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The double n–space property for contractible n–manifolds

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Motivated by a recent paper of Gabai (J. Topol. 4 (2011) 529–534) on the Whitehead contractible 3–manifold, we investigate contractible manifolds M^n which decompose or *split* as $M^n = A \cup_C B$ with $A, B, C \approx \mathbb{R}^n$ or $A, B, C \approx \mathbb{B}^n$. Of particular interest to us is the case $n = 4$. Our main results exhibit large collections of 4–manifolds that split in this manner.

57N13; 57N15

1 Introduction and acknowledgements

Introduction

Our results will generally be in the topological category, but because the spaces involved are so nice we are able to work in both the piecewise linear and smooth categories in our effort to obtain them. Rourke and Sanderson [16] is a good source for the piecewise linear theory we will employ.

Definition 1.1 We will write $A \cup_C B$ to indicate a union $A \cup B$ with intersection given by $A \cap B = C$. We say a manifold M^n *splits* if $M^n = A \cup_C B$ with $A, B, C \approx \mathbb{B}^n$ or $A, B, C \approx \mathbb{R}^n$. In the former case we say M *splits into closed balls* or M is a *closed splitter* and write $M^n = \mathbb{B}^n \cup_{\mathbb{B}^n} \mathbb{B}^n$. In the latter case we say M *splits into open balls* or M is an *open splitter* and write $M^n = \mathbb{R}^n \cup_{\mathbb{R}^n} \mathbb{R}^n$.

We are interested in contractible manifolds M^n which are open or closed splitters. We introduce a 4–manifold M containing a spine, which we call a jester's hat, that can be written as $A \cup_C B$ with A, B and C all collapsible. We'll show that this implies M is a closed splitter. Using M as a model we obtain a countably infinite collection of distinct 4–manifolds all of which are closed splitters.

Theorem 1.2 *There exists an infinite collection of topologically distinct splittable compact contractible 4–manifolds. The interiors of these are topologically distinct contractible splittable open 4–manifolds.*

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By combining the above examples with an infinite connected sum operation, we will then prove the following.

Theorem 1.3 *There exists an uncountable collection of contractible open 4–manifolds which split as* $\mathbb{R}^4 \cup_{\mathbb{R}^4} \mathbb{R}^4$.

Our motivation comes from David Gabai's result [6] that the Whitehead 3–manifold, Wh^3, splits into open 3–balls:

$$\mathrm{Wh}^3 = \mathbb{R}^3 \cup_{\mathbb{R}^3} \mathbb{R}^3.$$

Other terminology in use which is synonymous with open splitting includes *double n–space property* and *Gabai splitting*. Garity, Repovš and Wright [7] have recently discovered uncountable collections of both 3–dimensional contractible open splitters and 3–dimensional contractible nonsplitters (see Theorems 2.4 and 2.5).

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2 Background and history

2.1 Elementary results

It is clear that the unit ball $\mathbb{B}^n$ splits into two "subballs" overlapping in a n–ball. Likewise, euclidean space splits into two euclidean spaces meeting in a euclidean space. More generally, we have the following (which was assumed without proof in [9]). A proof can be found in [17, Proposition 1.2.1].

Proposition 2.1 *If* M^n *splits as* $M^n = \mathbb{B}^n \cup_{\mathbb{B}^n} \mathbb{B}^n$, *then* $\operatorname{int} M^n$ *splits as* $\operatorname{int} M^n = \mathbb{R}^n \cup_{\mathbb{R}^n} \mathbb{R}^n$.

2.2 History and current work

Some classical knowledge about manifold splitting was provided by Glaser.

Theorem 2.2 (a) *For each* $n \geq 4$ *there exists a compact contractible PL* n–*manifold with boundary,* W^n, *not homeomorphic to* $\mathbb{B}^n$, *such that* $W^n \approx \mathbb{B}^n \cup_{\mathbb{B}^n} \mathbb{B}^n$.

(b) *For each* $n \geq 3$ *there exist an open contractible* n–*manifold* O^n, *not homeomorphic to* $\mathbb{R}^n$, *such that* $O^n \approx \mathbb{R}^n \cup_{\mathbb{R}^n} \mathbb{R}^n$.

For the compact case, Glaser shows the existence of a contractible $(n-2)$–complex piecewise linearly embedded in S^n which has a nonball regular neighborhood which splits. The $n \geq 5$ case was shown in [9] and the $n = 4$ case was shown in [10].

For the noncompact $n \geq 4$ case he takes the interiors of the compact splitters found in (a). For the noncompact $n = 3$ case, Glaser shows that the complement of a certain embedding of a double Fox–Artin arc in S^3 splits and is not an (open) ball [10].

In [6], Gabai asks:

Question 2.3 Is there a reasonable characterization of open contractible 3–manifolds that are the union of two embedded submanifolds each homeomorphic to $\mathbb{R}^3$ and that intersect in a copy of $\mathbb{R}^3$?

Renewed interest in this topic, motivated by Gabai's splitting of the Whitehead manifold and the resulting question above, has led to the following recent results.

Theorem 2.4 [7] *There exist uncountably many distinct contractible 3–manifolds that are open splitters.*

Theorem 2.5 [7] *There are uncountably many distinct contractible 3–manifolds that are not open splitters.*

Note 2.6 In dimension 3, the Poincaré conjecture gives that every compact contractible manifold is homeomorphic to $\mathbb{B}^3$, so the question of closed splitters in this case is uninteresting.

Earlier work of Ancel and Guilbault [1] and more recent work of Ancel, Guilbault and Sparks [2] provides a great deal of information about splitters in dimensions greater than or equal to 5.

