spaces

We are interested in surfaces that minimize Thurston norm that represent relative second homology classes of an orientable Seifert fibered space M. Such surfaces are necessarily essential in M. Let M be a compact, orientable Seifert fibered space and let F be a two-sided incompressible surface in M. If F is everywhere transverse to the fibers of M, then F is said to be *horizontal*. If every fiber that meets F is entirely contained in F, then F is said to be *vertical*. Incompressible surfaces in Seifert fibered spaces were studied by several authors; see for instance Burde and Zieschang [4], Gordon and Heil [8], Hempel and Jaco [11], Jaco and Shalen [13], Tollefson [23] and Waldhausen [24].

Theorem 13 (Jaco [12, Theorem VI.34]) *Let M be an orientable Seifert fibered space. If F is a connected, two-sided, incompressible surface in M, then one of the following alternatives holds:*

(i) *F is a disk or an annulus and F is parallel into ∂M.*

(ii) *F does not separate M and F is a fiber in a fibration of M as a surface bundle over the circle.*

(iii) *F does separate M and $M = M_1 \cup M_2$, where $\partial M_i = F$ and M_i is a twisted I–bundle over a compact surface.*

(iv) *F is an annulus or a torus and F is saturated, ie consists of fibers, in some Seifert fibration of M.*

It is important to note that case (i) describes surfaces that are inessential and case (iii) describes surfaces that are trivial in second homology. Moreover, in the proof of Jaco's theorem, the Seifert fibration of M is fixed and a presentation of $\pi_1(M)$ is computed with respect to this fibration. Case (ii) results only for surfaces whose fundamental group does not contain the group C, whereas case (iv) results only for surfaces whose fundamental group contains C. In particular, the surfaces in case (ii) cannot be vertical with respect to the given fibration and the surfaces in case (iv) cannot be horizontal. Thus, F does not realize these two cases simultaneously with respect to the given fibration.

In case (ii), the structure of M as a surface bundle over the circle relates to the Seifert fibration of M. For instance if F is $(\text{torus}) \times \{\text{points}\}$ in the three-torus $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{S}^1$, then the fibers of the Seifert fibration are of the form $\{p\} \times \mathbb{S}^1$. Clearly, this is not the only Seifert fibration for the three-torus, but it is the one that gives rise to case (ii).

On the other hand, consider the case where F is a torus fiber in a nontrivial torus bundle over the circle. Theorem VI.26 in [12] describes the surfaces and surface bundles in case (ii) in more detail. In particular, the gluing map ϕ for the surface bundle is of finite order, say n. Thus, as M is obtained from $F \times [0,1]$ via the gluing map ϕ, setting $(p,1) = (\phi(p),0)$, we see that intervals of the form

$\{p\}\times[0,1]\cup\{\phi(p)\}\times[0,1]\cup\cdots\cup\{\phi^{n-1}(p)\}\times[0,1]$ match up to form the fibers of a Seifert fibration. It is with respect to this Seifert fibration that F is horizontal. It is also with respect to this Seifert fibration that the subgroup C is not contained in the fundamental group of F. Thus, given a Seifert fibered space with a particular fibration, the surfaces in case (ii) are horizontal and the surfaces in case (iv) are vertical.

Lemma 14 *Let M be an orientable Seifert fibered space and let F be a two-sided essential surface in M. If F is connected and horizontal with respect to a given Seifert fibration, then it is neither isotopic nor homologous to a vertical surface with respect to this fibration. Likewise, if F is connected and vertical with respect to a given Seifert fibration, then it is neither isotopic nor homologous to a horizontal surface with respect to this fibration.*

This follows from the proofs of Theorems VI.26 and VI.34 in [12].

Definition 15 Let M be an orientable Seifert fibered space. A Kakimizu complex of M is *horizontal* (*vertical*, respectively) with respect to a given Seifert fibration of M if one vertex of the complex is realized by a horizontal (vertical, respectively) surface with respect to the given Seifert fibration of M.

Remark 16 By Lemma 14, a Kakimizu complex of the Seifert fibered space M is either horizontal or vertical and not both with respect to a given Seifert fibration of M.

5 Horizontal Kakimizu complexes

In light of Theorem 13, horizontal Kakimizu complexes are easily shown to be trivial.

