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Topology of holomorphic Lefschetz pencils on the four-torus

NORIYUKI HAMADA
KENTA HAYANO

We discuss topological properties of holomorphic Lefschetz pencils on the four-torus. Relying on the theory of moduli spaces of polarized abelian surfaces, we first prove that, under some mild assumptions, the (smooth) isomorphism class of a holomorphic Lefschetz pencil on the four-torus is uniquely determined by its genus and divisibility. We then explicitly give a system of vanishing cycles of the genus-3 holomorphic Lefschetz pencil on the four-torus due to Smith, and obtain those of holomorphic pencils with higher genera by taking finite unbranched coverings. One can also obtain the monodromy factorization associated with Smith's pencil in a combinatorial way. This construction allows us to generalize Smith's pencil to higher genera, which is a good source of pencils on the (topological) four-torus. As another application of the combinatorial construction, for any torus bundle over the torus with a section we construct a genus-3 Lefschetz pencil whose total space is homeomorphic to that of the given bundle.

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1 Introduction

Lefschetz pencils on smooth four-manifolds are closely related to symplectic structures by Donaldson's construction [6] of Lefschetz pencils on symplectic manifolds and Gompf's generalization [10] of Thurston's construction [25] of symplectic structures on surface bundles. Moreover, Kas [14] and Matsumoto [20] gave a combinatorial interpretation of isomorphism classes of Lefschetz fibrations, in particular their results enable us to construct Lefschetz fibrations (and symplectic four-manifolds) using simple closed curves on oriented surfaces (these results are generalized to that for Lefschetz pencils in Baykur and Hayano [2]). For these reasons Lefschetz pencils and fibrations have attracted a lot of interest from four-dimensional topologists in the last two decades. On the other hand, Lefschetz originally introduced Lefschetz pencils as generic pencils (ie linear 1–systems) of very ample line bundles in order to study the topology of algebraic varieties (see eg Lamotke [16]). It is therefore natural to pay

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attention to holomorphic Lefschetz pencils as well as smooth ones. In this paper we study holomorphic Lefschetz pencils on the four-torus from a topological point of view.

In order to explain our main result, we first introduce two invariants for Lefschetz pencils. The *genus* of a Lefschetz pencil is the genus of the closure of a regular fiber, and the *divisibility* of a Lefschetz pencil is the maximum integer by which we can divide the integral homology class represented by the closure of a regular fiber. Two Lefschetz pencils on the same four-manifold have the same genus and divisibility if they are isomorphic, but the converse does not hold in general (the reader can find a counterexample for the converse in Baykur and Hayano [3], for example). Our first result states that the converse becomes true for holomorphic Lefschetz pencils on the four-torus under some assumptions:

Theorem 1.1 *Let f_0, f_1 be holomorphic Lefschetz pencils on the four-torus. Suppose either that the genus of f_0 is greater than 5 or that the divisibility of f_0 is greater than 1. Then f_0 and f_1 are isomorphic if and only if they have the same genus and divisibility.*

Note that in this paper we define a Lefschetz pencil to be holomorphic if there is *some* complex structure on the source manifold of the pencil with respect to which the pencil is holomorphic (see Section 2.1). In particular, *two Lefschetz pencils f_0 and f_1 in Theorem 1.1 might not be holomorphic with respect to the same complex structure.* One of the significant points of Theorem 1.1 is that it still holds for such f_0 and f_1. If f_0 and f_1 are pencils of a fixed polarization on the four-torus, the proof is much easier (see Remark 3.14).

The condition on the genus or the divisibility of f_0 in Theorem 1.1 is needed for some technical reasons and we believe that the theorem still holds without it (see the last paragraph of Section 3).

As we mentioned in the first paragraph, Lefschetz pencils are not only objects in complex geometry but are also related to symplectic topology. It is especially important to find out how smooth Lefschetz pencils differ from holomorphic ones, which is related to the difference between complex (or Kähler) surfaces and symplectic four-manifolds. Since there exist noncomplex symplectic four-manifolds, we can easily obtain Lefschetz pencils on noncomplex four-manifolds using Donaldson's construction [6]. While it is in general hard to obtain monodromy factorizations of Lefschetz pencils coming from Donaldson's construction, several ingenious techniques, such as fiber-sum operations and substitution operations, have been employed in order to give nonholomorphic Lefschetz pencils and fibrations (on possibly noncomplex four-manifolds) with explicit monodromy factorizations (see eg Baykur and Korkmaz [1; 4; 15], Fintushel and Stern [8], Hamada, Kobayashi and Monden [13; 21], Ozbagci and Stipsicz [22] and Smith [23]). Furthermore, Li [19] constructed nonholomorphic Lefschetz pencils on minimal Kähler surfaces of general type. The construction in [19] relies not only on Donaldson's result [6] but also on the differences between cohomology Kähler cones and symplectic cones. Since the cohomology Kähler cone of the four-torus coincides with its symplectic cone (see [19, Proposition 4.10]), this construction cannot give the affirmative answer to the following question:

Problem 1.2 Does there exist a nonholomorphic Lefschetz pencil on the four-torus?

Problem 1.2 is also important in complex geometry since it might be related to the existence of non-Kähler symplectic forms on the four-torus. (Here, a symplectic form ω is said to be *non-Kähler* if there do not exist complex structures compatible with ω.) Indeed, for a holomorphic Lefschetz pencil we can take a symplectic form on the total space taming the complex structure by using [10, Theorem 2.11(b)]. Such a symplectic form is Kähler if it is further compatible with the complex structure. Theorem 1.1, together with explicit examples we will construct, gives rise to several constraints on monodromy factorizations of holomorphic Lefschetz pencils on the four-torus; in particular, it might be possible to construct a nonholomorphic Lefschetz pencil on the four-torus using Theorem 1.1 (see Remark 5.10).

As we mentioned earlier, a system of vanishing cycles of a Lefschetz pencil completely determines its isomorphism class. Thus we can find a nonholomorphic Lefschetz pencil on the four-torus using Theorem 1.1 once we can get vanishing cycles of a holomorphic Lefschetz pencil on the four-torus with sufficiently large genus or divisibility, and find another system of simple closed curves (associated with a Lefschetz pencil on the four-torus) which is not Hurwitz equivalent to the system of the vanishing cycles. In this paper we first analyze the simplest example of a holomorphic pencil on the four-torus: a genus–3 Lefschetz pencil due to Smith [24].

Theorem 1.3 *The simple closed curves in Figure 6 are vanishing cycles of a genus-*3 *Lefschetz pencil constructed in* [24].

We can obtain holomorphic Lefschetz pencils on the four-torus with larger genera and divisibilities using finite unbranched coverings. The composition of a Lefschetz pencil and a finite unbranched covering of its total space is again a Lefschetz pencil, and any finite unbranched covering of the four-torus is also the four-torus. We will indeed prove that any holomorphic Lefschetz pencil on the four-torus with odd genus satisfying the assumption in Theorem 1.1 is isomorphic to the composition of the genus-3 Lefschetz pencil in [24] and a finite unbranched covering. (See Lemma 4.7 and the observation following it.)

As we will observe in the beginning of Section 4, we can obtain vanishing cycles of a Lefschetz pencil arising from composition of an unbranched covering once we know vanishing cycles of the original pencil (see Lemma 4.1). We will indeed give vanishing cycles of such pencils explicitly in Example 4.8 and Section 5.1.

Baykur [1] constructed genus-3 Lefschetz pencils on symplectic Calabi–Yau four-manifolds (ie symplectic manifolds with trivial canonical classes) with positive b_1 relying on combinatorial techniques. The family of Lefschetz pencils given in [1] covers all possible rational homology types of symplectic Calabi–Yau four-manifolds with $b_1 > 0$ (see Li [18]), in particular it contains a four-manifold *homeomorphic* to the four-torus. We will also construct a genus-3 Lefschetz pencil in a similar manner (by giving vanishing cycles; see Figure 14) and prove that our pencil is isomorphic to both the pencil with vanishing cycles in Figure 6 — that is, Smith's pencil — and the pencil given by Baykur [1] (see Lemma 5.1 and Remark 5.2). Our construction of the genus-3 pencil can be generalized to that of a genus-g symplectic Calabi–Yau Lefschetz pencil f_g for any $g \geq 3$. We will prove that the pencils with odd genera are compositions of Smith's pencil with finite unbranched coverings, and thus these are holomorphic pencils on the four-torus (Lemma 5.3). We further expect that the family of pencils $\{f_g \mid g-1 \text{ is prime}\}$ is a candidate for the family of *all essential* holomorphic Lefschetz pencils on the four-torus, where the tentative term "essential" means that they cannot be decomposed as the composition of a holomorphic pencil and a finite unbranched covering of the four-torus (Conjectures 5.8 and 5.9). Applying a combinatorial operation to our genus-3 pencil, we will obtain a family of genus-3 Lefschetz pencils $\{f_{\alpha,\beta}\}$ parametrized by $\alpha, \beta \in \mathrm{Mod}(\Sigma_1^1; U)$ with $[\alpha, \beta] = 1$, where $U = \{u\} \subset \partial\Sigma_1^1$ and $\mathrm{Mod}(\Sigma_1^1; U)$ is the mapping class group of the one-holed torus Σ_1^1 fixing U (for the precise definition of this mapping class group, see Section 2.2).

Theorem 1.4 *The total space of $f_{\alpha,\beta}$ is homeomorphic to that of the torus bundle over the torus with a section whose monodromy representation sends two elements generating $\pi_1(T^2)$ to α and β.*

We will give a monodromy factorization of $f_{\alpha,\beta}$ explicitly in (5-5). Note that Smith [24] observed that any torus bundle over the torus with a section admits a genus-3 Lefschetz pencil. We believe that this pencil is isomorphic to ours, and in particular the total space of $f_{\alpha,\beta}$ is *diffeomorphic* to that of a torus bundle over the torus.

The constructions of Lefschetz pencils in the previous paragraph are related to the smooth classification problem of symplectic four-manifolds with Kodaira dimension 0, which is one of the central concerns in symplectic topology. It is conjectured that any Kodaira dimension 0 symplectic manifold is diffeomorphic to one of the K3 surface, the Enriques surface or a torus bundle over the torus. The family of symplectic Calabi–Yau manifolds given in [1] contains potential counterexamples of the conjecture. Furthermore, we would obtain a new symplectic four-manifold with Kodaira dimension 0 if we could apply partial conjugations to any of the Lefschetz pencils in the previous paragraph so that the fundamental group of the total space of the resulting pencil was not a 4–dimensional solvmanifold group (see [1, Remark 18]).

In Section 3 we will prove Theorem 1.1 relying on the theory of moduli spaces of polarized abelian surfaces. In Section 4 we will first prove Theorem 1.3, that is, we will obtain vanishing cycles of the genus-3 holomorphic Lefschetz pencil due to Smith [24]. We will then discuss compositions of this pencil with finite unbranched coverings. In Section 5 we will first reconstruct Smith's pencil from a combinatorial point of view, and generalize the construction to obtain Lefschetz pencils with higher genera. Utilizing the technique in the appendix we will prove that the divisibilities of these Lefschetz pencils are all 1; see Lemma 5.6. We will further modify Smith's pencil to prove Theorem 1.4.

2 Preliminaries

Throughout the paper, we will always assume that all manifolds are smooth, oriented and connected unless otherwise noted.

2.1 Lefschetz pencils and fibrations

Let X be a closed 4–manifold and $B \subset X$ a nonempty discrete set. A smooth map $f\colon X \setminus B \to \mathbb{CP}^1$ is called a Lefschetz pencil if it satisfies the following conditions:

(1) The restriction $f|_{\mathrm{Crit}(f)}$ is injective, where $\mathrm{Crit}(f)$ is the set of all critical points of f.

(2) Each $x \in \mathrm{Crit}(f)$ is of *Lefschetz type*, that is, there exists a complex coordinate $(U, \varphi\colon U \to \mathbb{C}^2)$ (resp. $(V, \psi\colon V \to \mathbb{C})$) of x (resp. $f(x)$) compatible with the orientation such that $\psi \circ f \circ \varphi^{-1}(z, w)$ is equal to $z^2 + w^2$.

(3) For any $b \in B$ there exist a complex coordinate (U, φ) of x compatible with the orientation and an orientation-preserving self-diffeomorphism $\xi\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ such that $\xi \circ f \circ \varphi^{-1}(z, w)$ is equal to $[z : w] \in \mathbb{CP}^1$.

Each point in B is called a *base point* of f. A smooth map $f\colon X \to \mathbb{CP}^1$ satisfying the conditions (1) and (2) above is called a *Lefschetz fibration*. When we deal with a Lefschetz pencil and fibration at the same time we sometimes write $f\colon X \setminus B \to \mathbb{CP}^1$ to mean a Lefschetz fibration assuming B to be the empty set. A Lefschetz pencil or fibration f is said to be *holomorphic* if there exists a complex structure of X such that f is holomorphic and we can take biholomorphic φ, ψ and ξ in the conditions above.

Remark 2.1 Since a Lefschetz singularity germ has infinite $\mathcal{A}_e$–codimension as a real germ, it is not finitely determined in the smooth category, and thus it is basically hard to determine whether a given smooth germ is of Lefschetz type or not. However, in the complex category there is a useful criterion for a critical point to be of Lefschetz type: a critical point $x \in \mathbb{C}^2$ of a holomorphic function $f\colon \mathbb{C}^2 \to \mathbb{C}$ is of Lefschetz type if and only if the *complex Hessian*

$$\mathrm{Hess}(f)_x = \det\Bigl(\Bigl(\frac{\partial^2 f}{\partial z_k \partial z_l}(x)\Bigr)_{1 \le k,l \le 2}\Bigr)$$

is not equal to 0 (see [26, Lemma 2.11]).

For a Lefschetz pencil or fibration $f\colon X \setminus B \to \mathbb{CP}^1$, the genus of the closure $\overline{f^{-1}(*)}$ of a regular fiber is called the *genus* of f, which is denoted by $g(f)$. Using a regular fiber, we further define the number

$$d(f) = \sup\bigl\{n \in \mathbb{Z} \mid [\overline{f^{-1}(*)}] = n\alpha \text{ for some } \alpha \in H_2(X; \mathbb{Z})\bigr\} \in \mathbb{Z}_{>0} \cup \{\infty\},$$

called the *divisibility* of f. Two Lefschetz pencils or fibrations $f_0\colon X_0 \setminus B_0 \to \mathbb{CP}^1$ and $f_1\colon X_1 \setminus B_1 \to \mathbb{CP}^1$ are said to be *isomorphic* if there exist orientation-preserving diffeomorphisms $\Phi\colon X_0 \to X_1$ and $\phi\colon \mathbb{CP}^1 \to \mathbb{CP}^1$ which make the following diagram commute:

$$\begin{tikzcd} X_0 \setminus B_0 \arrow[r, "\Phi"] \arrow[d, "f_0"'] & X_1 \setminus B_1 \arrow[d, "f_1"] \\ \mathbb{CP}^1 \arrow[r, "\phi"] & \mathbb{CP}^1 \end{tikzcd}$$

Obviously two isomorphic Lefschetz pencils or fibrations have the same numbers of base points and critical points, the same genus and divisibility, but the converse does not hold in general (a pair $f_{(2,2)}$ and $f_{(3,1)}$ in [3], for example, is a counterexample).

2.2 Monodromy factorizations of Lefschetz fibrations/pencils

Let $\Sigma = \Sigma_g^p$ be a compact genus-g surface with p boundary components. We take points $u_1, \dots, u_p \in \partial\Sigma$ from each of the components of $\partial\Sigma$ and let $\delta_i \subset \mathrm{Int}(\Sigma)$ be a simple closed curve parallel to the boundary component containing u_i. Let U be the set $\{u_1, \dots, u_p\}$ and $\mathrm{Diff}(\Sigma; U)$ the group of orientation-preserving diffeomorphisms of Σ which preserve the set U. We call the set $\pi_0(\mathrm{Diff}(\Sigma; U))$ the *mapping class group of* Σ and denote it by $\mathrm{Mod}(\Sigma; U)$. An element of $\mathrm{Mod}(\Sigma; U)$ is the isotopy class of an element in $\mathrm{Diff}(\Sigma; U)$, where isotopies fix the set U. The group structure of $\mathrm{Mod}(\Sigma; U)$ is induced by compositions of maps, that is, $[\varphi_1]\cdot[\varphi_2] = [\varphi_1 \circ \varphi_2]$ for $\varphi_1, \varphi_2 \in \mathrm{Diff}(\Sigma; U)$.

Now let $f\colon X \setminus B \to \mathbb{CP}^1$ be a Lefschetz pencil or fibration with n critical points. Set $f(\mathrm{Crit}(f)) = \{a_1, \dots, a_n\}$, and take paths $\alpha_1, \dots, \alpha_n \subset \mathbb{CP}^1$ with a common initial point $a_0 \in \mathbb{CP}^1 \setminus f(\mathrm{Crit}(f))$ such that

- $\alpha_1, \dots, \alpha_n$ are mutually disjoint except at a_0,
- α_i connects a_0 with a_i,
- $\alpha_1, \dots, \alpha_n$ are ordered counterclockwise around a_0, ie there exists a small loop around a_0 oriented counterclockwise, hitting each α_i only once in the given order.

We take a loop $\widetilde{\alpha}_i$ with the base point a_0 by connecting α_i with a small counterclockwise circle with center a_i. We call a system of paths $\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_n$ obtained by the procedure above a *Hurwitz path system* of f. For each $b \in B$, let D_b be a sufficiently small 4–ball neighborhood of b and νB the disjoint union $\bigsqcup_{b\in B} D_b$. For each b we take a section $S_b \subset \partial D_b$ of $f|_{f^{-1}(E)}$, where $E \subset \mathbb{CP}^1$ is a disk containing $f(\mathrm{Crit}(f))$ and all the loops $\widetilde{\alpha}_1, \dots, \widetilde{\alpha}_n$. Let $\mathcal{H}$ be a horizontal distribution of $f|_{f^{-1}(E)\setminus(\nu B\cup \mathrm{Crit}(f))}$, that is, $\mathcal{H} = \{\mathcal{H}_x\}_{x\in f^{-1}(E)\setminus(\nu B\cup\mathrm{Crit}(f))}$ is a plane field such that $\mathrm{Ker}(df_x) \oplus \mathcal{H}_x = T_xX$ for any $x \in f^{-1}(E) \setminus (\nu B \cup \mathrm{Crit}(f))$. We assume that $\mathcal{H}_x = T_xS_b$ for any $x \in S_b$ and $\mathcal{H}_x \subset T_x\partial D_b$ for any $x \in \partial D_b$. Using $\mathcal{H}$, we can take a lift of the direction vector field of $\widetilde{\alpha}_i$ and a flow of this lift gives rise to a self-diffeomorphism of $f^{-1}(a_0)$. We call this diffeomorphism a *parallel transport* of $\widetilde{\alpha}_i$ and its isotopy class a *local monodromy* around a_i. Note that a local monodromy does not depend on the choice of $\mathcal{H}$.

Under an identification of the pair $\left(f^{-1}(a_0)\setminus \nu B, f^{-1}(a_0)\cap \bigsqcup_{b\in B} S_b\right)$ with the pair (Σ_g^p, U), we can regard a parallel transport as a diffeomorphism in $\mathrm{Diff}(\Sigma_g^p; U)$, and thus, a local monodromy as a mapping class in $\mathrm{Mod}(\Sigma_g^p; U)$. A local monodromy around a_i is a Dehn twist t_{c_i} along some simple closed curve $c_i \subset \mathrm{Int}\,\Sigma_g^p$ (see [14]). The curve c_i is called a *vanishing cycle* of f. Since the concatenation $\widetilde{\alpha}_1\cdots\widetilde{\alpha}_n$ is nullhomotopic in $\mathbb{CP}^1\setminus f(\mathrm{Crit}(f))$ and the restriction $f|_{\partial D_b}$ is the Hopf fibration, the composition $t_{c_n}\cdots t_{c_1}$ is equal to $t_{\delta_1}\cdots t_{\delta_p}$ in $\mathrm{Mod}(\Sigma_g^p; U)$ (which is the identity if $B=\varnothing$). The factorization

$$t_{c_n}\cdots t_{c_1} = t_{\delta_1}\cdots t_{\delta_p}$$

is called a *monodromy factorization* of f. Two factorizations $t_{c_n}\cdots t_{c_1} = t_{d_n}\cdots t_{d_1} = t_{\delta_1}\cdots t_{\delta_p}$ are said to be *Hurwitz equivalent* if one can be obtained from the other by successive applications of the following two kinds of moves:

- **Elementary transformation** $t_{c_n}\cdots t_{c_{i+1}} t_{c_i}\cdots t_{c_1} \to t_{c_n}\cdots t_{t_{c_{i+1}}(c_i)} t_{c_{i+1}}\cdots t_{c_1}$.
- **Simultaneous conjugation** $t_{c_n}\cdots t_{c_1} \to t_{\varphi(c_n)}\cdots t_{\varphi(c_1)}$ for $\varphi\in\mathrm{Mod}(\Sigma_g^p; U)$.

Theorem 2.2 [14; 20; 2] *Assume that $2-2g-p$ is negative. Two Lefschetz pencils or fibrations of genus g with p base points are isomorphic if and only if the corresponding monodromy factorizations are Hurwitz equivalent.*

2.3 Moduli spaces of polarized abelian surfaces

By an *abelian surface*, we mean a complex torus of dimension 2 which can be holomorphically embedded into $\mathbb{CP}^N$ for sufficiently large N. For a complex torus T, a *polarization* of T is a cohomology class $H\in H^2(T;\mathbb{Z})$ which is the first Chern class of an ample line bundle. Let $\overline{\Lambda}\subset\mathbb{C}^2$ be a lattice and $T=\mathbb{C}^2/\overline{\Lambda}$. We can canonically identify the group $H_1(T;\mathbb{Z})$ with the lattice $\overline{\Lambda}$. Using this identification we can regard polarizations of T as integer-valued alternating forms on $\overline{\Lambda}$. For any polarization E we can take a basis $\mu_1,\mu_2,\lambda_1,\lambda_2$ of $\overline{\Lambda}$ such that E is represented by the following matrix with respect to this basis:

$$E=\begin{pmatrix} 0 & D\\ -D & 0\end{pmatrix},$$

where $D=(\boldsymbol{d}_1,\boldsymbol{d}_2)=\left(\begin{smallmatrix} d_1 & 0\\ 0 & d_2\end{smallmatrix}\right)$ for $d_i>0$, $d_1\mid d_2$. We call the pair (d_1,d_2) or the matrix D the *type* of the polarization E.

We denote by $\mathfrak{H}_2$ the set of all symmetric complex 2×2 matrices with positive-definite imaginary part, which is a connected complex manifold of dimension 3. For $Z = (\boldsymbol{z}_1, \boldsymbol{z}_2) \in \mathfrak{H}_2$, we denote the ordered set $(\boldsymbol{z}_1, \boldsymbol{z}_2, \boldsymbol{d}_1, \boldsymbol{d}_2)$ by Λ_Z. The set Λ_Z is a basis of a lattice in $\mathbb{C}^2$, which we denote by $\overline{\Lambda}_Z$. In particular, Λ_Z gives rise to a complex torus $T_Z = \mathbb{C}^2/\overline{\Lambda}_Z$. Let H_Z be the imaginary part of a hermitian form represented by the matrix $\mathrm{Im}(Z)^{-1}$ with respect to the standard basis of $\mathbb{C}^2$. The form H_Z is a real-valued alternating form on $\mathbb{C}^2$. It is easy to check that the representation matrix of $H_Z|_{\overline{\Lambda}_Z}$ with respect to the basis Λ_Z is $\left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right)$. Thus, H_Z is a (d_1, d_2)–polarization of T_Z. Conversely, any polarized abelian surface can be obtained by the construction above. More precisely, it is known that for any complex torus $T = \mathbb{C}^2/\overline{\Lambda}$, its polarization H and a basis Λ of the lattice $\overline{\Lambda}$ with respect to which the representation of $H|_{\overline{\Lambda}}$ is $\left(\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}\right)$, there exists a matrix $Z \in \mathfrak{H}_2$ and a biholomorphic map $\Psi\colon \mathbb{C}^2 \to \mathbb{C}^2$ such that two triples (T, H, Λ) and (T_Z, H_Z, Λ_Z) correspond by Ψ (see [17, Section 8.1]). In particular, $\mathfrak{H}_2$ is a moduli space of (d_1, d_2)–polarized abelian surfaces with a symplectic basis of the lattice. The following are basic properties of this moduli space which will be used in this paper.