Theorem 2.7 *If C^n $(n \geq 5)$ is a compact, contractible n–manifold, then C^n splits as $\mathbb{B}^n \cup_{\mathbb{B}^n} \mathbb{B}^n$.*

Corollary 2.8 *For $n \geq 5$,*

(1) *the interior of every compact contractible n–manifold is an open splitter, and*

(2) *there are uncountably many nonhomeomorphic n–manifolds which are open splitters.*

Theorem 2.9 *For $n \geq 5$, every Davis n–manifold is an open splitter.*

3 The Mazur and jester's manifolds

3.1 The Mazur manifold

In [14], Barry Mazur described what are now often called *Mazur manifolds*. Starting with an $S^1 \times \mathbb{B}^3$, one adds a 2–handle $h^{(2)} \approx \mathbb{B}^2 \times \mathbb{B}^2$ along the curve Γ shown in Figure 1. That is,

$$\mathrm{Ma}^4_\Phi = (S^1 \times \mathbb{B}^3) \cup_\Phi (\mathbb{B}^2 \times \mathbb{B}^2)$$

is a *Mazur manifold*. Here Φ is a framing $\Phi\colon S^1 \times \mathbb{B}^2 \to T_\Gamma$, where T_Γ is a tubular neighborhood of Γ in $\partial(S^1 \times \mathbb{B}^3)$ and the domain $S^1 \times \mathbb{B}^2$ is the first term in the union

$$(S^1 \times \mathbb{B}^2) \cup (\mathbb{B}^2 \times S^1) = \partial(\mathbb{B}^2 \times \mathbb{B}^2).$$

For each Dehn twist of the $S^1 \times S^1 = \partial(S^1 \times \mathbb{B}^2)$ sending $S^1 \times p$ $(p \in S^1)$ to a closed curve (that is, an integer number of full twists), there exists a framing Φ. Thus the number of framings is infinite. Mazur chose a specific framing φ yielding a specific manifold, which we'll denote Ma^4, for which he showed that $\partial\,\mathrm{Ma}^4 \not\approx S^3$, so $\mathrm{Ma}^4 \not\approx \mathbb{B}^4$. The chosen framing corresponds to a parallel copy of Γ, say $\Gamma' = \varphi(S^1 \times p)$, which lies at the "top" (the up direction is perpendicular to the page, toward the viewer) of $S^1 \times \mathbb{B}^2$. Thus there are no twists with this framing.

Here we'll describe our interpretation of his argument for the nontriviality of $\pi_1(\partial\,\mathrm{Ma}^4)$. The details of this calculation will play a key role in our proof of Theorem 5.8. Starting with the link $\Gamma \cup \zeta$ in S^3 pictured in Figure 2, we obtain the corresponding Wirtinger presentation (see [15, page 56] for a treatment of Wirtinger presentations). This gives a presentation with exactly one generator for each arc in the link diagram. These generators correspond to the loops in S^3 which start at the viewer's nose (the basepoint), travel under the arc, and then return home (to the nose). Thus in our picture the generators are the x_i as pictured. The relators in the presentation

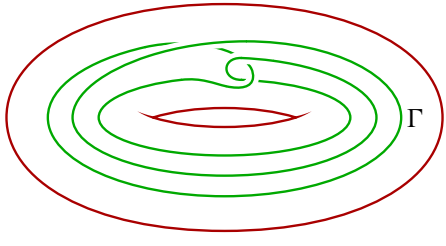


Figure 1: $\Gamma \subset \partial(S^1 \times \mathbb{B}^3)$

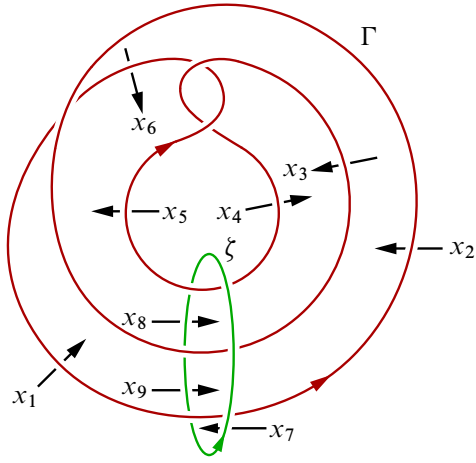


Figure 2: Wirtinger diagram of the Mazur link

correspond to the undercrossings of pairs of arcs. As there are nine undercrossings, the Wirtinger presentation of this link diagram has nine generators and nine relators: $\langle x_1, \dots, x_9 \mid r_1, \dots, r_9 \rangle$. We then perform a Dehn drilling on a tubular neighborhood, $N(\zeta) \approx \mathbb{B}^2 \times S^1$, of ζ. That is, we remove int $N(\zeta)$. Next, we perform a Dehn filling by sewing in $N(\zeta)$ backwards (ie sewing in an $S^1 \times \mathbb{B}^2$) along $\partial N(\zeta)$. This Dehn surgery on $S^3 \approx (S^1 \times \mathbb{B}^2) \cup_{S^1 \times S^1} (\mathbb{B}^2 \times S^1)$ results in an $(S^1 \times \mathbb{B}^2) \cup_{S^1 \times \partial \mathbb{B}^2} (S^1 \times \mathbb{B}^2) \approx S^1 \times S^2$ with Γ embedded as in Figure 1. This surgery exchanges the meridian of $N(\zeta)$ with the longitude. Thus the group element corresponding to following around ζ is killed, and we must add in a relator, say $r_\zeta = x_5 x_2^{-1} x_1^{-1} = 1$, to our presentation to adjust for this.

