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Modulo 2 counting of Klein-bottle leaves in smooth taut foliations

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We prove a modulo 2 invariance for the number of Klein-bottle leaves in taut foliations. Given two smooth cooriented taut foliations, assume that every Klein-bottle leaf has nontrivial linear holonomy, and assume that the two foliations can be smoothly deformed to each other through taut foliations. We prove that the numbers of Klein-bottle leaves in these two foliations must have the same parity.

57M50, 57R30, 57R57

1 Introduction

Given a smooth cooriented foliation on a three-manifold, it was proved in Bonatti and Firmo [1] that after a generic smooth perturbation, there is no closed leaf with genus greater than 1. This article explores the other side of the story, and proves the deformation invariance of the parity for the number of Klein-bottle leaves in taut foliations. As a corollary, one can construct a taut foliation such that every smooth deformation of it through taut foliations has at least one Klein-bottle leaf.

Let $\mathcal{L}$ be a smooth cooriented 2–dimensional foliation on a smooth three-manifold Y. The foliation $\mathcal{L}$ and the manifold Y are allowed to be nonorientable. By definition, the foliation $\mathcal{L}$ is called a taut foliation if, for every point $p \in Y$, there exists an embedded circle in Y passing through p and being transverse to $\mathcal{L}$.

Let K be a leaf of $\mathcal{L}$ and let $\gamma\colon S^1 \to K$ be a closed oriented curve on K. The holonomy of $\mathcal{L}$ along γ is defined as follows: Take a map $i\colon S^1 \times (-1,1) \to Y$ such that, for every $x \in S^1$, $i(x,0) = \gamma(x)$ and the image of $\{x\} \times (-1,1)$ is transverse to $\mathcal{L}$. The intersection of the image of i with $\mathcal{L}$ then defines a horizontal direction field on $S^1 \times (-1,1)$, and the integration of the direction field defines a map $h_\gamma\colon (-\epsilon,\epsilon) \to (-1,1)$ for ϵ sufficiently small. Up to conjugations, the germ of h_γ at 0 is well-defined and is independent of the choice of i. The holonomy of $\mathcal{L}$ along γ is defined to be the germ of h_γ at 0. The value $h'_\gamma(0)$ is called the linear holonomy of $\mathcal{L}$ along γ.

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Definition 1.1 Let $K \subset Y$ be a closed leaf of $\mathcal{L}$; then K is said to have nontrivial linear holonomy if there exists a closed curve γ on K such that the linear holonomy of $\mathcal{L}$ along γ is not equal to 1.

Let K be a closed 2–dimensional submanifold of Y. If K is cooriented, one can define an element $\mathrm{PD}[K] \in \mathrm{Hom}(H_1(Y;\mathbb{Z});\mathbb{Z})$ as follows. Let $[\gamma]$ be a homology class represented by a closed curve γ; then $\mathrm{PD}[K]$ maps $[\gamma]$ to the oriented intersection number of γ and K. Since $\mathrm{Hom}(H_1(Y;\mathbb{Z});\mathbb{Z}) \cong H^1(Y;\mathbb{Z})$, the element $\mathrm{PD}[K]$ can be viewed as an element of $H^1(Y;\mathbb{Z})$. If both Y and K are oriented and if the orientations of Y and K are compatible with the coorientation of K, then $\mathrm{PD}[K]$ is equal to the Poincaré dual of the fundamental class of K.

Definition 1.2 Let $A \in H^1(Y;\mathbb{Z})$. A closed leaf K of $\mathcal{L}$ is said to have homology class A if $\mathrm{PD}[K] = A$. The foliation $\mathcal{L}$ is called A–admissible if every Klein-bottle leaf of $\mathcal{L}$ in the class A has nontrivial linear holonomy.

The following result is the main theorem of this article:

Theorem 1.3 *Let $A \in H^1(Y;\mathbb{Z})$. Let $\mathcal{L}_s$, $s \in [0,1]$ be a smooth family of coorientable taut foliations on Y. Suppose $\mathcal{L}_0$ and $\mathcal{L}_1$ are both A–admissible. For $i = 0, 1$, let n_i be the number of Klein-bottle leaves in the class A. Then n_0 and n_1 have the same parity.*

Notice that if there is no Klein-bottle leaf of $\mathcal{L}$ in the homology class A, then $\mathcal{L}$ is automatically A–admissible. Therefore, the following result follows immediately:

Corollary 1.4 *Let $A \in H^1(Y;\mathbb{Z})$, and let $\mathcal{L}$ be an A–admissible smooth coorientable taut foliation on Y. Assume that $\mathcal{L}$ has an odd number of Klein-bottle leaves in the class A. Then every smooth deformation of $\mathcal{L}$ through taut foliations has at least one Klein-bottle leaf in the class A.* □

Remark 1.5 It would be interesting to understand whether a similar result holds for torus leaves. Suppose $\mathcal{L}_0$ and $\mathcal{L}_1$ are two cooriented taut foliations on Y that can be smoothly deformed to each other through taut foliations, and suppose that there is a homology class A such that every closed torus leaf of $\mathcal{L}_0$ and $\mathcal{L}_1$ in class A has nontrivial linear holonomy. Is it true that the numbers of torus leaves of $\mathcal{L}_0$ and $\mathcal{L}_1$ in class A always have the same parity? At the time of writing, the answer is not clear to the author.

This article is organized as follows. Sections 2 and 3 build up the necessary tools for the proof of Theorem 1.3. Section 4 proves the theorem, while some technical details are left to Section 5. Section 6 gives an explicit example for Corollary 1.4, and constructs a taut foliation such that every taut deformation of it has at least one Klein-bottle leaf.

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2 Moduli spaces of J–holomorphic tori

This section recalls some properties of the moduli space of J–holomorphic tori in a symplectic manifold. Many results in this section are essentially special cases of Taubes's theory on Gromov invariants [7].

Let X be a smooth 4–manifold. To avoid complications caused by spherical bubbles, assume throughout this section that $\pi_2(X)=0$. This will be enough for the proof of Theorem 1.3. Let J be a smooth almost-complex structure on X.

Consider an immersed closed J–holomorphic curve C in X. Let N be the normal bundle of C; the fiber of N then inherits an almost-complex structure from J. Let $\pi\colon N\to C$ be the projection from N to C. Choose a diffeomorphism φ from a neighborhood of the zero section of N to a neighborhood of C in X such that φ maps the zero section of N to C. The map φ can be chosen in such a way that the tangent map is $\mathbb{C}$–linear on the zero section of N. Every closed immersed J–holomorphic curve that is C^1–close to C is the image of a section of N. Fix an arbitrary connection ∇_0 on N and let $\bar\partial_0$ be the $(0,1)$–part of ∇_0. If s is a section of N near the zero section, the equation for $\varphi(s)$ to be a J–holomorphic curve in X can be schematically written as

$$\bar\partial_0 s+\tau(s)(\nabla_0(s))+Q(s)(\nabla_0(s),\nabla_0(s))+T(s)=0. \tag{2-1}$$

Here τ is a smooth section of $\pi^*(\mathrm{Hom}_{\mathbb{R}}(T^*C\otimes_{\mathbb{R}}N, T^{0,1}C\otimes_{\mathbb{C}}N))$, and Q is a smooth section of $\pi^*(\mathrm{Hom}_{\mathbb{R}}(T^*C\otimes_{\mathbb{R}}T^*C\otimes_{\mathbb{R}}N\otimes_{\mathbb{R}}N, T^{0,1}C\otimes_{\mathbb{C}}N))$, and T is a smooth section of $\pi^*(T^{0,1}C\otimes_{\mathbb{C}}N)$. The values of τ, Q and T are defined pointwise by the values of J in an algebraic way, and τ, Q and T are zero when $s=0$. The linearized equation of (2-1) at $s=0$ is $\bar\partial_0(s)+\frac{\partial T}{\partial s}(s)=0$. Define

$$L(s):=\bar\partial_0(s)+\frac{\partial T}{\partial s}(s). \tag{2-2}$$

Notice that L is only an $\mathbb{R}$–linear operator. The curve C is called nondegenerate if L is surjective as a map from $L^2_1(N)$ to $L^2(T^{0,1}C\otimes_{\mathbb{C}}N)$. By elliptic regularity, if C is nondegenerate, then the operator L is also surjective as a map from $L^2_k(N)$ to $L^2_{k-1}(T^{0,1}C\otimes_{\mathbb{C}}N)$ for every $k\geq 1$. The index of the operator L is given by

$$\operatorname{ind}L=\langle c_1(N),[C]\rangle-\langle c_1(T^{0,1}X),[C]\rangle. \tag{2-3}$$

It follows from the definition that nondegeneracy only depends on the 1–jet of J on C. Namely, if there is another almost-complex structure J' such that $(J-J')|_C=0$ and $(\nabla(J-J'))|_C=0$, then C is nondegenerate as a J–holomorphic curve if and only if it is nondegenerate as a J'–holomorphic curve.

For a homology class $e\in H_2(X;\mathbb{Z})$, define

$$d(e)=e\cdot e-\langle c_1(T^{0,1}X),e\rangle.$$

By (2-3), $d(e)$ is the formal dimension of the moduli space of embedded pseudoholomorphic curves in X in the homology class e. By the adjunction formula, the genus g of such a curve satisfies

$$e\cdot e+2-2g=-\langle c_1(T^{0,1}X),e\rangle.$$

Therefore, $d(e)=2(g-\langle c_1(T^{0,1}X),e\rangle-1)$. In general, the formal dimension of the moduli space of J–holomorphic *maps* from a genus g curve to X (not necessarily embedded) in the homology class e, modulo self-isomorphisms of the domain, is also given by $2(g-\langle c_1(T^{0,1}X),e\rangle-1)$.

Now assume X has a symplectic structure ω. Recall that an almost-complex structure J is compatible with ω if $\omega(\cdot\,,J\cdot)$ defines a Riemannian metric. Let $\mathcal{J}(X,\omega)$ be the set of smooth almost-complex structures compatible with ω. For a closed surface Σ and a map $\rho\colon\Sigma\to X$, define the topological energy of ρ to be $\int_\Sigma\rho^*(\omega)$.

Definition 2.1 Let (X,ω) be a symplectic manifold. Let $E>0$ be a constant. An almost-complex structure $J\in\mathcal{J}(X,\omega)$ is called E–admissible if the following conditions hold:

(1) Every embedded J–holomorphic torus C with topological energy less than or equal to E and with $d([C])=0$ is nondegenerate.