Lemma 2.3 *We fix the pair (d_1, d_2) and we regard $\mathfrak{H}_2$ as a moduli space of (d_1, d_2)–polarized abelian surfaces as explained above. Then:*

(1) *The subset $S_0 = \{Z \in \mathfrak{H}_2 \mid \mathrm{NS}(T_Z) \ncong \mathbb{Z}\}$ is contained in a countable union of proper analytic subsets of $\mathfrak{H}_2$, where*

$$\mathrm{NS}(T_Z) = \mathrm{Im}(c_1\colon H^1(T_Z; \mathcal{O}^*_{T_Z}) \to H^2(T_Z; \mathbb{Z}))$$

is the Néron–Severi group of T_Z.

(2) *The subset $S_1 = \{Z \in \mathfrak{H}_2 \mid T_Z \cong E_1 \times E_2$ for some elliptic curves $E_1, E_2\}$ is contained in S_0.*

Proof The first statement is in [17, Exercise 8.1] and the details are left to the reader. In order to prove the second one, assume that there exist elliptic curves E_1, E_2 such that T_Z is biholomorphic to $E_1 \times E_2$. The cohomology classes represented by the divisors $E_1 \times \{0\}$ and $\{0\} \times E_2$ are both contained in $\mathrm{NS}(T_Z)$. Thus the rank of $\mathrm{NS}(T_Z)$ is at least two. $\square$

For a holomorphic line bundle L we denote the set of holomorphic sections by $\Gamma(L)$, which is a finite-dimensional complex vector space. In the rest of this subsection we will construct an ample line bundle L_Z with $c_1(L_Z) = H_Z$ and a basis of $\Gamma(L_Z)$

explicitly (for more systematic constructions of line bundles on complex tori and their sections, see [17, Chapters 2 and 3]). For $Z \in \mathfrak{H}_2$ we denote the submodules $\langle \boldsymbol{z}_1, \boldsymbol{z}_2 \rangle$ and $\langle \boldsymbol{d}_1, \boldsymbol{d}_2 \rangle$ of the lattice $\overline{\Lambda}_Z$ by Λ_Z^1 and Λ_Z^2, respectively. Let V_Z^i be the real subspace of $\mathbb{C}^2$ generated by Λ_Z^i. It is easy to see that $\mathbb{C}^2$ is equal to the direct sum $V_Z^1 \oplus V_Z^2$. Using this decomposition we define a map $\chi_Z \colon \mathbb{C}^2 \to S^1$ as

$$\chi_Z(v_1 + v_2) = \exp(\pi i H_Z(v_1, v_2)),$$

where $v_i \in V_Z^i$. We further define a map $a_Z \colon \overline{\Lambda}_Z \times \mathbb{C}^2 \to \mathbb{C}^\times$ by

$$a_Z(\lambda, v) = \chi_Z(\lambda) \exp\bigl(\pi \operatorname{Im}(Z)^{-1}(v, \lambda) + \tfrac{\pi}{2} \operatorname{Im}(Z)^{-1}(\lambda, \lambda)\bigr),$$

where $\operatorname{Im}(Z)^{-1}$ is regarded as a hermitian form on $\mathbb{C}^2$. We then define a line bundle

$$L_Z = (\mathbb{C}^2 \times \mathbb{C})/\sim,$$

where the equivalence relation $\sim$ is generated by the relation

$$(v + \lambda, z) \sim (v, a(\lambda, v)z)$$

for $\lambda \in \overline{\Lambda}_Z$ and $v \in \mathbb{C}^2$. By assumption the alternating form H_Z is trivial on V_Z^2. Thus the restriction $\operatorname{Im}(Z)^{-1}|_{V_Z^2}$ is symmetric. Since the $\mathbb{C}$–extension of V_Z^2 is the whole space $\mathbb{C}^2$, we can define a symmetric form B_Z on $\mathbb{C}^2$ by extending $\operatorname{Im}(Z)^{-1}|_{V_Z^2}$. We define a holomorphic map $\vartheta_Z^{00} \colon \mathbb{C}^2 \to \mathbb{C}$, called a *Theta function*, by

$$\vartheta_Z^{00}(v) = \exp\bigl(\tfrac{\pi}{2} B_Z(v, v)\bigr) \times \sum_{\lambda \in \Lambda_Z^1} \exp\bigl(\pi(\operatorname{Im}(Z)^{-1} - B_Z)(v, \lambda) - \tfrac{\pi}{2}(\operatorname{Im}(Z)^{-1} - B_Z)(\lambda, \lambda)\bigr).$$

We can verify that the map $T_Z \ni [v] \mapsto [(v, \vartheta_Z^{00}(v))] \in L_Z$ is well-defined, in particular ϑ_Z^{00} gives rise to a section of L_Z (see [17, Lemma 3.2.4]). For two integers $0 \le i < d_1$ and $0 \le j < d_2$ we define the map $\vartheta_Z^{ij} \colon \mathbb{C}^2 \to \mathbb{C}$ by

$$\vartheta_Z^{ij}(v) = a_Z(w_{ij}, v)^{-1} \vartheta_Z^{00}(v + w_{ij}),$$

where $w_{ij} = (i/d_1)\boldsymbol{z}_1 + (j/d_2)\boldsymbol{z}_2$. This also gives rise to a section of L_Z for each pair i, j (see [17, Corollary 3.2.6]).

Theorem 2.4 [17, Theorem 3.2.7] *The set $\{\vartheta_Z^{ij} \in \Gamma(L_Z) \mid 0 \le i < d_1, 0 \le j < d_2\}$ is a basis of $\Gamma(L_Z)$.*

3 Uniqueness of holomorphic Lefschetz pencils on the four-torus

In this section we prove Theorem 1.1. Let L be a holomorphic line bundle on T^4. Throughout the paper we will use a broken arrow $\dashrightarrow$ to represent a meromorphic map. For $s_0, s_1 \in \Gamma(L)$, we define a meromorphic map $[s_0 : s_1]\colon T^4 \dashrightarrow \mathbb{CP}^1$ as follows: for $x \in T^4$, take a trivialization $\pi_L^{-1}(U) \cong U \times \mathbb{C}$ and regard the restriction $s_i|_U$ as a holomorphic function, and define $[s_0 : s_1](x) = [s_0(x) : s_1(x)]$. It is easy to see that a point $[s_0(x) : s_1(x)] \in \mathbb{CP}^1$ does not depend on the choice of a trivialization of L around x. The map $[s_0 : s_1]$ is defined on the complement of $s_0^{-1}(0) \cap s_1^{-1}(0)$.

Lemma 3.1 *For any holomorphic Lefschetz pencil f on T^4, there exists an ample line bundle L and sections s_0, s_1 such that f is equal to $[s_0 : s_1]$.*

Proof Let $V_i = \{[z_0 : z_1] \in \mathbb{CP}^1 \mid z_i \neq 0\}$ and let $\psi_i\colon V \to \mathbb{C}$ be a map defined by $\psi_i([z_0 : z_1]) = z_j/z_i$ for $j \neq i$. For each $b \in B$ we take a 4–ball neighborhood D_b and a biholomorphic map $\Phi_b\colon D_b \to \mathbb{C}^2$ so that D_b and $D_{b'}$ are disjoint if $b \neq b'$ and $f \circ \Phi_b^{-1}(z, w)$ is equal to $[z : w]$. We put $\Phi_b(x) = (\Phi_b^0(x), \Phi_b^1(x))$. We define a space L by

$$L = (f^{-1}(V_0) \times \mathbb{C}) \sqcup (f^{-1}(V_1) \times \mathbb{C}) \sqcup \bigsqcup_{b \in B} (D_b \times \mathbb{C})/\sim,$$

where the equivalence relation $\sim$ is defined by

$$(x, z) \sim \begin{cases} (x, \psi_1(f(x))z) & \text{for } x \in f^{-1}(V_0 \cap V_1), \\ (x, \Phi_b^0(x)z) & \text{for } x \in f^{-1}(V_0) \cap D_b, \\ (x, \Phi_b^1(x)z) & \text{for } x \in f^{-1}(V_1) \cap D_b. \end{cases}$$

It is easy to see that L together with the projection $\pi_f\colon L \to T^4$ onto the first component is a holomorphic line bundle on T^4. We define two sections $s_0, s_1\colon T^4 \to L$ of L by

$$s_0(x) = \begin{cases} (x, 1) \in f^{-1}(V_0) \times \mathbb{C} & \text{for } x \in f^{-1}(V_0), \\ (x, \psi_1(f(x))) \in f^{-1}(V_1) \times \mathbb{C} & \text{for } x \in f^{-1}(V_1), \\ (x, \Phi_b^0(x)) \in D_b \times \mathbb{C} & \text{for } x \in D_b, \end{cases}$$

$$s_1(x) = \begin{cases} (x, \psi_0(f(x))) \in f^{-1}(V_0) \times \mathbb{C} & \text{for } x \in f^{-1}(V_0), \\ (x, 1) \in f^{-1}(V_1) \times \mathbb{C} & \text{for } x \in f^{-1}(V_1), \\ (x, \Phi_b^1(x)) \in D_b \times \mathbb{C} & \text{for } x \in D_b. \end{cases}$$

It is easy to verify that the map $[s_0 : s_1]$ is equal to f. The line bundle L has nontrivial holomorphic sections s_0, s_1 and $c_1^2(L) = |B| > 0$. Thus by [17, Proposition 4.5.2], L is ample. □

Remark 3.2 The type of the polarization $c_1(L)$ for L in Lemma 3.1 is not equal to $(1,1)$ since s_0 and s_1 are linearly independent (see Theorem 2.4).

We can easily prove the following lemma using the inverse function theorem for holomorphic maps.

Lemma 3.3 *Let $x \in T^4$ be a base point of $h = [s_0 : s_1]$. Then the condition (3) in Section 2.1 holds at x if and only if $(ds_0)_x$ and $(ds_1)_x$ are linearly independent.*

Remark 3.4 For a polarization H of a complex torus T, the following map is surjective by [17, Corollary 2.5.4]:

$$T \to c_1^{-1}(H) \subset H^1(T; \mathcal{O}_L^*), \quad v \mapsto t_v^* L,$$

where L is an ample line bundle with $c_1(L) = H$ and $t_v \colon T \to T$ is the translation $x \mapsto x + v$. In particular the set of isomorphism classes of holomorphic Lefschetz pencils obtained from an ample line bundle L depends only on the class $c_1(L)$. Thus, any holomorphic Lefschetz pencil on T^4 is isomorphic to a pencil obtained from a pair of sections of a line bundle L_Z we constructed in Section 2.3.

3.1 A condition for pencils to be Lefschetz

As we explained in Section 2.3, $\mathfrak{H}_2$ is a moduli space of (d_1, d_2)–polarized abelian surfaces with a symplectic basis of the associated lattice for each type (d_1, d_2). Using a holomorphic function ϑ_Z^{ij} for $0 \le i < d_1$ and $0 \le j < d_2$ on $\mathbb{C}^2$ we define a map $\varphi_Z \colon T_Z \dashrightarrow \mathbb{CP}^N$, with $N = d_1 d_2 - 1$, by

$$\varphi_Z(\bar{x}) = [\cdots : \vartheta_Z^{ij}(x) : \cdots],$$

where $\bar{x} \in T_Z$ is a point represented by $x \in \mathbb{C}^2$. This map is well-defined by double-periodicity of ϑ_Z^{ij} and is defined on the complement of the intersection $\bigcap_{i,j} (\vartheta_Z^{ij})^{-1}(0)$. We denote the set of all hyperplanes in $\mathbb{CP}^N$ by $(\mathbb{CP}^N)^*$, which is canonically biholomorphic to $\mathbb{CP}^N$. For any projective line $P \subset (\mathbb{CP}^N)^*$, we define a pencil $f_P \colon T_Z \dashrightarrow P$ by

$$f_P(\bar{x}) = H \in P \quad \text{if } \bar{x} \in \varphi_Z^{-1}(H).$$

Let $H_0, H_1 \in P$ be distinct hyperplanes and $\sum_{i,j} a_{ij}^k X_{ij}$ a defining polynomial of H_k. It is easy to verify that f_P is defined on the complement of $\varphi_Z^{-1}(H_0 \cap H_1)$ and is isomorphic to $[s_0 : s_1]$, where $s_k = \sum_{i,j} a_{ij}^k \vartheta_Z^{ij} \in \Gamma(L_Z)$. Thus, by Lemma 3.1

any holomorphic Lefschetz pencil on T^4 is isomorphic to $f_P\colon T_Z \dashrightarrow P$ for some $P \subset (\mathbb{CP}^N)^*$. In this subsection, we will discuss when f_P becomes a Lefschetz pencil. Note that the arguments in this subsection are quite similar to those in [26, Section 2.1.1], in which generic pencils in linear systems of very ample line bundles are discussed, while we will discuss ample (but not necessarily very ample) line bundles on the four-torus. For this reason, we will omit details of some of the proofs in this subsection.

Lemma 3.5 *If $d_1 = 1$ and f_P is a Lefschetz pencil for some $P \subset (\mathbb{CP}^N)^*$, then T_Z is not biholomorphic to a product of elliptic curves.*

Proof Suppose that T_Z were a product $E_1 \times E_2$. The line bundle L_Z would be a tensor product $p_1^* L_1 \otimes p_2^* L_2$, where p_i is the projection onto the i^{th} component, L_1 is a line bundle on E_1 of degree 1 and L_2 is a line bundle on E_2 of degree d. The space $\Gamma(L_Z)$ would be generated by $s\cdot t_1, \dots, s\cdot t_d$, where $\Gamma(L_1) = \langle s\rangle$ and $\Gamma(L_2) = \langle t_1, \dots, t_d\rangle$. Thus, for any line $P \subset (\mathbb{CP}^N)^*$ the base locus of a pencil f_P would contain $s^{-1}(0)$, and in particular f_P would not be a Lefschetz pencil, contradicting the assumption. □

In what follows, we assume that T_Z is not a product of elliptic curves if $d_1 = 1$. Note that a generic $Z \in \mathfrak{H}_2$ satisfies this assumption by Lemma 2.3. For a homogeneous linear polynomial $q \in \mathbb{C}[\{X_{ij}\}_{0\le i<d_1, 0\le j<d_2}]$ we denote the zero-set of q by $H_q \in (\mathbb{CP}^N)^*$. We define a subset $W_Z \subset T_Z \times (\mathbb{CP}^N)^*$ by

$$W_Z = \left\{(\bar{x}, H_{\sum l_{ij}X_{ij}}) \in T_Z \times (\mathbb{CP}^N)^* \;\middle|\; \sum_{i,j} l_{ij}\vartheta_Z^{ij}(x) = 0,\ \sum_{i,j} l_{ij}\frac{\partial\vartheta_Z^{ij}}{\partial z_k}(x) = 0 \text{ for } k = 1, 2\right\}.$$

We can prove the following lemma by direct calculation.

Lemma 3.6 *Let $P \subset (\mathbb{CP}^N)^*$ be a line. Suppose that $\bar{x}$ is not a base point of f_P. The following conditions are equivalent:*

(1) $(\bar{x}, H_{\sum l_{ij}X_{ij}}) \in W_Z$.

(2) $f_P(\bar{x}) = H_{\sum l_{ij}X_{ij}}$ *and $\bar{x}$ is a critical point of f_P.*

(3) φ_Z *is not transverse to* $H_{\sum l_{ij}X_{ij}}$ *at* $\bar{x}$.

In what follows, we assume that $d_1 d_2$ is greater than or equal to 3. In this case φ_Z is defined on T_Z (see [17, Section 10.1]). For $i = 0, 1, 2$ we define

$$R_i = \{x \in T_Z \mid \operatorname{rank}(d\varphi_Z)_x = i\}. \tag{3-1}$$

We denote the union $\bigcup_{j \le i} R_j$ by S_i. The set S_i is an analytic subset of T_Z, and in particular the dimension of S_i makes sense.

Lemma 3.7 *The dimension of S_1 is at most 1. Furthermore, if $\mathrm{NS}(T_Z) \cong \mathbb{Z}$ and $S_0 \ne \varnothing$, then $\dim(S_0)$ is equal to 0.*

Proof Since T_Z is compact, the image $\varphi_Z(T_Z)$ is an analytic set by [5, Theorem 5.8]. Assume that $\dim(S_1) = 2$. Since T_Z is irreducible, S_1 is equal to T_Z. Thus the dimension of $\varphi_Z(T_Z)$ is 1 by the rank theorem (see [5, Theorem A2.2.2]). By Chow's theorem (see [5, Theorem 7.1]) $\varphi_Z(T_Z)$ is an algebraic curve. If the degree of $\varphi_Z(T_Z)$ were 1, then $\varphi_Z(T_Z)$ would be contained in some $H_{\sum l_{ij} X_{ij}} \in (\mathbb{CP}^N)^*$, but that would imply that the section $\sum l_{ij} \vartheta_Z^{ij}$ is the zero-section, contradicting the fact that $\{\vartheta_Z^{ij}\}_{i,j}$ is a basis of $\Gamma(L_Z)$. Thus the degree of $\varphi_Z(T_Z)$ is at least 2, and in particular $\varphi_Z(T_Z)$ intersects a generic hyperplane in $\mathbb{CP}^N$ at more than one point. This would imply that a generic divisor in $|L_Z|$ is reducible, contradicting [17, Theorem 4.3.5].

The map φ_Z is constant on each component of S_0. Since φ_Z is not a constant map, $\dim(S_0)$ is less than 2. Suppose that $\dim(S_0)$ were equal to 1. Take a one-dimensional component C of S_0 and denote the point in $\varphi_Z(C)$ by $c \in \mathbb{CP}^N$. Since there exists a hypersurface $H \in (\mathbb{CP}^N)^*$ away from c, the intersection number $[C] \cdot H_Z$ is equal to 0. On the other hand, the self-intersection H_Z^2 is positive. Since both the class $[C]$ and H_Z are contained in $\mathrm{NS}(T_Z)$, by assumption $[C]$ should be equal to 0, but it cannot happen since C is an algebraic curve. □

In what follows we assume that $\dim(S_0)$ is equal to 0 if S_0 is not empty. By Lemmas 2.3 and 3.7 this assumption holds for generic $Z \in \mathfrak{H}_2$. Note also that any pencil f_P would not be a Lefschetz pencil if $\dim(S_0) > 0$. Indeed, any point in a one-dimensional component of S_0 is either a base point or a critical point of f_P for any P.

Lemma 3.8 *The dimension of W_Z is at most $N - 1$.*

Proof of Lemma 3.8 Let $p_1\colon W_Z \to T_Z$ be the projection onto the first component. By Lemma 3.6, the restriction $p_1|_{p_1^{-1}(R_i)}$ is a fiber bundle with fiber $\mathbb{CP}^{N-1-i}$. Since the dimension of $R_0 = S_0$ is 0, it is a finite set (see [5, Proposition 3.4]). Thus $p_1^{-1}(R_0)$ is a manifold and its dimension is $N-1$ if it is not empty. Since $R_2 \subset T_Z$ is open, $p_1^{-1}(R_2)$ is also a manifold and its dimension is $N-1$ provided that $p_1^{-1}(R_0)$ is not empty. Suppose that the dimension of the locally analytic set $p_1^{-1}(R_1)$ is greater than $N-1$. There exists an open set $U \subset T_Z \times (\mathbb{CP}^N)^*$ such that the intersection $p_1^{-1}(R_1) \cap U$ is a manifold with dimension greater than $N-1$. Since $\mathrm{Sing}(R_1)$ is nowhere dense in R_1, the set $U \cap p_1^{-1}(\mathrm{reg}(R_1))$ is not empty. Since $\mathrm{reg}(R_1)$ is open in R_1, the set $U \cap p_1^{-1}(\mathrm{reg}(R_1))$ is a manifold with dimension greater than $N-1$. However, this is impossible since $p_1^{-1}(\mathrm{reg}(R_1))$ is a fiber bundle over $\mathrm{reg}(R_1)$, which is a 1–dimensional manifold if it is not empty, with fiber $\mathbb{CP}^{N-2}$. Thus the dimension of $p_1^{-1}(R_1)$ is at most $N-1$. Since W_Z is the union $p_1^{-1}(R_0) \cup p_1^{-1}(R_1) \cup p_1^{-1}(R_2)$, its dimension is at most $N-1$. □

Let $p_2\colon W_Z \to (\mathbb{CP}^N)^*$ be the projection onto the second component and $\mathcal{D}_Z$ the image of p_2. Since p_2 is a proper map, $\mathcal{D}_Z$ is an analytic set of dimension at most $\dim(W_Z)$; see [5, Theorem 5.8].

Lemma 3.9 *The dimensions of $\mathcal{D}_Z$ and W_Z are both $N-1$.*

Proof Since $\dim(\mathcal{D}_Z) \le \dim(W_Z)$, it is enough to prove $\dim(\mathcal{D}_Z) = N-1$ by Lemma 3.8. Since $\dim(\mathcal{D}_Z)$ is at most $N-1$, there exists a point $H \in (\mathbb{CP}^N)^*$ away from $\mathcal{D}_Z$. Let $\pi_H\colon (\mathbb{CP}^N)^* \setminus \{H\} \to \mathbb{CP}^{N-1}$ be the projection from H. The image $\pi_H(\mathcal{D}_Z)$ is an analytic set since the restriction $\pi_H|_{\mathcal{D}_Z}$ is proper. If $\dim(\mathcal{D}_Z)$ were less than $N-1$, the dimension of $\pi_H(\mathcal{D}_Z)$ would also be less than $N-1$. Thus we could take a point $x \in \mathbb{CP}^{N-1}$ away from $\pi_H(\mathcal{D}_Z)$. We denote the closure $\overline{\pi_H^{-1}(x)}$ by P_x, which is a line in $(\mathbb{CP}^N)^*$. Using Lemma 3.3 we can verify that f_{P_x} is a Lefschetz pencil on T_Z without critical points. This would imply that a blow-up of T_Z admits a surface bundle over $\mathbb{CP}^1$, which is impossible. □

We define the subset $W_Z^0 \subset W_Z$ as

$$W_Z^0 = \left\{ (\overline{x}, H_{\sum l_{ij} X_{ij}}) \in W_Z \;\middle|\; \det\left(\left(\sum_{i,j} l_{ij} \frac{\partial^2 \vartheta_Z^{ij}}{\partial z_k \partial z_l}(x)\right)_{1 \le k,l \le 2}\right) \neq 0 \right\}.$$

The next two lemmas follow from the same arguments as those in [26, Section 2.1.1].

Lemma 3.10 *Suppose that $\overline{x} \in T_Z$ is not a base point of f_P for a line $P \subset (\mathbb{CP}^N)^*$. Then the following conditions are equivalent:*

(1) $(\bar{x}, H) \in W_Z^0$.

(2) $f_P(\bar{x}) = H$ and $\bar{x}$ is a Lefschetz-type critical point of f_P.

Moreover, if $(\bar{x}, H) \in W_Z^0$ *then* W_Z *is regular at* $(\bar{x}, H)$ *with* $\dim_{(\bar{x},H)} W_Z = N-1$, *and* p_2 *is an immersion at* $(\bar{x}, H)$.

Lemma 3.11 *Let* $(\bar{x}, H) \in W_Z^0$ *and* $P \subset (\mathbb{CP}^N)^*$ *be a line containing* H. *Then* $T_H P$ *is contained in* $(dp_2)_{(\bar{x},H)}(T_{(\bar{x},H)} W_Z)$ *if and only if* $\varphi_Z^{-1}(H')$ *contains* $\bar{x}$ *for any* $H' \in P$.

We define two subsets $\mathcal{D}'_Z$ and $\mathcal{D}''_Z$ of $\mathcal{D}_Z$ as follows:

$$\mathcal{D}'_Z = p_2(W_Z \setminus W_Z^0), \quad \mathcal{D}''_Z = \{H \in \mathcal{D}_Z \setminus \mathcal{D}'_Z \mid \sharp(p_2^{-1}(H)) \neq 1\}.$$

Since $W_Z \setminus W_Z^0$ is analytic and p_2 is proper, $\mathcal{D}'_Z$ is analytic by [5, Theorem 5.8]. Furthermore, it is easy to see that $\mathcal{D}''_Z$ is contained in $\mathrm{Sing}(\mathcal{D}_Z)$, which is an analytic set with dimension at most $N-2$ by [5, Theorem 5.2.2].

Theorem 3.12 *For a line* $P \subset (\mathbb{CP}^N)^*$, *the following conditions are equivalent:*

(1) *The map* f_P *is a Lefschetz pencil.*

(2) *The line* P *is away from* $\mathcal{D}'_Z \cup \mathcal{D}''_Z$ *and for any* $(\bar{x}, H) \in p_2^{-1}(\mathcal{D}_Z \cap P)$ *we have* $(dp_2)_{(\bar{x},H)}(T_{(\bar{x},H)} W_Z) + T_H P = T_H(\mathbb{CP}^N)^*$.

We can prove Theorem 3.12 in the same way as in the proof of [26, Proposition 2.9]. By Lemma 3.10 the set $\mathcal{D}_Z^0 = \mathcal{D}_Z \setminus (\mathcal{D}'_Z \cup \mathcal{D}''_Z)$ is a submanifold of $(\mathbb{CP}^N)^*$ of dimension $N-1$. We can easily deduce the following corollary from Theorem 3.12.