Adding a 2–handle along Γ (and throwing out its portion of the interior of Ma^4) gives our $\partial \mathrm{Ma}^4 = (S^1 \times S^2 - \operatorname{int} N(\Gamma)) \cup_{\partial N(\Gamma)} (\mathbb{B}^2 \times S^1)$. We describe the gluing of $\mathbb{B}^2 \times S^1$ in two steps. We first glue in a thickened meridional disc D, which kills off the curve Γ' to which it is it is attached. Thus to our Wirtinger presentation we introduce a relator $r_\Gamma = x_7^{-1} x_5^{-1} x_7 x_3^{-1} x_2^{-1} x_7^{-1} = 1$. We next glue on the rest of $\mathbb{B}^2 \times S^1$. The closed complement of D in $\mathbb{B}^2 \times S^1$ is a 3–ball and it is attached along its entire boundary. Adding such does not change the fundamental group and thus $\pi_1(\partial \mathrm{Ma}^4) \cong \langle x_1, \dots, x_9 \mid r_1, \dots, r_9, r_\zeta, r_\Gamma \rangle$.

Proceeding as in [14], let $\beta = x_7$, $\lambda = x_2$ (see Figure 2) and $\alpha = \beta\lambda$. Via Tietze transformations (see [8, page 79] for a treatment of Tietze transformations), it was

shown in [14] that

$$\pi_1(\partial \mathrm{Ma}^4) \cong \langle \alpha, \beta \mid \beta^5 = \alpha^7,\ \beta^4 = \alpha^2\beta\alpha^2 \rangle,$$
$$G := \pi_1(\partial \mathrm{Ma}^4)/\mathrm{nc}\{\beta^5 = 1\} \cong \langle \beta, \gamma \mid \gamma^7 = \beta^5 = (\beta\gamma)^2 = 1 \rangle,$$

where $\gamma = \alpha^2$. We claim G maps nontrivially into the subgroup of the isometries of the hyperbolic plane generated by reflections in the geodesics containing the edges of a triangle with angles $\frac{\pi}{7}$, $\frac{\pi}{5}$, and $\frac{\pi}{2}$. That is, there exists a homomorphism

$$h\colon G \to \mathrm{Isom}(\mathbb{H}^2)$$

such that $\mathrm{Im}\, h$ can be generated by rotations with centers at the vertices of a triangle ΔABC with angles $\frac{\pi}{7}$, $\frac{\pi}{5}$, and $\frac{\pi}{2}$. Here $h(\beta) =$ rotation with angle $\frac{-2\pi}{5}$ at C and $h(\gamma) =$ rotation with angle $\frac{2\pi}{7}$ at A.

We'll show the relator $h((\beta\gamma)^2) = 1$ is satisfied. Let r_{XY} be reflection in the geodesic containing X and Y. Then

$$h(\beta) = r_{BC} \circ r_{AC} \quad \text{and} \quad h(\gamma) = r_{AC} \circ r_{AB},$$

so that

$$h(\beta)h(\gamma) = r_{BC} \circ r_{AC} \circ r_{AC} \circ r_{AB} = r_{BC} \circ r_{AB}.$$

This last isometry is a rotation at B with angle $-\pi$ and $h(\beta\gamma)$ is shown to have order 2. This shows $\mathrm{Im}\, h$ is nontrivial. Hence $\pi_1(\partial \mathrm{Ma}^4)$ is nontrivial and thus $\partial \mathrm{Ma}^4 \not\approx S^3$.

We will use the following proposition in Section 5.2.

Proposition 3.1 *Let m_Γ be the meridian of the torus ∂T_Γ. Then m_Γ is nontrivial in $S^1 \times S^2 - \mathrm{int}(T_\Gamma)$.*

Proof We choose x_5 as our representative of m_Γ. By the relator

$$r_9 : x_1 = x_7^{-1} x_2 x_7 = \beta^{-1}\lambda\beta = \beta^{-1}(\beta^{-1}\alpha)\beta$$

we get $x_1 = \beta^{-2}\alpha\beta$. By $r_\zeta : x_5 = x_1 x_2$ we obtain

$$x_5 = (\beta^{-2}\alpha\beta)(\beta^{-1}\alpha) = \beta^{-2}\alpha^2 = \beta^{-2}\gamma.$$

Thus

$$\begin{aligned} h(x_5) &= h(\beta^{-2}\gamma) \\ &= h(\beta^{-2})h(\gamma) \\ &= \big(\text{rotation of } \tfrac{4\pi}{5} \text{ at } C\big)\big(\text{rotation of } \tfrac{2\pi}{7} \text{ at } A\big) \\ &\neq 1_{\mathbb{H}^2} \qquad \text{(since } A \text{ is not fixed).} \end{aligned}$$

Thus x_5 is not trivial in $\partial \mathrm{Ma}^4$. Hence x_5 is nontrivial in $S^1 \times S^2 - \mathrm{int}(T_\Gamma)$. This concludes the proof of Proposition 3.1. □

The following question is still open.

Question 3.2 Does Ma^4 split into closed balls?

Question 3.3 Are there infinitely many closed 4–dimensional splitters?

We will answer this question in Section 5.2.

