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A trivial tail homology for non-A–adequate links

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We prove a conjecture of Rozansky's concerning his categorification of the tail of the colored Jones polynomial for an A–adequate link. We show that the tail homology groups he constructs are trivial for non-A–adequate links.

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1 Introduction

Let D be a diagram of a link K in S^3. A *Kauffman state* is a choice of replacing every crossing of D by the $A-$ or B–resolution as in Figure 1, with the dashed segment recording the location of the crossing before the replacement.

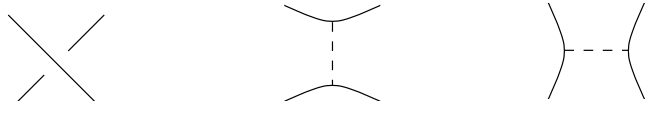


Figure 1: A–resolution (center) and B–resolution (right) of a crossing

Applying a Kauffman state results in a set of disjoint circles called *state circles*. We form a σ–*state graph* $s_\sigma(D)$ for each Kauffman state σ by letting the resulting state circles be vertices and the segments be edges. The *all-A* state graph $s_A(D)$ comes from the Kauffman state which chooses the A–resolution at every crossing of D.

Definition 1.1 A link diagram D is A–*adequate* if its all-A state graph has no one-edged loops.

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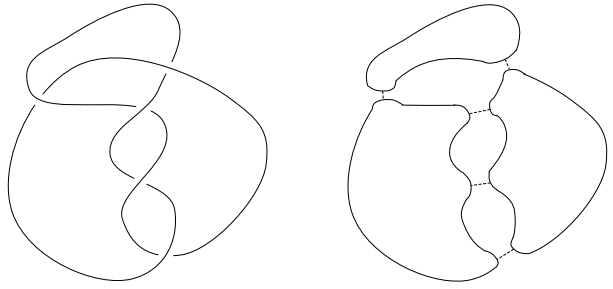


Figure 2: An adequate diagram and its all-A state

A link K is said to be *A–adequate* if it admits an A–adequate diagram. Every link diagram of a non-A–adequate link is not A–adequate. A diagram is *B–adequate* if its mirror image is A–adequate.

Definition 1.2 A link K is *semi-adequate* (*A– or B–adequate*) if it admits a diagram that is A– or B–adequate.

First studied by Lickorish and Thistlethwaite [12], *semi-adequate* links form a rich class of links which includes alternating links. Denote the colored Jones polynomial of a link $K \subset S^3$ by $\{J_K(q;n)\}_{n=1}^{\infty}$, where $J_K(q;n) \in \mathbb{Z}[q,q^{-1}]$ and $J_K(q;1)$ is the unreduced Jones polynomial of K. See Definition 2.2 for our normalization convention. Armond [1] and Garoufalidis and Lê [8] have independently shown the following result. It states a stability property, first conjectured by Dasbach and Lin [6] with partial evidence, of the colored Jones polynomial of a semi-adequate link.

Let $d(n)$ be the minimum degree of the n^{th} colored Jones polynomial $J_K(q;n)$.

Theorem 1.3 [1; 8] *For $i > 1$, let β_i be the coefficient of $q^{d(i)+2(i-2)}$ of $J_K(q;i)$. If K admits an A–adequate diagram, then the coefficient of $q^{d(n)+2(i-2)}$ of $J_K(q;n)$ is equal to β_i for all $n \geq i$.*

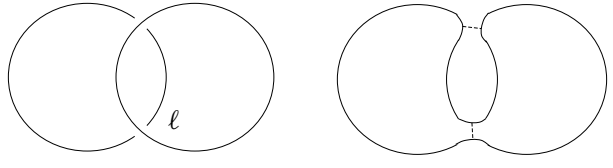


Figure 3: A non-A–adequate diagram of the 2–component unlink and its all-A state. A crossing corresponding to a one-edged loop in $s_A(D)$ is marked ℓ.