Theorem 17 *Every horizontal Kakimizu complex of an orientable Seifert fibered space with a given fibration is trivial.*

Proof Let M be an orientable Seifert fibered space and let $\mathrm{Kak}(M,\alpha)$ be a horizontal Kakimizu complex of M. Let F be a horizontal surface in M representing a vertex of $\mathrm{Kak}(M,\alpha)$. Recall case (ii) of Theorem 13. By Theorem 7, $\mathrm{Kak}(M,\alpha)$ is connected, hence it suffices to show that the link of the vertex corresponding to F is empty. A surface representing a vertex in this link would have to be disjoint from F. Since the complement of F is homeomorphic to $F\times(0,1)$, such a surface would be isotopic to F. Thus, this link is empty. □

Corollary 18 *If F and F' are homologous horizontal surfaces in an orientable Seifert fibered space, then F and F' are isotopic.*

Corollary 19 *More generally, suppose that M is a surface bundle over the circle with fiber F and is also a surface bundle over the circle with fiber F'. If F and F' are homologous, then they are isotopic.*

Both corollaries are special cases of Theorem 4 in [22].

6 Orbifolds and Baer's theorem concerning homotopy and isotopy of curves in surfaces

In what follows we will be interested in simple closed curves in 2–dimensional orbifolds. A classical theorem of Reinhold Baer establishes that, for simple closed curves in a surface, homotopy implies isotopy. We prove an analogous result for simple closed curves in orbifolds. In order to do so, we will need to define the relevant concepts on orbifolds. The orbifolds relevant to our discussion are good (covered by a surface) aspherical (not covered by the sphere) orbifolds without boundary. From here forward, all orbifolds will be good aspherical orbifolds without boundary.

Theorem 20 (Baer) *Two simple closed curves in a surface are isotopic if and only if they are homotopic.*

Theorem 20 is also known as the Baer–Epstein theorem, due to Epstein's generalization of Baer's theorem (see [1, Satz 1, page 106; 6, Theorem 4.1]).

Definition 21 A simple closed curve in an orbifold B is said to be *regular* if it is disjoint from the singular set of B.

Definition 22 Let b and b' be regular simple closed curves in an orbifold B. By abuse of notation, b denotes both the map $b\colon \mathbb{S}^1 \to B$ and its image and b' denotes both the map $b'\colon \mathbb{S}^1 \to B$ and its image. An *orbifold-homotopy* between b and b' is a map

$$H\colon \mathbb{S}^1 \times [0,1] \to B$$

such that:

(1) $H(s,0) = b(s)$ and $H(s,1) = b'(s)$ for all $s \in \mathbb{S}^1$.

(2) H lifts to a map $\widetilde{H}\colon \mathbb{R} \times [0,1] \to \widetilde{B}$ for $\widetilde{B}$ the universal cover of B.

Definition 23 Let b and b' be as above. An *orbifold-isotopy* between b and b' is a map

$$H\colon \mathbb{S}^1 \times [0,1] \to B$$

such that:

(1) $H(s,0) = b(s)$ and $H(s,1) = b'(s)$ for all $s \in \mathbb{S}^1$.

(2) $H(\cdot\,,t)$ is injective for all $t \in [0,1]$.

(3) The image of H is disjoint from the singular points of B.

Definition 24 A closed curve in an orbifold is *inessential* in the orbifold B if it is orbifold-homotopic to a point or orbifold-homotopic into ∂B. Otherwise, it is *essential*.

Proposition 25 *Suppose that B is a good aspherical orbifold without boundary. Let b and b' be regular essential simple closed curves in B. Then b and b' are orbifold-isotopic if and only if they are orbifold-homotopic.*

Lemma 26 *If b and b' are orbifold-isotopic, then they are orbifold-homotopic.*

Proof Let b and b' be regular essential simple closed curves in B. Suppose that b and b' are orbifold-isotopic in B. By definition, they are isotopic when restricted to the surface $B\backslash\{\text{singular set}\}$. The isotopy lifts to the universal cover of $B\backslash\{\text{singular set}\}$, a subsurface of $\widetilde{B}$. Thus, the orbifold-isotopy is an orbifold-homotopy. □

Lemma 27 *Proposition 25 holds when $b \cap b' = \varnothing$.*

Proof By Lemma 26, orbifold-isotopy implies orbifold-homotopy. To prove the converse, suppose that b and b' are orbifold-homotopic via an orbifold-homotopy,

$$H\colon \mathbb{S}^1 \times [0,1] \to B.$$

The universal cover, $\widetilde{B}$, of B, is a plane. In fact, with very few exceptions, $\widetilde{B}$ is the hyperbolic plane. We prove the lemma in this case. In the remaining cases, the proof is similar. Abusing notation slightly, we continue to denote the standard unit disk compactification of $\widetilde{B}$ by $\widetilde{B}$. (In the case of the Euclidean plane, we also compactify to a closed unit disk.)