3.2 The jester's manifolds

As an initial step towards constructing 4–dimensional splitters, we describe a collection of 4–manifolds similar to Mazur's. Start with an $S^1 \times \mathbb{B}^3$ and within its $S^1 \times S^2$ boundary select a curve C as follows. Let T be a tubular neighborhood of C in our $S^1 \times S^2$. We have chosen C so that it is the preimage of the Mazur curve Γ under the standard double covering map $p\colon S^1 \times \mathbb{B}^3 \to S^1 \times \mathbb{B}^3$; see Figure 3.

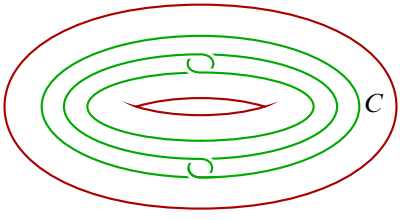


Figure 3: $C \subset \partial(S^1 \times \mathbb{B}^3)$

Then, given a framing $\Psi\colon S^1 \times \mathbb{B}^2 \to T$, define

$$M_\Psi = (S^1 \times \mathbb{B}^3) \cup_\Psi (\mathbb{B}^2 \times \mathbb{B}^2),$$

where the domain is the $S^1 \times \mathbb{B}^2$ factor in the boundary of our 2–handle $h^{(2)} \approx \mathbb{B}^2 \times \mathbb{B}^2$. We call such an M_Ψ a *jester's manifold.*

(In Section 4, we will expand our definition of jester's manifold to include analogous attachments using *pseudohandles*.)

Remark 3.4 Initially, we had hoped that, by altering the framings, we could prove the existence of an infinite collection of these jester's manifolds. Unfortunately, the group-theoretic calculations proved too complicated. Fortunately, however, we were able to get around this problem by employing a technique of David Wright's (see Section 5.2). We are still interested in the following question.

Question 3.5 Does there exist a jester's manifold that is not homeomorphic to a ball? Are there infinitely many jester's manifolds?

4 Spines

4.1 Collapses

We borrow our definitions of collapse from [4, pages 3–4, 14–15]. We denote the cone over a simplicial complex A with cone point a by aA.

Definition 4.1 If K and L are finite simplicial complexes, we say that there is an *elementary simplicial collapse* from K to L, and write $K \searrow^e L$, if L is a subcomplex of K and $K = L \cup aA$, where a is a vertex of K, A and aA are simplexes of K, and $aA \cap L = a(\partial A)$. We call such an A a *free face* of K.

Observe that a free face completely specifies an elementary simplicial collapse.

Definition 4.2 Suppose that (K, L) is a finite CW pair. Then $K \searrow^e L$ — that is, K *collapses to* L *by an elementary collapse* — if and only if

(1) $K = L \cup e^{n-1} \cup e^n$ where e^n and e^{n-1} are not in L, and

(2) there exists a ball pair $(Q^n, Q^{n-1}) \approx (\mathbb{B}^n, \mathbb{B}^{n-1})$ and a map $\varphi\colon Q^n \to K$ such that

  (a) φ is a characteristic map for e^n,

  (b) $\varphi|Q^{n-1}$ is a characteristic map for e^{n-1},

  (c) $\varphi(P^{n-1}) \subset L^{n-1}$, where $P^{n-1} \equiv \mathrm{cl}(\partial Q^n - Q^{n-1})$.

In both the simplicial and CW cases we define:

Definition 4.3 K *collapses* to L, denoted $K \searrow L$, if there is a finite sequence of elementary collapses

$$K = K_0 \searrow^e K_1 \searrow^e K_2 \searrow^e \cdots \searrow^e K_l = L.$$

If K collapses to a point we say K is *collapsible* and write $K \searrow 0$.

Definition 4.4 Suppose M is a compact PL manifold. If K is a subcomplex of M contained in $\operatorname{int} M$ with $M \searrow K$, we say K is a *spine* of M.

We will make use of the following regular neighborhood theory, due to J H C Whitehead. The following four results, 4.5–4.8, can be found in [16, pages 40–41].

Proposition 4.5 *Suppose $M \supset M_1$ are PL n–manifolds with $M \searrow M_1$. Then there exists a homeomorphism $h\colon M \to M_1$.*

Theorem 4.6 *Suppose $X \subset M$, where M is a PL manifold, X is a compact polyhedron, and $X \searrow Y$. Then a regular neighborhood of X in M collapses to a regular neighborhood of Y in M.*

Thus if K is a spine of M then for any regular neighborhood $N(K)$ of K in M we have $N(K) \approx M$.

Proposition 4.7 *If $X \searrow 0$ then a regular neighborhood of X is a ball.*

Corollary 4.8 *Suppose M is a manifold with a spine K and $K \searrow 0$. Then M is a ball.*

Proposition 4.9 *Suppose W is a PL manifold and A and B are simplicial complexes contained in the interior of W. If $W \searrow A \cup B$ with $A, B, A \cap B \searrow 0$, then W splits into closed balls.*

Proof Let A, B, and C be such that $W \searrow A \cup_C B$ with $A, B, C \searrow 0$. Regular neighborhoods of collapsible subcomplexes are piecewise linear balls. So, given a triangulation of W with A and B as subcomplexes, we construct (with respect to this triangulation) regular neighborhoods N_A of A and N_B of B and we have that N_A and N_B are balls and $N_A \cap N_B$ is a regular neighborhood of C, and as such is also a ball. $N_A \cup N_B$ is a regular neighborhood of $A \cup B$, a spine of W, so $N_A \cup N_B$ is homeomorphic to W. □

4.2 The dunce hat

The *dunce hat* D is defined as the quotient space obtained by identifying the edges of a triangular region as pictured in Figure 4.