Corollary 3.13 *Suppose that* $H \in (\mathbb{CP}^N)^*$ *is away from* $\mathcal{D}_Z$. *Denote the projection from* H *by* $\pi_H \colon \mathcal{D}_Z \to \mathbb{CP}^{N-1}$. *The map*

$$f_{\overline{\pi_H^{-1}(x)}} \colon T_Z \dashrightarrow \overline{\pi_H^{-1}(x)}$$

is a Lefschetz pencil if and only if $\pi_H^{-1}(x)$ *is away from* $\mathcal{D}'_Z \cup \mathcal{D}''_Z$ *and* x *is a regular value of* $\pi_H|_{\mathcal{D}_Z^0}$.

Remark 3.14 Using Theorem 3.12 we can prove that for a generic $Z \in \mathfrak{H}_2$ the set of lines in $(\mathbb{CP}^N)^*$ giving rise to Lefschetz pencils on T_Z is connected. We can thus deduce that two Lefschetz pencils on T_Z coming from the same polarization are isomorphic (see the proofs of Lemma 3.16 and Corollary 3.17). Still, Theorem 1.1 does not follow from this fact since the Lefschetz pencils f_0 and f_1 in Theorem 1.1 do not necessarily give the same polarization on the total spaces.

We denote the set of all projective lines in $(\mathbb{CP}^N)^*$ by $\mathcal{L}$, which can be identified with the Grassmannian manifold $G_2(\mathbb{C}^{N+1})$. Since $\pi_H(\mathrm{Sing}(\mathcal{D}_Z))$ is an analytic set of dimension at most $N-2$, we can deduce the following corollary.

Corollary 3.15 *Suppose that $f_P\colon T_Z \dashrightarrow P$ satisfies the conditions (2) and (3) in Section 2.1. For any open neighborhood $U \subset \mathcal{L}$ of P, there exists a line $P' \in U$ such that $f_{P'}$ is a Lefschetz pencil.*

3.2 Existence of paths connecting two Lefschetz pencils

Let d_1, d_2 be positive integers with $d_1 \mid d_2$. Throughout this subsection, we assume that $d_1 d_2 \geq 3$. As we observed in the beginning of Section 3.1, any holomorphic Lefschetz pencil on T^4 with genus $d_1 d_2 + 1$ and divisibility d_1 is isomorphic to $f_P\colon T_Z \dashrightarrow P$ for $Z \in \mathfrak{H}_2$ and $P \in \mathcal{L}$. The aim of this subsection is to take a path in $\mathfrak{H}_2 \times \mathcal{L}$ which connects two points associated with two given Lefschetz pencils.

Lemma 3.16 *The subset of $\mathfrak{H}_2 \times \mathcal{L}$ consisting of points which yield Lefschetz pencils is open.*

Proof Let $S_2(\mathbb{C}^{N+1})$ be the space of all pairs of $\mathbb{C}$–linearly independent vectors in $\mathbb{C}^{N+1}$ endowed with the relative topology of $(\mathbb{C}^{N+1})^2$; that is, $S_2(\mathbb{C}^{N+1})$ is a noncompact Stiefel manifold. Since the quotient map $\pi\colon S_2(\mathbb{C}^{N+1}) \to \mathcal{L}$ is continuous and open, it is enough to show that the set

$$\{(Z,(v_0,v_1)) \in \mathfrak{H}_2 \times S_2(\mathbb{C}^{N+1}) \mid (Z,\pi(v_0,v_1)) \text{ yields a Lefschetz pencil}\}$$

is an open subset.

For $(Z,(v_0,v_1)) \in \mathfrak{H}_2 \times S_2(\mathbb{C}^{N+1})$, define diffeomorphisms $\phi_{v_0,v_1}\colon \mathbb{CP}^1 \to \pi(v_0,v_1)$ and $\psi_Z\colon \mathbb{R}^4 \to \mathbb{C}^2$ by

$$\phi_{v_0,v_1}([l_0:l_1]) = H_{\sum(l_0 v_0^{ij} + l_1 v_1^{ij})X_{ij}}, \qquad \psi_Z(x) = (Z,D)x,$$

where we put $v_k = (\dots, v_k^{ij}, \dots)$ and $D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$. We can deduce from Lemma 3.10 that $f_{\pi(v_0,v_1)}(\overline{\psi_Z(x)}) = \phi_{v_0,v_1}([l_0:l_1])$ and that $\overline{\psi_Z(x)}$ is a Lefschetz critical point if and only if the following three conditions are satisfied:

- $\displaystyle\sum_{i,j}(l_0 v_0^{ij} + l_1 v_1^{ij})\vartheta_Z^{ij}(\psi_Z(x)) = 0.$

- $$\sum_{i,j}(l_0 v_0^{ij} + l_1 v_1^{ij})\frac{\partial \vartheta_Z^{ij}}{\partial z_k}(\psi_Z(x)) = 0 \text{ for } k = 1, 2.$$
- $$\det\left(\left(\sum_{i,j}(l_0 v_0^{ij} + l_1 v_1^{ij})\frac{\partial^2 \vartheta_Z^{ij}}{\partial z_k \partial z_l}(\psi_Z(x))\right)_{1\le k,l\le 2}\right) \neq 0.$$

Furthermore, by Lemma 3.11 the following two conditions are equivalent:

- The point $(\overline{\psi_Z(x)}, \phi_{v_0,v_1}([l_0 : l_1]))$ is contained in W_Z^0 and
$$T_{\phi_{v_0,v_1}([l_0:l_1])}\pi(v_0, v_1) \subset (dp_2)_{(\overline{\psi_Z(x)},\phi_{v_0,v_1}([l_0:l_1]))}(T_{(\overline{\psi_Z(x)},\phi_{v_0,v_1}([l_0:l_1]))}W_Z).$$
- $(Z, (v_0, v_1))$ satisfies the three conditions above and $\sum_{i,j} v_k^{ij}\vartheta_Z^{ij}(\psi_Z(x)) = 0$ for $k = 0, 1$.

We define a map $\Phi(Z, (v_0, v_1))\colon \mathbb{R}^4 \times \mathbb{CP}^1 \to \mathbb{C}^6$ as follows:

$$\begin{aligned}
&\Phi(Z, (v_0, v_1))(x, [l_0 : l_1]) = \\
&\Bigg(\sum_{i,j}(l_0 v_0^{ij} + l_1 v_1^{ij})\vartheta_Z^{ij}(\psi_Z(x)), \sum_{i,j}(l_0 v_0^{ij} + l_1 v_1^{ij})\frac{\partial \vartheta_Z^{ij}}{\partial z_1}(\psi_Z(x)), \\
&\quad \sum_{i,j}(l_0 v_0^{ij} + l_1 v_1^{ij})\frac{\partial \vartheta_Z^{ij}}{\partial z_2}(\psi_Z(x)), \det\left(\left(\sum_{i,j}(l_0 v_0^{ij} + l_1 v_1^{ij})\frac{\partial^2 \vartheta_Z^{ij}}{\partial z_k \partial z_l}(\psi_Z(x))\right)\right), \\
&\qquad\qquad \sum_{i,j} v_0^{ij}\vartheta_Z^{ij}(\psi_Z(x)), \sum_{i,j} v_1^{ij}\vartheta_Z^{ij}(\psi_Z(x))\Bigg).
\end{aligned}$$

Let

$$\begin{aligned}
V_1 &= \{0\} \times \{0\} \times \mathbb{C}^2, \\
V_2 &= \{0\} \times \mathbb{C} \times \{0\} \subset \mathbb{C}^3 \times \mathbb{C} \times \mathbb{C}^2 = \mathbb{C}^6, \\
\Delta &= \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid |x_i| \le \tfrac{1}{2}\}.
\end{aligned}$$

We can deduce from Lemmas 3.6, 3.10 and Theorem 3.12, together with double-periodicity of the Theta functions, that $f_{\pi(v_0,v_1)}\colon T_Z \dashrightarrow \pi(v_0, v_1)$ satisfies the conditions (2) and (3) in Section 2.1 if and only if $\Phi(Z, (v_0, v_1))(\Delta \times \mathbb{CP}^1) \cap (V_1 \cup V_2)$ is empty. Since V_1 and V_2 are closed and $\Delta \times \mathbb{CP}^1$ is compact, the subset

$$W_0 = \{(Z, (v_0, v_1)) \in \mathfrak{H}_2 \times S_2(\mathbb{C}^{N+1}) \mid \Phi(Z, (v_0, v_1))(\Delta \times \mathbb{CP}^1) \cap (V_1 \cup V_2) = \varnothing\}$$

is open.

We take a point $(Z, (v_0, v_1)) \in W_0$ and suppose that $f_{\pi(v_0,v_1)}\colon T_Z \dashrightarrow \pi(v_0, v_1)$ is a Lefschetz pencil. We put $\phi_{v_0,v_1}^{-1}(f_{\pi(v_0,v_1)}(\mathrm{Crit}(f_{\pi(v_0,v_1)}))) = \{y_1, \ldots, y_n\}$, where

$n = 6d_1 d_2$ is the number of critical points of $f_{\pi(v_0,v_1)}$. For each i, we take a disk neighborhood D_i of y_i in $\mathbb{CP}^1$ such that $D_i \cap D_j = \varnothing$ if $i \neq j$. The first three components of $\Phi(Z,(v_0,v_1))(x,[l_0 : l_1])$ can never be equal to 0 simultaneously for $(x,[l_0 : l_1]) \in \Delta \times \left(\mathbb{CP}^1 \setminus \bigsqcup_i D_i\right)$. Since $\Delta \times \left(\mathbb{CP}^1 \setminus \bigsqcup_i D_i\right)$ is compact, there exists an open neighborhood $U \subset W_0$ of $(Z,(v_0,v_1))$ such that for any $(Z',(v_0',v_1')) \in U$ the first three components of $\Phi(Z',(v_0',v_1'))(x,[l_0 : l_1])$ will never be equal to 0 simultaneously for $(x,[l_0 : l_1]) \in \Delta \times \left(\mathbb{CP}^1 \setminus \bigsqcup_i D_i\right)$. Thus, all the critical values of $f_{\pi(v_0',v_1')}\colon T_{Z'} \dashrightarrow \pi(v_0',v_1')$ are contained in the disjoint union $\bigsqcup_i D_i$, where $\pi(v_0',v_1')$ is identified with $\mathbb{CP}^1$ via $\phi^{-1}_{v_0',v_1'}$. Furthermore, we can make U sufficiently small so that the conjugacy class of a local monodromy of $f_{\pi(v_0',v_1')}\colon T_{Z'} \dashrightarrow \pi(v_0',v_1')$ around D_i is independent of the choice of $(Z',(v_0',v_1')) \in U$. In particular for any $(Z',(v_0',v_1')) \in U$ and i, the preimage $f^{-1}_{\pi(v_0',v_1')}(D_i)$ contains at least one critical point of $f_{\pi(v_0',v_1')}\colon T_{Z'} \dashrightarrow \pi(v_0',v_1')$. Since $f_{\pi(v_0',v_1')}$ has genus $g = d_1 d_2 + 1$ and $b = 2d_1 d_2$ base points, $0 = \chi(T_{Z'})$ is equal to $4 - 4g + n' - b = n' - 6d_1 d_2$, where n' is the number of critical points of $f_{\pi(v_0',v_1')}$. Thus n' is equal to n and the set $f^{-1}_{\pi(v_0',v_1')}(D_i)$ contains exactly one critical point for each i, which implies that $f_{\pi(v_0',v_1')}$ satisfies condition (1) for any $(Z',(v_0',v_1')) \in U$. We can eventually conclude that the set of points in $\mathfrak{H}_2 \times S_2(\mathbb{C}^{N+1})$ giving rise to a Lefschetz pencil is open. □

The proof of Lemma 3.16 implies the following corollaries.

Corollary 3.17 *Let $\mathcal{W}$ be the set of $(Z,P) \in \mathfrak{H}_2 \times \mathcal{L}$ such that $f_P\colon T_Z \dashrightarrow P$ is a Lefschetz pencil. For $(Z_i,P_i) \in \mathcal{W}$ with $i = 0,1$ the two pencils f_{P_0} and f_{P_1} are isomorphic if (Z_0,P_0) and (Z_1,P_1) are contained in the same connected component of $\mathcal{W}$.*

Proof Suppose that (Z_0,P_0) and (Z_1,P_1) are contained in the same connected component of $\mathcal{W}$. The proof of Lemma 3.16 shows that the monodromy factorizations of f_{P_0} and f_{P_1} are Hurwitz equivalent. Thus f_{P_0} and f_{P_1} are isomorphic by Theorem 2.2. □

Corollary 3.18 *A genus $g \geq 4$ holomorphic Lefschetz pencil on T^4 does not have a reducible fiber.*

Proof Suppose that $f_P\colon T_Z \dashrightarrow P$ has a reducible fiber $F = F_1 + F_2$. By Lemma 2.3, Lemma 3.16 and Corollary 3.17, we may assume that $\mathrm{NS}(T_Z)$ is isomorphic to $\mathbb{Z}$

without loss of generality. Since $[F_i] \in \mathrm{NS}(T_Z)$ for $i = 1, 2$, $[F_i] = n_i\alpha$ for some $n_i \in \mathbb{Z}$ and $\alpha \in H^2(T_Z; \mathbb{Z})$. Since F_1 and F_2 intersect at one point (which is a Lefschetz singularity of f_P), we have that $n_1 n_2 \alpha^2$ is equal to 1. Thus $2g-2 = [F]^2$ must be equal to 4, which contradicts the assumption. □

Remark 3.19 We can deduce from Corollary 3.18 that a genus $g \geq 4$ Lefschetz pencil on the four-torus T^4 with reducible fibers cannot be holomorphic.

Lemma 3.20 *Suppose that the following condition, for (d_1, d_2), is satisfied:*

(∗) *The set $\{Z \in \mathfrak{H}_2 \mid \dim(\mathcal{D}'_Z) \geq N-1\}$ is contained in a countable union of analytic sets with positive codimensions.*

Then $\mathcal{W}$ defined in Corollary 3.17 is path-connected.

Proof For $(Z_i, P_i) \in \mathcal{W}$, $i = 0, 1$, we first take a path $\beta\colon [0,1] \to \mathfrak{H}_2$ which satisfies the following properties:

- $\beta(i) = Z_i$ for $i = 0, 1$.
- $\dim(\mathcal{D}'_{\beta(t)}) < N-1$ for any $t \in (0,1)$.
- The group $\mathrm{NS}(T_{\beta(t)})$ is the infinite cyclic group for any $t \in (0,1)$.

We can take such a path by the assumption and Lemma 2.3. We may further assume that $T_{\beta(t)}$ is not a product of elliptic curves by Lemma 2.3 and that an analytic set $S_0 \subset T_{\beta(t)}$, defined after Lemma 3.6, has dimension 0 by Lemma 3.7. By Lemma 3.16 there exists $\varepsilon > 0$ such that $f_{P_0}\colon T_{\beta(t)} \dashrightarrow P_0$ and $f_{P_1}\colon T_{\beta(1-t)} \dashrightarrow P_1$ are both Lefschetz pencils for $t \in [0, \varepsilon]$. We will prove that there exists a piecewise smooth path $\gamma\colon [t_0, t_1] \to \mathfrak{H}_2 \times \mathcal{L}$ which satisfies the following conditions:

(1) $\gamma_2(t_0) = P_0$.

(2) $f_{\gamma_2(t)}\colon T_{\gamma_1(t)} \dashrightarrow \gamma_2(t)$ is a Lefschetz pencil for any $t \in [t_0, t_1]$.

(3) There exists a monotone nondecreasing function $\delta\colon [t_0, t_1] \to [\varepsilon, 1-\varepsilon]$ such that $\delta(t_0) = \varepsilon$ and $\gamma_1(t) = \beta(\delta(t))$ for any $t \in [t_0, t_1]$.

(4) $\delta(t_1) = 1-\varepsilon$ and $\gamma_2(t_1) = P_1$.

Here $\gamma_i(t)$ is the i^{th} component of $\gamma(t)$. In order to prove existence of such a path, we define a value $T \leq 1-\varepsilon$ by

$$T = \sup\{t \in [0, 1-\varepsilon] \mid \exists\, \gamma\colon [t_0, t_1] \to \mathcal{L} \text{ satisfying (1)–(3) and } \gamma_1(t_1) = \beta(t)\}.$$

The value T is equal to $1-\varepsilon$. To see this, suppose that T is less than $1-\varepsilon$. Using Corollary 3.13 we can find a line $P \in \mathcal{L}$ such that $f_P\colon T_{\beta(T)} \dashrightarrow P$ is a Lefschetz pencil. By Lemma 3.16 we can take $\varepsilon' > 0$ such that $f_P\colon T_{\beta(t)} \dashrightarrow P$ is a Lefschetz pencil for any $t \in [T-\varepsilon', T+\varepsilon']$. By the definition of T, there exists a path $\gamma\colon [t_0, t_1] \to \mathcal{L}$ and $s \in (T-\varepsilon', T]$ such that γ satisfies the conditions (1)–(3) and $\gamma_1(t_1) = \beta(s)$. By the assumption we can take a path in $\mathcal{L}^0_{\beta(s)}$ which connects $\gamma_2(s)$ and P. We can then extend a path γ so that the extended path $\tilde{\gamma}$ satisfies the conditions (1)–(3) and $\tilde{\gamma}(t_1) = T + \varepsilon'$, which contradicts the definition of T. Thus we can conclude that $T = 1-\varepsilon$. In the same way as above, we can then take a path γ which satisfies the conditions (1)–(4). Eventually we can obtain a path connecting (Z_0, P_0) and (Z_1, P_1) by concatenating the three paths $t \mapsto (\beta(t), P_0)$ defined on $[0, \varepsilon]$, the path γ obtained above and the path $t \mapsto (\beta(t), P_1)$ defined on $[1-\varepsilon, 1]$. □

We can eventually deduce the following from Corollary 3.17 and Lemma 3.20.

Theorem 3.21 *Suppose that the condition (∗) in Lemma 3.20 is satisfied. Then any two holomorphic Lefschetz pencils on T^4 with genus $(d_1 d_2 + 1)$ and divisibility d_1 are isomorphic.*

3.3 The condition (∗) for a pair (d_1, d_2)

As we proved in the last subsection, any two holomorphic Lefschetz pencils with genus $(d_1 d_2 + 1)$ and divisibility d_1 are isomorphic *provided that the condition (∗) in Lemma 3.20 is satisfied*. In this subsection we discuss which pairs (d_1, d_2) satisfy this condition.

We first observe that if $d_1 d_2 \geq 5$, the set of $Z \in \mathfrak{H}_2$ such that L_Z is not very ample is contained in an algebraic set with positive codimension (see [17, Theorem 4.5.1, Section 10.1, Theorem 10.4.1]). Furthermore, by the same arguments as those in [26, Section 2.1.1], we can deduce that $\mathcal{D}'_Z$ has dimension at most $N-2$ when L_Z is very ample. We thus obtain:

Lemma 3.22 *The condition (∗) holds if $d_1 d_2 \geq 5$.*

The only remaining case covered in Theorem 1.1 is $(d_1, d_2) = (2, 2)$.

Lemma 3.23 *The condition (∗) holds for $(d_1, d_2) = (2, 2)$.*

Proof For $Z \in \mathfrak{H}_2$, the $(2,2)$–polarized abelian surface (T_Z, H_Z) is isomorphic to $(T_{Z'}, 2H_{Z'})$, where $Z' = Z/2 \in \mathfrak{H}_2$ and $(T_{Z'}, H_{Z'})$ is the $(1,1)$–polarized abelian surface corresponding to Z' (the isomorphism sends $\overline{x} \in T_Z$ to $\overline{x/2} \in T_{Z'}$). Suppose that T_Z is not a product of elliptic curves (this condition holds for generic Z by Lemma 2.3). The abelian surface $T_{Z'}$ is not also a product of elliptic curves. We denote the Kummer surface $T_Z/\langle -1 \rangle$ associated with T_Z by K_Z. Since $L_{Z'}$ is symmetric in the sense of [17, Section 4.6], there exists an embedding $\eta_Z\colon K_Z \to (\mathbb{CP}^3)^*$ such that the following diagram commutes (see [17, Section 4.8]):

$$\begin{tikzcd} T_Z \arrow[r, "\varphi_Z"] \arrow[d, "\pi"'] & (\mathbb{CP}^3)^* \\ K_Z \arrow[ur, "\eta_Z"'] & \end{tikzcd}$$

where $\pi\colon T_Z \to K_Z$ is the quotient map. Thus, the set $R_0 \subset T_Z$ defined in (3-1) consists of sixteen points which are preimages of singular points of K_Z under π, and $R_2 = T_Z \setminus R_0$. The preimage $p_1^{-1}(R_2) \subset W_Z$ is a manifold with dimension $N-1$. If the dimension of $p_2(p_1^{-1}(R_2))$ is less than $N-1$, that of $\mathcal{D}_Z \cap p_2(p_1^{-1}(R_2))$ is also less than $N-1$. Suppose that the dimension of $p_2(p_1^{-1}(R_2))$ is $N-1$. In the same way as in the proof of Lemma 3.22, we can verify that $p_1^{-1}(R_2) \cap (W_Z \setminus W_Z^0)$ is the set of points at which the restriction $p_2\colon p_1^{-1}(R_2) \to \mathcal{D}_Z$ is not an immersion, in particular $p_1^{-1}(R_2) \cap W_Z^0$ is not empty. Since $p_1^{-1}(R_2)$ is irreducible and $p_1^{-1}(R_2) \cap (W_Z \setminus W_Z^0)$ is locally analytic, the dimension of $p_1^{-1}(R_2) \cap (W_Z \setminus W_Z^0)$ is at most $N-2$.

For $\overline{x} \in R_0$, the preimage $p_1^{-1}(\overline{x})$ is a hyperplane in $\{\overline{x}\} \times (\mathbb{CP}^N)^*$. The proof of [17, Theorem 4.8.1] implies that the following map is an isomorphism for any $x \in \mathbb{C}^2$ representing $\overline{x} \in R_0$:

$$S^2(T_x\mathbb{C}^2) \to \mathrm{Hom}(\varphi_Z(\overline{x}), \mathbb{C}), \quad \sum a_{ij} \frac{\partial^2}{\partial z_i \partial z_j} \mapsto \Big(\vartheta \mapsto a_{ij} \frac{\partial^2 \vartheta}{\partial z_i \partial z_j}(x)\Big),$$

where $S^2(T_x\mathbb{C}^2)$ is the symmetric product of $T_x\mathbb{C}^2$ and we identify $\varphi_Z(\overline{x})$ in $(\mathbb{CP}^N)^*$ with a hyperplane in $\Gamma(L_Z)$ using the basis $\{\vartheta_Z^{ij}\}$ of $\Gamma(L_Z)$. We let $s_{ij} \in \mathrm{Hom}(\varphi_Z(\overline{x}), \mathbb{C})$ be the image of $\partial^2/\partial z_i \partial z_j$ under the map above. The set $\{s_{11}, s_{12}, s_{22}\}$ is a basis of $\mathrm{Hom}(\varphi_Z(\overline{x}), \mathbb{C})$, so we can take a dual basis $\{s_{11}^*, s_{12}^*, s_{22}^*\}$ of $\mathrm{Hom}(\varphi_Z(\overline{x}), \mathbb{C})^* \cong \varphi_Z(\overline{x})$. Let $\vartheta_0 = s_{11}^* + s_{22}^* \in \varphi_Z(\overline{x})$. This theta function satisfies $\partial^2\vartheta_0/\partial z_i \partial z_j(x) = \delta_{ij}$. In particular, $(\overline{x}, \vartheta_0)$ is contained in $W_Z^0 \cap p_1^{-1}(\overline{x})$. Since $p_1^{-1}(\overline{x})$ is irreducible, the dimension of $(W_Z \setminus W_Z^0) \cap p_1^{-1}(\overline{x})$ is at most $N-2$.

We can eventually conclude that the dimension of

$$W_Z \setminus W_Z^0 = (p_1^{-1}(R_2) \cap (W_Z \setminus W_Z^0)) \cup (p_1^{-1}(R_0) \cap (W_Z \setminus W_Z^0))$$

is at most $N-2$. Thus the dimension of the image $\mathcal{D}'_Z = p_2(W_Z \setminus W_Z^0)$ is also at most $N-2$. □

Theorem 1.1 immediately follows from Theorem 3.21 and Lemmas 3.22 and 3.23.