(2) For every homology class $e \in H_2(X;\mathbb{Z})$, if $\langle [\omega], e\rangle \leq E$, and if the formal dimension of the moduli space of J–holomorphic maps from a torus to X in the homology class e, modulo self-isomorphisms of the domain, is negative, then there is no somewhere-injective J–holomorphic map from a torus to X in the homology class e.

The next lemma is a special case of Proposition 7.1 in [8]. Recall that the C^∞–topology on $\mathcal{J}(X,\omega)$ is defined as the Fréchet topology induced by the distance function

$$d(j_1, j_2) = \sum_{n=1}^{\infty} 2^{-n} \cdot \frac{\|j_1 - j_2\|_{C^n}}{1 + \|j_1 - j_2\|_{C^n}}.$$

Lemma 2.2 *Let $E > 0$ be a constant. If (X,ω) is a compact symplectic manifold, the set of E–admissible almost-complex structures forms a dense subset of $\mathcal{J}(X,\omega)$ in the C^∞–topology.* □

For $e \in H_2(X;\mathbb{Z})$, define $\mathcal{M}(X, J, e)$ to be the moduli space of *embedded* J–holomorphic tori in X with fundamental class e. A homology class e is called primitive if e cannot be expressed as $n \cdot e'$ for any integer $n > 1$ and $e' \in H_2(X;\mathbb{Z})$.

Now consider smooth families of almost-complex structures. Assume ω_s for $s \in [0,1]$ is a smooth family of symplectic forms on X. For $i = 0, 1$, let $J_i \in \mathcal{J}(X,\omega_i)$. Define

$$\mathcal{J}(X, \{\omega_s\}, J_0, J_1)$$

to be the set of smooth families $\{J_s\}$ connecting J_0 and J_1 such that $J_s \in \mathcal{J}(X,\omega_s)$ for each $s \in [0,1]$.

Lemma 2.3 *Let X be a compact 4–manifold and let ω_s for $s \in [0,1]$ be a smooth family of symplectic forms on X. Let $e \in H_2(X;\mathbb{Z})$ be a primitive class with $\langle c_1(T^{0,1}X), e\rangle = 0$ and $e \cdot e = 0$, and let $E > 0$ be a constant such that $E \geq \langle [\omega_i], e\rangle$ for $i = 0, 1$. For $i \in \{0,1\}$, let $J_i \in \mathcal{J}(X,\omega_i)$ be an E–admissible almost-complex structure on X. Then there is an open and dense subset $\mathcal{U} \subset \mathcal{J}(X, \{\omega_s\}, J_0, J_1)$ in the C^∞–topology such that, for every element $\{J_s\} \in \mathcal{U}$, the moduli space $\mathcal{M}(X, \{J_s\}, e) = \coprod_{s\in[0,1]} \mathcal{M}(X, J_s, e)$ has the structure of a compact smooth 1–manifold with boundary $\mathcal{M}(X, J_0, e) \cup \mathcal{M}(X, J_1, e)$.*

Proof For general $(X, \{J_s\}, e)$, the moduli space $\mathcal{M}(X, \{J_s\}, e)$ may not be compact. In fact, the compactness of $\mathcal{M}(X, \{J_s\}, e)$ follows from the assumptions that $\pi_2(X) = 0$ and e is primitive. Since $\mathcal{M}(X, \{J_s\}, e)$ only consists of tori, Gromov's compactness theorem (see for example [9]) implies that, for every sequence $\{C_n\} \subset \mathcal{M}(X, \{J_s\}, e)$, there is a subsequence $\{C_{n_i}\}$ with $C_{n_i} \in \mathcal{M}(X, J_{s_i}, e)$ and $\lim_{i\to\infty} s_i = s_0$ such that the sequence C_{n_i} is convergent to the image of one of (1) a possibly branched multiple cover of a somewhere-injective J_{s_0}–holomorphic map, (2) a J_{s_0}–holomorphic map with at least one spherical component, or (3) a somewhere-injective J_{s_0}–holomorphic map from a torus. Case (1) is impossible since e is assumed to be a primitive class. Case (2) is impossible because there are no nonconstant J_{s_0}–holomorphic maps from a sphere to X. When case (3) happens, for the limit curve the adjunction formula states that $e \cdot e + 2 - 2g = -\langle c_1(T^{0,1}X), e\rangle + \kappa$, where κ depends on the behavior of singularities and self-intersections of the curve, and κ is always positive if the curve is not embedded (see [4]). Since $g = 1$, $e \cdot e = 0$ and $\langle c_1(T^{0,1}X), e\rangle = 0$, it follows that $\kappa = 0$, hence the limit curve is an embedded curve, namely it is an element of $\mathcal{M}(X, J_{s_0}, e)$. Therefore, the space $\mathcal{M}(X, \{J_s\}, e)$ is compact.

Since both J_0 and J_1 are E–admissible, the moduli space $\mathcal{M}(X, \{J_s\}, e)$ is cut out transversely at $s = 0$ and $s = 1$. Moreover, since $\mathcal{M}(X, \{J_s\}, e)$ consists of only embedded curves, the standard transversality argument (see for example Section 3.2 of [5]) shows that on a dense subset $\mathcal{V} \subset \mathcal{J}(X, \{\omega_s\}, J_0, J_1)$, the moduli space $\mathcal{M}(X, \{J_s\}, e)$ is a smooth 1–manifold with boundary $\mathcal{M}(X, J_0, e) \cup \mathcal{M}(X, J_1, e)$.

Since $\mathcal{M}(X, \{J_s\}, e)$ is always compact, the transversality condition is an open condition, therefore there exists an open set $\mathcal{U} \subset \mathcal{J}(X, \{\omega_s\}, J_0, J_1)$ such that $\mathcal{V} \subset \mathcal{U}$ and, for every $\{J_s\} \in \mathcal{U}$, the moduli space $\mathcal{M}(X, \{J_s\}, e)$ is a compact smooth 1–manifold with boundary $\mathcal{M}(X, J_0, e) \cup \mathcal{M}(X, J_1, e)$. □

With a little more effort one can generalize Lemma 2.3 to noncompact symplectic manifolds. To start, one needs the following definition:

Definition 2.4 Let (X, ω) be a symplectic manifold, not necessarily compact. Let $J \in \mathcal{J}(X, \omega)$. The pair (ω, J) defines a Riemannian metric g on X. The triple (X, ω, J) is said to have bounded geometry with bounding constant N if the following conditions hold:

(1) The metric g is complete.

(2) The norm of the curvature tensor of g is less than N.

(3) The injectivity radius of (X, g) is greater than $1/N$.

One says that a path $\{(X, \omega_s, J_s)\}$ has uniformly bounded geometry if each (X, ω_s, J_s) has bounded geometry and the bounding constant N is independent of s.

The following lemma is a well-known result:

Lemma 2.5 *Let (X,ω,J) be a triple with bounded geometry, with bounding constant N. Let $e\in H_2(X;\mathbb{Z})$ and let $E>0$ be a constant such that $E\geq\langle[\omega],e\rangle$. Then there is a constant $M(N,E)$, depending only on N and E, such that every connected J–holomorphic curve C with fundamental class e must have diameter less than $M(N,E)$ with respect to the metric defined by $\omega(\cdot\,,J\cdot)$.*

Proof By the monotonicity of area for J–holomorphic curves (see for example [3, Section 2.3. E_2']), the area of $B_p(1/N)\cap C$ is greater than or equal to π/N^2. Since C is connected, this implies that its diameter is bounded by $\mathrm{Area}(C)\cdot 2/N\cdot N^2/\pi$. Notice that the area of C equals $\langle[\omega],e\rangle$, which is bounded by E, hence the diameter is bounded by $2EN/\pi$. □

In the noncompact case, one needs to be more careful about the topology on the space of almost-complex structures. A topology on $\mathcal{J}(X,\omega)$ can be defined as follows. Cover X by countably many compact sets $\{A_i\}_{i\in\mathbb{Z}}$. For each A_i define the C^∞–topology on $\mathcal{J}(A_i,\omega)$. Endow the product space

$$\prod_{i\in\mathbb{Z}}\mathcal{J}(A_i,\omega)$$

with the box topology, and consider the map

$$\mathcal{J}(X,\omega)\hookrightarrow\prod_{i\in\mathbb{Z}}\mathcal{J}(A_i,\omega)$$

defined by restrictions. The topology on $\mathcal{J}(X,\omega)$ is then defined as the pullback of the box topology on the product space.

The topology on $\mathcal{J}(X,\omega)$ does not depend on the choice of the covering $\{A_i\}$. When X is noncompact, the topology on $\mathcal{J}(X,\omega)$ is not first-countable.

For $N>0$, define $\mathcal{J}(X,\omega,N)$ to be the set of almost-complex structures $J\in\mathcal{J}(X,\omega)$ such that (X,ω,J) has bounded geometry with bounding constant N. With the topology given above, the space $\mathcal{J}(X,\omega,N)$ is an open subset of $\mathcal{J}(X,\omega)$.

A topology on $\mathcal{J}(X,\{\omega_s\},J_0,J_1)$ can be defined in a similar way. Cover X by countably many compact sets $\{A_i\}_{i\in\mathbb{Z}}$. For each A_i define the C^∞–topology on $\mathcal{J}(A_i,\{\omega_s\},J_0,J_1)$. The topology on the space $\mathcal{J}(X,\{\omega_s\},J_0,J_1)$ is then defined to be the pullback of the box topology on the product space. This topology does not depend on the choice of the covering $\{A_i\}$.

For $N>0$, define the set $\mathcal{J}(X,\{\omega_s\},J_0,J_1,N)$ to be the set of families $\{J_s\}\in\mathcal{J}(X,\{\omega_s\},J_0,J_1)$ such that $\{(X,J_s,\omega_s)\}$ has uniformly bounded geometry with bounding constant N. Then the set $\mathcal{J}(X,\{\omega_s\},J_0,J_1,N)$ is an open subset of $\mathcal{J}(X,\{\omega_s\},J_0,J_1)$.

The following lemma is essentially a diagonal argument. It explains why the topologies defined above are convenient for the perturbation arguments in this article.