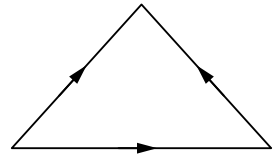


Figure 4: The dunce hat D

The dunce hat was one of the first examples of a contractible but not collapsible simplicial complex. A well-known result by Zeeman is that the Mazur manifold has a dunce hat spine [20]. That observation will become clear in the following subsection, when we identify a spine of a slightly more complicated example.

To the best of our knowledge the following question is open.

Question 4.10 Can the dunce hat be expressed as $D = A \cup_C B$ with $A, B, C \searrow 0$? If so, the answer to Question 3.2 is yes: $\mathrm{Ma}^4 \approx \mathbb{B}^4 \cup_{\mathbb{B}^4} \mathbb{B}^4$.

4.3 The jester's hat

We define the *jester's hat* J to be the quotient space obtained by gluing the hexagonal region of the plane as in Figure 5. We can also realize this space by attaching a disc to a circle with the attaching map in Figure 6. Since the attaching map is homotopic to the identity, J is contractible [12, page 16]. J is not collapsible as it has no free edge.

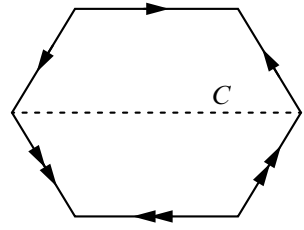


Figure 5: The jester's hat

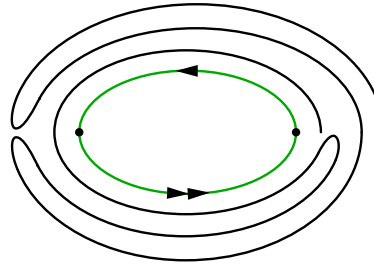


Figure 6: Attaching map for J

By cutting J open along the dashed arc in Figure 5, one can see that J can be decomposed into the union of collapsible subsets intersecting in C, another collapsible subset:

$$J = A \cup_C B \quad \text{with } A, B, C \searrow 0.$$

The interested reader can see [17, pages 17–18] for details.

Proposition 4.11 *Every jester's manifold has a jester's hat spine.*

Proof The proof is analogous to Zeeman's proof that Mazur's manifold has a dunce hat spine [20]. Let $M = M_\Psi$ be a jester's manifold for a given framing Ψ. We divide the S^1 of the $S^1 \times S^2$ in which C resides into four arcs I_1, I_2, I_3, I_4 so that $I_1 \times S^2$ and $I_2 \times S^2$ each contain a "clasp" of C (see Figure 7).

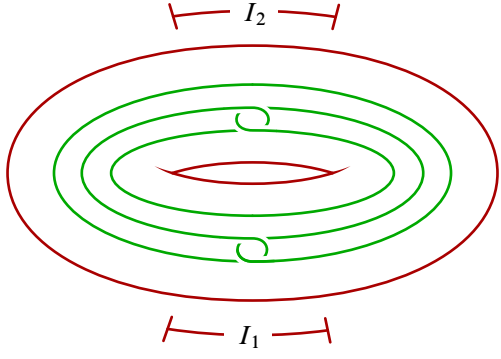


Figure 7: Intervals of S^1 and their clasps

For $i = 1, 2$, let $f_i\colon S^1 \to S^1$ be the map that shrinks I_i to a point, say p_i, and is a homeomorphism on the complement of I_i. Further, let $\pi\colon S^1 \times S^2 \to S^1$ be projection onto the first factor, $j\colon C \hookrightarrow S^1 \times S^2$ be the inclusion, $g = f_1 \circ f_2 \circ \pi\colon S^1 \times S^2 \to S^1$ and $h = g \circ j$. Let $M(g)$ and $M(h)$ be the mapping cylinders of g and h, respectively. That is,

$$M(g) = \big[(S^1 \times S^2 \times [0,1]) \sqcup S^1\big]/{\sim_g} \quad \text{and} \quad M(h) = \big[(C \times [0,1]) \sqcup S^1\big]/{\sim_h},$$

where $\sim_g$ and $\sim_h$ are generated by $(x, 0) \sim_g g(x)$ and $(y, 0) \sim_h h(y)$, respectively. The mapping cylinder $M(g)$ is homeomorphic to $S^1 \times \mathbb{B}^3$ [17, page 19]. Since $h = g|_C$, $M(h)$ is a subcylinder of $M(g)$, and by a result of Whitehead, $M(g) \searrow M(h)$ [18]. Further, the 2–handle $h^{(2)}$ viewed as $\mathbb{B}^2 \times \mathbb{B}^2$ in our construction of M collapses onto the union of its core with the attaching tube: $(\mathbb{B}^2 \times \{0\}) \cup (S^1 \times \mathbb{B}^2)$. Follow this with the collapse of $M(g)$ onto $M(h)$ to obtain the collapse:

$$M = S^1 \times \mathbb{B}^3 \cup_\Psi \mathbb{B}^2 \times \mathbb{B}^2 \searrow S^1 \times \mathbb{B}^3 \cup_\Psi \big[(\mathbb{B}^2 \times \{0\}) \cup (S^1 \times \mathbb{B}^2)\big] \searrow M(h) \cup_{\Psi|C} B^2.$$

But from the illustration of $M(h)$ (Figure 8) we can see that $M(h) \cup_{\Psi|C} B^2$ is our jester's hat J. □

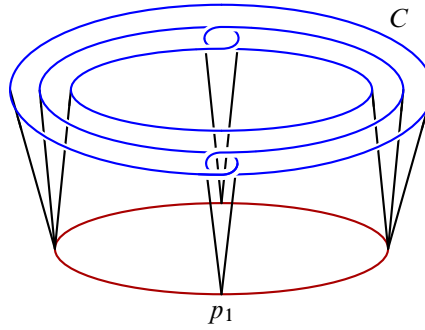


Figure 8: The mapping cylinder of h

Corollary 4.12 *The jester's manifolds split into closed 4–balls.*

Remark 4.13 While we now know that the M_Ψ's split into closed balls, we have not demonstrated that any M_Ψ is not just a ball. To deal with that issue we will modify the construction.