We thus far cannot guarantee that the assumption $(*)$ holds when $(d_1, d_2) = (1, 3)$ or $(1, 4)$. Furthermore, the arguments in this section do not work when $(d_1, d_2) = (1, 2)$ (note that we assumed that $d_1 d_2$ is at least 3 in the paragraph preceding Lemma 3.7). Still, we believe the following conjecture holds:

Conjecture 3.24 Two holomorphic Lefschetz pencils on the four-torus (with any genera and divisibilities) are isomorphic if and only if their genera and divisibilities coincide.

4 Examples of Lefschetz pencils on the four-torus

We can deduce from Corollary 3.13 and Lemmas 3.22 and 3.23 that there exists a holomorphic Lefschetz pencil on T^4 with genus $(d_1 d_2 + 1)$ and divisibility d_1 for any d_1, d_2 with $d_1 \mid d_2$ and either $d_1 d_2 \geq 5$ or $d_1 \geq 2$. Such a pencil can be obtained by taking a generic pencil in the complete linear system of an ample line bundle, in particular it is in general difficult to determine its monodromy factorization. Note that so far we have not yet proved the existence of holomorphic Lefschetz pencils corresponding to $(d_1, d_2) = (1, 2)$, $(1, 3)$ and $(1, 4)$, while the case of $(1, 1)$ is already excluded (cf Remark 3.2). In this section we will first obtain a genus-3 holomorphic pencil on T^4 with $(d_1, d_2) = (1, 2)$ following the construction due to Smith [24], and determine vanishing cycles of it. We will further obtain holomorphic pencils with higher genera (and their vanishing cycles) by composing unbranched coverings.

We begin by observing the relation between (possibly branched) coverings and monodromies of mappings. Let X be a closed four-manifold, Σ a closed surface and $f\colon X \to \Sigma$ a smooth map with discrete critical value set. As in Section 2.2, we can define a local monodromy by taking a loop around a critical value of f and a parallel transport along this loop with respect to a horizontal distribution of the submersion $f|_{X \setminus \mathrm{Crit}(f)}$. Let $q\colon \widetilde{X} \to X$ be a covering branched at a (possibly disconnected and empty) immersed surface S with transverse self-intersections (the reader can refer to [11, Chapter 7], for example, for coverings branched at such surfaces). We denote the set of self-intersections of S by $D(S) \subset X$ and the critical point set of the restriction $f|_{S\setminus D(S)}$ by $T_f(S)$. In what follows we assume that the image $f(T_f(S))$ is a discrete set. It is easy to see that the critical value set of the composition $f \circ q \colon \widetilde{X} \to \Sigma$ is contained in the image $f(\mathrm{Crit}(f) \cup D(S) \cup T_f(S))$, which is a discrete set by assumption. In particular, we can define a local monodromy of each critical value of $f \circ q$. We will discuss the relation of monodromies of f and $f \circ q$ below.

Let $a_0 \in \Sigma$ be a point away from $f(\mathrm{Crit}(f) \cup D(S) \cup T_f(S))$. By assumption the fiber $f^{-1}(a_0)$ is a submanifold of X and intersects S transversely. In particular the intersection $f^{-1}(a_0) \cap S$ is a finite set. Using this we can identify $f^{-1}(a_0)$ with a genus g closed surface Σ_g with $p = \sharp(f^{-1}(a_0) \cap S)$ marked points, which we denote by $\Sigma_{g,p}$. For a point $a \in f(\mathrm{Crit}(f) \cup D(S) \cup T_f(S))$ we take a path α from a_0 to a. We further take a loop $\widetilde{\alpha}$ by connecting α with a small circle around a. Let $\mathcal{H}$ be a horizontal distribution of the restriction $f|_{X\setminus(\mathrm{Crit}(f)\cup D(S)\cup T_f(S))}$ such that $\mathcal{H}_x$ coincides with T_xS for any $x \in S \setminus (\mathrm{Crit}(f) \cup D(S) \cup T_f(S))$. Using the identification $f^{-1}(a_0) \cong \Sigma_{g,p}$ we can regard the parallel transport $T_{\widetilde{\alpha},\mathcal{H}}$ along $\widetilde{\alpha}$ with respect to $\mathcal{H}$ as a self-diffeomorphism of $\Sigma_{g,p}$ preserving the marked points setwise.

By assumption the restriction $q|_{f^{-1}(a_0)}$ is a covering which is branched at the finite set $f^{-1}(a_0) \cap S$. In particular we can take an identification of a fiber $(f \circ q)^{-1}(a_0)$ with a marked surface $\Sigma_{\widetilde{g},p}$, which is a covering of $\Sigma_{g,p}$ branched at p marked points. Since $\mathcal{H}$ is tangent to S at any point in $S \setminus (\mathrm{Crit}(f) \cup D(S) \cup T_f(S))$, we can take a lift $\widetilde{\mathcal{H}}$ of $\mathcal{H}$ by the branched covering q, which is a horizontal distribution of $f \circ q$. It is easy to verify that the parallel transport $T_{\widetilde{\alpha},\widetilde{\mathcal{H}}}$ is a lift of $T_{\widetilde{\alpha},\mathcal{H}}$ by q; that is, the following diagram commutes:

$$\begin{array}{ccc} \Sigma_{\widetilde{g},p} & \xrightarrow{T_{\widetilde{\alpha},\widetilde{\mathcal{H}}}} & \Sigma_{\widetilde{g},p} \\ {\scriptstyle q}\big\downarrow & & \big\downarrow{\scriptstyle q} \\ \Sigma_{g,p} & \xrightarrow{T_{\widetilde{\alpha},\mathcal{H}}} & \Sigma_{g,p} \end{array}$$

We eventually obtain the following lemma.

Lemma 4.1 *Let $\mathrm{Mod}(\Sigma_{g,p})$ be the group of isotopy classes of orientation-preserving self-diffeomorphisms of $\Sigma_{g,p}$ which preserve the marked points setwise, and let $\varphi_{\widetilde{\alpha}} \in \mathrm{Mod}(\Sigma_{g,p})$ be the monodromy of f along $\widetilde{\alpha}$. Then the monodromy of $f \circ q$ along $\widetilde{\alpha}$ is represented by a lift of a representative of $\varphi_{\widetilde{\alpha}}$ by q.*

Remark 4.2 Lemma 4.1 *does not* uniquely determine the monodromy of $f \circ q$ along $\widetilde{\alpha}$: a lift of a representative of $\varphi_{\widetilde{\alpha}}$ by q is unique *up to covering transformations of* q. Still, such an ambiguity will not cause any problems in the following subsections. Indeed, monodromies we will deal with must satisfy some additional conditions, which determine them uniquely.

4.1 Genus-3 holomorphic pencils due to Smith

In [24] Smith gave a way to construct a genus-3 holomorphic Lefschetz pencil on T^4 by taking a branched covering of a singular projective variety. Although Smith showed that we can obtain a holomorphic pencil by his construction, he neither carried out the construction in practice nor obtained vanishing cycles of the resulting pencil (but mentioned the symplectic representation of the monodromy). In this subsection, we will construct a genus-3 holomorphic pencil of T^4 following the construction in [24] and determine the vanishing cycles of the pencil.

We begin with a brief review of Smith's construction. For homogeneous polynomials $q_1, \dots, q_n$ we denote their zero-set by $V(q_1, \dots, q_n) \subset \mathbb{CP}^m$. We put

$$Q = V(x^2 + y^2 + z^2) \subset \mathbb{CP}^3, \quad S = V(x^2 + y^2 + z^2) \subset \mathbb{CP}^2.$$

The set Q is a singular variety with an A_1–singularity $[1:0:0:0] \in Q$, and S is a sphere in $\mathbb{CP}^2$. Let $\pi\colon Q \setminus \{[1:0:0:0]\} \to S$ be the restriction of the projection $[t:x:y:z] \mapsto [x:y:z]$. We take six conics $C_1, \dots, C_6$ with the following properties:

- Each C_i is away from the singularity $[1:0:0:0]$.
- Two spheres C_i and C_j intersect at two points transversely for $i \neq j$ with $i, j \leq 4$, or $(i,j) = (5,6)$.
- Two spheres C_5 and C_6 are tangent to C_i at one point for $i \leq 4$.

Let $q_1\colon Z \to Q$ be a double covering branched at $C_1 \cup \dots \cup C_4$. The space Z has two A_1–singularities in the preimage $q_1^{-1}([1:0:0:0])$. Let $r\colon Z_{\mathrm{sm}} \to Z$ be the resolution of these singularities. The space Z_{sm} is a manifold obtained by replacing neighborhoods of the two singularities of Z with two disk bundles over the sphere with degree -2. In particular, Z_{sm} has two spheres S_1, S_2 with self-intersection -2. Since $C_1 \cup \dots \cup C_4$ has twelve transverse double points, Z_{sm} has another twelve spheres $S_3, \dots, S_{14}$ with self-intersection -2. Furthermore, the preimage $q_1^{-1}(C_5 \cup C_6)$ contains two disjoint sphere S_{15} and S_{16} with self-intersection -2. We can take a double covering $q_2\colon \widetilde{T} \to Z_{\mathrm{sm}}$ branched at the disjoint union $\bigsqcup_{i=1}^{16} S_i$. The space $\widetilde{T}$

has 16 exceptional spheres in the preimage $q_2^{-1}\left(\bigsqcup_i S_i\right)$. We denote the blow-down of $\widetilde{T}$ along these spheres by T. The composition $\pi \circ q_1 \circ r \circ q_2$ factors through T and defines a pencil $f\colon T \dashrightarrow S$ with four base points which satisfies the conditions (2) and (3) in Section 2.1. If we take conics $C_1, \dots, C_6$ so that the restriction of π on the set of double points of $C_1 \cup \dots \cup C_4$ is injective, the resulting pencil f becomes Lefschetz.

In what follows, we consider the following conics in Q:

$$\begin{aligned}
C_1 &= V(x^2+y^2+z^2, t), & C_2 &= V(x^2+y^2+z^2, t+x),\\
C_3 &= V\left(x^2+y^2+z^2, t+\tfrac{1}{2}x+\tfrac{1}{2}y\right), & C_4 &= V\left(x^2+y^2+z^2, t+\tfrac{1}{2}x-\tfrac{1}{2}y\right),\\
C_5 &= V\left(x^2+y^2+z^2, t+\tfrac{1}{2}x+\tfrac{1}{2}iz\right), & C_6 &= V\left(x^2+y^2+z^2, t+\tfrac{1}{2}x-\tfrac{1}{2}iz\right).
\end{aligned}$$

It is easy to verify that these conics satisfy the three conditions in the previous paragraph. We denote the set of double points of $C_1 \cup \dots \cup C_6$ by D. Using Lemma 4.1 we can obtain vanishing cycles of f once we can calculate the monodromies of π around the image $\pi(D)$. Let $S_0 = \{[x:y:z] \in S \mid z \neq 0\}$. We define a holomorphic map $\varphi\colon S_0 \to \mathbb{C}^2$ by $\varphi([x:y:z]) = (x/z, y/z)$. Since the image $\varphi(S_0)$ is equal to $\{(X,Y) \in \mathbb{C}^2 \mid (X+iY)(X-iY) = -1\}$, the composition $\psi \circ \varphi\colon S_0 \to \mathbb{C}^\times$ is biholomorphic, where $\psi\colon \mathbb{C}^2 \to \mathbb{C}^\times$ is defined by $\psi(X,Y) = X+iY$. Furthermore, this map can be extended to a biholomorphic map $S \to \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which sends $[1:i:0]$ and $[1:-i:0]$ to 0 and ∞, respectively. Using this map, we will identify S with $\overline{\mathbb{C}}$ throughout this subsection. The following lemma can be deduced easily by direct calculation.

Lemma 4.3 *The image $\pi(D)$ is contained in $\{\xi^n \in \overline{\mathbb{C}} \mid n = 0, \dots, 7\} \cup \{0, \infty\}$, where $\xi = \exp\left(\frac{\pi i}{4}\right)$. Furthermore, the intersection $C_5 \cap C_6$ is contained in $\pi^{-1}(\{0,\infty\})$.*

For any $w \in \mathbb{C} = \overline{\mathbb{C}} \setminus \{\infty\}$, the map $\pi^{-1}(w) \to \mathbb{C}$ defined as $[t:x:y:z] \to t/(x-iy)$ is biholomorphic. Using this, we will identify the fiber $\pi^{-1}(w)$ with $\mathbb{C}$ for any $w \in \mathbb{C}$. With this identification in hand, we can define a path $\gamma^{(i)}\colon J \to \mathbb{C}$, where $i = 1, \dots, 6$, for any path $\gamma\colon J \to \mathbb{C}$ (where $J \subset \mathbb{R}$), by

$$\gamma^{(i)}(t) = z \in \pi^{-1}(\gamma(t)) \cap C_i \subset \mathbb{C}.$$

The value of this path is indeed determined uniquely since C_i is a section of π. Let α be an oriented path in $\mathbb{C}$ which intersects $\pi(D)$ only at its terminal point. The corresponding paths $\alpha^{(1)}, \dots, \alpha^{(6)}$ are also oriented paths in $\mathbb{C}$ two of which, say $\alpha^{(i)}$ and $\alpha^{(j)}$, intersect at their common terminal point. We denote by $\widetilde{\alpha}$ the oriented loop based at the initial point of α obtained by connecting α with a small counterclockwise circle around the terminal point of α. We can easily verify that the parallel transport along $\widetilde{\alpha}$ is isotopic to a composition of the point-pushing self-diffeomorphism of $\mathbb{C}$ along the paths $\alpha^{(1)},\dots,\alpha^{(6)}$, the m_{ij}^{th} power of the local full-twist around the common terminal point of $\alpha^{(i)}$ and $\alpha^{(j)}$, and the inverse of the point-pushing self-diffeomorphism, where m_{ij} is the multiplicity of the intersection of C_i and C_j in the fiber on the terminal point of α, which is 1 if $i,j\le 4$ or $i,j\ge 5$, and 2 otherwise.

Let α_k, where $k=1,2,3,4$, and β be paths in $\mathbb{C}$ defined by

$$\alpha_k\colon [-1,1]\to\mathbb{C},\quad \alpha_k(s)=s\xi^{k-1},$$
$$\beta\colon [-\pi,\pi]\to\mathbb{C},\quad \beta(\theta)=\varepsilon\exp(i\theta),$$

where $\varepsilon>0$ is a sufficiently small real number. In order to determine monodromies of π, we first calculate the paths $\alpha_k^{(j)}$ and $\beta^{(j)}$. Under the identification $S\cong\overline{\mathbb{C}}$, $\alpha_1(s)=s$ corresponds to $[s^2-1:-i(s^2+1):2s]$. Thus, $\alpha_1^{(j)}(s)=t/(x-iy)$ can be calculated as follows:

$$\alpha_1^{(1)}(s)=\frac{0}{x-iy}=0,$$
$$\alpha_1^{(2)}(s)=\frac{-x}{x-iy}=\frac{s^2-1}{2}=-\frac{1}{2}+\frac{1}{2}s^2,$$
$$\alpha_1^{(3)}(s)=\frac{-x/2-y/2}{x-iy}=\frac{s^2-1}{4}-i\frac{s^2+1}{4}=\frac{\xi^{-3}}{2\sqrt{2}}+\frac{\xi^{-1}}{2\sqrt{2}}s^2,$$
$$\alpha_1^{(4)}(s)=\frac{-x/2+y/2}{x-iy}=\frac{s^2-1}{4}+i\frac{s^2+1}{4}=\frac{\xi^{3}}{2\sqrt{2}}+\frac{\xi}{2\sqrt{2}}s^2,$$
$$\alpha_1^{(5)}(s)=\frac{-x/2-iz/2}{x-iy}=\frac{s^2-1}{4}+i\frac{s}{2}=-\frac{1}{4}(1-is)^2,$$
$$\alpha_1^{(6)}(s)=\frac{-x/2+iz/2}{x-iy}=\frac{s^2-1}{4}-i\frac{s}{2}=-\frac{1}{4}(1+is)^2.$$

In the same way as above, we can also calculate the other paths as follows:

$$\alpha_2^{(1)}(s)=0,\qquad \alpha_2^{(2)}(s)=-\tfrac{1}{2}+\tfrac{i}{2}s^2,$$
$$\alpha_2^{(3)}(s)=\tfrac{1}{2\sqrt{2}}\xi^{-3}+\tfrac{1}{2\sqrt{2}}\xi s^2,\qquad \alpha_2^{(4)}(s)=\tfrac{1}{2\sqrt{2}}\xi^{3}+\tfrac{1}{2\sqrt{2}}\xi^3 s^2,$$
$$\alpha_2^{(5)}(s)=-\tfrac{1}{4}(1+\xi^{-1}s)^2,\qquad \alpha_2^{(6)}(s)=-\tfrac{1}{4}(1+\xi^{3}s)^2,$$
$$\alpha_3^{(1)}(s)=0,\qquad \alpha_3^{(2)}(s)=-\tfrac{1}{2}-\tfrac{1}{2}s^2,$$

$$\alpha_3^{(3)}(s) = \tfrac{1}{2\sqrt{2}}\xi^{-3} + \tfrac{1}{2\sqrt{2}}\xi^{3}s^2, \qquad \alpha_3^{(4)}(s) = \tfrac{1}{2\sqrt{2}}\xi^{3} + \tfrac{1}{2\sqrt{2}}\xi^{-3}s^2,$$
$$\alpha_3^{(5)}(s) = -\tfrac{1}{4}(1+s)^2, \qquad \alpha_3^{(6)}(s) = -\tfrac{1}{4}(1-s)^2,$$
$$\alpha_4^{(1)}(s) = 0, \qquad \alpha_4^{(2)}(s) = -\tfrac{1}{2} - \tfrac{i}{2}s^2,$$
$$\alpha_4^{(3)}(s) = \tfrac{1}{2\sqrt{2}}\xi^{-3} + \tfrac{1}{2\sqrt{2}}\xi^{-3}s^2, \qquad \alpha_4^{(4)}(s) = \tfrac{1}{2\sqrt{2}}\xi^{3} + \tfrac{1}{2\sqrt{2}}\xi^{-1}s^2,$$
$$\alpha_4^{(5)}(s) = -\tfrac{1}{4}(1+\xi s), \qquad \alpha_4^{(6)}(s) = -\tfrac{1}{4}(1+\xi^{-3}s),$$
$$\beta^{(1)}(\theta) = 0, \qquad \beta^{(2)}(\theta) = -\tfrac{1}{2} + \tfrac{1}{2}\varepsilon^2 e^{2i\theta},$$
$$\beta^{(3)}(\theta) = \tfrac{1}{2\sqrt{2}}\xi^{-3} + \tfrac{1}{2\sqrt{2}}\xi^{-1}\varepsilon^2 e^{2i\theta}, \qquad \beta^{(4)}(\theta) = \tfrac{1}{2\sqrt{2}}\xi^{3} + \tfrac{1}{2\sqrt{2}}\xi\varepsilon^2 e^{2i\theta},$$
$$\beta^{(5)}(\theta) = -\tfrac{1}{4}(1+\varepsilon e^{i(\theta-\frac{\pi}{2})})^2, \qquad \beta^{(6)}(\theta) = -\tfrac{1}{4}(1+\varepsilon e^{i(\theta+\frac{\pi}{2})})^2.$$

We can draw the paths $\alpha_i^{(j)}$ in the plane $\mathbb{C}$ as shown in Figure 1. In each of the figures, the five dots are the points $\alpha_k^{(1)}(0), \dots, \alpha_k^{(6)}(0)$ (note that $\alpha_k^{(5)}(0) = \alpha_k^{(6)}(0)$), while $\alpha_k^{(1)}$ is the constant path, the bold line describes the path $\alpha_k^{(2)}$, the dotted lines (which are colored in red) describe the paths $\alpha_k^{(3)}$ and $\alpha_k^{(4)}$ (the denser one is $\alpha_k^{(4)}$, while the other one is $\alpha_k^{(3)}$) and the semi-dotted lines (which are colored in light blue) describe the paths $\alpha_k^{(5)}$ and $\alpha_k^{(6)}$ (the denser one is $\alpha_k^{(5)}$, while the other one is $\alpha_k^{(6)}$). Moreover, at each of the transverse crossings except for the terminal points, the path going over the other path goes through the crossing point after the other path comes to the point as the parameter s increases.

For $k = 1, \dots, 8$ we define a path γ_k in $\mathbb{C}$ as follows:

- For $k \le 4$, let γ_k be the concatenation of $\beta|_{[0,(k-1)\pi i/4]}$ and $\alpha_k|_{[\varepsilon,1]}$.
- For $k \ge 5$, let γ_k be the concatenation of $\alpha_{k-4}|_{[-1,-\varepsilon]}$ and $\beta|_{[-(9-k)\pi i/4,0]}$ with the opposite orientation.

According to the arguments above, the monodromy of π along the path $\widetilde{\gamma}_k$ is the product of the full-twists along the paths shown in Figure 2 and the squares of the full-twists along other paths, which have either $\alpha_k^{(5)}(0)$ or $\alpha_k^{(6)}(0)$ as end points and are not described in Figure 2. (Note that we only need local monodromies derived from the double points in $C_1 \cup \dots \cup C_4$ in order to obtain vanishing cycles of $f\colon T \to S$.)

By Lemma 4.1 we can obtain local monodromies of the genus-1 Lefschetz fibration $\pi \circ q_1 \circ r\colon Z_{\mathrm{sm}} \to S$ by taking a lift of the full-twists along the paths in Figure 2 under the double covering of $\overline{\mathbb{C}} \cong \overline{\pi^{-1}(\varepsilon)}$ branched at the set $\overline{\pi^{-1}(\varepsilon)} \cap (C_1 \cup \dots \cup C_4)$ of four points (ie the four nested dots in Figure 2). The resulting local monodromies are the

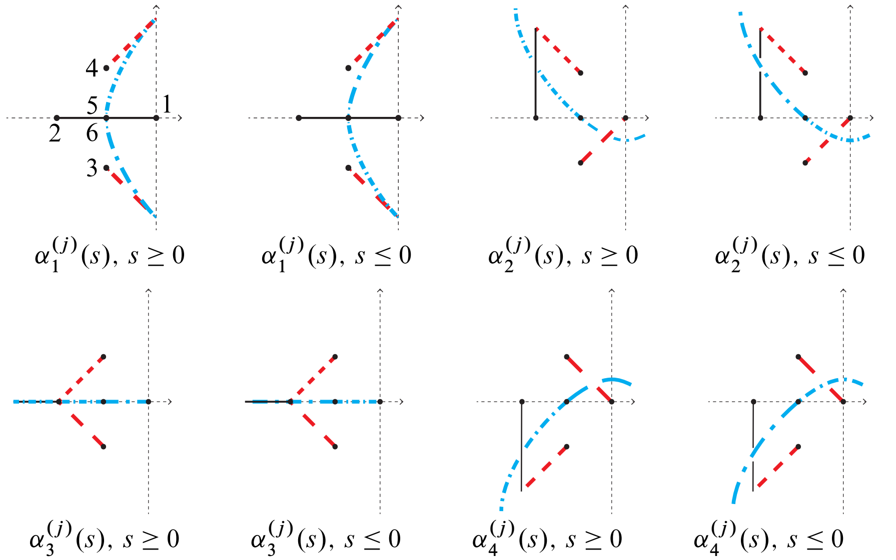

Figure 1: The paths $\alpha_i^{(j)}$ in $\mathbb{C} \cong \mathbb{R}^2$

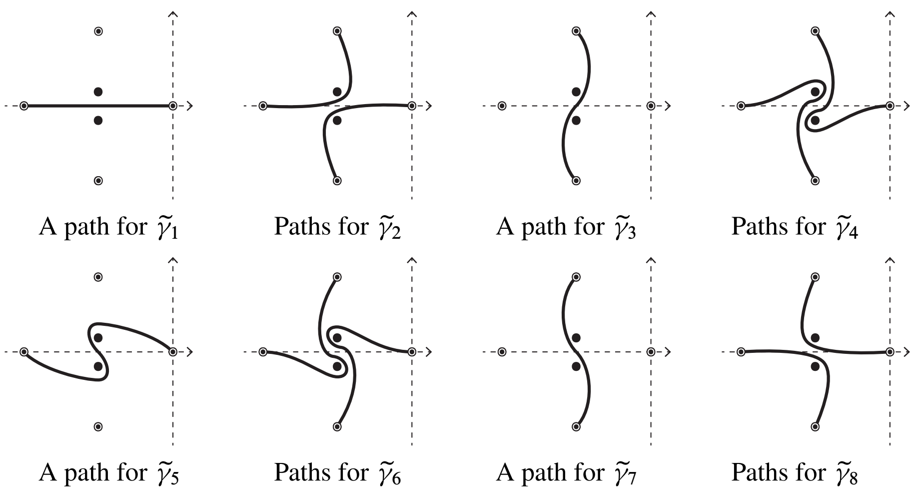

Figure 2: The monodromy along $\widetilde{\gamma}_i$ is the product of the full-twists along the paths above and other twists which are less important

squares of Dehn twists along the curves in Figure 3. Here, two of the four marked points in Figure 3 are the points in the preimage of $\infty \in \overline{\mathbb{C}} \cong \overline{\pi^{-1}(\varepsilon)}$, while the other two marked points are two points in the preimage of the intersection $\pi^{-1}(\varepsilon) \cap (C_5 \cup C_6)$. All the marked points describe sections of the Lefschetz fibration $\pi \circ q_1 \circ r$ with self-intersection (-2).