For an A–adequate link K, one defines a power series

$$T_K(q) = \sum_{i=2}^{\infty} \beta_i q^{2(i-2)}, \tag{1}$$

called a *tail* of the colored Jones polynomial of K. For a B–adequate link with a B–adequate diagram D, the mirror image $\overline{D}$ is A–adequate, and we may apply Theorem 1.3 to obtain a *head* of the colored Jones polynomial, since $J_K(q;n) = J_{\overline{K}}(q^{-1};n)$.

Subsequent to Armond [1] and Garoufalidis and Lê [8], Rozansky [15] has shown that stability behaviors also exist in the categorification of the colored Jones polynomial. More precisely, let $\{H^{\mathrm{Kh}}_{i,j}(D,n)\}$ be the set of bigraded chain groups constructed by Rozansky [14] which categorifies the colored Jones polynomial, ie

$$J_K(q;n) = ((-1)^n q^{\frac{1}{2}(n^2+2n)})^{\omega(D)} \sum_{i,j} (-1)^j q^{i+j} \dim H^{\mathrm{Kh}}_{i,j}(D,n). \tag{2}$$

See Cooper and Krushkal [4] and Frenkel, Khovanov and Stroppel [7] for other and previous constructions of the categorification of the polynomial. In Section 2.4, we describe the convention used in this paper for the categorification of the colored Jones polynomial adapted from [15]. In [15], Rozansky studies a shifted version of the categorifying chain groups, $\{\widetilde{H}^{\mathrm{Kh}}_{i,j}(D,n)\}$, defined as follows.

$$\widetilde{H}^{\mathrm{Kh}}_{i,j}(D,n) := \boldsymbol{h}^{\frac{1}{2}n^2 c(D)} \boldsymbol{q}^{n|s_A(D)|} H^{\mathrm{Kh}}_{i,j}(D,n),$$

where $c(D)$ is the number of crossings of D, $|s_A(D)|$ is the number of state circles in the all-A state of D, and $\boldsymbol{h}$ and $\boldsymbol{q}$ indicate the shifts to the homological and quantum grading, respectively. See Section 3.1 for the equivalence of this definition to his. He shows that there exists a directed system of degree-preserving maps

$$\widetilde{H}^{\mathrm{Kh}}(D,n) \xrightarrow{f_n} \widetilde{H}^{\mathrm{Kh}}(D,n+1),$$

where f_n is an isomorphism on $\widetilde{H}^{\mathrm{Kh}}_{i,*}(D,n)$ for $i \le n-1$; see Section 3 for details. This implies the existence of a *tail homology* $H^{\infty}(D)$, which is defined as the direct limit of the directed system.

Let

$$J_{D,\infty}(q) := \sum_{i,j} (-1)^j q^{i+j} \dim H^{\infty}_{i,j}(D)$$

be the graded Euler characteristic of $H^{\infty}(D)$. For A–adequate links, Rozansky shows that the tail homology categorifies $T_K(q)$ since $J_{D,\infty}(q)$ determines the lower powers

of $J_K(q;n)$ for D an A–adequate diagram of K. See Theorem 3.6 in Section 3.2 for the precise statement.

He makes the following conjecture, which we adjust for the convention used in this paper regarding the directed system thus constructed.

Conjecture 1.4 [15, Conjecture 2.14] *If a diagram D is not A–adequate, then $H^\infty(D)$ is trivial.*

We are motivated by the following result, which gives partial evidence for Conjecture 1.4.

Let

$$h_n(D) = -\tfrac{1}{2}n^2c(D) - n|s_A(D)| + \tfrac{1}{2}\omega(D)(n^2+2n), \tag{3}$$

where $\omega(D)$ is the writhe of the diagram D.