Lemma 2.6 *Let $\{A_n\}_{n\geq 1}$ be a countable, locally finite cover of X by compact subsets. Let ω be a symplectic form on X and let ω_s be a smooth family of symplectic forms on X. Let $N>0$ be a constant. Let $J_i\in\mathcal{J}(X,\omega_i,N)$, where $i=0$ or 1.*

(1) *Let $\varphi\colon\mathcal{J}(X,\omega)\hookrightarrow\prod_n\mathcal{J}(A_n,\omega)$ be the embedding map given by restrictions. For every n, let $\mathcal{U}_n$ be an open and dense subset of $\mathcal{J}(A_n,\omega)$; then $\varphi^{-1}\left(\prod_n\mathcal{U}_n\right)$ is an open and dense subset of $\mathcal{J}(X,\omega)$.*

(2) *Let $\varphi\colon\mathcal{J}(X,\{\omega_s\},J_0,J_1)\hookrightarrow\prod_n\mathcal{J}(A_i,\{\omega_s\},J_0,J_1)$ be the embedding map given by restrictions. For every n, let*

$$\mathcal{U}_n\subset\mathcal{J}(A_n,\{\omega_s\},J_0,J_1)$$

be an open and dense subset; then $\varphi^{-1}\left(\prod_n\mathcal{U}_n\right)$ is an open and dense subset of $\mathcal{J}(X,\{\omega_s\},J_0,J_1)$.

Proof For part (1), the set $\varphi^{-1}\left(\prod_n\mathcal{U}_n\right)$ is open by the definition of the box topology. To prove that $\varphi^{-1}\left(\prod_n\mathcal{U}_n\right)$ is dense, let J be an element of $\mathcal{J}(X,\omega)$. Let $J_n=J|_{A_n}\in\mathcal{J}(A_n,\omega)$. For every n, let $\mathcal{V}_n\subset\mathcal{J}(A_n,\omega)$ be a given open neighborhood of J_n. One needs to find an element $J'\in\mathcal{J}(X,\omega)$ such that $J'|_{A_n}\in\mathcal{V}_n\cap\mathcal{U}_n$. For each n, let D_n be an open neighborhood of A_n such that the family $\{D_n\}$ is still a locally finite cover of X. One obtains the desired J' by perturbing J on the open sets $\{D_n\}$ one by one. To start, perturb the section J on D_1 to obtain a section J_1. Since $\mathcal{U}_1$ is dense it is possible to find a perturbation such that $J_1|_{A_1}\in\mathcal{U}_1\cap\mathcal{V}_1$. Now assume that after perturbations on $D_1,D_2,\dots,D_k$, one obtains a section J_k such that $J_k|_{A_i}\in\mathcal{U}_j\cap\mathcal{V}_j$ for $j=1,2,\dots,k$. Then a perturbation of J_k on D_{k+1} gives a section J_{k+1} such that $J_{k+1}|_{A_{k+1}}\in\mathcal{U}_{k+1}\cap\mathcal{V}_{k+1}$. One can choose the perturbation on D_{k+1} to be sufficiently small that $J_{k+1}|_{A_j}\in\mathcal{U}_j\cap\mathcal{V}_j$ for $j=1,2,\dots,k$. Since $\{D_n\}$ is a locally finite cover of X, on each compact subset of X the sequence $\{J_k\}$ stabilizes for sufficiently large k. The limit $\lim_{k\to\infty}J_k$ then gives the desired J'.

The proof for part (2) is exactly the same; one only needs to change the notation $\mathcal{J}(\cdot,\omega)$ to $\mathcal{J}(\cdot,\{\omega_s\},J_0,J_1)$. □

Remark 2.7 Lemma 2.6 is essentially a result on the box topology, as it does not use any specific property of symplectic topology or almost-complex geometry. Since the lemma above is already sufficient for the purpose of this article, we will not try to give the most general statement here.

Lemma 2.8 *Let X be a 4–manifold and let $e \in H_2(X;\mathbb{Z})$ be a primitive class with $\langle c_1(T^{0,1}X), e\rangle = 0$ and $e\cdot e = 0$. Assume ω_s $(s\in[0,1])$ is a smooth family of symplectic forms on X. Let E be a positive constant such that $E > \langle[\omega_i], e\rangle$ for $i = 0,1$. For $i = 0,1$, assume $J_i \in \mathcal{J}(X,\omega_i,N)$ is E–admissible. If the set $\mathcal{J}(X,\{\omega_s\},J_0,J_1,N)$ is not empty, then there is an open and dense subset $\mathcal{U}\subset \mathcal{J}(X,\{\omega_s\},J_0,J_1,N)$ such that, for each $\{J_s\}\in\mathcal{U}$, the moduli space*

$$\mathcal{M}(X,\{J_s\},e) = \coprod_{s\in[0,1]} \mathcal{M}(X,J_s,e)$$

has the structure of a smooth 1–manifold with boundary $\mathcal{M}(X,J_0,e)\cup\mathcal{M}(X,J_1,e)$. Moreover, if $f\colon X\to\mathbb{R}$ is a smooth proper function on X, then the function defined as

$$\mathfrak{f}\colon \mathcal{M}(X,\{J_s\},e)\to\mathbb{R},\quad C\mapsto \fint_C f\,dA,$$

is a smooth proper function on $\mathcal{M}(X,\{J_s\},e)$. (Here, for $C\in\mathcal{M}(X,J_s,e)$, the form dA is the area form on C defined by J_s and ω_s.)

Proof One first proves that $\mathfrak{f}$ is a proper function. For any constant $z>0$, take a sequence of curves $C_n\in\mathcal{M}(X,\{J_s\},e)$ such that $|\mathfrak{f}(C_n)|<z$. By the definition of $\mathfrak{f}$, there exists a sequence of points $p_n\in C_n$ such that $|f(p_n)|<z$. Since f is a proper function on X, the sequence p_n is bounded on X. By Lemma 2.5, this implies that the curves C_n stay in a bounded subset of X. By the argument for the compact case (Lemma 2.3), the sequence $\{C_n\}$ has a subsequence that converges to another point in $\mathcal{M}(X,\{J_s\},e)$, hence the function $\mathfrak{f}$ is proper.

It remains to prove that there is an open and dense subset $\mathcal{U}\subset\mathcal{J}(X,\{\omega_s\},J_0,J_1,N)$ such that, for every $\{J_s\}\in\mathcal{U}$, the moduli space $\mathcal{M}(X,\{J_s\},e)$ is a smooth 1–dimensional manifold. Let g_s be the metric on X compatible with J_s and ω_s. Let g be a complete metric on X such that $g_s\ge g$ for every s. From now on, the distance function on X is defined by the metric g. By Lemma 2.5, there exists a constant $M>0$ such that the diameter of every J_s–holomorphic curve with topological energy no greater than E is bounded by M. Let $\{B_n\}$ be a countable locally finite cover of X by open balls of radius 1. For every n, let A_n be the closed ball with the same center as B_n and with radius $M+1$. The family $\{A_n\}$ is also a locally finite cover of X. For each n, let $\mathcal{M}_n(X,\{J_s\},e)$ be the set of curves $C\in\mathcal{M}(X,\{J_s\},e)$ such that $C\subset A_n$, and let $\mathcal{M}'_n(X,\{J_s\},e)$ be the set of curves $C\cap\mathcal{M}(X,\{J_s\},e)$ such that $C\cap B_n\neq\varnothing$. By the diameter bound, $\mathcal{M}'_n(X,\{J_s\},e)\subset\mathcal{M}_n(X,\{J_s\},e)$. Take $\mathfrak{f}$ to be the distance function to the center of B_n; it was proved in the previous paragraph that the corresponding function $\mathfrak{f}$ on the moduli space is proper. Since $\mathfrak{f}$ is bounded on $\mathcal{M}'_n(X,\{J_s\},e)$, this implies that $\mathcal{M}'_n(X,\{J_s\},e)$ is a compact set, therefore the transversality condition on $\mathcal{M}'_n(X,\{J_s\},e)$ is open. As a result, there is an open and dense subset $\mathcal{U}_n\subset\mathcal{J}(A_n,\{\omega_s\},J_0,J_1,N)$ such that if $\{J_s\}|_{A_n}\in\mathcal{U}_n$, then the set $\mathcal{M}'_n(X,\{J_s\},e)\subset\mathcal{M}_n(X,\{J_s\},e)$ is a smooth 1–dimensional manifold. Notice that $\{\mathcal{M}'_n(X,\{J_s\},e)\}_{n\geq 1}$ is an open cover of $\mathcal{M}(X,\{J_s\},e)$. It then follows from Lemma 2.6(2) that there is an open and dense subset $\mathcal{U}\subset\mathcal{J}(X,\{\omega_s\},J_0,J_1,N)$ such that for every element $\{J_s\}\in\mathcal{U}$ the set $\mathcal{M}(X,\{J_s\},e)$ is a smooth 1–manifold. □

3 Symplectization of taut foliations

This section discusses a symplectization of *oriented and cooriented* taut foliations. It is the main ingredient for the proof of Theorem 1.3.

Let M be a smooth 3–manifold, let $\mathcal{F}$ be a smooth oriented and cooriented taut foliation on M. Since $\mathcal{F}$ is cooriented, it can be written as $\mathcal{F}=\ker\lambda$ such that λ is a smooth 1–form and is positive in the positive normal direction of $\mathcal{F}$. Since $\mathcal{F}$ is taut, there exists a smooth closed 2–form ω such that $\omega\wedge\lambda>0$ everywhere on M. Choose a metric g_0 on M such that $*_{g_0}\lambda=\omega$. By the Frobenius theorem, $d\lambda=\mu\wedge\lambda$ for a unique 1–form μ satisfying $\langle\mu,\lambda\rangle_{g_0}=0$. Locally, write $\omega=e^1\wedge e^2$, where e^1 and e^2 are orthonormal with respect to the metric g_0. Consider the 2–form $\Omega=\omega+d(t\lambda)$ on $\mathbb{R}\times M$ and the metric g defined by

$$g=\frac{1}{1+t^2}(dt+t\mu)^2+(1+t^2)\lambda^2+(e^1)^2+(e^2)^2.$$

The 2–form Ω is a symplectic form on $\mathbb{R}\times M$, and the metric g is independent of the choice of $\{e^1,e^2\}$ and is compatible with Ω. Let J be the almost-complex structure compatible with (Ω,g). To simplify notation, let X be the manifold $\mathbb{R}\times M$.

Lemma 3.1 [10, Lemma 2.1] *The triple (X,Ω,J) has bounded geometry.* □

Locally, let $\{e_0,e_1,e_2\}$ be the basis of TM dual to $\{\lambda,e^1,e^2\}$, and extend them to $\mathbb{R}$–translation-invariant vector fields on $\mathbb{R}\times M$. Let $\hat{e}_1=e_1-t\mu(e_1)\frac{\partial}{\partial t}$ and

$\hat{e}_2 = e_2 - t\mu(e_2)\frac{\partial}{\partial t}$. The almost-complex structure J is then given by

$$J\frac{\partial}{\partial t} = \frac{1}{1+t^2}e_0, \quad J\hat{e}_1 = \hat{e}_2.$$

Define $\widetilde{\mathcal{F}} = \operatorname{span}\{\hat{e}_1, \hat{e}_2\}$; it is a J–invariant plane field on X.