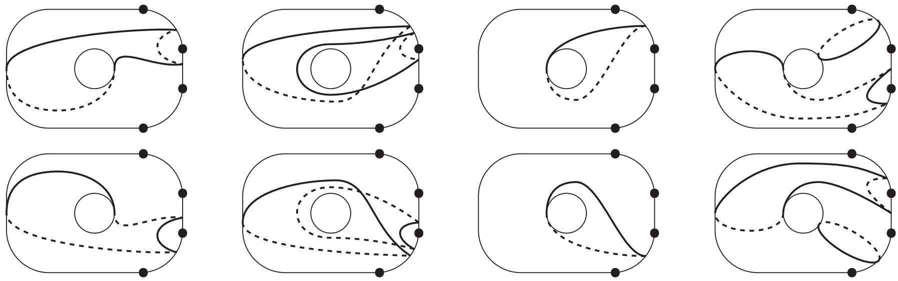

Figure 3: Vanishing cycles of the Lefschetz fibration $\pi \circ q_1 \circ r\colon Z_{\mathrm{sm}} \to S$, associated with (top row, left to right) γ_1, γ_2, γ_3, γ_4 and (bottom row, left to right) γ_5, γ_6, γ_7, γ_8

To obtain the pencil $f\colon T \dashrightarrow S$, we further take a double covering $q_2\colon \widetilde{T} \to Z_{\mathrm{sm}}$ branched at 16 spheres with self-intersection (-2). Four of these spheres are the sections of $\pi \circ q_1 \circ r$, and the other twelve spheres are contained in singular fibers of $\pi \circ q_1 \circ r$, each of which is an irreducible component of a fiber containing two Lefschetz singularities with parallel vanishing cycles.

Lemma 4.4 *We denote the fiber $(\pi \circ q_1 \circ r)^{-1}(\varepsilon)$ by $F \subset Z_{\mathrm{sm}}$. The preimage of each vanishing cycle in Figure 3 under the restriction $q_2|_{q_2^{-1}(F)}$ is connected.*

Proof We first observe that there is a one-to-one correspondence between the set of isomorphism classes of double coverings of a four-manifold X branched at $B \subset X$ and the set of homomorphisms $\varphi\colon H_1(X \setminus B; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ sending a meridian of each component of B to 1. Furthermore, for a given double covering $q\colon \widetilde{X} \to X$ branched at B, the corresponding homomorphism $\varphi_q\colon H_1(X \setminus B; \mathbb{Z}/2\mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$ can be obtained as follows: for a simple closed curve c, the value $\varphi_q([c])$ is 1 (resp. 0) if the preimage $q^{-1}(c)$ is connected (resp. disconnected).

Let $S \subset Z_{\mathrm{sm}}$ be one of the twelve spheres in singular fibers of $\pi \circ q_1 \circ r$ and $N_1 \subset Z_{\mathrm{sm}}$ a tubular neighborhood of S. The restriction of $\pi \circ q_1 \circ r$ on N_1 has two Lefschetz singularities and a regular fiber of this restriction is an annulus. According to [11, Section 8.2] we can draw a handlebody picture of the closure $\overline{N}_1$ which reflects the configuration of the two singularities as shown in Figure 4 on the left (two (-1)–framed knots correspond to the two Lefschetz singularities). Moreover, applying the algorithm in [11, Section 6.2] to our situation, we can obtain a diagram of the complement $\overline{N}_1 \setminus N_2$ of a smaller tubular neighborhood N_2 of S as shown in Figure 4 on the right (the bold handles and the two 3–handles in the figure correspond to handles of S).

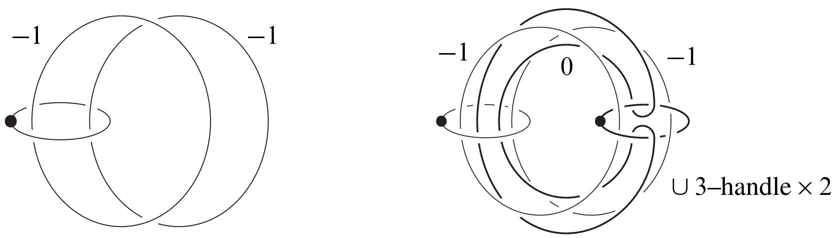

Figure 4: Handlebody pictures of (left) the closure $\overline{N}_1$ of a neighborhood of S and (right) the complement $\overline{N}_1 \setminus N_2$ of a smaller neighborhood of S in $\overline{N}_1$

Let $c_1, \dots, c_{12} \subset \Sigma_{1,4}$ be the vanishing cycles in the four-punctured torus $\Sigma_{1,4}$ described in Figure 3. It is easy to verify (by drawing a handlebody picture using the observation above) that the first homology of the complement of the sixteen spheres in Z_{sm} is isomorphic to the group

$$\left(H_1(\Sigma_1^4; \mathbb{Z}/2\mathbb{Z}) \oplus \bigoplus_{i=1}^{12} (\mathbb{Z}/2\mathbb{Z} e_i)\right) \Big/ \langle \{c_i + e_i \mid i = 1, \dots, 12\} \rangle,$$

where the e_i coincide with the meridians of the spheres in singular fibers. As we observed above, the homomorphism φ_{q_2} associated with the branched covering q_2 must send each e_i to 1. Since c_i is equal to e_i in the group above, the preimage $q_2^{-1}(c_i)$ is connected. □

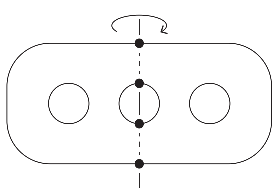

Figure 5: The involution η, which is the rotation by π around the indicated axis.

Since the vanishing cycles $c_1, \dots, c_{12}$ span the homology group $H_1(T^2; \mathbb{Z}/2\mathbb{Z})$, the argument in the proof of Lemma 4.4 also shows that a double covering of T^2 branched at the marked points by which each loop c_i cannot be lifted is unique up to isomorphism. In particular, we can obtain vanishing cycles of the pencil $f\colon T \dashrightarrow S$ once we can find such a branched covering, which is obtained by dividing Σ_3 by the involution η shown in Figure 5. Taking the preimage of the vanishing cycles in Figure 3 by the branched covering induced by η, we can eventually obtain vanishing cycles $\widetilde{c}_1, \dots, \widetilde{c}_{12}$ of $f\colon T \dashrightarrow S$ as shown in Figure 6, and thus the monodromy factorization associated with f,

$$t_{\widetilde{c}_{12}} \cdots t_{\widetilde{c}_1} = t_{\delta_1} \cdots t_{\delta_4}. \tag{4-1}$$

As the pencil f has genus 3, it gives rise to a $(1,2)$–polarization on T (see Lemma 3.1).

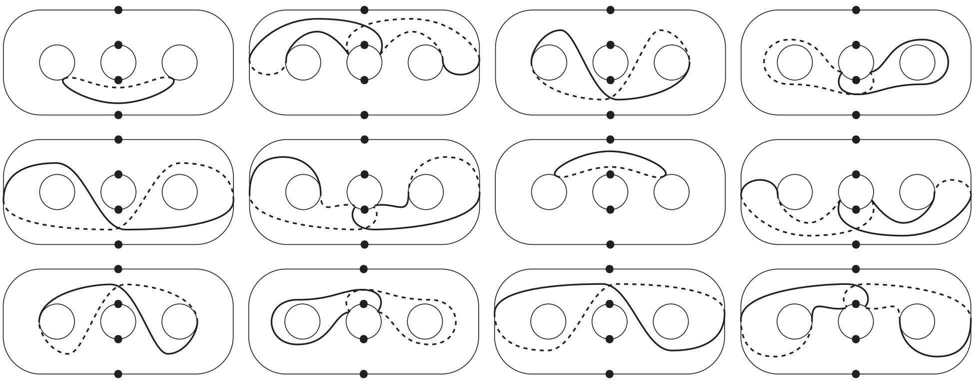

Figure 6: Vanishing cycles of a holomorphic genus-3 pencil on T^4 due to Smith. From left to right, the curves (top row) $\widetilde{c}_{12}$, $\widetilde{c}_{11}$, $\widetilde{c}_{10}$, $\widetilde{c}_9$; (middle row) $\widetilde{c}_8$, $\widetilde{c}_7$, $\widetilde{c}_6$, $\widetilde{c}_5$; (bottom row) $\widetilde{c}_4$, $\widetilde{c}_3$, $\widetilde{c}_2$, $\widetilde{c}_1$.

Remark 4.5 To be precise, the pencil we have constructed here is not a Lefschetz pencil yet since it does not satisfy the condition (1) in Section 2.1. However, we can obtain a holomorphic genus-3 Lefschetz pencil on T^4 by perturbing the conics $C_1, \dots, C_6$ so that the restriction of π on the set of double points of $C_1 \cup \cdots \cup C_4$ becomes injective. We can further check (using Mathematica) that the monodromy factorization of the Lefschetz pencil obtained in this way is Hurwitz equivalent to that of our pencil.

Remark 4.6 Recently, Baykur [1] has also constructed a genus-3 symplectic Calabi–Yau Lefschetz pencil whose total space is *homeomorphic* to the standard four-torus T^4, but its diffeomorphism type was unknown. In addition, the geometric structure of the pencil is not clear since his construction is based on a purely combinatorial method in terms of relations among Dehn twists. In Section 5, we will see that his pencil is in fact isomorphic to the pencil corresponding to (4-1) (see Remark 5.2) after observing some arguments on combinatorial structures of the factorization (4-1). Thus, we now understand in detail the geometric structure of Baykur's pencil; in particular, his pencil is not only homeomorphic but also *diffeomorphic* to the standard T^4, and the pencil may be considered *holomorphic*.

4.2 Holomorphic Lefschetz pencils with higher genera

As observed in the beginning of this section, for any integers $d_1, d_2 > 0$ with $d_1 \mid d_2$ and either $d_1 d_2 \geq 5$ or $d_1 \geq 2$, there exists a genus-$(d_1 d_2+1)$ holomorphic Lefschetz pencil on T^4 with divisibility d_1. In this subsection we will explain how to obtain monodromy factorizations of some of these Lefschetz pencils. We will also find a holomorphic Lefschetz pencil corresponding to $(d_1, d_2) = (1, 4)$, whose existence is not verified so far.

Let $c \in H^2(T^4;\mathbb{Z})$ be a (d_1, d_2)–polarization of T^4. The cohomology class c is equal to $d_1\alpha_1 \cup \beta_1 + d_2\alpha_2 \cup \beta_2$ for some generating system $\alpha_1, \beta_1, \alpha_2, \beta_2$ of $H^1(T^4;\mathbb{Z})$. Let $a_i, b_j \in H_1(T^4;\mathbb{Z})$ be, respectively, duals of α_i, β_j with respect to the Kronecker product. We take an unbranched covering map $q\colon \widetilde{T} \to T^4$ corresponding to the subgroup of $H_1(T^4;\mathbb{Z})$ generated by $n_1 a_1, b_1, n_2 a_2, b_2$ for some positive integers n_1, n_2. It is easy to see that $\widetilde{T}$ is again a 4–torus and the set

$$\left\{\frac{q^*(\alpha_1)}{n_1}, q^*(\beta_1), \frac{q^*(\alpha_2)}{n_2}, q^*(\beta_2)\right\}$$

generates $H^1(\widetilde{T};\mathbb{Z})$. In particular the pull-back $q^*(c) \in H^2(\widetilde{T};\mathbb{Z})$ is a $(\widetilde{d}_1, \widetilde{d}_2)$–polarization of $\widetilde{T}$, where $\widetilde{d}_1 = \gcd(n_1 d_1, n_2 d_2)$ and $\widetilde{d}_2 = n_1 n_2 d_1 d_2/\widetilde{d}_1$. Since d_2 is divisible by d_1, we have that $\widetilde{d}_1$ is also divisible by d_1. Furthermore, $\widetilde{d}_2$ is divisible by d_2 since $n_1 d_1/\widetilde{d}_1$ must be an integer. Conversely, for any positive integers l_1, l_2 with $(l_1 d_1) | (l_2 d_2)$, the pull-back $\overline{q}^*(c)$ by an unbranched covering map $\overline{q}$ corresponding to the subgroup $\langle l_1 a_1, b_1, l_2 a_2, b_2\rangle \subset H_1(T^4;\mathbb{Z})$ is an $(l_1 d_1, l_2 d_2)$–polarization of $\widetilde{T}$. We thus obtain the following:

Lemma 4.7 *Let $c \in H^2(T^4;\mathbb{Z})$ be a (d_1, d_2)–polarization of T^4. For any positive integers $\widetilde{d}_1, \widetilde{d}_2$ such that $\widetilde{d}_1 | \widetilde{d}_2$ and $d_i | \widetilde{d}_i$ for $i = 1, 2$, there exists an unbranched covering map $q\colon \widetilde{T} \to T^4$ such that the pull-back $q^*(c)$ is a $(\widetilde{d}_1, \widetilde{d}_2)$–polarization of $\widetilde{T}$. If q is an n–fold unbranched covering, $\widetilde{d}_1 \widetilde{d}_2$ is equal to $n d_1 d_2$.*

Let $f\colon T^4 \dashrightarrow \mathbb{CP}^1$ be a holomorphic pencil associated with a (d_1, d_2)–polarization $c \in H^2(T^4;\mathbb{Z})$ and $q\colon \widetilde{T} \to T^4$ a finite unbranched covering map. It is easy to verify that the composition $f \circ q\colon \widetilde{T} \dashrightarrow \mathbb{CP}^1$ is also a holomorphic pencil associated with the polarization $q^*(c)$. By Lemma 4.7, for any d_1, d_2 with $d_1 d_2$ even we can obtain a holomorphic pencil on T^4 associated with a (d_1, d_2)–polarization by composing a finite unbranched covering map with the genus-3 pencil in the preceding subsection, which is associated with a $(1, 2)$–polarization. We can perturb the resulting pencil so that no fiber contains more than one critical point; and hence it becomes a Lefschetz pencil. We can further obtain the vanishing cycles of the pencil $f \circ q$ using Lemma 4.1 once we find out how the deck transformations of q act on a reference fiber of $f \circ q$. In what follows we will apply the above procedure to obtain two holomorphic pencils on T^4 with the same genus but distinct divisibilities.

Example 4.8 Let $f \colon T^4 \dashrightarrow \mathbb{CP}^1$ be the holomorphic pencil obtained in the preceding subsection. According to Lemma 4.7, for a double unbranched covering $q \colon \widetilde{T} \to T^4$ the type of a polarization associated with the composition $f \circ q$ is either $(1,4)$ or $(2,2)$. We will give two double unbranched coverings which yield both types of polarization.

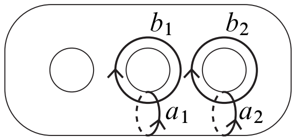

Figure 7: The curves generating $H_1(T^4;\mathbb{Z})$

It is easy to see that $H_1(T^4;\mathbb{Z})$ is generated by the elements represented by the curves a_1, b_1, a_2, b_2 shown in Figure 7 (which are contained in a reference fiber of f). Let $q_1 \colon \widetilde{T}_1 \to T^4$ be a double unbranched covering corresponding to the subgroup $\langle a_1, 2b_1, a_2, b_2 \rangle \subset H_1(T^4;\mathbb{Z})$. The restriction of q_1 on the preimage of a reference fiber of f is the quotient map by the involution η_1 shown in Figure 8, in particular the restriction of the deck transformation of q_1 is equal to η_1.

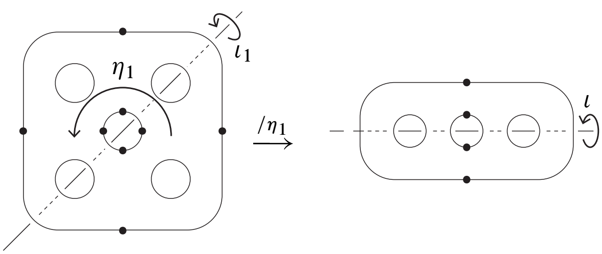

Figure 8: The involutions η_1 and ι_1. Both of them are rotations by π.

Let ι_1 be the involution of a fiber of $f \circ q$ shown in Figure 8, which is a lift of the hyperelliptic involution ι of the genus-3 fiber. For any $i = 1, \dots, 6$ we take a lift d_i of $\widetilde{c}_i$ in Figure 6 under the unbranched covering map q_1 as shown in Figure 9. The other lift of $\widetilde{c}_i$ is $\eta_1(d_i)$, which is also given in Figure 9. Since $\widetilde{c}_{i+6}$ is equal to $\iota(\widetilde{c}_i)$,

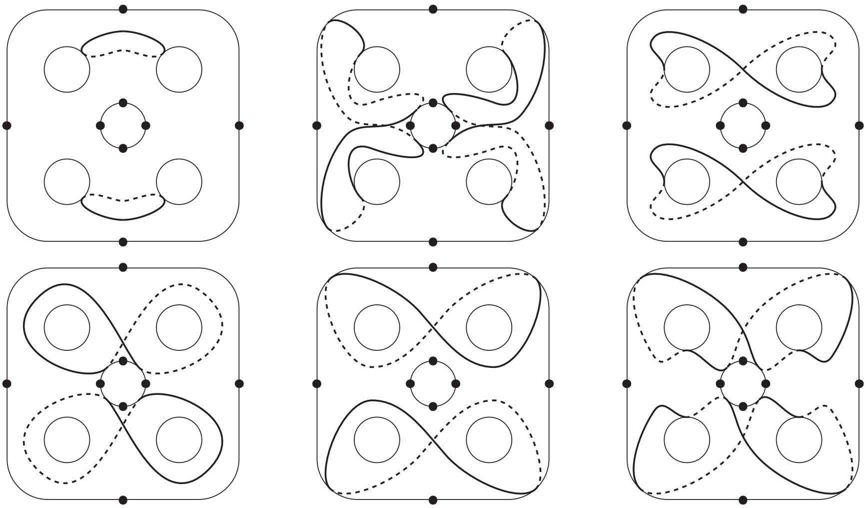

Figure 9: Vanishing cycles of the Lefschetz pencil $f \circ q_1$. The curves d_i and $\eta_1(d_i)$ for (top, left to right) $i = 6, 5, 4$ and (bottom, left to right) $i = 3, 2, 1$.

the lifts of the curve $\widetilde{c}_{i+6}$ are $\iota_1(d_i)$ and $\eta_1(\iota_1(d_i))$. Thus a monodromy factorization of the pencil $f \circ q_1$ is

$$t_{\eta_1(\iota_1(d_6))} t_{\iota_1(d_6)} \cdots t_{\eta_1(\iota_1(d_1))} t_{\iota_1(d_1)} \cdot t_{\eta_1(d_6)} t_{d_6} \cdots t_{\eta_1(d_1)} t_{d_1} = t_{\delta_1} \cdots t_{\delta_8}.$$

Applying the algorithm given in the appendix, we can calculate the divisibility of $f \circ q_1$ (using Mathematica), which is equal to 1. Thus the type of polarization associated with $f \circ q_1$ is $(1,4)$.

Let $q_2\colon \widetilde{T}_2 \to T^4$ be a double unbranched covering corresponding to the subgroup $\langle a_1, b_1, a_2, 2b_2 \rangle \subset H_1(T^4;\mathbb{Z})$. We take involutions η_2 and ι_2 of a genus-5 surface as shown in Figure 10. It is easily verified that the restriction of q_2 to the preimage of a reference fiber of f is the quotient map by η_2, and ι_2 is a lift of ι under this map. By Lemma 4.1, we can obtain vanishing cycles of $f \circ q_2$ by taking lifts of the $\widetilde{c}_i$, which are denoted by e_i and given in Figure 11 together with $\eta_2(e_i), \iota_2(e_i), \eta_2(\iota_2(e_i))$ for $i = 1, \dots, 6$. We eventually obtain a monodromy factorization of $f \circ q_2$ given by

$$t_{\eta_2(\iota_2(e_6))} t_{\iota_2(e_6)} \cdots t_{\eta_2(\iota_2(e_1))} t_{\iota_2(e_1)} \cdot t_{\eta_2(e_6)} t_{e_6} \cdots t_{\eta_2(e_1)} t_{e_1} = t_{\delta_1} \cdots t_{\delta_8}.$$

Applying the algorithm in the appendix (using Mathematica), we can verify that the divisibility of $f \circ q_2$ is equal to 2. Thus, the type of polarization associated with $f \circ q_2$ is $(2,2)$.

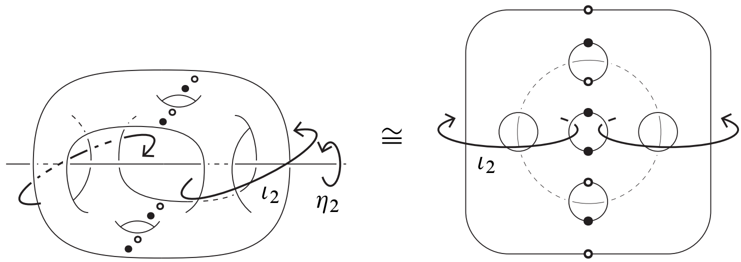

Figure 10: The involutions η_2 and ι_2. Left: the punctured dots are on opposite sides of the surface. Right: another description of the surface, where the involution ι_2 becomes the rotation by π about the dotted circular axis.

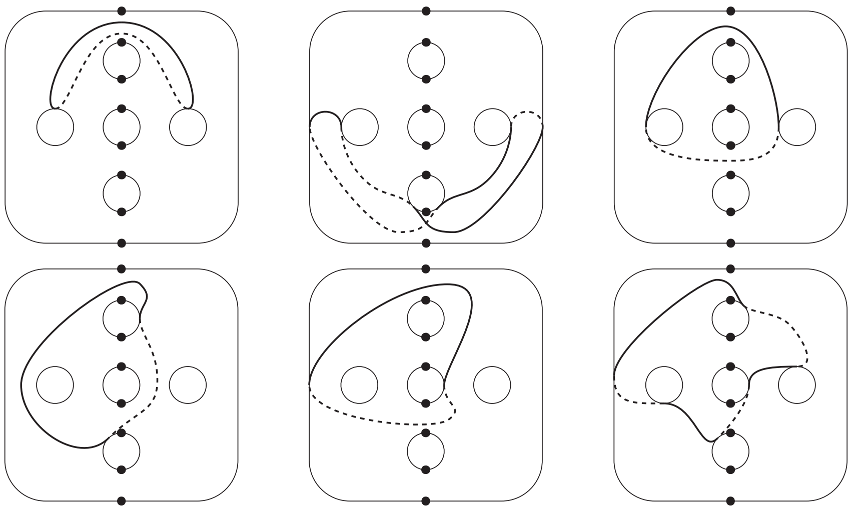

Figure 11: Vanishing cycles of $f \circ q_2$. From left to right, the curves (top row) e_6, e_5, e_4; (bottom row) e_3, e_2, e_1.

Remark 4.9 By Corollary 3.15 we can obtain two holomorphic Lefschetz pencils by perturbing $f \circ q_1$ and $f \circ q_2$. These Lefschetz pencils are not isomorphic, since they have distinct divisibilities. As far as the authors know, this pair is the first example of a pair of nonisomorphic *holomorphic* Lefschetz pencils on the same four-manifold with the same genus, the same number of base points and explicit monodromy factorizations.

5 Combinatorial approach and its applications

In this section we will observe a combinatorial aspect of our pencils. We can reconstruct the factorization (4-1) in a combinatorial way by utilizing a lift to $\mathrm{Mod}(\Sigma_2^4; U)$ of

Matsumoto's factorization in $\mathrm{Mod}(\Sigma_2)$, which was given in [12]. In [1], Baykur independently gave a very similar construction to obtain a genus-3 Lefschetz pencil whose total space is homeomorphic to T^4. In fact, it turns out that his factorization is Hurwitz equivalent to the factorization (4-1) (see Remark 5.2). Although the combinatorial construction has been already presented in [1], we repeat it here in a slightly different way (more symmetric) for completeness. Our combinatorial construction of Smith's pencil is pretty useful for obtaining two new families of symplectic Calabi–Yau Lefschetz pencils: one is a generalization of Smith's pencil to higher genera, the other consists of pencils on four-manifolds homeomorphic to the total spaces of torus-bundles over the torus admitting sections.