Theorem 1.5 [9] *Suppose that a link diagram D is not A–adequate; then*

$$d(n) \geq h_n(D) + 2(n-1) \quad \text{for } n > 1.$$

Let $h(n)$ be the maximum of $h_n(D)$ taken over all diagrams D of a link. Theorem 1.5 shows that the difference between the lower bound $h(n)$ and the actual degree of the n^{th} colored Jones polynomial increases linearly with n, which would be a consequence of Conjecture 1.4 for large n.

1.1 Main result

In this paper we prove Conjecture 1.4 and, therefore, the categorical analogue of Theorem 1.5. The result sheds some light on the behavior of the categorification of the colored Jones polynomial when the link is not A–adequate. Prior to Theorem 1.5, it was not known whether a link may still achieve the lower bound of $h(n)$ when it is not A–adequate. The result of this paper shows that the categorification has a similar gap, linear with respect to n, between the lower bound $-\frac{1}{2}n^2c(D)$ of the minimum homological degree of the complex and the actual minimum homological degree.

The precise implication for the colored Jones polynomial is given by the following corollary:

Corollary 1.6 *The graded Euler characteristic of the tail homology determines the lower powers of q in the unicolored Jones polynomial of a non-A–adequate link K:*

$$J_K(q;n) = \big((-1)^n q^{\frac{1}{2}(n^2+2n)}\big)^{\omega(D)} q^{-\frac{1}{2}n^2c(D)-n|s_A(D)|} \times \big(J_{D,\infty}(q) + O(q^{\frac{1}{2}n-\frac{1}{2}c(D)-\frac{3}{2}c(D)^\ell})\big),$$

where $c(D)^\ell$ is the number of non-A–adequate crossings of D whose corresponding segments in $s_A(D)$ are one-edged loops. If D is a diagram realizing the lower bound

$$h(n) := \max_{D \text{ of } K} \big\{-\tfrac{1}{2}n^2c(D) - n|s_A(D)| + \tfrac{1}{2}\omega(D)(n^2+2n)\big\},$$

then

$$J_K(q;n) = (-1)^{n(\omega(D))} q^{h(n)} \big(J_{D,\infty}(q) + O(q^{\frac{1}{2}n-\frac{1}{2}c(D)-\frac{3}{2}c(D)^\ell})\big).$$

This is a straightforward consequence of the general homological bounds in Theorem 3.5 by Rozansky. Conjecture 1.4 then implies that $J_{D,\infty}(q) = 0$ for a non-A–adequate link, and we recover Theorem 1.5 in the sense that the gap between the actual degree $d(n)$ of the polynomial and the lower bound $h_n(D)$ increases linearly with respect to n for sufficiently large n.

Given a non-A–adequate diagram, which necessarily has a crossing ℓ corresponding to a one-edged loop in $s_A(D)$, the key to the argument is localizing the behavior of the tail homology to the categorification of a skein $\mathcal{S}^k$ containing the $n+1$–cable of ℓ, shown in Figure 4. With the help of the homological machinery developed by Rozansky [15], the homology of this skein is approximated by that of the unknot with a left-hand twist, which has trivial homology in homological gradings $\leq n-1$.

1.2 Relation to a general tail

As for the question of stability behaviors for all links along the lines of Theorem 1.3, a tail may be defined for a general link, if it exists, as follows from Armond and Dasbach [2] with our conventions.

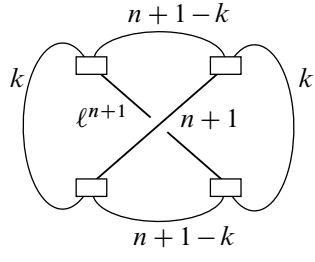


Figure 4: The skein $\mathcal{S}^k$. The notation ℓ^{n+1} indicates the set of $(n+1)^2$ crossings corresponding to ℓ in the $n+1$–blackboard cable of D.