5 More jester's manifolds

For this section we let $M = M_\Psi$ be an arbitrary jester's manifold. Recall Ψ is the framing $\Psi\colon S^1 \times \mathbb{B}^2 \to T$ and T is a tubular neighborhood of the curve C in $\partial(S^1 \times \mathbb{B}^3)$.

5.1 Pseudo-2–handles

Using M as a model, we apply a construction due to Wright to obtain a collection of manifolds $\{W_i\}$, as follows [19]. To construct W_i, we start with the $S^1 \times \mathbb{B}^3$ of the jester's manifold construction and attach a "pseudo-2–handle", a $\mathbb{B}^4$, along K_i, the connected sum of i trefoils in the boundary of $\mathbb{B}^4$, to the curve C in $\partial(S^1 \times \mathbb{B}^3)$. (See Figure 9.) That is,

$$W_i = (S^1 \times \mathbb{B}^3) \cup_{\Psi_i} H.$$

Here Ψ_i is a homeomorphism from a tubular neighborhood T_i of K_i in $\partial\mathbb{B}^4$ to T.

We define the *core of the pseudohandle* to be the cone of K_i with cone point the center of $\mathbb{B}^4$. The core is then a 2–disc whose interior lies in $\operatorname{int}\mathbb{B}^4$.

Proposition 5.1 *Each W_i collapses to J.*

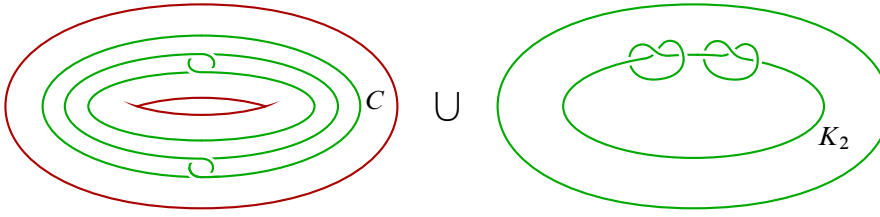


Figure 9: The union of $S^1 \times \mathbb{B}^3$ with a degree-2 pseudo-2–handle

Proof The proof that every jester's manifold collapses to J (Proposition 4.11) goes through with the pseudo-2–handle collapsing to its core. H collapses to the union of its core with its attaching tube, defined as $\Psi_i(T_i)$. The mapping cylinder $M(g)$ again collapses to $M(h)$, with the attaching tube collapsing to the attaching sphere: $\Psi_i(K_i) = C$. □

Corollary 5.2 *Each W_i decomposes as $W_i = \mathbb{B}^4 \cup_{\mathbb{B}^4} \mathbb{B}^4$.*

5.2 A theorem of Wright

Applying the following theorem will yield an infinite collection of distinct W_i. Before we state the theorem we'll need some definitions.

Definition 5.3 A 3–manifold is *irreducible* if every embedded S^2 bounds a $\mathbb{B}^3$.

Definition 5.4 A torus S in a 3–manifold X is said to be *incompressible in X* if the homomorphism $\pi_1(S) \to \pi_1(X)$ induced by inclusion is injective.

Definition 5.5 A group G is *indecomposable* if for all subgroups A, B such that $G \approx A * B$, either $A = 1$ or $B = 1$. (That is, G contains no nontrivial free factors.)

Theorem 5.6 [19] *Suppose X is a compact 4–manifold which is obtained from the 4–manifold N by adding a 2–handle H. If $\mathrm{cl}(\partial X - H)$ is an orientable irreducible 3–manifold with incompressible boundary, then there exists a countably infinite collection of compact 4–manifolds M_i such that*

(1) *$\pi_1(\partial M_i) \ncong \mathbb{Z}$ and $\pi_1(\partial M_i)$ is indecomposable, and*

(2) *$\pi_1(\partial M_i) \ncong \pi_1(\partial M_j)$ for $i \neq j$,*

and hence $\mathrm{int}(M_i)$ is not homeomorphic to $\mathrm{int}(M_j)$.

Wright constructs the infinite collection of manifolds $\{M_i\}$ of the theorem as follows. For each $i = 1, 2, \ldots$ he constructs a manifold by attaching to N a pseudo-2–handle along K_i. From this sequence he exhibits a subsequence $\{M_{i_j}\}$ each term of which has a distinct boundary.

For the proof of the following theorem we'll employ the loop theorem [15, page 101].

Theorem 5.7 (loop theorem) *If X is a 3–manifold with boundary and the induced inclusion homomorphism $\pi_1(\partial X) \to \pi_1(X)$ has nontrivial kernel, then there exists an embedding of a disc D in X such that ∂D lies in ∂X and represents a nontrivial element of $\pi_1(\partial X)$.*

Theorem 5.8 *There exists an infinite collection of closed 4–dimensional splitters. The fundamental groups of their boundaries are distinct, indecomposable, and noncyclic.*

Proof We'll show M meets the hypotheses of Theorem 5.6, thus yielding a subsequence of $\{W_i\}$ as our desired collection. Recall T is the tubular neighborhood of the attaching sphere C in the construction of the jester's manifold so that $\partial T = \partial\,\mathrm{cl}(\partial M - h^{(2)})$. It suffices to show:

Claim *∂T is incompressible in* $\mathrm{cl}(\partial M - h^{(2)}) = S^1 \times S^2 - \mathrm{int}(T)$.