In this section we will freely use elementary transformations, especially commutativity relations, and permutations in the calculations. Given a Dehn twist factorization $W = t_{a_1} \cdots t_{a_k}$ (which is not necessarily equal to the identity or the boundary twist) and a mapping class ϕ, we denote the simultaneous conjugation $\phi W \phi^{-1} = t_{\phi(a_1)} \cdots t_{\phi(a_k)}$ by ${}_{\phi}(W)$ throughout the section.

5.1 Smith's pencil and its generalization

As we mentioned, in order to combinatorially construct the factorization (4-1) we make use of a lift of Matsumoto's factorization. Matsumoto's factorization is well known as a factorization of a genus-2 Lefschetz fibration on $T^2 \times S^2 \# 4\overline{\mathbb{CP}^2}$; see [20]. In [12], the first author found several lifts of the factorization to $\mathrm{Mod}(\Sigma_2^4)$, each of which gives four (-1)–sections of Matsumoto's Lefschetz fibration. One of them is

$$t_{B_{0,1}} t_{B_{1,1}} t_{B_{2,1}} t_{C_1} t_{B_{0,2}} t_{B_{1,2}} t_{B_{2,2}} t_{C_2} = t_{\delta_1} \cdots t_{\delta_4}, \tag{5-1}$$

where the curves are as shown in Figure 12.

We first modify the relation to make it match our scheme. Consider a 3–chain relation $(t_c t_a t_b)^4 = t_{\delta_3} t_{C_1}$ as in the top right of Figure 12. Substituting for t_{C_1} in (5-1) we have

$$\begin{aligned}
t_{\delta_1} \cdots t_{\delta_4} &= t_{B_{0,1}} t_{B_{1,1}} t_{B_{2,1}} \cdot (t_c t_a t_b)^4 t_{\delta_3}^{-1} \cdot t_{B_{0,2}} t_{B_{1,2}} t_{B_{2,2}} t_{C_2} \\
&= t_{B_{0,1}} t_{B_{1,1}} t_{B_{2,1}} \cdot (t_c t_a t_b)^2 \underline{(t_c t_a t_b)^2 \cdot t_{B_{0,2}} t_{B_{1,2}} t_{B_{2,2}}} t_{\delta_3}^{-1} t_{C_2} \\
&= \underline{t_{B_{0,1}} t_{B_{1,1}} t_{B_{2,1}} \cdot (t_c t_a t_b)^2} \cdot t_{B'_{0,2}} t_{B'_{1,2}} t_{B'_{2,2}} (t_c t_a t_b)^2 t_{\delta_3}^{-1} t_{C_2} \\
&= (t_c t_a t_b)^2 t_{B'_{0,1}} t_{B'_{1,1}} t_{B'_{2,1}} \cdot t_{B'_{0,2}} t_{B'_{1,2}} t_{B'_{2,2}} (t_c t_a t_b)^2 t_{\delta_3}^{-1} t_{C_2} \\
&= t_{B'_{0,1}} t_{B'_{1,1}} t_{B'_{2,1}} t_{B'_{0,2}} t_{B'_{1,2}} t_{B'_{2,2}} (t_c t_a t_b)^2 t_{\delta_3}^{-1} t_{C_2} (t_c t_a t_b)^2 \\
&= t_{B'_{0,1}} t_{B'_{1,1}} t_{B'_{2,1}} t_{B'_{0,2}} t_{B'_{1,2}} t_{B'_{2,2}} \cdot (t_c t_a t_b)^4 t_{\delta_3}^{-1} \cdot t_{C_2},
\end{aligned}$$

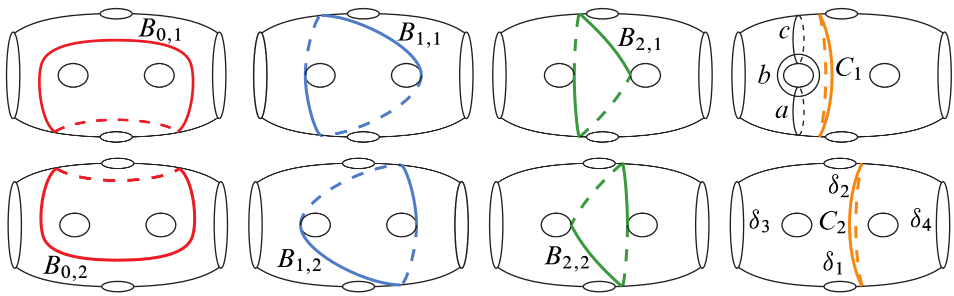

Figure 12: A lift of Matsumoto's factorization. The curves $B_{i,j}$ for $i = 0, 1, 2$ and $j = 1, 2$ are as labeled. Top right: the curve C_1 and a 3–chain configuration. Bottom right: the curve C_2 and the boundary curves.

where $B'_{i,1} = (t_c t_a t_b)^{-2}(B_{i,1})$ and $B'_{i,2} = (t_c t_a t_b)^2(B_{i,2})$ for $i = 0, 1, 2$, which are as depicted in Figure 13. (Note that the geometric action of the mapping class $(t_c t_a t_b)^{\pm 2} t_{\delta_3}^{\mp 1}$ on the surface rotates the subsurface between δ_3 and C_1 by ± 180 degrees with respect to the horizontal axis, while holding δ_3 and C_1.) By resubstituting the 3–chain relation in the reverse way, we obtain

(5-2) $$t_{B'_{0,1}} t_{B'_{1,1}} t_{B'_{2,1}} t_{B'_{0,2}} t_{B'_{1,2}} t_{B'_{2,2}} t_{C_1} t_{C_2} = t_{\delta_1} \cdots t_{\delta_4}.$$

This expression has a nice symmetry: namely, each $B'_{i,j}$ is preserved by the 180 degree rotation with respect to the vertical axis, while C_1 and C_2 switch.

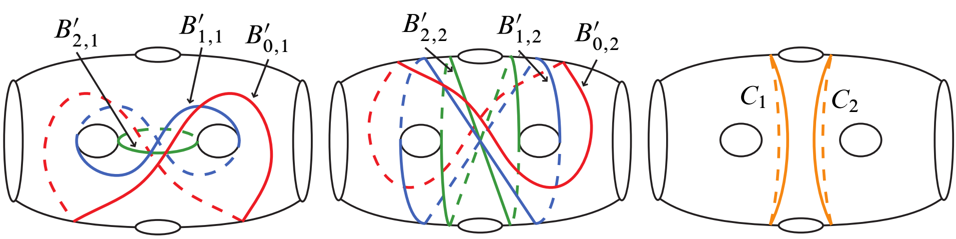

Figure 13: Modified lift of Matsumoto's factorization

To construct Smith's pencil, we now consider a 4–holed genus-3 surface and two configurations for the relation (5-2) as in Figure 14, which give

$$t_{d_1} t_{d_2} t_{d_3} t_{d_4} t_{d_5} t_{d_6} \cdot t_{s_1} t_{s_2} = t_{\delta_1} t_{\delta_2} \cdot t_{s_3} t_{s_4},$$
$$t_{d_7} t_{d_8} t_{d_9} t_{d_{10}} t_{d_{11}} t_{d_{12}} \cdot t_{s_3} t_{s_4} = t_{\delta_3} t_{\delta_4} \cdot t_{s_1} t_{s_2}.$$

We rewrite them as

$$t_{d_1} t_{d_2} t_{d_3} t_{d_4} t_{d_5} t_{d_6} = t_{\delta_1} t_{\delta_2} \cdot t_{s_3} t_{s_4} t_{s_1}^{-1} t_{s_2}^{-1},$$
$$t_{d_7} t_{d_8} t_{d_9} t_{d_{10}} t_{d_{11}} t_{d_{12}} = t_{\delta_3} t_{\delta_4} \cdot t_{s_1} t_{s_2} t_{s_3}^{-1} t_{s_4}^{-1}.$$

Combining them and canceling out $t_{s_3}t_{s_4}t_{s_1}^{-1}t_{s_2}^{-1}$ and $t_{s_1}t_{s_2}t_{s_3}^{-1}t_{s_4}^{-1}$, we then obtain

$$t_{d_1}t_{d_2}t_{d_3}t_{d_4}t_{d_5}t_{d_6}t_{d_7}t_{d_8}t_{d_9}t_{d_{10}}t_{d_{11}}t_{d_{12}} = t_{\delta_1}t_{\delta_2}t_{\delta_3}t_{\delta_4}. \tag{5-3}$$

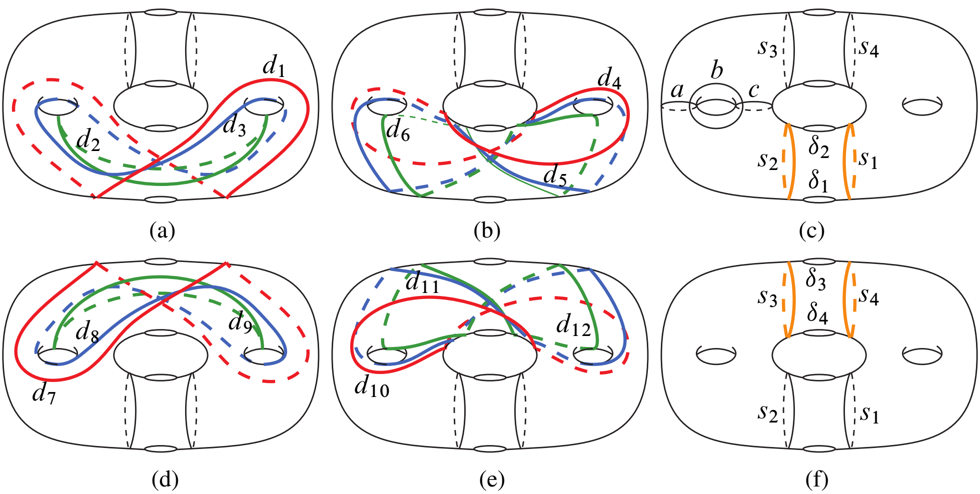

Figure 14: Combinatorial construction of Smith's Lefschetz pencil

Lemma 5.1 *The factorization (4-1) is Hurwitz equivalent to the factorization (5-3).*

Proof Noticing that we already have $\widetilde{c}_1 = d_{11}$, $\widetilde{c}_3 = d_{10}$, $\widetilde{c}_5 = d_6$, $\widetilde{c}_6 = d_9$, $\widetilde{c}_7 = d_5$, $\widetilde{c}_9 = d_4$, $\widetilde{c}_{11} = d_{12}$ and $\widetilde{c}_{12} = d_3$, we have

$$\begin{aligned}
t_{\delta_1}\cdots t_{\delta_4} &= t_{\widetilde{c}_{12}}t_{\widetilde{c}_{11}}t_{\widetilde{c}_{10}}t_{\widetilde{c}_9}t_{\widetilde{c}_8}t_{\widetilde{c}_7}t_{\widetilde{c}_6}t_{\widetilde{c}_5}t_{\widetilde{c}_4}t_{\widetilde{c}_3}t_{\widetilde{c}_2}t_{\widetilde{c}_1}\\
&= \underline{t_{d_3}t_{d_{12}}}t_{\widetilde{c}_{10}}t_{d_4}t_{\widetilde{c}_8}t_{d_5}\underline{t_{d_9}t_{d_6}}t_{\widetilde{c}_4}t_{d_{10}}t_{\widetilde{c}_2}t_{d_{11}}\\
&\sim t_{d_3}t_{\widetilde{c}_{10}}\underline{t_{d_4}t_{\widetilde{c}_8}}t_{d_5}t_{d_6}\cdot t_{d_9}t_{\widetilde{c}_4}\underline{t_{d_{10}}t_{\widetilde{c}_2}}t_{d_{11}}t_{d_{12}}\\
&\sim t_{d_3}\underline{t_{\widetilde{c}_{10}}t_{\widetilde{c}_8}}t_{d_4}t_{d_5}t_{d_6}\cdot t_{d_9}\underline{t_{\widetilde{c}_4}t_{\widetilde{c}_2}}t_{d_{10}}t_{d_{11}}t_{d_{12}}\\
&\sim \underline{t_{d_3}t_{t_{\widetilde{c}_{10}}(\widetilde{c}_8)}t_{\widetilde{c}_{10}}}t_{d_4}t_{d_5}t_{d_6}\cdot \underline{t_{d_9}t_{t_{\widetilde{c}_4}(\widetilde{c}_2)}t_{\widetilde{c}_4}}t_{d_{10}}t_{d_{11}}t_{d_{12}}\\
&\sim t_{t_{d_3}t_{\widetilde{c}_{10}}(\widetilde{c}_8)}t_{t_{d_3}(\widetilde{c}_{10})}t_{d_3}t_{d_4}t_{d_5}t_{d_6}\cdot t_{t_{d_9}t_{\widetilde{c}_4}(\widetilde{c}_2)}t_{t_{d_9}(\widetilde{c}_4)}t_{d_9}t_{d_{10}}t_{d_{11}}t_{d_{12}}\\
&= t_{d_1}t_{d_2}t_{d_3}t_{d_4}t_{d_5}t_{d_6}t_{d_7}t_{d_8}t_{d_9}t_{d_{10}}t_{d_{11}}t_{d_{12}},
\end{aligned}$$

where the last equality follows from observing that $t_{d_3}t_{\widetilde{c}_{10}}(\widetilde{c}_8) = d_1$, $t_{d_3}(\widetilde{c}_{10}) = d_2$, $t_{d_9}t_{\widetilde{c}_4}(\widetilde{c}_2) = d_7$ and $t_{d_9}(\widetilde{c}_4) = d_8$. □

Remark 5.2 As mentioned earlier, the Lefschetz pencil constructed by Baykur [1] whose total space is homeomorphic to T^4 is isomorphic to the pencil corresponding to (5-3), hence (4-1). To see this we first take the simultaneous conjugation of (5-3) by $(t_c t_a t_b)^2$, where the curves a, b, c are as shown in Figure 14(c). Letting d_i' be the resulting curve of d_i mapped by $(t_c t_a t_b)^2$, we see

$$\begin{aligned}
t_{\delta_1}\cdots t_{\delta_4} &= t_{d_1}t_{d_2}t_{d_3}t_{d_4}t_{d_5}t_{d_6}t_{d_7}t_{d_8}t_{d_9}t_{d_{10}}t_{d_{11}}t_{d_{12}} \\
&\sim t_{d_1'}t_{d_2'}t_{d_3'}t_{d_4'}t_{d_5'}t_{d_6'}\underline{t_{d_7'}t_{d_8'}t_{d_9'}}t_{d_{10}'}t_{d_{11}'}t_{d_{12}'} \\
&\sim t_{d_1'}t_{d_2'}t_{d_3'}t_{d_4'}t_{d_5'}t_{d_6'}t_{d_8'}t_{d_9'}\underline{t_{t_{d_9'}^{-1}t_{d_8'}^{-1}(d_7')}t_{d_{10}'}}t_{d_{11}'}t_{d_{12}'} \\
&\sim t_{d_1'}t_{d_2'}t_{d_3'}t_{d_4'}t_{d_5'}t_{d_6'}\underline{t_{d_8'}t_{d_9'}t_{d_{10}'}}t_{t_{d_9'}^{-1}t_{d_8'}^{-1}(d_7')}t_{d_{11}'}t_{d_{12}'} \\
&\sim t_{d_1'}t_{d_2'}t_{d_3'}t_{d_4'}t_{d_5'}t_{d_6'}t_{t_{d_8'}t_{d_9'}(d_{10}')}t_{d_8'}t_{d_9'}t_{t_{d_9'}^{-1}t_{d_8'}^{-1}(d_7')}t_{d_{11}'}t_{d_{12}'}.
\end{aligned}$$

The last factorization is exactly the same as Baykur's factorization.

The construction of the factorization (5-3) can be generalized to higher genera, which provides a new family of symplectic Calabi–Yau Lefschetz pencils. We consider the surface Σ_g^{2g-2} of genus $g \geq 3$ with $2g-2$ boundary components in a circular position and the $\left(\frac{2\pi}{g-1}\right)$–rotation ϕ around the center as shown in the top left of Figure 15. Take the configuration for the relation (5-2) as in the figures in the top right, bottom left and bottom right of Figure 15. Then, as before, we have

$$t_{d_{1,1}}t_{d_{2,1}}t_{d_{3,1}}t_{d_{4,1}}t_{d_{5,1}}t_{d_{6,1}} = t_{\delta_1}t_{\delta_2}\cdot t_{s_0}t_{s_3}t_{s_1}^{-1}t_{s_2}^{-1}.$$

We put $W_1 = t_{d_{1,1}}t_{d_{2,1}}t_{d_{3,1}}t_{d_{4,1}}t_{d_{5,1}}t_{d_{6,1}}$ (as a factorization) and for $i = 1, \dots, g-1$ take the simultaneous conjugation $W_i := {}_{\phi^{i-1}}(W_1)$ of W_1 by the rotation map ϕ^{i-1}:

$$W_i = t_{d_{1,i}}t_{d_{2,i}}t_{d_{3,i}}t_{d_{4,i}}t_{d_{5,i}}t_{d_{6,i}} = t_{\delta_{2i-1}}t_{\delta_{2i}}\cdot t_{s_{2i-2}}t_{s_{2i+1}}t_{s_{2i-1}}^{-1}t_{s_{2i}}^{-1},$$

where $d_{j,i} = \phi^{i-1}(d_{j,1})$, $s_{2i-2} = \phi^{i-1}(s_0)$, $s_{2i-1} = \phi^{i-1}(s_1)$, $s_{2i} = \phi^{i-1}(s_2)$, $s_{2i+1} = \phi^{i-1}(s_3)$, $\delta_{2i-1} = \phi^{i-1}(\delta_1)$ and $\delta_{2i} = \phi^{i-1}(\delta_2)$. Note that $s_{2g-2} = s_0$ and $s_{2g-1} = s_1$. When we combine $W_1, \dots, W_{g-1}$, a similar canceling process as before works well, so that we obtain

$$W_1\cdots W_{g-1} = \prod_{i=1}^{g-1} t_{d_{1,i}}t_{d_{2,i}}t_{d_{3,i}}t_{d_{4,i}}t_{d_{5,i}}t_{d_{6,i}} = t_{\delta_1}\cdots t_{\delta_{2g-2}}. \tag{5-4}$$

This factorization gives a genus-g Lefschetz pencil with $2g-2$ base points and $6g-6$ irreducible critical points for $g \geq 3$. We denote this Lefschetz pencil by $f_g\colon X_g \setminus B_g \to \mathbb{CP}^1$ (for $g \geq 3$). Note that f_3 is nothing but the Lefschetz pencil corresponding to the factorization (5-3), ie Smith's pencil. It is straightforward to

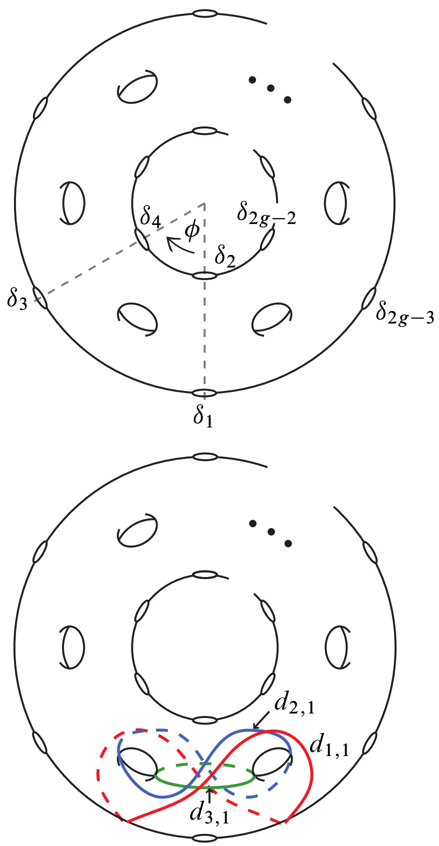

Figure 15: Construction of a generalization of Smith's pencil. Top: the surface Σ_g^{2g-2} and the rotation ϕ (left) and the curves $s_1 = s_{2g-1}$, s_2, s_3, $s_0 = s_{2g-2}$, δ_1 and δ_2 (right). Bottom: the curves $d_{1,1}$, $d_{2,1}$, $d_{3,1}$ (left) and the curves $d_{4,1}$, $d_{5,1}$, $d_{6,1}$ (right).

see that the Euler characteristic of X_g is 0. The signature of the relation (5-1) is 0 (see [12]) and the signature of a braid relation is also 0 in the sense of Endo–Nagami [7]. Since the relation (5-4) is constructed by a combination of copies of the relation (5-1) and braid relations, the signature of (5-4) is 0, hence the signature of X_g is 0. It is also easy to verify that the fundamental group of the total space X_g of f_g is $\mathbb{Z}^4$. By a theorem of Baykur and Hayano [3, Theorem 4.1], the Lefschetz pencil f_g is symplectic Calabi–Yau. Since a symplectic Calabi–Yau manifold whose fundamental group is isomorphic to $\pi_1(T^4) = \mathbb{Z}^4$ is indeed homeomorphic to T^4 (see [9, Corollary 3.3]), so is the total space X_g. In fact, for odd g we can even show that X_g is diffeomorphic to the T^4 and f_g is holomorphic.

Lemma 5.3 *For odd g, the Lefschetz pencil f_g can be obtained by perturbing $f_3 \circ q$, where $q\colon T^4 \to T^4$ is a $\left(\frac{g-1}{2}\right)$–fold unbranched covering.*

In order to prove Lemma 5.3 we first observe that, in general, the order among $W_1, \dots, W_{g-1}$ in (5-4) does not matter.

Lemma 5.4 *The factorization $W_{\sigma(1)} W_{\sigma(2)} \cdots W_{\sigma(g-1)} = t_{\delta_1} \cdots t_{\delta_{2g-2}}$ is Hurwitz equivalent to $W_1 W_2 \cdots W_{g-1} = t_{\delta_1} \cdots t_{\delta_{2g-2}}$ for any permutation σ of $\{1, \dots, g-1\}$.*

Proof We will show that $W_i W_j$ in the factorization can switch to $W_j W_i$ for $i \neq j$. We only need to consider the cases $|i-j| = 1$ and $\{i, j\} = \{1, g-1\}$, otherwise $W_i W_j$ obviously switches since the supporting subsurfaces for W_i and W_j are disjoint. Recalling that $W_i = t_{\delta_{2i-1}} t_{\delta_{2i}} t_{s_{2i-2}} t_{s_{2i+1}} t_{s_{2i-1}}^{-1} t_{s_{2i}}^{-1}$ as a mapping class, and noticing that the only curve among them that intersects with the curves of $W_{i+1} = t_{d_{1,i+1}} t_{d_{2,i+1}} t_{d_{3,i+1}} t_{d_{4,i+1}} t_{d_{5,i+1}} t_{d_{6,i+1}}$ is s_{2i+1}, and in addition, s_{2i+1} is away from any other curve in W_j for $j \neq i+1$, we get

$$\begin{aligned} t_{\delta_1} \cdots t_{\delta_{2g-2}} &= W_{\sigma(1)} \cdots W_i W_{i+1} \cdots W_{\sigma(g-1)} \\ &\sim W_{\sigma(1)} \cdots {}_{W_i}(W_{i+1}) W_i \cdots W_{\sigma(g-1)} \\ &= W_{\sigma(1)} \cdots {}_{t_{s_{2i+1}}}(W_{i+1}) W_i \cdots W_{\sigma(g-1)} \\ &\sim W_{\sigma(1)} \cdots W_{i+1} W_i \cdots W_{\sigma(g-1)}, \end{aligned}$$

where the last equivalence is achieved by taking a simultaneous conjugation by $t_{s_{2i+1}}^{-1}$. The same argument works when W_i is replaced by W_1 and W_{i+1} by W_g. □

Proof of Lemma 5.3 By Lemma 5.4 the factorization (5-4) is Hurwitz equivalent to the following factorization for odd g:

$$\prod_{i=1}^{(g-1)/2} t_{d_{1,2i-1}} t_{d_{2,2i-1}} \cdots t_{d_{6,2i-1}} \cdot \prod_{i=1}^{(g-1)/2} t_{d_{1,2i}} t_{d_{2,2i}} \cdots t_{d_{6,2i}} = t_{\delta_1} \cdots t_{\delta_{2g-2}}.$$

Then, by only using commutativity relations, it can be reformulated as

$$\begin{aligned} &\prod_{i=1}^{(g-1)/2} t_{d_{1,2i-1}} \cdots \prod_{i=1}^{(g-1)/2} t_{d_{6,2i-1}} \cdot \prod_{i=1}^{(g-1)/2} t_{d_{1,2i}} \cdots \prod_{i=1}^{(g-1)/2} t_{d_{6,2i}} \\ &\qquad = \prod_{i=1}^{(g-1)/2} t_{\delta_{4i-3}} \prod_{i=1}^{(g-1)/2} t_{\delta_{4i-2}} \prod_{i=1}^{(g-1)/2} t_{\delta_{4i-1}} \prod_{i=1}^{(g-1)/2} t_{\delta_{4i}}. \end{aligned}$$

Each of the subfactorizations $\prod_i t_{d_{j,2i-1}}$, $\prod_i t_{d_{j,2i}}$, $\prod_i t_{\delta_{4i-3}}$, $\prod_i t_{\delta_{4i-2}}$, $\prod_i t_{\delta_{4i-1}}$ and $\prod_i t_{\delta_{4i}}$ is preserved by the $\left(\frac{4\pi}{g-1}\right)$–rotation ϕ^2, which acts freely on the surface Σ_g^{2g-2}. Now we can take the quotient by ϕ^2, which gives the surface Σ_3^4 and the factorization (5-3). In this way, we can think of the factorization (5-4) for odd g as a $\left(\frac{g-1}{2}\right)$–fold unbranched covering of the factorization (5-3), ie an unbranched covering of Smith's pencil f_3. □

Remark 5.5 Lemma 5.3 can be easily generalized to the claim that for g_1, g_2 such that $(g_1-1) \mid (g_2-1)$, the pencil f_{g_2} is obtained as a $((g_2-1)/(g_1-1))$–fold unbranched covering of f_{g_1}. On the other hand, for g such that $g-1$ is prime the pencil f_g cannot be obtained as a finite unbranched covering of any Lefschetz pencil of lower genus since the surface Σ_g of such a genus g cannot be the total space of an unbranched covering of any surface of lower genus other than 2, which is easily excluded in any case.