Lemma 3.2 *The plane field $\widetilde{\mathcal{F}}$ is a foliation on X. Under the projection $\mathbb{R}\times M \to M$, the leaves of $\widetilde{\mathcal{F}}$ project to the leaves of $\mathcal{F}$.*

Proof Since $d\mu \wedge \lambda = d(d\lambda) = 0$, there is a μ_1 such that $d\mu = \mu_1 \wedge \lambda$. Therefore, one has $d(dt + t\mu) = (dt + t\mu)\wedge\mu + t\mu_1 \wedge \lambda$, and $d\lambda = \mu\wedge\lambda$. By the Frobenius theorem, the plane field $\widetilde{\mathcal{F}} = \ker(dt + t\mu) \cap \ker\lambda$ is a foliation. The tangent planes of $\widetilde{\mathcal{F}}$ projects isomorphically to the tangent planes of $\mathcal{F}$ pointwise, thus the leaves of $\widetilde{\mathcal{F}}$ project to the leaves of $\mathcal{F}$. □

It turns out that every closed J–holomorphic curve in X is a closed leaf of $\widetilde{\mathcal{F}}$.

Lemma 3.3 *Let $\rho\colon \Sigma \to X$ be a J–holomorphic map from a closed Riemann surface to X. Then either ρ is a constant map, or it is a branched cover of a closed leaf of $\widetilde{\mathcal{F}}$.*

Proof Since ρ is J–holomorphic, $\rho^*((dt + t\mu)\wedge\lambda) \geq 0$ pointwise on Σ. On the other hand,

$$\int_\Sigma \rho^*((dt + t\mu)\wedge\lambda) = \int_\Sigma \rho^*(d(t\lambda)) = 0.$$

Therefore, $\rho(\Sigma)$ is tangent to $\ker(dt + t\mu)\cap\ker\lambda$, hence either ρ is a constant map, or it is a branched cover of a closed leaf of $\widetilde{\mathcal{F}}$. □

Lemma 3.4 *Let L be a leaf of $\mathcal{F}$ and γ a closed curve on L. Let $\pi\colon \mathbb{R}\times M \to M$ be the projection map. The foliation $\widetilde{\mathcal{F}}$ is then transverse to $\pi^{-1}(\gamma)$ and gives a horizontal foliation on $\pi^{-1}(\gamma) \cong \mathbb{R}\times\gamma$. The holonomy of this foliation along γ is given by the multiplication of $l(\gamma)^{-1}$, where $l(\gamma)$ is the linear holonomy of $\mathcal{F}$ along γ.*

Proof Recall that locally (λ, e^1, e^2) is an orthonormal basis of T^*M and (e_0, e_1, e_2) is its dual basis. Let $(-\epsilon,\epsilon)\times L \subset M$ be a tubular neighborhood of L in M and let z be the first coordinate function on $(-\epsilon,\epsilon)\times L$. The parametrization of the tubular neighborhood can be chosen such that $\frac{\partial}{\partial z} = e_0$. Now λ has the form $\lambda = dz + \nu(z)$, where $\nu(z)$ is a 1–form on L depending on z and $\nu(0)=0$. The condition that $\ker\lambda$ is a foliation is equivalent to

$$d\nu+\frac{\partial\nu}{\partial z}\wedge\nu=0.$$

The 1–form μ satisfies $d\lambda=\mu\wedge\lambda$, therefore $\mu|_L=-\frac{\partial\nu}{\partial z}|_{z=0}$.

Suppose γ is a closed curve on L parametrized by $u\in[0,1]$. Let $(t(u),\gamma(u))$ be a curve in $\mathbb{R}\times M$ that is a lift of γ and tangent to $\tilde{\mathcal{F}}$. Then the function $t(u)$ satisfies $\dot{t}+t\mu(\dot\gamma)=0$. Therefore,

$$t(1)=\exp\biggl(-\int_0^1\mu(\dot\gamma)\,du\biggr)\cdot t(0)=\exp\biggl(\int_0^1\frac{\partial\nu}{\partial z}(0)(\dot\gamma(u))\,du\biggr)\cdot t(0).$$

Now compute the linear holonomy of $\mathcal{F}$ along γ. If $(z(u),\gamma(u))$ is a curve in $(-\epsilon,\epsilon)\times L$ tangent to $\mathcal{F}$, then

$$\dot z+\nu(z)(\dot\gamma)=0. \tag{3-1}$$

If $z_s(u)$ for $s\in[0,\epsilon)$ is a smooth family of solutions to (3-1) with $z_0(u)=0$, then the linearized part $l(u)=\partial z_s/\partial s|_{s=0}(u)$ satisfies

$$\dot l+l\cdot\frac{\partial\nu}{\partial z}\bigg|_{z=0}(\dot\gamma)=0.$$

Therefore, the linear holonomy of $\mathcal{F}$ along γ is

$$\exp\biggl(-\int_0^1\frac{\partial\nu}{\partial z}(0)(\dot\gamma(u))\,du\biggr),$$

hence the linear holonomy of $\mathcal{F}$ along γ is inverse to the holonomy on $\pi^{-1}(\gamma)$ given by $\tilde{\mathcal{F}}$. □

The following result follows immediately from Lemmas 3.3 and 3.4.

Corollary 3.5 *Let C be a closed embedded J–holomorphic curve on X. Then either $C\subset M\times\{0\}$ and C is a closed leaf of $\mathcal{F}$, or C does not intersect the slice $M\times\{0\}$ and it projects diffeomorphically to a closed leaf of $\mathcal{F}$ with trivial linear holonomy.* □

The next lemma studies J–holomorphic tori on X.

Lemma 3.6 *Suppose T is a torus leaf of $\mathcal{F}$ with nontrivial linear holonomy. Then $T\times\{0\}$ is a nondegenerate J–holomorphic curve in X.*

Proof Notice that $d([T])=0$, thus the index of the deformation operator is zero, and one only needs to prove that the operator L on T defined by (2-2) has a trivial kernel.

Let $T_0 = T \times \{0\}$ be the torus in X. As in Lemma 3.4, let (e_0, e_1, e_2) be the dual basis of (λ, e^1, e^2). Let $(-\epsilon, \epsilon) \times T \subset M$ be a tubular neighborhood of T, let z be the first coordinate function and choose a parametrization such that $\frac{\partial}{\partial z} = e_0$. Then on this neighborhood, λ has the form $\lambda = dz + \nu(z)$, where $\nu(z)$ is a 1–form on T depending on z and $\nu(0) = 0$. The condition that $\ker\lambda$ is a foliation is equivalent to

$$d\nu + \frac{\partial \nu}{\partial z} \wedge \nu = 0.$$

Let $\beta = \frac{\partial \nu}{\partial z}\big|_{z=0}$. Apply $\frac{\partial}{\partial z}$ on the equation above at $z = 0$; one obtains $d\beta = 0$. Extend β to $(-\epsilon, \epsilon) \times T$ by pulling back from the second factor. Let $\lambda' = dz + z \cdot \beta$; then $\ker\lambda'$ defines another foliation near T. Let $\mu' = -\beta$.

Let e_1' and e_2' be vector fields on $(-\epsilon, \epsilon) \times T$ such that they are tangent to $\ker\lambda'$ and that their projections to T form a positive orthonormal basis. Consider $\mathbb{R} \times (-\epsilon, \epsilon) \times T$, let t be the coordinate function of the $\mathbb{R}$–component and extend e_1' and e_2' to a neighborhood of T_0 in X by translations in the t–direction. Define an almost-complex structure J' on $\mathbb{R} \times (-\epsilon, \epsilon) \times T$ by

$$J' \frac{\partial}{\partial t} = \frac{\partial}{\partial z}, \quad J'\Big(e_1' - t\mu'(e_1')\frac{\partial}{\partial t}\Big) = e_2' - t\mu'(e_2')\frac{\partial}{\partial t}.$$

One claims that the deformation equation (2-1) for J'–holomorphic curves near T_0 is a linear equation. In fact, let

$$(f, g)\colon T \to \mathbb{R} \times (-\epsilon, \epsilon)$$

be the parametrization of a curve C near T_0. For $p \in T \cong T_0$, let $v = e_1'(p)$ and $w = e_2'(p)$; then the tangent space of C at $(f(p), g(p), p)$ is spanned by $(\partial_v f, \partial_v g, v)$ and $(\partial_w f, \partial_w g, w)$. Notice that

$$\mu'(e_1')(f(p), p) = -\beta(v), \quad \mu'(e_2')(f(p), p) = -\beta(w).$$

Therefore,

$$\begin{aligned} J'_{(f(p),g(p),p)}(\partial_v f, \partial_v g, v) &= (-\partial_v g + \beta(w) f, \partial_v f - \beta(v) f, w), \\ J'_{(f(p),g(p),p)}(\partial_w f, \partial_w g, w) &= (-\partial_w g - \beta(v) f, \partial_w f - \beta(w) f, -v). \end{aligned}$$

Hence, C is J'–holomorphic at $(f(p), g(p), p)$ if and only if

$$\begin{aligned} \beta(w) f &= \partial_v g + \partial_w f, \\ \beta(v) f &= \partial_v f - \partial_w g. \end{aligned}$$

This shows that (2-1) is linear for curves near T_0.

On the other hand, since T has nontrivial linear holonomy, the same arguments as in Lemmas 3.3 and 3.4 show that T_0 is the only embedded J'–holomorphic torus in a neighborhood of T_0. Since (2-1) is linear for T_0, this implies that T_0 is nondegenerate as a J'–holomorphic curve. Recall that J' and J agree up to first-order derivatives along the curve T_0, therefore T_0 is nondegenerate with respect to J. □

4 Proof of Theorem 1.3

Now let $\mathcal{L}$ be a cooriented smooth taut foliation on a smooth 3–manifold Y. Consider its orientation double cover $\widetilde{\mathcal{L}}$. It is an oriented and cooriented taut foliation on the orientation double cover $\widetilde{Y}$ of Y. Let $p\colon \widetilde{Y} \to Y$ be the covering map. If K is a Klein-bottle leaf of $\mathcal{L}$, then $p^{-1}(K)$ is a torus leaf of $\widetilde{L}$. Recall that in the beginning of Section 1, a homology class $\mathrm{PD}[K] \in H^1(Y;\mathbb{Z})$ was defined for every embedded cooriented surface in Y.