Choose a lift, $\widetilde{b}$, of b to $\widetilde{B}$. Then $\widetilde{b}$ is an embedded arc. Indeed, to see that the endpoints of $\widetilde{b}$ cannot coincide, consider the simple closed curve b in B. It is orbifold-homotopic

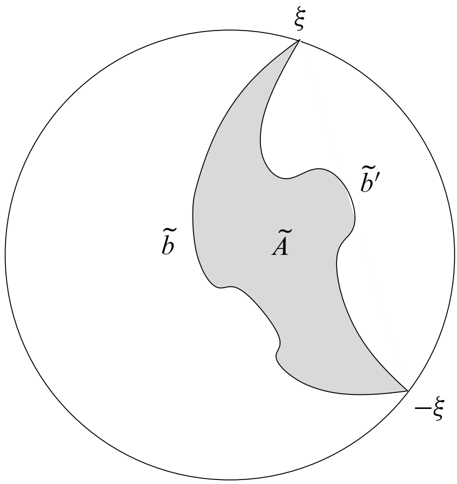

Figure 3: A lune

to a geodesic l. The orbifold-homotopy lifts to a homotopy $\widetilde{H}\colon \mathbb{R}\times[0,1]\to\widetilde{B}$ where $\widetilde{H}|_{\mathbb{R}\times\{0\}}=\widetilde{b}$. Denote the simple curve $\widetilde{H}|_{\mathbb{R}\times\{1\}}$ by $\tilde{l}$. Then $\tilde{l}$ is finite distance from $\widetilde{b}$. Geodesics in the hyperbolic plane (as well as the Euclidean plane) have distinct ends. Since $\widetilde{b}$ is finite distance from $\tilde{l}$, its endpoints coincide with those of $\tilde{l}$. In particular, they do not coincide with each other. We denote the endpoints by $\pm\xi$.

The orbifold-homotopy between b and b' also lifts to a homotopy $\widetilde{G}\colon \mathbb{R}\times[0,1]\to\widetilde{B}$ with $\widetilde{G}|_{\mathbb{R}\times\{0\}}=\widetilde{b}$. Denote the simple curve $\widetilde{G}|_{\mathbb{R}\times\{1\}}$ by $\widetilde{b}'$. Since $\widetilde{b}$ and $\widetilde{b}'$ are finite distance apart, the endpoints of $\widetilde{b}'$ are also $\pm\xi$. Thus, $\widetilde{b}$ and $\widetilde{b}'$ cobound a lune, $\widetilde{A}$, in $\widetilde{B}$. See Figure 3.

A priori it is possible that lifts of b' other than $\widetilde{b}'$ lie in $\widetilde{A}$. In this case, we replace $\widetilde{b}'$ by an innermost lift of b', ie a lift of b' such that no other lifts of b' lie in the lune cobounded by $\widetilde{b}$ and the given lift of b'. Abusing notation, we continue to denote this lift of b' by $\widetilde{b}'$ and the (smaller) lune cobounded by $\widetilde{b}$ and $\widetilde{b}'$ by $\widetilde{A}$. (It is a subtle fact that in the case where $\widetilde{B}$ is the hyperbolic plane, the lift of b' obtained from the lifted homotopy is, in fact, necessarily innermost.)

We wish to show that $\widetilde{A}$ projects to an annulus (with no singular points) in B cobounded by b and b'. In particular, this will ensure that b and b' are orbifold-isotopic. To this end we are interested in the action of the group of covering transformations on $\widetilde{A}$.