Lemma 5.6 *The divisibility of f_g is 1.*

Proof Let a, b, a_i, b_i for $i = 1, \dots, g-1$ and δ_j for $j = 1, \dots, 2g-2$ be oriented simple closed curves in Σ_g^{2g-2} as shown in Figure 16. We take points $q_1, \dots, q_{2g-2}$ in $\partial\Sigma_g^{2g-2}$ so that the natural map $\pi_0(\{q_1, \dots, q_{2g-2}\}) \to \pi_0(\partial\Sigma_g^{2g-2})$ is bijective. Let $D \subset \mathrm{Int}(\Sigma_g^{2g-2})$ be a disk sufficiently close to δ_1, and δ the simple closed curve ∂D with suitable orientation. Let $\Sigma_g^{2g-1} = \Sigma_g^{2g-2} \setminus \mathrm{Int}(D)$ and $q \in \partial D$, and write $Q = \{q, q_1, \dots, q_{2g-2}\}$. We denote the homology classes in $H_1(\Sigma_g^{2g-1}, Q; \mathbb{Z})$ represented by $a, b, a_i, b_i, \delta, \delta_j$ by the same symbols $a, b, a_i, b_i, \delta, \delta_j$, respectively.

It is easy to verify that the following equalities hold in $H_1(\Sigma_g^{2g-1}, Q; \mathbb{Z})$:

$$\begin{aligned}
d_{1i} &= b_i - b_{i-1} + \delta_{2i-1} + \delta_{1,i}\delta, \\
d_{2i} &= a_i + b_i - a_{i-1} - b_{i-1} + \delta_{2i-1} + \delta_{1,i}\delta, \\
d_{3i} &= a_i - a_{i-1} + \delta_{2i-1} + \delta_{1,i}\delta, \\
d_{4i} &= b_i - b_{i-1} + \delta_{2i} + \delta_{1,i}\delta, \\
d_{5i} &= a_i + b_i - a_{i-1} - b_{i-1} + \delta_{2i} + \delta_{1,i}\delta, \\
d_{6i} &= a_i - a_{i-1} + \delta_{2i} + \delta_{1,i}\delta,
\end{aligned}$$

where $a_g = a_1$, $b_g = b_1$ and $\delta_{1,i} \in \{1, 0\}$ denotes the Kronecker delta. We can take a handle decomposition of the blow-up $\widetilde{X}_g$ of the total space X_g of f_g by applying the procedure explained in the appendix. Let $\widetilde{d_{ij}} \in C_2$ be the chain corresponding

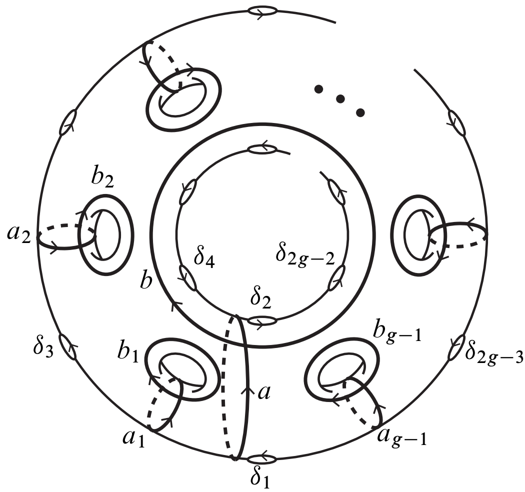

Figure 16: The oriented curves in Σ_g^{2g-2}. The curves $\delta_1, \dots, \delta_{2g-2}$ are on boundary components.

to the vanishing cycle d_{ij}, let $f \in C_2$ be the chain corresponding to a regular fiber, and $\sigma_i \in C_2$ the chain represented by the 2–handle in a neighborhood of the section corresponding to the boundary component δ_i. It is easy to see that the cycle group Z_2 is generated by the elements

$$Z_1^i = \widetilde{d}_{1i} - \widetilde{d}_{4i}, \quad Z_2^i = \widetilde{d}_{2i} - \widetilde{d}_{5i}, \quad Z_3^i = \widetilde{d}_{3i} - \widetilde{d}_{6i}, \quad W_i = d_{2i} - d_{1i} - d_{3i}$$

for $i = 1, \dots, g-1$, and the elements

$$X = \sum_{i=1}^{g-1} d_{1i}, \quad Y = \sum_{i=1}^{g-1} d_{3i}, \quad f, \quad \sigma_1, \dots, \sigma_{2g-2}.$$

As we explain in the appendix, each 3–handle in the handle decomposition of $\widetilde{X}_g$ corresponds to a 1–handle of Σ_g. Take a handle decomposition of Σ_g so that each boundary component of Σ_g^{2g-2} corresponds to a 0–handle, the $a_i, b_i \in H_1(\Sigma_g^{2g-1}, Q; \mathbb{Z})$ are represented by 1–handles, and a regular neighborhood of a path $\gamma_i \subset \Sigma_g^{2g-1} \setminus \bigl(\bigcup_i (a_i \cup b_i)\bigr)$ connecting δ_1 with δ_{i+1} is a 1–handle. Let $A_i, B_i \in C_3$ be the chains represented by the 3–handles corresponding to a_i, b_i respectively, and $\widetilde{\gamma}_i \in C_3$ the chain represented by the 3–handle corresponding to γ_i. Using Lemma A.1, we can calculate the images of the 3–chains under the boundary operator ∂_3 as follows:

$$\partial_3(A_i) = Z_1^i - Z_2^i - Z_1^{i+1} + Z_2^{i+1},$$
$$\partial_3(B_i) = -Z_2^i + Z_3^i + Z_2^{i+1} - Z_3^{i+1},$$

$$\begin{aligned}
\partial_3(A_g) &= \partial_3(B_g) = 0,\\
\partial_3(\widetilde{\gamma}_{2j}) &= W_1 - W_{j+1} - f + \sigma_1 - \sigma_{j+1},\\
\partial_3(\widetilde{\gamma}_{2j-1}) &= W_1 - W_j + Z_1^j - Z_2^j + Z_3^j - f + \sigma_1 - \sigma_{j+1}
\end{aligned}$$

for $i = 1 \dots, g-1$ and $j = 1 \dots, g-2$, where $Z_k^g = Z_k^1$ for $k = 1, 2, 3$. Thus, the following set is a basis of the cycle group Z_2:

$$\big\{\partial_3(A_1), \partial_3(B_1), \dots, \partial_3(A_{g-2}), \partial_3(B_{g-2}), \partial_3(\widetilde{\gamma}_1), \dots, \partial_3(\widetilde{\gamma}_{2g-3}),\\ Z_1^1, Z_2^1, X, Y, f, W_1, \sigma_1, \dots, \sigma_{2g-2}\big\}.$$

The homology group $H_2(X_g;\mathbb{Z})$ is isomorphic to $\mathbb{Z}^6$ and $\{Z_1^1, Z_2^1, X, Y, f, W_1\}$ is a basis of $H_2(X_g;\mathbb{Z})$. Since f is represented by a regular fiber of f_g, the divisibility of f_g is 1. □

Combining Lemmas 5.3 and 5.6, we eventually obtain:

Theorem 5.7 *For g odd, f_g is a holomorphic Lefschetz pencil on T^4 associated with a $(1, g-1)$–polarization.*

According to Lemma 5.6 and the observation preceding Lemma 5.3, it is natural to expect that the following conjecture holds:

Conjecture 5.8 For g even, f_g is a holomorphic Lefschetz pencil on T^4 associated with a $(1, g-1)$–polarization.

Note that in order to prove Conjecture 5.8 it is sufficient to prove that f_g is holomorphic for g such that $g-1$ is prime, by Remark 5.5. If Conjectures 5.8 and 3.24 hold, we can deduce the following from Lemma 4.7.

Conjecture 5.9 Let $f\colon T^4 \dashrightarrow \mathbb{CP}^1$ be a holomorphic Lefschetz pencil. There exists an unbranched covering $q\colon T^4 \to T^4$ such that f is isomorphic to the composition $f_g \circ q$ with $g-1$ prime.

Note that this conjecture holds under the following assumptions:

- The genus of f is odd.
- The genus of f is greater than 5 or the divisibility of f is greater than 1.

In this case we can take q so that g is equal to 3 (see the observation following Lemma 4.7).

Remark 5.10 If Conjecture 5.9 holds, it is theoretically possible to obtain monodromy factorizations of all the holomorphic Lefschetz pencils on the four-torus. In particular, a Lefschetz pencil on the four-torus is not holomorphic if the associated monodromy factorization is not Hurwitz equivalent to any of them (see also Remark 3.19).

5.2 Symplectic Calabi–Yau Lefschetz pencils on homotopy T^2–bundles over T^2

We have seen explicit monodromy factorizations of the genus-3 Lefschetz pencil on *the four-torus* T^4 constructed by Smith in [24], geometrically in Section 4.1 and combinatorially in Section 5.1. Smith also mentioned that by modifying the pencil on T^4 one can construct Lefschetz pencils on the total spaces of T^2*–bundles over* T^2, provided that the bundles admit sections. In this subsection we will follow Smith's idea in a combinatorial way; for any $\alpha, \beta \in \mathrm{Mod}(\Sigma_1^1; U)$ with $[\alpha, \beta] = 1$, we will construct a genus-3 Lefschetz pencil $f_{\alpha,\beta}$ by modifying the factorization (5-3), and prove the following theorem, which was also stated in Section 1:

Theorem 1.4 *The total space of $f_{\alpha,\beta}$ is homeomorphic to that of the torus bundle over the torus with a section whose monodromy representation sends two elements generating $\pi_1(T^2)$ to α and β.*

We first observe presentations for the fundamental groups of T^2–bundles over T^2 with sections. Let $p\colon X \to T^2$ be a torus bundle over the torus which has a section $S \subset X$, and $D \subset T^2$ a small disk. We take a meridian m and a longitude l of the base T^2. We denote the monodromy along m and l by α and β, respectively. Since p has a section, α and β can be considered as elements in $\mathrm{Mod}(\Sigma_1^1; U)$, where $U = \{u\} \subset \partial\Sigma_1^1$. A tubular neighborhood νS can be decomposed into a 0–handle contained in $p^{-1}(D)$, two 1–handles a, b whose cores are lifts of m and l, and a 2–handle. The preimage $p^{-1}(D)$ also admits a handle decomposition with the 0–handle of νS, two 1–handles c, d whose cores are a longitude and a meridian of a regular fiber T, and a 2–handle (we take c for the longitude and d for the meridian for convenience in later calculations). The total space X can be obtained from the union $p^{-1}(D) \cup \nu S$ by attaching four 2–handles, four 3–handles and a 4–handle. Two of the 2–handles are contained in the preimage of a neighborhood of m, while the other 2–handles are contained in the preimage of a neighborhood of l. We eventually obtain a handle decomposition of X and the associated cell decomposition of X. We denote the 1–cells corresponding to a, b, c, d endowed with suitable orientations by the same symbols. Since X has only one 0–cell, the 1–cells a, b, c, d represent elements in $\pi_1(X)$. Furthermore, c and d also represent elements in $\pi_1(T)$, in particular we can describe $\alpha(c), \alpha(d), \beta(c), \beta(d)$ as words consisting of c and d. Analyzing attaching maps of the 2–cells, we can easily prove the following:

Lemma 5.11 *The fundamental group $\pi_1(X)$ has the following presentation:*

$$\pi_1(X) = \big\langle a, b, c, d \mid [a,b],\ ca\alpha(c)^{-1}a^{-1},\ cb\beta(c)^{-1}b^{-1},\ [c,d],\ da\alpha(d)^{-1}a^{-1},\ db\beta(d)^{-1}b^{-1}\big\rangle.$$

In order to modify the factorization (5-3) we need a key observation about a symmetrical property of some subwords in (5-3) as mapping classes. We set $X_1 = t_{d_1}t_{d_2}t_{d_3}$, $Y_1 = t_{d_4}t_{d_5}t_{d_6}$, $X_2 = t_{d_7}t_{d_8}t_{d_9}$ and $Y_2 = t_{d_{10}}t_{d_{11}}t_{d_{12}}$ in $\mathrm{Mod}(\Sigma_3^4; U)$ and consider their actions on the curves L_i, R_i (for $i=1,2$) and S_L, S_R on Σ_3^4, as depicted on the left of Figure 17. The actions can be read off from similar actions on curves on the surface Σ_2^4. Set $X = t_{B'_{0,1}}t_{B'_{1,1}}t_{B'_{2,1}}$ and $Y = t_{B'_{0,2}}t_{B'_{1,2}}t_{B'_{2,2}}$ in $\mathrm{Mod}(\Sigma_2^4; U)$ and consider the curves A_i, B_i (for $i=1,2$) on Σ_2^4, as depicted on the left of Figure 17.

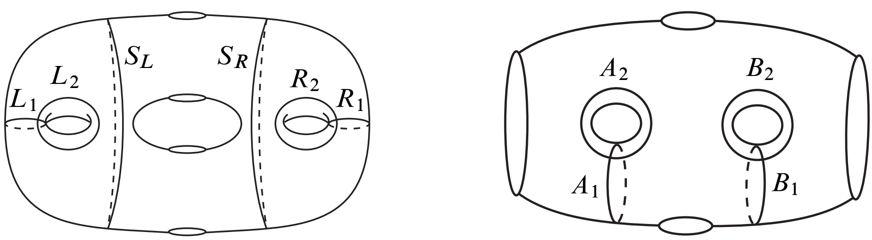

Figure 17: Actions of X, Y, X_1, Y_1, X_2 and Y_2. Left: the curves L_1, L_2, S_L, R_1, R_2, S_R on Σ_3^4. Right: the curves A_1, A_2, B_1, B_2 on Σ_2^4.

Lemma 5.12 (1) *As a mapping class, each of X and Y in* $\mathrm{Mod}(\Sigma_2^4; U)$ *maps the 4–tuple of simple closed curves (A_1, A_2, B_1, B_2) to (B_1, B_2, A_1, A_2) on Σ_2^4.*

(2) *Each of X_1, Y_1, X_2 and Y_2 in* $\mathrm{Mod}(\Sigma_3^4; U)$ *maps $(L_1, L_2, S_L, R_1, R_2, S_R)$ to $(R_1, R_2, S_R, L_1, L_2, S_L)$ on Σ_3^4.*

Proof It is simply routine work to check (1). To see (2), we recall the embedding of Σ_2^4 to Σ_3^4 with which we dealt in Figure 14(a), (b), (c). We can identify X, Y, A_i, B_i with X_1, Y_1, L_i, R_i, respectively. Thus from (1), each of X_1 and Y_1 maps (L_1, L_2, R_1, R_2) to (R_1, R_2, L_1, L_2). Since S_L and S_R are the boundaries of the regular neighborhoods of $L_1 \cup L_2$ and $R_1 \cup R_2$, respectively, X_1 and Y_1 also switch S_L and S_R. By considering the other embedding dealt with in Figure 14(d), (e), (f), by which we can identify X, Y, A_i, B_i with X_2, Y_2, R_i, L_i, respectively, we can verify the claims for X_2 and Y_2 in a similar manner. □

Now we construct a monodromy factorization as a Lefschetz pencil corresponding to a given T^2–bundle over T^2 with an explicit monodromy factorization as a bundle. We assume that the bundle has a section. It is known that the section has to be of self-intersection number 0.[1] Hence, the monodromy factorization has the form

$$[\alpha, \beta] = t_\delta^0 = 1$$

in $\mathrm{Mod}(\Sigma_1^1; U)$, where α and β are the monodromies along the meridian a and the longitude b of the base torus, respectively, and δ is the boundary of the one-holed torus Σ_1^1.

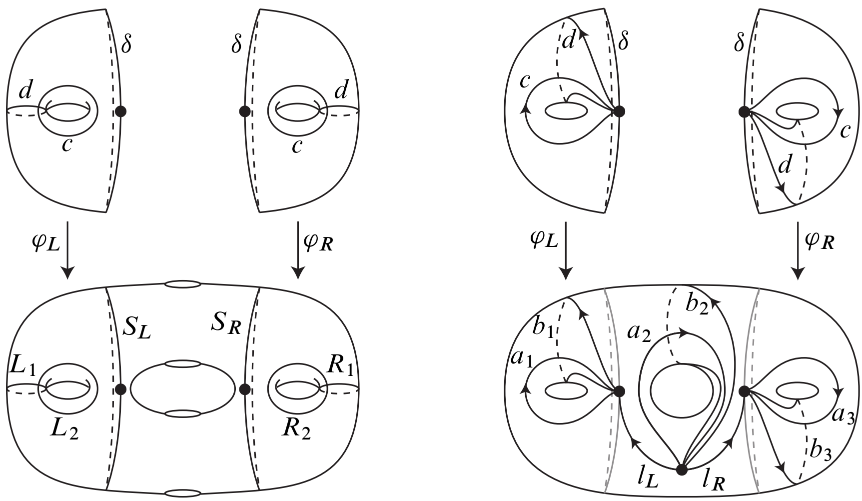

Figure 18: The embeddings φ_L and φ_R. Left: the correspondence between the free loops. Right: the correspondence between the based loops. Here a_1 stands for the loop $l_L \cdot \varphi_L(c) \cdot l_L^{-1}$. The loops b_1, a_3 and b_3 are similarly defined.

We consider the two symmetrical embeddings φ_L and φ_R of Σ_1^1 into Σ_3^4 as shown in Figure 18, one of which takes the meridian d to L_1 and the longitude c to L_2 and the

[1] One way to verify this is the following. Since a T^2–bundle over T^2 is a symplectic Calabi–Yau manifold, its canonical class K is a torsion. Let S be a section, which can be made symplectic. By the adjunction equality, we obtain $[S]^2 = 2 \cdot 1 - 2 + K(S) = 0$.

other of which takes d to R_1 and c to R_2. Then we can regard α and β as elements in $\mathrm{Mod}(\Sigma_3^4; U)$ via those embeddings; for instance, take $\varphi_L \circ \alpha \circ \varphi_L^{-1}$ on the image of φ_L and extend it as the identity map on the complement $\Sigma_3^4 \setminus \varphi_L(\Sigma_1^1)$. Let α_L denote the resulting mapping class in $\mathrm{Mod}(\Sigma_3^4; U)$. Similarly we have α_R corresponding to α via φ_R, and β_L, β_R corresponding to β via φ_L, φ_R, respectively. Note that we can deduce from the commutativity of α and β that α_L and β_L commute and α_R and β_R commute. Obviously any other pair among α_L, α_R, β_L and β_R also commutes. Since t_c and t_d are generators of $\mathrm{Mod}(\Sigma_1^1; U)$, the monodromies α and β may be written as words in t_c and t_d. Fix such word expressions. Then α_L and β_L are written as the words in t_{L_1} and t_{L_2} corresponding to the fixed expressions, while α_R and β_R are written as the corresponding words in t_{R_1} and t_{R_2}. By Lemma 5.12, the conjugation of t_{L_i} by any of X_1, Y_1, X_2, Y_2 is t_{R_i}, hence the conjugation of the fixed word for α_L by X_i or Y_i is exactly the fixed word for α_R. This simply means that the conjugation of α_L by X_i or Y_i is α_R. We can apply similar arguments to the conjugations of α_R, β_L and β_R. In summary, we have the following *switching property*: for $i = 1, 2$,

$$X_i\alpha_L X_i^{-1} = \alpha_R, \quad X_i\alpha_R X_i^{-1} = \alpha_L, \quad X_i\beta_L X_i^{-1} = \beta_R, \quad X_i\beta_R X_i^{-1} = \beta_L,$$
$$Y_i\alpha_L Y_i^{-1} = \alpha_R, \quad Y_i\alpha_R Y_i^{-1} = \alpha_L, \quad Y_i\beta_L Y_i^{-1} = \beta_R, \quad Y_i\beta_R Y_i^{-1} = \beta_L.$$

In order to create a desired factorization, we modify the factorization (5-3) by using α_R, α_L, β_R and β_L as follows:

$$\begin{aligned}
t_{\delta_1}\cdots t_{\delta_4} = t_{d_1}\cdots t_{d_{12}} = X_1Y_1X_2Y_2 &= \alpha_R\cdot X_1Y_1\cdot\alpha_R^{-1}\underline{\alpha_R\cdot X_2\cdot\alpha_R^{-1}}\ \underline{\alpha_R\cdot Y_2\cdot\alpha_R^{-1}} \\
&= \underline{\alpha_R X_1}Y_1\alpha_R^{-1}\,{}_{\alpha_R}(X_2)\,{}_{\alpha_R}(Y_2) \\
&= X_1\underline{\alpha_L Y_1}\alpha_R^{-1}\,{}_{\alpha_R}(X_2)\,{}_{\alpha_R}(Y_2) \\
&= X_1Y_1\alpha_R\alpha_R^{-1}\,{}_{\alpha_R}(X_2)\,{}_{\alpha_R}(Y_2) \\
&= X_1Y_1\,{}_{\alpha_R}(X_2)\,{}_{\alpha_R}(Y_2) \\
&= X_1\cdot\beta_R^{-1}\underline{\beta_R\cdot Y_1\cdot\beta_R^{-1}}\beta_R\cdot{}_{\alpha_R}(X_2)\,{}_{\alpha_R}(Y_2) \\
&= \underline{X_1\beta_R^{-1}}\,{}_{\beta_R}(Y_1)\underline{\beta_R\,{}_{\alpha_R}(X_2)}\,{}_{\alpha_R}(Y_2) \\
&= \beta_L^{-1}X_1\,{}_{\beta_R}(Y_1)\underline{\beta_R\alpha_R X_2\alpha_R^{-1}}\,{}_{\alpha_R}(Y_2) \\
&= X_1\,{}_{\beta_R}(Y_1)\underline{\alpha_R\beta_R X_2\alpha_R^{-1}}\,{}_{\alpha_R}(Y_2)\beta_L^{-1} \\
&= X_1\,{}_{\beta_R}(Y_1)\underline{\alpha_R X_2\beta_L\alpha_R^{-1}}\,{}_{\alpha_R}(Y_2)\beta_L^{-1} \\
&= X_1\,{}_{\beta_R}(Y_1)\underline{\alpha_R X_2\alpha_R^{-1}}\ \underline{\beta_L\,{}_{\alpha_R}(Y_2)\beta_L^{-1}} \\
&= X_1\,{}_{\beta_R}(Y_1)\,{}_{\alpha_R}(X_2)\,{}_{\beta_L\alpha_R}(Y_2),
\end{aligned}$$

where the $=$ sign above means equality as a mapping class, not as a factorization. Here we freely used the switching property explained above as well as the commutativity among α_L, α_R, β_L, β_R (and $t_{d_1},\dots,t_{d_4}$). In other words, we have obtained the following factorization:

$$t_{\delta_1}t_{\delta_2}t_{\delta_3}t_{\delta_4}=t_{d_1}t_{d_2}t_{d_3}t_{\beta_R(d_4)}t_{\beta_R(d_5)}t_{\beta_R(d_6)}t_{\alpha_R(d_7)}t_{\alpha_R(d_8)}t_{\alpha_R(d_9)}t_{\beta_L\alpha_R(d_{10})}\cdot t_{\beta_L\alpha_R(d_{11})}t_{\beta_L\alpha_R(d_{12})}. \tag{5-5}$$

Let $f_{\alpha,\beta}\colon X_{\alpha,\beta}\setminus B_{\alpha,\beta}\to\mathbb{CP}^1$ be the Lefschetz pencil corresponding to the monodromy factorization (5-5). The pencil $f_{\alpha,\beta}$ has 12 critical points and 4 base points, hence the Euler characteristic of $X_{\alpha,\beta}$ is 0. The signature of $X_{\alpha,\beta}$ is also 0 since we modified the factorization (5-3), whose corresponding pencil has the signature 0, by only using braid relations, which do not change the signature [7].