Lemma 4.1 *Let K be an embedded cooriented surface in Y; then $p^{-1}(K)$ is cooriented and hence inherits an orientation from $\widetilde{Y}$. Let $\mathrm{PD}[p^{-1}(K)]$ be the Poincaré dual of the fundamental class of $p^{-1}(K)$; then $p^*(\mathrm{PD}[K]) = \mathrm{PD}[p^{-1}(K)]$.*

Proof Let γ be a closed curve in $\widetilde{Y}$. Use $I(\cdot,\cdot)$ to denote the intersection number. Then

$$\begin{aligned}\langle \mathrm{PD}[p^{-1}(K)], [\gamma]\rangle &= I(p^{-1}(K),\gamma)\\ &= I(K, p(\gamma)) = \langle \mathrm{PD}[K], p_*[\gamma]\rangle = \langle p^*(\mathrm{PD}[K]), [\gamma]\rangle.\end{aligned}$$

Therefore, $p^*(\mathrm{PD}[K]) = \mathrm{PD}[p^{-1}(K)]$. □

Lemma 4.2 *The pullback map $p^*\colon H^1(Y;\mathbb{Z}) \to H^1(\widetilde{Y};\mathbb{Z})$ is injective.*

Proof Every element in $\ker p^*$ is represented by an element $\alpha \in \mathrm{Hom}(\pi_1(Y),\mathbb{Z})$ such that α is zero on the image of $p_*\colon \pi_1(\widetilde{Y}) \to \pi_1(Y)$. Since $\mathrm{Im}\, p_*$ is a normal subgroup of $\pi_1(Y)$ with index 2, the map α is decomposed as

$$\alpha\colon \pi_1(Y) \to \pi_1(Y)/\pi_1(\widetilde{Y}) \cong \mathbb{Z}/2 \to \mathbb{Z},$$

which has to be zero. Therefore, p^* is injective. □

By Lemmas 4.1 and 4.2, a Klein-bottle leaf K has $\mathrm{PD}[K] = A$ if and only if $\mathrm{PD}([p^{-1}(K)]) = p^*(A)$. The next lemma shows that for every Klein-bottle leaf K of $\mathcal{L}$, the fundamental class $[p^{-1}(K)]$ is a primitive class.

Lemma 4.3 *Let $\mathcal{F}$ be an oriented and cooriented taut foliation on a smooth three-manifold M; then the fundamental class of every closed leaf of $\mathcal{F}$ is a primitive class.*

Proof Let L be a closed leaf of $\mathcal{F}$. Take a point $p \in L$. By the definition of tautness, there exists an embedded circle γ passing through p and transverse to the foliation. Let $\gamma\colon [0,1] \to M$ with $\gamma(0) = \gamma(1) = p$ be a parametrization of γ. By transversality, $\gamma^{-1}(L)$ is a finite set. Let t_0 be the minimum value of $t > 0$ such that $\gamma(t_0) \in L$. Then, for ϵ sufficiently small, one can slide the part of γ on $(t_0 - \epsilon, t_0 + \epsilon)$ along the foliation so that the resulting curve is still transverse to $\mathcal{F}$, and so that, after sliding, $\gamma(t_0) = p$. Now $\gamma|_{[0,t_0]}$ defines a circle whose intersection number with L equals 1. The existence of such a curve implies that the fundamental class of L is primitive. □

With the preparations above, one can now prove Theorem 1.3.

Proof of Theorem 1.3 Let $A \in H^1(Y;\mathbb{Z})$. Suppose $\mathcal{L}_0$ and $\mathcal{L}_1$ are two smooth A–admissible taut foliations on Y that can be deformed to each other by a smooth family of taut foliations $\mathcal{L}_s$ for $s \in [0,1]$. Let $\widetilde{Y}$ be the orientation double cover of Y. Then the orientation double covers $\widetilde{\mathcal{L}}_s$ of $\mathcal{L}_s$ form a smooth family of oriented and cooriented taut foliations on $\widetilde{Y}$.

Let $\widetilde{\sigma}\colon \widetilde{Y} \to \widetilde{Y}$ be the deck transformation of the orientation double cover. Then the map $\widetilde{\sigma}$ preserves the coorientation of $\widetilde{\mathcal{L}}_s$ and reverses its orientation for each s.

There exists a smooth family of 1–forms λ_s and closed 2–forms ω_s on $\widetilde{Y}$ such that $\widetilde{\mathcal{L}}_s = \ker \lambda_s$ and $\lambda_s \wedge \omega_s > 0$. By changing λ_s to $\frac{1}{2}(\lambda_s + \widetilde{\sigma}^*\lambda_s)$ and changing ω_s to $\frac{1}{2}(\omega_s - \widetilde{\sigma}^*\omega_s)$, one can assume that $\widetilde{\sigma}^*\lambda_s = \lambda_s$, and $\widetilde{\sigma}^*\omega_s = -\omega_s$. Let (Ω_s, J_s) be the corresponding symplectic structures and almost-complex structures on $X = \mathbb{R} \times \widetilde{Y}$ as defined in Section 3. Define

$$\sigma\colon X \to X, \quad (t,x) \mapsto (-t, \widetilde{\sigma}(x)).$$

Then $\sigma^*(\Omega_s) = -\Omega_s$ and $\sigma^*(J_s) = -J_s$. The family $\{(X, \Omega_s, J_s)\}$ has uniformly bounded geometry. This means that there is a constant $N > 0$ such that $J_s \in \mathcal{J}(X, \Omega_s, N)$ for each s.

If neither $\mathcal{L}_0$ nor $\mathcal{L}_1$ has any Klein-bottle leaf in the class A, the statement of Theorem 1.3 is trivially true. From now on assume that either $\mathcal{L}_0$ or $\mathcal{L}_1$ has at least one Klein-bottle leaf in the class A. This implies either $\widetilde{\mathcal{L}}_0$ or $\widetilde{\mathcal{L}}_1$ has at least one torus leaf. By a theorem of Novikov [6] (or see for example Theorem 9.1.7 of [2]), $\pi_2(X)=\pi_2(\widetilde{Y})=0$. Let e be the pushforward of $\mathrm{PD}(p^*(A))\in H_2(\widetilde{Y};\mathbb{Z})$ to $H_2(X;\mathbb{Z})$ via the inclusion map $\widetilde{Y}\cong\{0\}\times\widetilde{Y}\hookrightarrow X$. The class e then satisfies $\sigma_*(e)=-e$. By Lemma 4.3, e is a primitive class.

Take a positive constant E such that $E\geq\langle[\Omega_i],e\rangle$ for $i=0,1$. For every $J'\in\mathcal{J}(X,\Omega_s,N)$ and every $C\in\mathcal{M}(X,J',e)$, Lemma 2.5 gives a diameter bound of C by $M(N,E)$.

Let $t_0>0$ be a fixed positive number. For $i=0,1$, the union of torus leaves L in $\widetilde{\mathcal{L}}_i$ in the homology class $p^*(A)$ such that (1) $\int_L\omega_i\leq E$, and (2) L is not the lift of any Klein-bottle leaf, form a compact set $\widetilde{B}_i$. The set $\widetilde{B}_i$ satisfies $\widetilde{\sigma}(\widetilde{B}_i)=\widetilde{B}_i$. Let $\widetilde{U}_i$ be a neighborhood of $\widetilde{B}_i$ such that $\widetilde{\sigma}(\widetilde{U}_i)=\widetilde{U}_i$ and the closure of $\widetilde{U}_i$ does not intersect the lift of any Klein-bottle leaf of $\mathcal{L}_i$. Let

$$
\begin{aligned}
V&=((-\infty,-t_0)\cup(t_0,\infty))\times\widetilde{Y},\\
U_i&=(\mathbb{R}\times\widetilde{U}_i)\cup((-\infty,-t_0)\cup(t_0,\infty))\times\widetilde{Y}.
\end{aligned}
$$

V and U_i are open subsets of X. The following two lemmas will be proved in Section 5.

Lemma 4.4 *For $i=0,1$, the almost-complex structure J_i can be perturbed to $J_i'\in\mathcal{J}(X,\Omega_i,N)$ such that $J_i'=J_i$ near the lifts of Klein-bottle leaves, and J_i' is E–admissible. Moreover, one can choose J_i' such that the following are satisfied:*

(1) *One has $\sigma^*(J_i')=-J_i'$ on U_i.*

(2) *Every J_i'–holomorphic torus of X in the homology class e is either contained in U_i or is the lift of a Klein-bottle leaf in $\mathcal{L}_i$ in the class A.*

(3) *If C is a J_i'–holomorphic curve in the homology class e contained in U_i, then $\sigma(C)\neq C$.*

Lemma 4.5 *The almost-complex structures J_0' and J_1' given by Lemma 4.4 can be connected by a smooth family of almost-complex structures*

$$J_s'\in\mathcal{J}(X,\Omega_s,N),$$

such that $\sigma^*(J_s')=-J_s'$ on V, and the moduli space

$$\mathcal{M}(X,\{J_s'\},e)=\coprod_{s\in[0,1]}\mathcal{M}(X,J_s',e)$$

has the structure of a smooth 1–manifold with boundary $\mathcal{M}(X, J'_0, e) \cup \mathcal{M}(X, J'_1, e)$. Moreover, let $t\colon X \to \mathbb{R}$ be the projection of $X = \mathbb{R} \times \widetilde{Y}$ to $\mathbb{R}$; then the function

$$\mathfrak{f}\colon \mathcal{M}(X, \{J'_s\}, e) \to \mathbb{R}, \quad C \mapsto \int_C t\, dA,$$

is a smooth proper function on $\mathcal{M}(X, \{J'_s\}, e)$, where for $C \in \mathcal{M}(X, J'_s, e)$, the form dA is the area form on C given by g_s.

Let $\{J'_s\}$ be the family of almost-complex structures given by the lemmas above. By the bound on geometry and the diameter bound, there exists a sufficiently large $t_1 > 0$ such that for every J'_s–holomorphic torus C in the homology class e, if $|\mathfrak{f}(C)| > t_1$, then C is contained in V. Take a constant $t_2 > t_1$ such that both t_2 and $-t_2$ are regular values of $\mathfrak{f}$, and that $t_2 \notin \mathfrak{f}(\mathcal{M}(X, J'_0, e) \cup \mathcal{M}(X, J'_1, e))$. Let $S_i = \mathcal{M}(X, J'_i, e) \cap \mathfrak{f}^{-1}([-t_2, t_2])$. The set $\mathfrak{f}^{-1}(t_2) \cup \mathfrak{f}^{-1}(-t_2) \cup S_0 \cup S_1$ is the boundary of the compact 1–manifold $\mathfrak{f}^{-1}([-t_2, t_2])$, hence it has an even number of elements.