We will show that

$$\ker\big(\pi_1(\partial T) \to \pi_1(S^1 \times S^2 - \mathrm{int}(T))\big) = 1.$$

Recall T_Γ is the tubular neighborhood of the Mazur curve Γ in the $S^1 \times S^2$ in the construction of the Mazur manifold Ma^4 (see Section 3.1). Recall further Proposition 3.1: Let m_Γ be the meridian of the torus ∂T_Γ. Then m_Γ is nontrivial in $S^1 \times S^2 - \mathrm{int}(T_\Gamma)$.

By construction, $(S^1 \times S^2) - \mathrm{int}(T)$ is a double cover of $(S^1 \times S^2) - \mathrm{int}(T_\Gamma)$. Call the associated covering map p, and let m be a lift of m_Γ, so m is a meridian of ∂T. Then $p_*([m]) = [m_\Gamma] \neq 1$ gives $[m] \neq 1$. Suppose by way of contradiction that there exists an embedded disc D in $(S^1 \times S^2) - \mathrm{int}(T)$ with ∂D being a nontrivial loop in ∂T. Choose a longitude l on ∂T and let $\mu = [m]$ and $\lambda = [l]$ in $\pi_1(\partial T)$ so that

$$[\partial D] = \mu^k \lambda^j \quad \text{in } \pi_1(\partial T) \text{ for some } k, j \in \mathbb{Z}.$$

As C has algebraic index 1 in $S^1 \times S^2$, a nonzero j would imply $[\partial D]$ nontrivial in $\pi_1(S^1 \times S^2 - \mathrm{int}(T))$. Thus $[\partial D] = \mu^k$. But any loop going around meridionally more than once and longitudinally not at all will not be embedded. See [17, page 25] for an

illustration. Then it must be that $[\partial D]=[m]^{\pm 1}$. Since m is nontrivial in $S^1\times S^2-\operatorname{int} T$, such a D cannot exist, and by the loop theorem we conclude that

$$\ker\big(\pi_1(\partial T)\to \pi_1(S^1\times S^2-\operatorname{int}(T))\big)=1. \qquad \square$$

Definition 5.9 We call any M_i as yielded by the theorem when applied to any M_Ψ a *jester's manifold*.

Note that for a given knot K_i, different choices of framing homeomorphism potentially yield different manifolds. So the variety of distinct jester's manifolds produced by this construction is potentially much greater than we have shown.

We conclude this section with a restatement of our first main result which we have now demonstrated.

Theorem 1.2 *There exists an infinite collection of topologically distinct splittable compact contractible 4–manifolds. The interiors of these are topologically distinct contractible splittable open 4–manifolds.*

6 Sums of splitters

In this concluding section, we will exhibit an uncountable collection of contractible open 4–dimensional splitters. We will do so by considering the interiors of infinite boundary-connected sums of our jester's manifolds. These open manifolds can also be constructed as the connected sum at infinity of the interiors of the same sequence of manifolds See [3, pages 1807–1813] for a description of connected sum at infinity. We will demonstrate a splitting for such manifolds, and then, by applying a result of Curtis and Kwun, we will show that two such sums are topologically distinct if some jester's manifold appears more often as a summand in one sum than in the other.

We describe what we mean by the *induced orientation* of the boundary of an oriented manifold X^n. Given a collar neighborhood of ∂X which we identify as $\partial X\times[0,1]$ (∂X identified with $\partial X\times\{0\}$) and a map $h\colon \mathbb{B}^{n-1}\to\partial X$, we define $\bar h$ as

$$\bar h\colon \mathbb{B}^n\to\partial X\times(0,1],\quad \bar h(x_1,x_2,\dots,x_n)=\Big(h(x_1,x_2,\dots,x_{n-1}),\frac{3+x_n}{4}\Big).$$

(To be precise, the codomain of $\bar h$ should be $\operatorname{int} X$.) If $h\colon \mathbb{B}^{n-1}\to\partial X$ and $\bar h$ is a representative of the orientation of X then the ambient isotopy class of h is the *induced orientation* of ∂X; see [16, page 45].

Definition 6.1 Let M^n and N^n be connected oriented n–manifolds with nonempty boundaries. Orient ∂M^n and ∂N^n with their induced orientations and let B_M and B_N be tame $(n-1)$–balls in ∂M^n and ∂N^n, respectively. Let $\phi\colon B_M \to B_N$ be an orientation-reversing homeomorphism. Then $M^n \cup_\phi N^n$ is called a *boundary connected sum* (*BCS*) and is denoted $M^n \diamond N^n$.

Proposition 6.2 [13, page 97] *The boundary connected sum is a connected oriented manifold which, provided* $\operatorname{Bd} M$ *and* $\operatorname{Bd} N$ *are connected, does not depend on the choices of* B_i *or* ϕ_i. *Furthermore, the set of connected oriented* n*–dimensional manifolds with connected boundaries is, under the operation of connected sum, a commutative monoid (that is, associative and contains an identity), the identity being* $\mathbb{B}^n$.