Lemma 5.13 *The fundamental group $\pi_1(X_{\alpha,\beta})$ of the total space of the Lefschetz pencil $f_{\alpha,\beta}$ is isomorphic to that of the total space of the T^2–bundle over T^2 associated with the monodromy factorization $[\alpha,\beta]=1$ in $\mathrm{Mod}(\Sigma_1^1;U)$.*

Proof It is a standard fact that the fundamental group $\pi_1(X)$ of the total space X of a genus-g Lefschetz pencil with a monodromy factorization $t_{c_n}\cdots t_{c_1}=t_{\delta_1}\cdots t_{\delta_p}$ is isomorphic to the quotient $\pi_1(\Sigma_g)/\langle c_1,\dots,c_n\rangle$, where $\langle c_1,\dots,c_n\rangle$ is the normal subgroup generated by the curves $c_1,\dots,c_n$. Let us begin with the easiest case that $\alpha=\beta=\mathrm{id}$, which is Smith's pencil itself. We give an explicit presentation of $\pi_1(X_{\mathrm{id},\mathrm{id}})$, which is of course $\pi_1(T^4)=\mathbb{Z}^4$, by deriving from the monodromy factorization (5-3). Starting from the standard generators $a_1,b_1,\dots,a_3,b_3$ of $\pi_1(\Sigma_3)$ as depicted on the right of Figure 18, we get a presentation of $\pi_1(X_{\mathrm{id},\mathrm{id}})$ with the same generators and the following defining relations:

$$[a_1,b_1][a_2,b_2][a_3,b_3]=1, \tag{R–0}$$

$$a_1[b_1,a_1]a_3^{-1}=1, \tag{R–d_1}$$

$$a_1a_1b_1^{-1}a_1^{-1}b_3a_3^{-1}=1, \tag{R–d_2}$$

$$a_1b_1^{-1}a_1^{-1}b_3=1, \tag{R–d_3}$$

$$a_1a_2b_2a_2^{-1}a_3^{-1}b_2^{-1}[a_3,b_3]=1, \tag{R–d_4}$$

$$a_1b_1^{-1}a_2b_2a_2^{-1}b_3a_3^{-1}b_2^{-1}[a_3,b_3]=1, \tag{R–d_5}$$

$$b_1[b_3,a_3]b_2b_3^{-1}a_2b_2^{-1}a_2^{-1}=1, \tag{R–d_6}$$

$$\text{(R–}d_7\text{)} \qquad a_1a_2[a_3,b_3]a_3^{-1}a_2^{-1}=1,$$
$$\text{(R–}d_8\text{)} \qquad a_1b_1^{-1}a_2a_3b_3a_3^{-1}a_3^{-1}a_2^{-1}=1,$$
$$\text{(R–}d_9\text{)} \qquad b_1a_2a_3b_3^{-1}a_3^{-1}a_2^{-1}=1,$$
$$\text{(R–}d_{10}\text{)} \qquad a_1a_2b_2^{-1}[a_3,b_3]a_3^{-1}b_2a_2^{-1}=1,$$
$$\text{(R–}d_{11}\text{)} \qquad a_1b_1^{-1}a_2b_2^{-1}a_3b_3a_3^{-1}a_3^{-1}b_2a_2^{-1}=1,$$
$$\text{(R–}d_{12}\text{)} \qquad b_1a_2b_2^{-1}a_3b_3^{-1}a_3^{-1}b_2a_2^{-1}=1.$$

Here each relation (R–d_i) comes from the vanishing cycle d_i. By substituting (R–d_3) into (R–d_2) we obtain the relation $a_1=a_3$, with which (R–d_1) implies that $[b_1,a_1]=1$. Then (R–d_3) gives that $b_1=b_3$. Now we know that $[a_1,b_1]=[a_3,b_3]=1$, hence (R–0) reduces to $[a_2,b_2]=1$. Note that we so far have the set of the relations $a_1=a_3$, $b_1=b_3$, $[a_1,b_1]=[a_2,b_2]=1$, and that this is indeed equivalent to the set of the relations (R–0), (R–d_1), (R–d_2), (R–d_3). With those new relations in mind, the relation (R–d_4) becomes $[a_1,b_2]=1$, the relation (R–d_6) becomes $[b_1,b_2]=1$, the relation (R–d_7) becomes $[a_1,a_2]=1$ and the relation (R–d_9) becomes $[b_1,a_2]=1$. None of the other defining relations gives a new relation among a_i and b_j. Therefore, by renaming $a=a_2$, $b=b_2$, $c=a_1=a_3$, $d=b_1=b_3$, it follows that $\pi_1(X_{\mathrm{id},\mathrm{id}})$ is the free abelian group generated by a, b, c and d.

We can modify the presentation of $\pi_1(X_{\mathrm{id},\mathrm{id}})$ to obtain a presentation of $\pi_1(X_{\alpha,\beta})$ with generators $a_1,b_1,\dots,a_3,b_3$ and with defining relations (R–0), (R–d_1), (R–d_2), (R–d_3) and the following ones:

$$\text{(R–}\beta_R(d_4)\text{)} \qquad a_1a_2b_2a_2^{-1}\beta_R(a_3)^{-1}b_2^{-1}[\beta_R(a_3),\beta_R(b_3)]=1,$$
$$\text{(R–}\beta_R(d_5)\text{)} \qquad a_1b_1^{-1}a_2b_2a_2^{-1}\beta_R(b_3)\beta_R(a_3)^{-1}b_2^{-1}[\beta_R(a_3),\beta_R(b_3)]=1,$$
$$\text{(R–}\beta_R(d_6)\text{)} \qquad b_1[\beta_R(b_3),\beta_R(a_3)]b_2\beta_R(b_3)^{-1}a_2b_2^{-1}a_2^{-1}=1,$$
$$\text{(R–}\alpha_R(d_7)\text{)} \qquad a_1a_2[\alpha_R(a_3),\alpha_R(b_3)]\alpha_R(a_3)^{-1}a_2^{-1}=1,$$
$$\text{(R–}\alpha_R(d_8)\text{)} \qquad a_1b_1^{-1}a_2\alpha_R(a_3)\alpha_R(b_3)\alpha_R(a_3)^{-1}\alpha_R(a_3)^{-1}a_2^{-1}=1,$$
$$\text{(R–}\alpha_R(d_9)\text{)} \qquad b_1a_2\alpha_R(a_3)\alpha_R(b_3)^{-1}\alpha_R(a_3)^{-1}a_2^{-1}=1,$$
$$\text{(R–}\beta_L\alpha_R(d_{10})\text{)} \qquad \beta_L(a_1)a_2b_2^{-1}[\alpha_R(a_3),\alpha_R(b_3)]\alpha_R(a_3)^{-1}b_2a_2^{-1}=1,$$
$$\text{(R–}\beta_L\alpha_R(d_{11})\text{)} \qquad \beta_L(a_1)\beta_L(b_1)^{-1}a_2b_2^{-1}\alpha_R(a_3)\alpha_R(b_3)\alpha_R(a_3)^{-1}\cdot\alpha_R(a_3)^{-1}b_2a_2^{-1}=1,$$
$$\text{(R–}\beta_L\alpha_R(d_{12})\text{)} \qquad \beta_L(b_1)a_2b_2^{-1}\alpha_R(a_3)\alpha_R(b_3)^{-1}\alpha_R(a_3)^{-1}b_2a_2^{-1}=1.$$

Again the relation (R–$*$) corresponds to each vanishing cycle. As we discussed above, the relations (R–0), (R–d_1), (R–d_2), (R–d_3) imply that $a_1 = a_3$, $b_1 = b_3$ and $[a_1, b_1] = [a_2, b_2] = 1$. We regard $\pi_1(\Sigma_1)$ as the quotient $\pi_1(\Sigma_1^1)/\langle\delta\rangle = \langle c, d\rangle/\langle[c, d]\rangle$ and then construct a homomorphism $\varphi_{L*}\colon \pi_1(\Sigma_1) \to \pi_1(X_{\alpha,\beta})$ as follows: for an element $g \in \pi_1(\Sigma_1^1)$ we define $\varphi_{L*}([g])$ to be $l_L \cdot \varphi_L(g) \cdot l_L^{-1}$, where $[g] \in \pi_1(\Sigma_1) = \pi_1(\Sigma_1^1)/\langle\delta\rangle$ is an element represented by g. The map φ_{L*} is well-defined since $l_L \cdot \varphi_L(c) \cdot l_L^{-1} = a_1$, $l_L \cdot \varphi_L(d) \cdot l_L^{-1} = b_1$ and $[a_1, b_1] = 1$. Similarly we can define another homomorphism $\varphi_{R*}\colon \pi_1(\Sigma_1) \to \pi_1(X_{\alpha,\beta})$ as $[g] \mapsto l_R \cdot \varphi_R(g) \cdot l_R^{-1}$. However, φ_{L*} and φ_{R*} in fact coincide since $\varphi_{L*}(c) = a_1 = a_3 = \varphi_{R*}(c)$ and $\varphi_{L*}(d) = b_1 = b_3 = \varphi_{R*}(d)$. It follows that $\alpha_L(a_1) = \varphi_{L*}(\alpha(c)) = \varphi_{R*}(\alpha(c)) = \alpha_R(a_3)$, and similarly $\alpha_L(b_1) = \alpha_R(b_3)$, $\beta_L(a_1) = \beta_R(a_3)$ and $\beta_L(b_1) = \beta_R(b_3)$. Therefore we can identify c with $a_1 = a_3$ and d with $b_1 = b_3$ in a way that $\alpha(c) = \alpha_L(a_1) = \alpha_R(a_3)$ as a word in $c = a_1 = a_3$ and $d = b_1 = b_3$, and $\alpha(d) = \alpha_L(b_1) = \alpha_R(b_3)$, and so on. We also rename $a = a_2$ and $b = b_2$. Then the relation (R–$\beta_R(d_4)$) becomes $cb\beta(c)^{-1}b^{-1} = 1$, the relation (R–$\beta_R(d_6)$) becomes $db\beta(d)^{-1}b^{-1} = 1$, the relation (R–$\alpha_R(d_7)$) becomes $ca\alpha(c)^{-1}a^{-1} = 1$ and the relation (R–$\alpha_R(d_9)$) becomes $da\alpha(d)^{-1}a^{-1} = 1$. No other defining relations give a new relation. In conclusion, $\pi_1(X_{\alpha,\beta})$ is isomorphic to the group described in Lemma 5.11, hence to the fundamental group of the T^2–bundle over T^2 associated with the monodromy factorization $[\alpha, \beta] = 1$, as desired. □

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4 The fundamental group of a T^2–bundle over T^2 cannot be isomorphic to that of a rational or ruled surface. The theorem by Baykur–Hayano [3, Theorem 4.1] implies that the Lefschetz pencil $f_{\alpha,\beta}$ is symplectic Calabi–Yau. Furthermore, we can deduce from [9, Corollary 3.3] that a symplectic Calabi–Yau four-manifold M is homeomorphic to the total space X of a T^2–bundle $X \to T^2$ with a section if and only if $\pi_1(M)$ is isomorphic to $\pi_1(X)$. By Lemma 5.13 we can conclude that $X_{\alpha,\beta}$ is homeomorphic to the total space of T^2–bundle over T^2 with a section whose monodromy representation sends two elements generating $\pi_1(T^2)$ to α and β. □

We end this subsection with the following conjecture:

Conjecture 5.14 The pencil $f_{\alpha,\beta}$ is isomorphic to that constructed by Smith [24]. In particular, the total space $X_{\alpha,\beta}$ is *diffeomorphic* to that of a T^2–bundle over T^2.

Appendix: Homology groups of Lefschetz pencils from monodromy factorizations

Let $f\colon X \setminus B \to \mathbb{CP}^1$ be a genus-g Lefschetz pencil and $t_{c_n} \cdots t_{c_1} = t_{\delta_1} \cdots t_{\delta_p}$ a monodromy factorization of f. As in the proof of Lemma 5.13, the fundamental group $\pi_1(X)$ is calculated from the vanishing cycles $c_1, \dots, c_n$. In particular, we can easily calculate $H_1(X;\mathbb{Z})$, which is isomorphic to the abelianization of $\pi_1(X)$. Since the Euler characteristic of X is equal to $4-4g+n-p$, we can deduce from the universal coefficient theorem that $H_2(X;\mathbb{Z})$ is isomorphic to $\mathbb{Z}^r \oplus T(H_1(X;\mathbb{Z}))$, where $r = 2-4g+n-p+2\operatorname{rank}(H_1(X;\mathbb{Z}))$ and $T(H_1(X;\mathbb{Z}))$ is the torsion part of $H_1(X;\mathbb{Z})$. However, the above observation does not give any information on the fiber class of f, especially the divisibility of f. In this appendix, we will explain how to obtain the divisibility of f by calculating the second homology of X from a handlebody structure associated with f.

Let $\widetilde{X}$ be the blow-up of X at the points in B and $\tilde{f}$ the Lefschetz fibration on $\widetilde{X}$ derived from f. The manifold $\widetilde{X}$ can be decomposed as

$$\widetilde{X} = D^2 \times \Sigma_g \cup (2\text{–handles}) \cup D^2 \times \Sigma_g, \tag{A-1}$$

where the two copies of $D^2 \times \Sigma_g$ are tubular neighborhoods of regular fibers and each 2–handle corresponds to a Lefschetz singularity of $\tilde{f}$ and are attached along a vanishing cycle with framing (-1) with respect to the fiber framing (see [14]). We denote the 2–handle in the above decomposition corresponding to the vanishing cycle c_i by h_L^i. The two copies of $D^2 \times \Sigma_g$ above can be further decomposed as follows:

- The former $D^2 \times \Sigma_g$ can be decomposed into a 0–handle, $2g$ 1–handles and a 2–handle, which we denote by h_F.
- The latter $D^2 \times \Sigma_g$ can be decomposed into p 2–handles, $2g+p-1$ 3–handles and a 4–handle, where the cores of the 2–handles are contained in the exceptional spheres arising in the blow-up. We denote the 2–handles in this decomposition by $h_S^1, \dots, h_S^p$.

The decompositions above give rise to a handlebody structure of $\widetilde{X}$, in particular we can obtain a chain complex $\mathcal{C} = \{(C_i, \partial_i)\}_i$ such that C_i is a free abelian group generated by the i–handles above. The homology group $H_i(\mathcal{C})$ is isomorphic to $H_i(\widetilde{X};\mathbb{Z})$, with the fiber class of $\tilde{f}$ (resp. the exceptional classes) represented by h_F (resp. $h_S^1, \dots, h_S^p$).

The group C_1 can be identified with $H_1(\Sigma_g;\mathbb{Z})$ in the obvious way. Under this identification, $\partial_2\colon C_2 \to C_1$ sends h_L^i to the homology class of the corresponding vanishing cycle. Since both h_F and h_S^j are contained in $Z_2=\ker\partial_2$, we obtain a generating set $\{z_1,\dots,z_m,h_F,h_S^1,\dots,h_S^p\}$ of Z_2 such that each z_i is a linear combination of $h_L^1,\dots,h_L^n$.

The decomposition of the latter $D^2\times\Sigma_g$ in (A-1) is induced from the handle decomposition of Σ_g, which consists of p 0–handles, $2g+p-1$ 1–handles and a 2–handle. Indeed, we can regard an i–handle of this $D^2\times\Sigma_g$ as the product of D^2 and an $(i-2)$–handle of Σ_g. Let Σ_g^p be the surface obtained by removing the p 0–handles of Σ_g. We take points $q_1,\dots,q_p\in\partial\Sigma_g^p$ so that the map $\pi_0(\{q_1,\dots,q_p\})\to\pi_0(\partial\Sigma_g^p)$ induced by the inclusion is bijective. Let δ_i be the simple closed curve in Σ_g^p parallel to the boundary component containing q_i. We take a disk $D\subset\mathrm{Int}(\Sigma_g^p)$, a point $q\in\partial D$ and a path $\alpha\subset\Sigma_g^p\setminus\mathrm{Int}(D)$ connecting q_1 and q. Let Σ_g^{p+1} be the surface $\Sigma_g^p\setminus\mathrm{Int}(D)$ and Q the set $\{q_1,\dots,q_p,q\}$. It is easy to see that the homology group $H_1(\Sigma_g^{p+1},Q;\mathbb{Z})$ is isomorphic to $\mathbb{Z}^{2g+2p}$ and is generated by elements represented by the cores of 1–handles of Σ_g, $\delta_1,\dots,\delta_p$ and α. We can define an intersection pairing

$$\varrho\colon H_1(\Sigma_g^{p+1},Q;\mathbb{Z})\times H_1(\Sigma_g^{p+1};\mathbb{Z})\to\mathbb{Z}$$

which assigns $(\alpha,\beta)\in H_1(\Sigma_g^{p+1},Q;\mathbb{Z})\times H_1(\Sigma_g^{p+1};\mathbb{Z})$ to the algebraic intersection between a union of oriented paths in $\Pi(\Sigma_g^{p+1},Q)$ representing α and a closed curve representing β, where $\Pi(\Sigma_g^{p+1},Q)$ is the set of paths whose edges are points in Q.

Lemma A.1 *Let η be a 3–handle in the latter $D^2\times\Sigma_g$ in (A-1) and η' the 1–handle of Σ_g corresponding to η. Suppose that η' is attached to two 0–handles corresponding to h_S^i and h_S^j. We inductively take $k_l\in\mathbb{Z}$ and $\eta_l\in H_1(\Sigma_g^{p+1},Q;\mathbb{Z})$ as follows:*

- $\eta_0=\eta'$,
- $k_l=\varrho(\eta_{l-1},c_l)$ *and* $\eta_l=\eta_{l-1}-k_lc_l$,

where we identify two regular fibers in the former and latter $D^2\times\Sigma_g$ via the attaching map. Then the element η_n is equal to $\eta'+\delta_i-\delta_j-k_F[\partial D]$ for some $k_F\in\mathbb{Z}$, and the following equality holds:

$$\partial_3(\eta)=\sum_{l=1}^{n}k_lh_L^l+k_Fh_F+h_S^i-h_S^j.$$

Proof Note first that the coefficients of $\partial_3(\eta)$ are equal to the algebraic intersections between the attaching sphere of η and the belt spheres of the corresponding 2–handles. The attaching sphere of η is the union of

- the product of D^2 and the edges of the core of η', and
- the product of ∂D^2 and the core of η'.

The former part intersects the belt spheres of h_S^i and h_S^j geometrically once (and the signs of these intersections are opposite). It is easy to see that the latter part intersects the belt sphere of h_L^l algebraically k_l times. Let $\mathrm{Mod}(\Sigma_g^{p+1})$ be the set of isotopy classes of self-diffeomorphisms of Σ_g^{p+1} *which are not necessarily the identity map on the boundary*. The product $t_{c_n} \cdots t_{c_1}$ in $\mathrm{Mod}(\Sigma_g^{p+1})$ is a diffeomorphism obtained by pushing the disk D. We can verify that the path $t_{c_n} \cdots t_{c_1}$ (the core of η') goes through the center of D algebraically k_F times when we move it to η' by an isotopy of Σ_g^p. This observation implies that the product of ∂D^2 and the core of η' intersects the belt sphere of h_F algebraically k_F times. $\square$

Let $\{\alpha_1, \dots, \alpha_{2g+p-1}\}$ be a generating set of C_3. Using Lemma A.1 we can obtain the representation matrix A_0 of ∂_3 with respect to the generating sets $\{\alpha_1, \dots, \alpha_{2g+p-1}\}$ and $\{z_1, \dots, z_m, h_F, h_S^1 \dots, h_S^p\}$:

$$(\partial_3(\alpha_1), \dots, \partial_3(\alpha_{2g+p-1})) = (z_1, \dots, z_m, h_F, h_S^1, \dots, h_S^p) A_0.$$

Furthermore, applying fundamental operations to A_0 we can find another generating set $\{\alpha_1', \dots, \alpha_{2g+p-1}'\}$ of C_3 and elements $z_1', \dots, z_{m+1}'$ of Z_2 such that:

- z_i' is a linear combination of $z_1, \dots, z_m, h_F$.
- $\{z_1', \dots, z_{m+1}', h_S^1, \dots, h_S^p\}$ is a generating set of C_2.
- The first $m+1$ rows of the representation matrix A_1 of ∂_3 with respect to $\{\alpha_1', \dots, \alpha_{2g+p-1}'\}$ and $\{z_1', \dots, z_{m+1}', h_S^1, \dots, h_S^p\}$ has nonzero entries only on its diagonal part.
- Let $e_1, \dots, e_{m+1}$ be each of the first $m+1$ diagonal entries of A_1. There exists a positive integer k such that $e_l = 0$ for any $l > k$ and $e_i | e_{i+1}$ for any $i < k$.

Lemma A.2 *For any $i \le k$ and $j \ge m+2$, all the entries in the j^{th} column of A_1 are equal to 0 and the (j,i)–entry of A_1 is divisible by e_i.*

Proof Let a_{ij} be the (i,j)–entry of A_1. The cycle $\sum_{i \ge m+2} a_{ij} h_S^{i-m-1}$ is a boundary for any $j \ge m+2$. Since the handle h_S^l represents the exceptional class, the classes $[h_S^1], \dots, [h_S^p]$ are linearly independent in $H_2(\widetilde{X})$. Thus, $a_{ij} = 0$ if $j \ge m+2$. Since the cycle $e_i z_i' + \sum_{j \ge m+2} a_{ji} h_S^{j-m-1}$ is a boundary when $i \le k$, the class $e_i[z_i']$ is equal in $H_2(\mathcal{C})$ to $-\sum_{j \ge m+2} a_{ji}[h_S^{j-m-1}]$. Since $[h_S^{j-m-1}]$ is the exceptional class, the divisibility of $\sum_{j \ge m+2} a_{ji}[h_S^{j-m-1}]$ in $H_2(\widetilde{X})$ is equal to $\gcd(\{a_{ji}\}_{j \ge m+2})$. On the other hand, the divisibility of $e_i[z_i']$ is divisible by e_i. Thus, $e_i | a_{ji}$ for any $i \le k$ and $j \ge m+2$. $\square$

Lemma A.2 implies that we can find cycles $z''_1, \dots, z''_{m+1}$ such that

- $\{z''_1, \dots, z''_{m+1}, h^1_S, \dots, h^p_S\}$ is a generating set of C_2,
- the representation matrix A_2 of ∂_3 with respect to $\{\alpha'_1, \dots, \alpha'_{2g+p-1}\}$ and $\{z''_1, \dots, z''_{m+1}, h^1_S, \dots, h^p_S\}$ has nonzero entries only on its diagonal part,
- the first k diagonal entries of A_2 are $e_1, \dots, e_k$ and the other entries are 0.

In particular, the second homology group $H_2(\widetilde{X};\mathbb{Z})$ is isomorphic to the group $\left(\bigoplus_i \mathbb{Z}/e_i\mathbb{Z}\right) \oplus \mathbb{Z}^{m+1-k} \oplus \mathbb{Z}^p$, where the former two components are generated by $z''_1, \dots, z''_{m+1}$, while the last component is generated by $h^1_S, \dots, h^p_S$. Furthermore, the natural homomorphism $H_2(\widetilde{X};\mathbb{Z}) \to H_2(X;\mathbb{Z})$ coincides with the projection $(\oplus_i \mathbb{Z}/e_i\mathbb{Z}) \oplus \mathbb{Z}^{m+1-k} \oplus \mathbb{Z}^p \to (\oplus_i \mathbb{Z}/e_i\mathbb{Z}) \oplus \mathbb{Z}^{m+1-k}$. The fiber class of $\tilde{f}$ is represented by h_F, which we can represent as a linear combination of $z''_1, \dots, z''_{m+1}, h^1_S, \dots, h^p_S$. Thus, the fiber class of f is the image of $[h_F]$ under the projection $H_2(\widetilde{X};\mathbb{Z}) \to H_2(X;\mathbb{Z})$.

In summary, the procedure above not only gives an isomorphism between $H_2(X;\mathbb{Z})$ and an abelian group, but also determines which element in the abelian group corresponds to the fiber class of f. In particular we can calculate the divisibility of f. Moreover, in order to apply this procedure in practice we need only (possibly tedious) linear-algebraic calculations, which we can do using a computer.

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Department of Mathematics and Statistics, University of Massachusetts
Amherst, MA, United States

Department of Mathematics, Keio University, Yagami Campus
Yokohama, Japan

hamada@math.umass.edu, k-hayano@math.keio.ac.jp

http://www.math.keio.ac.jp/~k-hayano/index_en.html

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