The construction of t_2 implies that every element in $\mathfrak{f}^{-1}(t_2) \cup \mathfrak{f}^{-1}(-t_2)$ is contained in V. The properties of $\{J'_s\}$ given by Lemma 4.5 state that $\sigma^*(J'_s) = -J'_s$ on V, thus σ maps $\mathfrak{f}^{-1}(t_2)$ to $\mathfrak{f}^{-1}(-t_2)$, hence the set $\mathfrak{f}^{-1}(t_2) \cup \mathfrak{f}^{-1}(-t_2)$ has an even number of elements. The properties given by Lemma 4.4 implies that σ acts on the set S_i, and the fixed-point set consists of tori in $\{0\} \times \widetilde{Y}$ which are lifts of Klein-bottle leaves of $\mathcal{L}_i$ in the homology class A. On the other hand, let $\mathcal{K}_i$ be the set of lifts of Klein-bottle leaves of $\mathcal{L}_i$ in the homology class A; then for every

$$C \in \{0\} \times (\mathcal{K}_0 \cup \mathcal{K}_1) \subset \mathcal{M}(X, J'_0, e) \cup \mathcal{M}(X, J'_1, e),$$

one has $\mathfrak{f}(C) = 0$, hence $C \in S_0 \cup S_1$ and it is fixed by σ.

The arguments above showed that the number of elements in $\mathfrak{f}^{-1}(t_2) \cup \mathfrak{f}^{-1}(-t_2) \cup S_0 \cup S_1$ has the same parity as the number of elements in $\mathcal{K}_0 \cup \mathcal{K}_1$. Therefore, the set $\mathcal{K}_0 \cup \mathcal{K}_1$ has an even number of elements, and the desired result is proved. □

5 Technical lemmas

The purpose of this section is to prove Lemmas 4.4 and 4.5. The proofs are routine and straightforward; they are given here for lack of a direct reference. Throughout this section X will be a smooth 4–manifold with $\pi_2(X) = 0$.

Definition 5.1 Let (X,ω) be a symplectic manifold. Let $B \subset X$ be a closed subset. Let $E, N > 0$ be constants. An almost-complex structure $J \in \mathcal{J}(X,\omega,N)$ is called (B,E)–admissible if the following conditions hold:

(1) Every embedded torus C with topological energy less than or equal to E, $d([C]) = 0$ and $[C]$ primitive, and satisfying $C \cap B \neq \varnothing$, is nondegenerate.

(2) For every primitive homology class $e \in H_2(X;\mathbb{Z})$, if $\langle [\omega], e\rangle \leq E$, and if the formal dimension of the moduli space of J–holomorphic maps from a torus to X in the homology class e, modulo self-isomorphisms of the domain, is negative, then there is no somewhere-injective J–holomorphic map ρ from a torus to X in the homology class e such that $\mathrm{Im}(\rho) \cap B \neq \varnothing$.

The next lemma follows from Gromov's compactness theorem and the diameter bound of Lemma 2.5.

Lemma 5.2 *Let (X,ω) be a symplectic manifold. Let $B \subset X$ be a closed subset and $E, N > 0$ be constants. The elements of $\mathcal{J}(X,\omega,N)$ that are (B,E)–admissible form an open and dense subset of $\mathcal{J}(X,\omega,N)$.*

Proof First consider the case when B is compact. The denseness of (B,E)–admissible almost-complex structures then follows from the standard transversality argument. One only needs to prove the openness. Let $M(N,E)$ be the upper bound of diameter given by Lemma 2.5. Suppose J is a (B,E)–admissible almost-complex structure; endow X with the metric given by (J,ω). Let A be a compact set containing B such that the distance between ∂A and B is greater than $M(N,E)+2$. Let $\mathcal{U}$ be a sufficiently small open neighborhood of $J|_A \in \mathcal{J}(A,\omega)$ such that, for every $J' \in \mathcal{J}(X,\omega,N)$, if $J'|_A \in \mathcal{U}$ then the distance between ∂A and B is greater than $M(N,E)+1$. One claims that there is a smaller neighborhood $\mathcal{V} \subset \mathcal{U}$ containing J such that for every $J' \in \mathcal{J}(X,\omega,N)$, if $J'|_A \in \mathcal{V}$ then J' is (B,E)–admissible. In fact, assume the claim is not true; since $\mathcal{J}(A,\omega)$ is first-countable, there is a sequence $\{J_n\} \subset \mathcal{J}(X,\omega,N)$ such that $J_n|_A \to J|_A$ in the C^∞–topology and that every J_n is not (B,E)–admissible. Therefore, for every n, there exists a J_n–holomorphic curve C_n with topological energy no greater than E such that $[C_n]$ is primitive and $C_n \cap B \neq \varnothing$. Moreover, either C_n is an embedded degenerate curve with index zero, or C_n is a curve with negative index. The diameter bound implies $C_n \subset A$ for each n. Gromov's compactness theorem then implies that there is a subsequence of C_n converging to a nonconstant J–holomorphic map, possibly with bubbles, nodal singularities and branched-cover components. Since it is assumed that $\pi_2(X) = 0$ and $[C_n]$ is primitive, the argument in Lemma 2.3 shows that the limit map has to be an embedded J–holomorphic torus. The torus given by the limit map has topological energy less than or equal to E, and it violates the assumption that J is (B,E)–admissible.

Now consider the case when B is not necessarily compact. Choose any $J_0 \in \mathcal{J}(X,\omega,N)$ and define a metric on X by (J_0,ω). Cover B by a locally finite family of compact subsets B_n, let A_n be the closed $(M(N,E)+2)$–neighborhood of B_n. By the arguments in the previous paragraph, for each n there is an open subset $\mathcal{V}_n \subset \mathcal{J}(A_n,\omega)$ such that $J_0|_{A_n} \in \overline{\mathcal{V}}_n$, and such that for every $J' \in \mathcal{J}(X,\omega,N)$ with $J'|_{A_n} \in \mathcal{V}_n$, the almost-complex structure J' is (B_n,E)–admissible. Moreover, if J_0 is (B_n,E)–admissible, then one can choose $\mathcal{V}_n$ such that $J_0 \in \mathcal{V}_n$. Since J' is (B,E)–admissible if and only if it is (B_n,E)–admissible for every n, the desired result follows from Lemma 2.6(1). □

Now let $\sigma\colon X \to X$ be a map that acts diffeomorphically on X such that $\sigma^2 = \mathrm{id}_X$ and the quotient map $X \to X/\sigma$ is a covering map.

Definition 5.3 Let (X,ω) be a symplectic manifold. Let $d, E, N > 0$ be constants. Let B be a closed subset of X such that $\sigma(B) = B$. An almost-complex structure $J \in \mathcal{J}(X,\omega,N)$ is called (d,E)–regular with respect to B if for every J–holomorphic map ρ from a torus to X with topological energy less than or equal to E, at least one of the following conditions hold:

(1) The distance between the sets $\mathrm{Im}(\rho)$ and $\sigma(\mathrm{Im}(\rho))$ is greater than d.

(2) The distance between $\mathrm{Im}(\rho)$ and B is greater than d.

Here the distance is defined by the metric $g_J = \omega(\cdot\,, J\cdot)$ on X.

Remark 5.4 Since the map ρ in the definition above is allowed to be a constant map, for a (d,E)–regular almost-complex structure J with respect to B, one has $\mathrm{dist}(p,\sigma(p)) > d$ for every $p \in B$.

The following result is another consequence of Gromov's compactness theorem, and the proof follows a similar strategy as Lemma 5.2.

Lemma 5.5 *Let $d, E, N > 0$ be constants and B be a closed subset of X such that $\sigma(B) = B$. The elements of $\mathcal{J}(X,\omega,N)$ that are (d,E)–regular with respect to B form an open subset of $\mathcal{J}(X,\omega,N)$.*

Proof First consider the case when B is compact. Let $M(N,E)$ be the upper bound of diameter given by Lemma 2.5. Suppose J is a (d,E)–regular almost-complex structure with respect to B; endow X with the metric given by (J,ω). Let A be a compact set containing B such that the distance between ∂A and B is greater than $M(N,E)+d+2$. Let $\mathcal{U}$ be a sufficiently small open neighborhood of $J|_A \in \mathcal{J}(A,\omega)$ such that, for every $J' \in \mathcal{J}(X,\omega,N)$, if $J'|_A \in \mathcal{U}$ then the distance between ∂A and B is greater than $M(N,E)+d+1$. One claims that there is a smaller neighborhood $\mathcal{V} \subset \mathcal{U}$ containing J such that for every $J' \in \mathcal{J}(X,\omega,N)$, if $J'|_A \in \mathcal{V}$ then J' is (d,E)–regular with respect to B. In fact, assume the claim is not true; since $\mathcal{J}(A,\omega)$ is first-countable, there is a sequence $\{J_n\} \subset \mathcal{J}(X,\omega,N)$ such that $J_n|_A \to J|_A$ in the C^∞–topology and that every J_n is not (d,E)–regular with respect to B. By the definition of (d,E)–regularity, there is a sequence of J_n–holomorphic maps ρ_n from the torus to X with topological energy less than or equal to E such that the distance of $\mathrm{Im}(\rho)$ to B with respect to the metric given by J_n is less than or equal to d, and the distance between $\mathrm{Im}(\rho)$ and $\sigma(\mathrm{Im}(\rho))$ with respect to the metric given by J_n is less than or equal to d. By the diameter bound, every curve C_n is contained in the set A. Gromov's compactness theorem then implies that there is a subsequence of ρ_n converging to a nonconstant J–holomorphic map possibly with bubbles, nodal singularities and branched-cover components. Since is it assumed that $\pi_2(X)=0$, the limit map has to be a possibly branched cover of a torus. The torus given by the limit map has topological energy less than or equal to E, and it violates the assumption that J is (d,E)–regular with respect to B.

Now consider the case when B is not necessarily compact. Let J be a (d,E)–regular almost-complex structure with respect to B. Endow X with the metric given by (J,ω). Cover B by a locally finite family of compact subsets B_n such that $\sigma(B_n)=B_n$ for each n. Let A_n be the closed $(M(N,E)+d+2)$–neighborhood of B_n. By the argument of the previous paragraph, for each n there is an open neighborhood $\mathcal{V}_n$ of $J|_{A_n}$ in $\mathcal{J}(A_n,\omega)$ such that, for every $J' \in \mathcal{J}(X,\omega,N)$, if $J'|_{A_n} \in \mathcal{V}_n$ then J' is (d,E)–regular with respect to B_n. Notice that J' is (d,E)–regular with respect to B if and only if it is (d,E)–regular with respect to every B_n. By the definition of the topology on $\mathcal{J}(X,\omega,N)$, this implies that J has an open neighborhood consisting of (d,E)–regular almost-complex structures with respect to B. $\square$

The following lemma is a 1–parametrized version of Lemma 5.5.