Consider a covering transformation ϕ. It permutes distinct lifts of b and, likewise, it permutes distinct lifts of b'. Hence, either $\phi(\widetilde{b})=\widetilde{b}$, or else $\phi(\widetilde{b})\cap\widetilde{b}=\varnothing$. Likewise, either $\phi(\widetilde{b}')=\widetilde{b}'$, or else $\phi(\widetilde{b}')\cap\widetilde{b}'=\varnothing$. Since $b\cap b'=\varnothing$, the lift $\widetilde{b}$ is disjoint from all lifts of b' and $\widetilde{b}'$ is disjoint from all lifts of b. In particular, $\widetilde{b}$ is disjoint from $\phi(\widetilde{b}')$ and $\widetilde{b}'$ is disjoint from $\phi(\widetilde{b})$. Since $\widetilde{A}$ is cobounded by $\widetilde{b}$ and $\widetilde{b}'$, its image, $\phi(\widetilde{A})$, is cobounded by $\phi(\widetilde{b})$ and $\phi(\widetilde{b}')$. The boundary components of $\phi(\widetilde{A})$ hence either coincide with or are disjoint from the boundary components of $\widetilde{A}$. Any lift of b' in the interior of $\phi(\widetilde{A})$ would pull back (via ϕ^{-1}) to a lift of b' in the interior of $\widetilde{A}$. Hence, since $\widetilde{A}$ is an innermost lune, $\phi(\widetilde{A})$ is also an innermost lune. It follows that either $\phi(\widetilde{A}) = \widetilde{A}$, or else $\phi(\widetilde{A})$ is disjoint from $\widetilde{A}$.

We are interested in covering transformations that map $\widetilde{A}$ to itself. So suppose $\phi(\widetilde{A}) = \widetilde{A}$. Consider $\pm\xi$. If ϕ interchanges $\pm\xi$, then, since it is an orientation-preserving map of the disk, it also interchanges $\widetilde{b}$ and $\widetilde{b}'$. But this is impossible. Hence, ϕ fixes ξ and $-\xi$.

The covering transformations that map $\widetilde{A}$ to itself form a subgroup, which we denote by $\mathrm{Stab}(\widetilde{A})$. A priori, if ϕ lies in $\mathrm{Stab}(\widetilde{A})$, then it can act on $\widetilde{A}$ by translation or rotation. However, a rotation has a fixed point x in the interior of $\widetilde{A}$, and therefore fixes the three points x, ξ and $-\xi$. A nontrivial rotation is therefore out of the question, hence $\mathrm{Stab}(\widetilde{A})$ acts on $\widetilde{A}$ only by translations.

To understand the action of $\mathrm{Stab}(\widetilde{A})$ on $\widetilde{A}$, we consider the restriction of $\mathrm{Stab}(\widetilde{A})$ to $\widetilde{b}$. Since the image of $\widetilde{b}$ under the covering map is the simple closed curve b, the restriction of $\mathrm{Stab}(\widetilde{A})$ to b is the infinite cyclic group. Moreover, the kernel of this restriction map must be trivial, since it consists of covering transformations that fix $\widetilde{b}$ pointwise. Therefore, $\mathrm{Stab}(\widetilde{A})$ is the infinite cyclic group. The quotient space,

$$A = \widetilde{A}/\sim_{\langle g \rangle}$$

is thus an annulus that embeds into B, whence b and b' cobound the annulus A, which contains no singular points. They are therefore orbifold-isotopic. □

Definition 28 Two transverse simple closed curves in a surface are said to be in *minimal position* if either they coincide, or they are transverse and intersect in the minimal possible number of points in their isotopy classes.

In the proof of Proposition 25 we will employ the *bigon criterion*; see for instance [7, Proposition 1.7].

Definition 29 Two transverse simple closed curves b and b' in a surface Σ form a *bigon* if there is an embedded disk in Σ whose boundary is the union of a subarc of b and a subarc of b' that meet in their endpoints. See Figure 4.

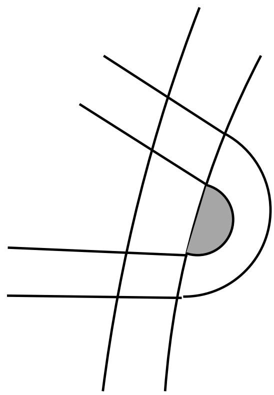

Figure 4: An innermost bigon

Criterion 30 (bigon criterion) *Two transverse simple closed curves in a surface Σ are in minimal position if and only if they do not form a bigon.*

Proof See [7, Proposition 1.7]. □

In establishing the bigon criterion, Farb and Margalit prove a key lemma for surfaces and their universal covers:

Lemma 31 [7, Lemma 1.8] *If transverse simple closed curves b and b' in a surface Σ do not form bigons, then in the universal cover of Σ, any pair of lifts $\widetilde{b}$ and $\widetilde{b}'$, of b and b', intersect in at most one point.*