Definition 6.3 Let $\{M_i^n\}_{i=1}^m$ (m possibly ∞) be oriented manifolds with nonempty connected boundaries and for each $i = 1, 2, \dots$ let $B_{i,L}$ and $B_{i,R}$ be disjoint tame $(n-1)$–balls in ∂M_i^n. For $i > 1$ let $\phi_i\colon B_{i,L} \to B_{i-1,R}$ be an orientation-reversing homeomorphism. Let $\phi\colon \bigsqcup_{i>1} B_{i,L} \to \bigsqcup_{i\geq 1} B_{i,R}$ with $\phi|_{B_{i,L}} = \phi_i$. Then $\big(\bigsqcup M_i\big)/\phi$ is called a *boundary connected sum* (*BCS*) and is denoted by $M_1 \diamond M_2 \diamond \cdots \diamond M_m$ (or $M_1 \diamond M_2 \diamond \cdots$ when $m = \infty$).

Proposition 6.4 *Let* $B_i \approx \mathbb{B}^n$. *Then*

(1) $B_1 \diamond B_2 \diamond \cdots \diamond B_m \approx \mathbb{B}^n$, *and*

(2) $B_1 \diamond B_2 \diamond \cdots \approx \mathbb{R}^n_+$.

Proof Let $B_i = [-1,1]^{n-1} \times [i-1, i]$ with $B_{i,L} = [-1,1]^{n-1} \times \{i-1\}$ and $B_{i,R} = [-1,1]^{n-1} \times \{i\}$. Then with each attaching map ϕ_i equal to the identity on $B_{i,L} = B_{i-1,R}$, we have

$$B_1 \diamond B_2 \diamond \cdots \diamond B_m = [-1,1]^{n-1} \times [0, m] \approx \mathbb{B}^n, \tag{1}$$

and

$$B_1 \diamond B_2 \diamond \cdots = [-1,1]^{n-1} \times [0, \infty) \approx \mathbb{R}^n_+. \tag{2}$$

□

Using this proposition and a proof similar to that of the following theorem, one can show that any finite boundary connected sum of closed splitters is also a closed splitter.

Theorem 6.5 *Suppose that* $M_i = A_i \cup_{C_i} B_i$ *with* $A_i, B_i, C_i \approx \mathbb{B}^n$. *Then* $M = M_1 \diamond M_2 \diamond \cdots = A \cup_C B$, *where* $A, B, C \approx \mathbb{R}^n_+$.

Proof Given $M_i = A_i \cup_{C_i} B_i$ with $A_i, B_i, C_i \approx \mathbb{B}^n$, we can take regular neighborhoods

of A_i and B_i in M_i yielding another splitting $A_i' \cup_{C_i'} B_i'$ of M_i, this one guaranteed to have a $(n-1)$–ball in the boundary of the intersection C_i'. Thus we can assume each ∂C_i contains disjoint $(n-1)$–balls $B_{i,L}$ and $B_{i,R}$. Forming $M = M_1 \diamond M_2 \diamond \cdots$ with these $B_{i,L}$'s and $B_{i,R}$'s and letting $A = \bigcup_i A_i \subset M$, we see that $A = A_1 \diamond A_2 \diamond \cdots$, which is $\mathbb{R}^n_+$ by Proposition 6.4. Likewise, $B = B_1 \diamond B_2 \diamond \cdots$ and $C = C_1 \diamond C_2 \diamond \cdots$ are both halfspaces and $M = A \cup_C B$. □

Proposition 6.6 *If $M = A \cup_C B$ with $A, B, C \approx \mathbb{R}^n_+$, then* $\operatorname{int} M = A' \cup_{C'} B'$ *with $A', B', C' \approx \mathbb{R}^n$.*

This proposition is proved similarly to Proposition 2.1; see [17, Proposition 1.2.1].

The following theorem and a discussion of group systems can be found in [5].

Theorem 6.7 *Let M and N be infinite boundary connected sums of compact connected n–manifolds ($n \geq 4$) with nonempty connected boundaries. If* $\operatorname{int} M \approx \operatorname{int} N$, *then the corresponding group systems are compatible. In particular, $\pi_1(\partial M) \cong \pi_1(\partial N)$.*

We are now ready to prove:

Theorem 1.3 *There exists an uncountable collection of contractible open 4–manifolds which split as $\mathbb{R}^4 \cup_{\mathbb{R}^4} \mathbb{R}^4$.*

Proof Given two sequences $\{M_i\}$ and $\{N_i\}$ of jester's manifolds such that some manifold X appears more times in $\{M_i\}$ than in $\{N_i\}$, the corresponding group systems of $M = M_1 \diamond M_2 \diamond \cdots$ and $N = N_1 \diamond N_2 \diamond \cdots$ are incompatible, since jester's manifolds have distinct and indecomposable boundary fundamental groups and this implies the weak limits are not isomorphic:

$$\pi_1(\partial M_1) * \pi_1(\partial M_2) * \cdots \ncong \pi_1(\partial N_1) * \pi_1(\partial N_2) * \cdots .$$

By Theorem 6.7, $\operatorname{int} M \not\approx \operatorname{int} N$. As there are an uncountable number of ways to form such boundary connected sums of jester's manifolds, we have our desired result. □

A result of Ancel and Siebenmann states that a Davis manifold generated by C is homeomorphic to the interior of an alternating boundary connected sum

$$\operatorname{int}(C \diamond -C \diamond C \diamond -C \diamond \cdots).$$

Here $-C$ is a copy of C with the opposite orientation [11]. We have now proved:

Corollary 6.8 *There exist (non-$\mathbb{R}^4$) 4–dimensional Davis manifold splitters.*

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