Lemma 5.6 *Let $d,E,N>0$ be constants and B be a closed subset of X such that $\sigma(B)=B$. Let ω_s for $s\in[0,1]$ be a smooth family of symplectic forms on X, and let $J_i \in \mathcal{J}(X,\omega_i,N)$. Then the set of elements $\{J_s\} \in \mathcal{J}(X,\{\omega_s\},J_0,J_1,N)$*

such that every J_s is (d,E)–regular with respect to B forms an open subset of $\mathcal{J}(X,\{\omega_s\},J_0,J_1,N)$.

Proof The proof is exactly the same as Lemma 5.5. One only needs to change the notation J to $\{J_s\}$ and change the notation $\mathcal{J}(X,\omega,N)$ to $\mathcal{J}(X,\{\omega_s\},J_0,J_1,N)$. □

Lemma 5.7 *Let (X,ω) be a symplectic manifold such that $\sigma^*(\omega)=-\omega$. Let $d,E,N>0$ be constants. Let B be a closed subset of X such that $\sigma(B)=B$. Assume $J\in\mathcal{J}(X,\omega,N)$ is (d,E)–regular with respect to B and assume that $\sigma^*(J)=-J$ on B. Then, for every open neighborhood $\mathcal{U}$ of J in $\mathcal{J}(X,\omega,N)$, there is an element J' such that J' is (d,E)–regular with respect to B and is E–admissible, and $\sigma^*(J')=-J'$ on B. Moreover, if there is a closed subset $H\subset X$ such that $\sigma(H)=H$ and J is (H,E)–admissible, then J' can be taken to be equal to J on the set H.*

Proof By shrinking the open neighborhood $\mathcal{U}$, one can assume that every element of $\mathcal{U}$ is (d,E)–regular with respect to B and that there is a complete metric g_0 on X such that $\sigma^*(g_0)=g_0$ and $g_0\le g_{J'}$ for every $J'\in\mathcal{U}$. For the rest of this proof, the metric on X is given by g_0.

Cover X by a locally finite family of closed balls with radius $\frac{d}{10}$. Say

$$X=\bigcup_{i=1}^{+\infty}B_i,$$

where $\{B_i\}$ are closed balls with radius $\frac{d}{10}$. Let D_i be the open $\frac{d}{10}$–neighborhood of B_i; then the diameter of D_i is less than $\frac{d}{2}$.

Let $A_j=\bigcup_{i\le j}B_j$; then $A_0=\varnothing$. The construction of J' follows from induction. Assume that J_j is already (A_j,E)–admissible with $\sigma^*(J_j)=-J_j$ on B; the following paragraph will perturb J_j to J_{j+1} such that J_{j+1} is (A_{j+1},E)–admissible with $\sigma^*(J_{j+1})=-J_{j+1}$ on B.

In fact, if $D_{j+1}\cap B=\varnothing$, then a generic perturbation on D_{j+1} will do the job. If $D_{j+1}\cap B\ne\varnothing$, choose a small perturbation on D_{j+1} such that the resulting almost-complex structure J'_{j+1} is still in $\mathcal{U}$ and is (B_{j+1},E)–admissible. Recall that every element in $\mathcal{U}$ is (d,E)–regular with respect to B; hence, by Remark 5.4 and the diameter bound on D_{j+1}, one has $\sigma(D_{j+1})\cap D_{j+1}=\varnothing$. Now make an additional perturbation on $\sigma(D_{j+1})$ such that the resulting almost-complex structure J_{j+1} satisfies $\sigma(J_{j+1})=-J_{j+1}$ on B. One can choose the perturbation on D_{j+1} to be small enough that J_{j+1} is also in $\mathcal{U}$.

Notice that J_{j+1} is (d,E)–regular with respect to B, the diameter of D_j is less than $\frac{d}{2}$ and $D_j \cap B \neq \varnothing$. One claims that there is no J_{j+1}–holomorphic map from a torus with topological energy less than or equal to E and passing through both D_{j+1} and $\sigma(D_{j+1})$. In fact, assume C passes through both D_{j+1} and $\sigma(D_{j+1})$; then the distance between C and $\sigma(C)$ is less than or equal to $\frac{d}{2}$. Since $D_j \cap B \neq \varnothing$, the distance between C and B is less than or equal to $\frac{d}{2}$. This is contradictory to the fact that J'_{j+1} is (d,E)–regular with respect to B.

Since a J_{j+1}–holomorphic map from a torus with topological energy less than or equal to E can never pass through both D_{j+1} and $\sigma(D_{j+1})$, the almost-complex structure J'_{j+1} being (D_{j+1},E)–admissible implies that J_{j+1} is (D_{j+1},E)–admissible. By Lemma 5.2, being (A_j,E)–admissible is an open condition, thus when the perturbation is sufficiently small J_{j+1} is also (A_j,E)–admissible. Therefore, one can choose J_{j+1} such that the almost-complex structure J_{j+1} is (A_{j+1},E)–admissible. Since the family $\{D_n\}$ is locally finite, on each compact set the sequence $\{J_j\}$ stabilizes for sufficiently large j. The desired J' can then be obtained by taking $\lim_{j\to\infty} J_j$. Moreover, if there is a closed subset $H \subset X$ such that $\sigma(H) = H$ and J is (H,E)–admissible, then each step of the perturbation can be taken to be outside of H. □

The following lemma is a 1–parametrized version of Lemma 5.7, and the proof is essentially the same.

Lemma 5.8 *Let $e \in H_1(Y;\mathbb{Z})$ be a primitive class. Let B be a closed subset of X such that $\sigma(B) = B$. Assume ω_s $(s \in [0,1])$ is a smooth family of symplectic forms such that $\sigma^*(\omega_s) = -\omega_s$ for each s. Let $d, N > 0$ be constants and let $E > 0$ be a constant such that $E \geq \langle e, [\omega_i]\rangle$ for $i = 0,1$. For $i = 0,1$, assume $J_i \in \mathcal{J}(X,\omega_i,N)$ is E–admissible and (d,E)–regular with respect to B. Assume $\{J_s\} \in \mathcal{J}(X,\{\omega_s\},J_0,J_1,N)$ is such that, for each s, the almost-complex structure J_s is (d,E)–regular with respect to B, and $\sigma^*(J_s) = -J_s$ on B. Then, for every open neighborhood $\mathcal{U}$ of $\{J_s\}$ in $\mathcal{J}(X,\{\omega_s\},J_0,J_1,N)$, there is an element $\{J'_s\}$ such that $\{J'_s\}$ is (d,E)–regular with respect to B, and the moduli space $\mathcal{M}(X,\{J'_s\},e) = \coprod_{s\in[0,1]} \mathcal{M}(X,J'_s,e)$ has the structure of a smooth 1–manifold with boundary $\mathcal{M}(X,J_0,e) \cup \mathcal{M}(X,J_1,e)$, and $\sigma^*(J'_s) = -J'_s$ on B for every s. Moreover, if there is a closed subset $H \subset X$ such that $\sigma(H) = H$ and $\{J_s\}$ is (H,E)–admissible for every s, then J'_s can be taken to be equal to J_s on the set H.*

Proof The proof follows verbatim as the proof of Lemma 5.7. One only needs to change the notation J to $\{J_s\}$ and change $\mathcal{J}(X,\omega,N)$ to $\mathcal{J}(X,\{\omega_s\},J_0,J_1,N)$. □

Combining the results above, one obtains the following lemma:

Lemma 5.9 *Let $e \in H_2(X;\mathbb{Z})$ be a primitive class. Let B be a closed subset of X such that $\sigma(B) = B$. Assume ω_s $(s \in [0,1])$ is a smooth family of symplectic forms on X such that $\sigma^*(\omega_s) = -\omega_s$ for each s. Let $d, N > 0$ be constants. Let E be a positive constant such that $E \geq \langle [\omega_i], e\rangle$ for $i = 0, 1$. For $i = 0, 1$, assume $J_i \in \mathcal{J}(X, \omega_i, N)$ is E–admissible and (d, E)–regular with respect to B. Let $\mathcal{J}$ be the subset of elements $\{J_s\}$ of $\mathcal{J}(X, \{\omega_s\}, J_0, J_1, N)$ such that, for each s, the almost-complex structure J_s is (d, E)–regular with respect to B, and $\sigma^*(J_s) = -J_s$ on B. If $\mathcal{J}$ is not empty, let $\mathcal{U} \subset \mathcal{J}$ be the subset of $\mathcal{J}$ such that, for every $\{J_s\} \in \mathcal{U}$, the moduli space $\mathcal{M}(X, \{J_s\}, e) = \coprod_{s\in[0,1]} \mathcal{M}(X, J_s, e)$ has the structure of a smooth 1–manifold with boundary $\mathcal{M}(X, J_0, e) \cup \mathcal{M}(X, J_1, e)$. Then $\mathcal{J}$ is open and $\mathcal{U}$ is dense in $\mathcal{J}$. Moreover, if $f\colon X \to \mathbb{R}$ is a smooth proper function on X, then the function defined as*

$$\mathfrak{f}\colon \mathcal{M}(X, \{J_s\}, e) \to \mathbb{R}, \quad C \mapsto \int_C f\, dA.$$

is a smooth proper function on $\mathcal{M}(X, \{J_s\}, e)$, where, for $C \in \mathcal{M}(X, J_s, e)$, the area form dA of C is given by (J_s, ω_s).

Proof The openness of $\mathcal{J}$ follows from Lemma 5.6. The fact that $\mathcal{U}$ is dense in $\mathcal{J}$ follows from Lemma 5.8. The properness of the function $\mathfrak{f}$ was proved in Lemma 2.8. □

The following lemma controls the location of pseudoholomorphic curves after perturbation of the almost-complex structure:

Lemma 5.10 *Let (X, ω) be a symplectic manifold and let $J \in \mathcal{J}(X, \omega, N)$. Let $E > 0$ be a positive constant and let B be a closed subset of X. Assume that there is no nonconstant J–holomorphic map ρ from a torus to X such that $\mathrm{Im}(\rho) \cap B$ is nonempty and the topological energy of ρ is no greater than E. Then there is an open neighborhood $\mathcal{U}$ of J in $\mathcal{J}(X, \omega, N)$ such that, for every $J' \in \mathcal{U}$, there is no embedded J'–holomorphic torus in X intersecting B with topological energy less than or equal to E.*

Proof Endow X with the metric given by (ω, J). Cover the set B by a locally finite family of compact subsets B_n. Let $M(N, E)$ be the upper bound given by Lemma 2.5 for geometry bound N and energy bound E. Let A_n be the closed $(M(N, E)+1)$–neighborhood of B_n.