The lemma is proved by establishing the contrapositive: if a pair of lifts $\widetilde{b}$ and $\widetilde{b}'$ of the transverse curves b and b' in the surface Σ form a bigon in the universal cover, $\widetilde{\Sigma}$, of Σ, then b and b' form a bigon. We are interested in orbifolds and their universal covers. In this setting, the proof of [7, Lemma 1.8] applies with a minor augmentation to establish an analogous result:

Lemma 32 *If a pair of lifts $\widetilde{b}$ and $\widetilde{b}'$ of transverse curves b and b' in an orbifold B form a bigon in the universal cover, $\widetilde{B}$, of B, then b and b' form a bigon containing no singular points.*

Proof Applied in the orbifold setting, the proof of [7, Lemma 1.8] establishes that, given an innermost bigon $\widetilde{D}$ formed by $\widetilde{b}$ and $\widetilde{b}'$, any covering transformation, ϕ, with fixed points in the interior of $\widetilde{D}$ is a rotation that takes $\widetilde{D}$ to itself. Given that ϕ takes the corners of $\widetilde{D}$ to themselves or each other, the rotation must be either through an angle of 2π (the identity) or an angle of π. Note, however, that a rotation through an angle of π would interchange $\widetilde{b}$ and $\widetilde{b}'$, but this is impossible. Hence, the covering map is injective on $\widetilde{D}$ and the conclusion holds. $\square$

Proof of Proposition 25 By Lemma 26, orbifold-isotopy implies orbifold-homotopy. We next proceed as in the proof of Lemma 27. In the argument establishing that orbifold-homotopy implies orbifold-isotopy, we choose lifts of b and b' to the compactified universal cover $\widetilde{B}$. When we proceed analogously, the resulting lifts $\widetilde{b}$ and $\widetilde{b}'$ are not necessarily disjoint. Unlike in the proof of Lemma 27, there is, at this stage, not necessarily an innermost choice of $\widetilde{b}'$, but this will be of no concern here.

Lemma 32 tells us that by applying an orbifold-isotopy to b and b', we can eliminate bigons formed by $\widetilde{b}$ and $\widetilde{b}'$. From here forward, we will assume that b and b' have been orbifold-isotoped into minimal position in the complement of the singular points.

As in the proof of Lemma 27, there is a covering transformation, g, that acts nontrivially on $\widetilde{b}$ and $\widetilde{b}'$.

Claim 1 $$\widetilde{b}\cap\widetilde{b}'=\varnothing.$$

Proof of Claim 1 Since $g(\widetilde{b})=\widetilde{b}$ and $g(\widetilde{b}')=\widetilde{b}'$, we have $g(\widetilde{b}\cap\widetilde{b}')=\widetilde{b}\cap\widetilde{b}'$. Since b and b' are essential, g has infinite order. Hence, either $\widetilde{b}\cap\widetilde{b}'=\varnothing$, or else the set $\widetilde{b}\cap\widetilde{b}'$ is infinite. Since there are no bigons, the latter is impossible. Thus, $\widetilde{b}\cap\widetilde{b}'=\varnothing$. This proves Claim 1. $\square$

Claim 2 *If ϕ is a covering transformation of $\widetilde{B}$, then $\widetilde{b}\cap\phi(\widetilde{b}')=\varnothing$.*

Proof of Claim 2 If ϕ fixes the set $\{\pm\xi\}$, then Claim 2 follows from the proof of Claim 1. In the remaining case, $\phi(\pm\xi)$ is disjoint from $\{\pm\xi\}$. In particular, since $\phi(\widetilde{b}')$ is just a different lift of b', the endpoints of $\phi(\widetilde{b}')$ lie on one side of $\widetilde{b}'$ and hence on one side of $\widetilde{b}$. Thus, if $\widetilde{b}\cap\phi(\widetilde{b}')\neq\varnothing$, then $\widetilde{b}$ and $\phi(\widetilde{b}')$ form a bigon. By Lemma 32, b and b' form a bigon that does not contain singular points. By applying the bigon criterion to $B\setminus\{\text{singular points}\}$, we see that this contradicts our assumption that b and b' are in minimal position. This proves Claim 2. $\square$

Claim 2 (strong form) *Any pair of lifts of b and b' to $\widetilde{B}$ are disjoint.*

Proof of Claim 2 (strong form) For suitable covering transformations ϕ and ϕ', the lifts of b and b' are $\phi(\widetilde{b})$ and $\phi'(\widetilde{b}')$. Claim 2 (strong form) thus follows by applying Claim 2 to $\widetilde{b}$ and $\phi^{-1} \circ \phi'(\widetilde{b}')$. This proves Claim 2 (strong form). □

It follows that b and b' are disjoint. Therefore, by Lemma 27, orbifold-homotopic regular simple closed curves are orbifold-isotopic. □

It is worthwhile to highlight Claim 1 in the proof above and state it as an independent lemma:

Lemma 33 *Suppose that the transverse simple closed curves b and b' in an orbifold B are orbifold-homotopic. If lifts $\widetilde{b}$ and $\widetilde{b}'$ of b and b' limit on the same points in an appropriate compactification to the universal cover $\widetilde{B}$ of B and do not form bigons, then $\widetilde{b}$ and $\widetilde{b}'$ are disjoint.*

It is also worthwhile to highlight Claim 2 (strong form) in the proof above and state it as an independent lemma:

Lemma 34 *Suppose that the transverse simple closed curves b and b' in an orbifold B are orbifold-homotopic and do not form bigons in the complement of the singular points. Then any pair of lifts of b and b' are disjoint.*

Remark 35 With suitable modifications, Proposition 25 holds when B has boundary and, moreover, b and b' are allowed to be arcs.

7 Vertical Kakimizu complexes

As it turns out, any vertical Kakimizu complex can be computed entirely in terms of the base orbifold of the Seifert fibered space. To simplify matters, we will only consider closed orientable Seifert fibered spaces. (The proof in the general case is similar, but involves more cases.) The orbifold over which such a Seifert fibered space fibers does not have boundary. Moreover, a Seifert fibered space that admits vertical Kakimizu complexes contains an essential vertical surface that is nontrivial in second homology. Such a Seifert fibered space cannot have a bad or spherical base orbifold. To summarize, the Seifert fibered spaces we consider here have base orbifolds that are, as in Section 6, good aspherical orbifolds without boundary.

Lemma 36 *Let M be an orientable Seifert fibered space with a fixed fibration and let α be a second homology class of M. If F is a vertical surface representing α, then F consists of regular fibers of M.*

Proof Let F be a vertical surface representing α and let f be a fiber of M that lies in F. Denote a fibered solid torus neighborhood of f by T and denote the collar of f in $F \cap T$ by $C(f)$. Then $C(f)$ consists of fibers. There are only two options: either these fibers are parallel to f or they wind around f twice. In the former case, f is a regular fiber. In the latter case, $C(f)$ is a Möbius band. However, since M and F are orientable, the latter case cannot occur. Thus, F consists of regular fibers. $\square$

Let M be an orientable Seifert fibered space with base orbifold B and let α be a second homology class of M generated by a vertical weighted multisurface (w, S). Let B^- be the surface obtained from B by removing neighborhoods of the singular points. Let β be the first homology class corresponding to the projection of (w, S) restricted to B^-.

We define a map

$$\Phi\colon \mathrm{Kak}(B^-, \beta) \to \mathrm{Kak}(M, \alpha)$$

as follows: Given a vertex v of $\mathrm{Kak}(B^-, \beta)$, let (w, b) be a representative of v. Set $S = p^{-1}(b)$ and $\Phi(v) = [(w, S)]$. To see that this vertex map is well-defined, consider another representative, (w, b') of v. By definition, b' is orbifold-isotopic to b. This isotopy extends to an isotopy between $S' = p^{-1}(b')$ and S.

Next, consider an edge $e = (v, v')$ of $\mathrm{Kak}(B^-, \beta)$. The existence of e guarantees that there are representatives (w, b) of v and (w', b') of v' such that a lift of $B^- \backslash b$ to the covering space of B^- associated with β intersects exactly two lifts of $B^- \backslash b'$. Note that the infinite cyclic covering space of B^- associated with β can be constructed from a countably infinite collection of copies of $B^- \backslash b$ (or of $B^- \backslash b'$) via suitable identifications. Likewise, for $S = p^{-1}(b)$ and $S' = p^{-1}(b')$, the infinite cyclic covering space of M associated with α can be constructed from a countably infinite collection of copies of $M \backslash S$ (or of $M \backslash S'$) via suitable identifications. It follows that a lift of $B^- \backslash b$ to the covering space of B^- associated with β intersects exactly two lifts of $B^- \backslash b'$ if and only if a lift of $M \backslash S$ to the covering space of M associated with α intersects exactly two lifts of $M \backslash S'$. We extend our vertex map to the 1–skeleton as follows:

$$\Phi((v, v')) = ([w, S], [w', S']).$$

We have defined Φ on the 1–skeleton of $\mathrm{Kak}(B^-,\beta)$. Since the complexes $\mathrm{Kak}(B^-,\beta)$ and $\mathrm{Kak}(M,\alpha)$ are flag complexes, Φ extends from the 1–skeleton to $\mathrm{Kak}(B^-,\beta)$.

Theorem 37 *Every vertical Kakimizu complex of an orientable Seifert fibered space with a given fibration is isomorphic to the corresponding Kakimizu complex of the surface obtained from the base orbifold by removing neighborhoods of the singular points.*

Specifically, let M be an orientable Seifert fibered space with base orbifold B and let α be a second homology class of M generated by a weighted vertical surface (w,S). Let B^- be the surface obtained from B by removing neighborhoods of the singular points. Let β be the first homology class corresponding to the projection of (w,S) restricted to B^-. Then the map $\Phi\colon \mathrm{Kak}(B^-,\beta)\to\mathrm{Kak}(M,\alpha)$, defined above, is an isomorphism.

Our challenge will be to show that Φ is injective. We first prove a couple of lemmas.

Lemma 38 *Let M be a closed orientable Seifert fibered space with base orbifold B. The projection map $p\colon M\to B$ lifts to a projection map $\widetilde{p}\colon \widetilde{M}\to\widetilde{B}$, where $\widetilde{M}$ and $\widetilde{B}$ are the universal covers of M and B.*

Proof Peter Scott proves this in the discussion leading up to [19, Lemma 3.1]. □

Lemma 39 *Let M be a closed orientable Seifert fibered space with base orbifold B and let B^- be as above. Let F and F' be connected essential vertical surfaces in M and denote $p(F)$ by b and $p(F')$ by b'. If F and F' are isotopic, then b and b' are orbifold-isotopic.*

Proof We choose an essential simple closed curve on the torus F that projects to b and denote it by f. We then denote the image of f under the isotopy between F and F' by f'. We denote the restriction of the isotopy between F and F' to an isotopy between f and f' by H. The isotopy H lifts to an isotopy $\widetilde{H}$ between lifts $\widetilde{f}$ and $\widetilde{f}'$ of f and f' to $\widetilde{M}$. Therefore, for $\widetilde{p}\colon \widetilde{M}\to\widetilde{B}$ as in Lemma 38, $\widetilde{p}\circ\widetilde{H}$ is a lift of $p\circ H$ to $\widetilde{B}$. Hence, $p\circ H$ lifts to a homotopy in $\widetilde{B}$. Thus, the homotopy $p\circ H$ between b and b' is an orbifold-homotopy. By Proposition 25, b and b' are orbifold-isotopic, hence isotopic in B^-. □

Lemma 40 *Φ is injective.*

Proof We first show that Φ is injective on vertices. Let $v=[(w,c)]$ and $v'=[(w',c')]$ be vertices of $\mathrm{Kak}(B^-,\beta)$ and suppose that $\Phi(v)=\Phi(v')$. Since $\Phi(v)=\Phi(v')$, the vertices are represented by isotopic vertical multisurfaces. This can only happen if the surfaces are isotopic componentwise. Let b and b' be components of c and c' such that $S=p^{-1}(b)$ and $S'=p^{-1}(b')$ are isotopic. By Lemma 39, b and b' are orbifold-isotopic, hence isotopic in B^-. Thus, Φ is injective on vertices. Since a pair of vertices spans at most one edge and, since $\mathrm{Kak}(B^-,\beta)$ is a simplicial complex, Φ is injective. □

Proof of Theorem 37 We wish to show that the map Φ is an isomorphism. Lemma 40 shows that Φ is injective. To see that Φ is surjective, suppose that $[(w,S)]\in\mathrm{Kak}(M,\alpha)$. By Lemma 36, S consists of regular fibers. Thus, the projection, b of S lies in B^-. Hence, $[(w,b)]\in\mathrm{Kak}(B^-,\beta)$ and $\Phi([(w,b)])=[(w,S)]$. Thus, Φ is surjective. □

Appendix: Existence of spanning submanifolds

In Lemmas 6.6–6.8 of [10], John Hempel described necessary and sufficient conditions for the existence of incompressible surfaces in 3–manifolds. Such arguments were also used by Réné Thom and Jean-Pierre Serre in their work related to Steenrod realization problems. These arguments can be tailored to prove the existence of Seifert surfaces and their generalizations to arbitrary dimensions. The resulting argument is sometimes referred to as "the canonical proof of Seifert's theorem" and is mentioned, for instance, in Martelli [15, Proposition 1.7.16]. We include it here for completeness.