One claims that there is an open neighborhood $\mathcal{U}_n$ of $J|_{A_n} \in \mathcal{J}(A_n, \omega)$ such that for every $J' \in \mathcal{J}(A_n, \omega, N)$, if $J'|_{A_n} \in \mathcal{U}_n$, then there is no embedded J'–holomorphic torus in X intersecting B_n with topological energy less than or equal to E. Assume the result does not hold, then there is a sequence of $J_n \subset \mathcal{J}(A, \omega, N)$ such that for each n there exists a J_n–holomorphic map ρ_n from a torus to X which intersects B and has topological energy less than or equal to E, and $J_n|_{A_n} \to J|_{A_n}$. For sufficiently large n, the distance between ∂A_n and B_n is greater than $M(N, E)$ with respect to the distance given by J_n, therefore the relevant J_n–holomorphic curve is contained in A_n. By Gromov's compactness theorem, a subsequence of ρ_n will give a nonconstant J–holomorphic map from a torus to A_n such that the intersection $\mathrm{Im}(\rho) \cap B$ is nonempty, and the topological energy of ρ is less than or equal to E, which is a contradiction. Therefore, the claim holds. The result of the lemma then follows from Lemma 2.6(1). □

With the preparations above, one can now prove Lemmas 4.4 and 4.5:

Proof of Lemma 4.4 Let g_i be the metric on X given by (Ω_i, J_i). By Corollary 3.5, every $C \in \mathcal{M}(X, J_i, e)$ either satisfies $\sigma(C) \cap C = \varnothing$, or C is the lift of a Klein-bottle leaf. Since the space of torus leaves in Y is compact, there exists a positive constant $d_i^{(1)} > 0$ such that when $\sigma(C) \cap C = \varnothing$, the distance between C and $\sigma(C)$ with respect to g_i is greater than $d_i^{(1)}$. Let $d_i^{(2)}$ be the distance from $\widetilde{U}_i$ to the union of the lifts of Klein-bottle leaves.

Recall that

$$U_i = (\mathbb{R} \times \widetilde{U}_i) \cup ((-\infty, -t_0) \cup (t_0, \infty)) \times \widetilde{Y},$$

and $\overline{U}_i$ is the closure of U_i. Let

$$d_i^{(3)} = \tfrac{1}{3} \inf_{p \in \overline{U}_i} d_{g_i}(p, \sigma(p)).$$

Let

$$d = \min_{\substack{i=0,1 \\ j=1,2,3}} d_i^{(j)}.$$

Fix a metric g_* on X such that $g_* \le g_i$ for $i = 0, 1$, and define (d, E)–regularity using the metric g_*. For every $E > 0$, the almost-complex structure J_i is (d, E)–regular with respect to $\overline{U}_i$. In fact, every J_i–holomorphic map from a torus to X is one of (1) a constant map, (2) a covering to the lift of a torus leaf, or (3) a covering to the lift of a Klein-bottle leaf. Let C be its image. In case (1), either the distance from C to $\overline{U}_i$ is at least $d_i^{(3)}$, or the distance from C to $\sigma(C)$ is at least $d_i^{(3)}$. In case (2), the distance from C to $\sigma(C)$ is at least $d_i^{(1)}$. In case (3), the distance from C to $\overline{U}_i$ is at least $d_i^{(2)}$. Choose E to be any positive constant such that $E \geq \max_i \langle \Omega_i, e \rangle$.

Apply Lemma 5.7 to $B = \overline{U}_i$; there is a perturbation

$$J_i' \in \mathcal{J}(X, \Omega_i, N)$$

of J_i such that J_i' is E–admissible and $\sigma^*(J_i') = -J_i'$ on $\overline{U}_i$. Let W_i be a small open neighborhood of the union of lifts of Klein-bottle leaves such that $\sigma(W_i) = W_i$ and J_i is $(\overline{W}_i, E)$–admissible. The almost-complex structure J_i' can then be taken to be equal to J_i on $\overline{W}_i$. By the definition of the set U_i, every J_i–holomorphic map from a torus to X in the homology class e is either a lift of a Klein-bottle leaf in $\{0\} \times Y$ or is mapped into the set U_i. Therefore, Lemma 5.10 shows that when the perturbation is sufficiently small, every J_i'–holomorphic torus with homology class e is either contained in U_i or is contained in W_i. In the latter case the curve is contained in $\widetilde{Y} \times \{0\}$ and it is a lift of a Klein-bottle leaf of $\mathcal{L}_i$ in class A. Since J_i' is (d, E)–regular with respect to $\overline{U}_i$, for every J_i' holomorphic torus C in U_i one has $\sigma(C) \neq C$. □

Proof of Lemma 4.5 The almost-complex structures J_0' and J_1' can be connected by a smooth family of almost-complex structures $J_s' \in \mathcal{J}(X, \Omega_s, J_0', J_1', N)$ such that $\sigma^*(J_s') = -J_s'$ on the closure of V. Using Lemma 5.9, the family J_s' can be further perturbed to satisfy the desired conditions. □

6 An example

This section gives an example of a taut foliation with an odd number of Klein-bottle leaves such that every closed leaf has nontrivial linear holonomy. By Corollary 1.4, every deformation of such a foliation via taut foliations has at least one Klein-bottle leaf.

Think of the torus $T_0 = S^1 \times S^1$ as a trivial S^1–bundle over S^1. Let $z_1, z_2 \in \mathbb{R}/2\pi$ be the coordinates of the two S^1 factors, where z_1 is the coordinate for the fiber and z_2 is the coordinate for the base. Let γ be a closed curve on the base that wraps the S^1 once in the positive direction. Take a horizontal foliation $\widehat{\mathcal{I}}$ on T_0 such that the holonomy along γ has two fixed points $z_1 = 0$ and $z_1 = \pi$, and that holonomy map has nontrivial linearization at these two points. Moreover, choose $\widehat{\mathcal{I}}$ so that it is invariant under the map $(z_1, z_2) \mapsto (z_1 + \pi, \pi - z_2)$ and the map $(z_1, z_2) \mapsto (z_1, z_2 + \pi)$. Figure 1 gives

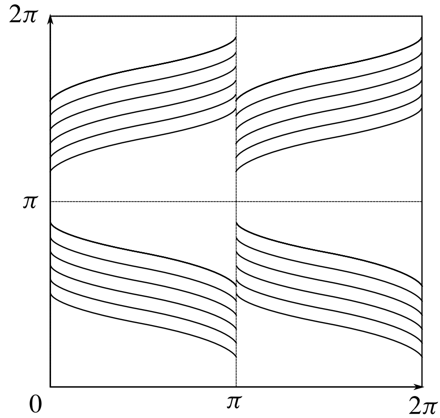

Figure 1: The foliation $\widehat{\mathcal{I}}$ on $S^1 \times S^1$

a picture for such a foliation $\widehat{\mathcal{I}}$, where z_2 is the horizontal coordinate, and z_1 is the vertical coordinate.

Consider the pullback of the foliation $\widehat{\mathcal{I}}$ to $T_0 \times S^1$. Let $z_3 \in \mathbb{R}/2\pi$ be the coordinate for the S^1 factor; then $\operatorname{span}\{\widehat{\mathcal{I}}, \partial/\partial z_3\}$ defines a foliation $\mathcal{I}$ on $T_0 \times S^1$. There are exactly two torus leaves in $\mathcal{I}$, and they are given by $z_1 = 0$ and $z_1 = \pi$.

The foliation $\mathcal{I}$ is invariant under the maps

$$\sigma_1\colon (z_1, z_2, z_3) \mapsto (z_1 + \pi, \pi - z_2, z_3),$$
$$\sigma_2\colon (z_1, z_2, z_3) \mapsto (z_1, z_2 + \pi, -z_3),$$
$$\sigma_3\colon (z_1, z_2, z_3) \mapsto (z_1 + \pi, -z_2, -z_3).$$

The set $V = \{\mathrm{id}, \sigma_1, \sigma_2, \sigma_3\}$ is a group acting freely and discontinuously on $T_0 \times S^1$ and it preserves the coorientation of $\mathcal{I}$. The two torus leaves in $T_0 \times S^1$ are identified under the quotient by V, and their images give the unique Klein-bottle leaf in $\mathcal{I}/V$. Moreover, the Klein-bottle leaf has nontrivial linear holonomy. Therefore, Corollary 1.4 implies the following result:

Proposition 6.1 *Every deformation of $\mathcal{I}/V$ through taut foliations must have at least one Klein-bottle leaf.* □

References

[1] **C Bonatti**, **S Firmo**, *Feuilles compactes d'un feuilletage générique en codimension* 1, Ann. Sci. École Norm. Sup. 27 (1994) 407–462 MR

[2] **A Candel**, **L Conlon**, *Foliations, II*, Graduate Studies in Mathematics 60, Amer. Math. Soc., Providence, RI (2003) MR

[3] **M Gromov**, *Pseudo holomorphic curves in symplectic manifolds*, Invent. Math. 82 (1985) 307–347 MR

[4] **D McDuff**, *The local behaviour of holomorphic curves in almost complex 4–manifolds*, J. Differential Geom. 34 (1991) 143–164 MR

[5] **D McDuff**, **D Salamon**, *J–holomorphic curves and symplectic topology*, 2nd edition, American Mathematical Society Colloquium Publications 52, Amer. Math. Soc., Providence, RI (2012) MR

[6] **S P Novikov**, *The topology of foliations*, Trudy Moskov. Mat. Obšč. 14 (1965) 248–278 MR In Russian; translated in Trans. Mosc. Math. Soc. 14 (1965) 268–304

[7] **C H Taubes**, *Counting pseudo-holomorphic submanifolds in dimension 4*, J. Differential Geom. 44 (1996) 818–893 MR

[8] **C H Taubes**, *$\mathrm{SW} \Rightarrow \mathrm{Gr}$: from the Seiberg–Witten equations to pseudo-holomorphic curves*, J. Amer. Math. Soc. 9 (1996) 845–918 MR

[9] **R Ye**, *Gromov's compactness theorem for pseudo holomorphic curves*, Trans. Amer. Math. Soc. 342 (1994) 671–694 MR

[10] **B Zhang**, *A monopole Floer invariant for foliations without transverse invariant measure*, preprint (2016) arXiv

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