Proposition 41 *Let M be a compact (possibly closed) connected oriented n–manifold and let α be an infinite element of $H_{n-1}(M,\partial M,\mathbb{Z})$. Then there is a properly embedded orientable $(n-1)$–dimensional submanifold Σ of M with $[\![\Sigma]\!]=\alpha$.*

Proof The isomorphism between $H_{n-1}(M,\partial M)$ and $H^1(M)$ identifies an element a^* of $H^1(M)$ corresponding to α. A fundamental relationship between singular cohomology and Eilenberg–Mac Lane spaces tells us that there is a natural bijection

$$T\colon \langle M,K(\mathbb{Z},1)\rangle\to H^1(M)$$

(see [9, Theorem 4.57]), where

$$T([f])=f^*(\gamma)$$

for a certain distinguished class $\gamma \in H^1(K(\mathbb{Z},1))$. Since $K(\mathbb{Z},1) = \mathbb{S}^1$, a^* corresponds to a map

$$f_a\colon M \to \mathbb{S}^1$$

such that

$$T([f_a]) = f_a^*(\gamma) = a^*.$$

Moreover, since $H^1(\mathbb{S}^1) = \mathbb{Z}$, we may assume that γ is the generator, $1 \in \mathbb{Z}$. (We will not use this fact immediately, but it will be relevant below.)

By replacing f_a by a smooth approximation, if necessary, we may assume that f_a is smooth. Let p be a regular value of f_a and denote $f^{-1}(p)$ by Σ. Since p is a regular value of f_a, Σ is a properly embedded orientable $(n-1)$–dimensional submanifold of M. Note that the cohomology class $\gamma = 1 \in H^1(\mathbb{S}^1)$ is realized by intersection with p.

Claim *The Poincaré dual of* $[\![\Sigma]\!]$ *is* a^*.

Consider a 1–cycle c. Then

$$\langle a^*, c\rangle = \langle f_a^*(\gamma), c\rangle = \langle \gamma, (f_a)_*(c)\rangle = I(p, f_a(c)) = \sum (\pm 1) f_a^{-1}(p) \cap c = I(\Sigma, c).$$

Thus, the cohomology class a^* is realized by intersection with Σ. This proves the claim.

Since a^* is realized by intersection with Σ, $[\![\Sigma]\!]$ is dual to a^* which is dual to α. Thus, $[\![\Sigma]\!] = \alpha$. □

Proposition 42 *Let M be a compact (possibly closed) connected oriented n–manifold and let α be an infinite element of $H_{n-1}(M, \partial M, \mathbb{Z})$. Then there is a properly embedded orientable $(n-1)$–dimensional weighted submanifold S of M with connected complement such that $[\![\Sigma]\!] = \alpha$.*

Proof It suffices to show that the submanifold Σ provided by Proposition 41 can be tailored to produce a homologous weighted submanifold S with connected complement. We proceed by induction on the number of components of the complement of Σ. If there is only one complementary component, then Σ has connected complement and the proposition follows.

Suppose there are $n > 1$ complementary components and let C be one such component. Partition the components of Σ that limit on C into ∂C_+ and ∂C_- according to whether the coorientation points into or out of C. Construct a new surface Σ^1 by replacing the components of ∂C_- by copies of ∂C_+ and then replacing parallel components of the resulting surface with appropriately weighted single components. Then Σ^1 has at least one fewer complementary component, yet $[\![\Sigma^1]\!] = [\![\Sigma]\!]$.

Proceeding in this manner, repeating the process until there is only one complementary component, we obtain a surface Σ^n with connected complement such that $[\![\Sigma^n]\!] = \alpha$. $\square$

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Department of Mathematics, University of California, Davis
Davis, CA, United States

jcs@math.ucdavis.edu

Received: 29 June 2017 Revised: 19 January 2018