Definition 1.7 For a Laurent polynomial $P_1(q)$ and a power series $P_2(q)$, we define

$$P_1(q) \doteq_{2n} P_2(q)$$

if $P_1(q)$ coincides with $P_2(q) \pmod{q^{2n}}$ up to multiplication with $\pm q^s$ for $s \in \mathbb{Z}$.

Definition 1.8 The *tail* of the colored Jones polynomial of a knot K, if it exists, is a series $T_K(q) = \sum_{j=2}^{\infty} a_j q^{2(j-2)}$ with

$$J_K(q;n) \doteq_{2n} T_K(q) \quad \text{for all } n > 1.$$

The existence and behavior of a tail $T_K(q)$ for a general link remains an interesting and important question. It is the hope that the techniques of this paper may be applied to the study of a categorification of $T_K(q)$. Note that Conjecture 1.4 and Theorem 1.5 do not provide information on the existence or the nature of $T_K(q)$, only that if $T_K(q)$ exists, it must occur at a distance, linear with respect to n, from the diagrammatic lower bound. For understanding $T_K(q)$, a persistent difficulty lies in determining $d(n)$ when a link is not A–adequate. In the case of torus knots, where the colored Jones polynomial is explicitly computed, it has been shown that a tail does not exist. Rather, multiple tails exist, which indicates a more complex stability behavior of the polynomial; see Armond and Dasbach [2]. Although this is also expected to be reflected in the categorification of the colored Jones polynomial of torus links, a potentially simpler class to study would be non-A–adequate links which are still expected to admit a single tail. In [10], the author and Roland van der Veen have determined $d(n)$ for many non-A–adequate 3–string pretzel knots. It is expected that a single tail exists for these knots, and we will address this topic for the polynomial and its categorification in the future using the techniques in this paper.

1.3 Organization

Sections 2 and 3 gather the necessary definitions for understanding the main result and the convention for the colored Jones polynomial and its categorification used in this paper. In Section 3, we describe Rozansky's definition of a tail homology. Conjecture 1.4 is proven in Section 4 as Theorem 4.16.

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2 Background

2.1 Skein theory

We will follow Lickorish [11] in defining the Temperley–Lieb algebra, except he uses A, and we make the substitution of variable $A^2 = q^{-1}$. This is for the convenience of not having to substitute q for A later and to avoid confusion with the "A" in A–adequacy.

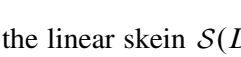
Let F be an orientable surface which has a finite (possibly empty) collection of points specified on ∂F if $\partial F \neq \varnothing$. A link diagram on F consists of finitely many arcs and closed curves on F such that:

- There are finitely many transverse crossings with an over-strand and an under-strand.
- The endpoints of the arcs form a subset of the specified points on ∂F. In other words, the arcs are *properly embedded*.

Two link diagrams on F are isotopic if they differ by a homeomorphism of F isotopic to the identity. The isotopy is required to fix ∂F.

Definition 2.1 Let q be a fixed complex number. The *linear skein* $\mathcal{S}(F)$ of F is the vector space of formal linear sums over $\mathbb{C}$ of isotopy classes of link diagrams in F quotiented by the relations

(i) $D \cup \bigcirc = (-q - q^{-1})D$, and

(ii)  $= q^{\frac{1}{2}}$) ($+ q^{-\frac{1}{2}}$.

We consider the linear skein $\mathcal{S}(D, n)$ of the disc D with $2n$ points specified on its boundary, which will be represented as a square with n marked points above and below. The square may be rotated with the marked points going from left to right. For $D_1, D_2 \in \mathcal{S}(D, n)$, there is a natural multiplication $D_1 \cdot D_2$ defined by identifying the top boundary of D_1 with the bottom boundary of D_2. This makes $\mathcal{S}(D, n)$ into an algebra TL_n, called the n^{th} *Temperley–Lieb algebra*. The algebra TL_n is generated by crossing-less matchings $1_n, e_n^1, \ldots, e_n^{n-1}$. See Figure 5 for an example.

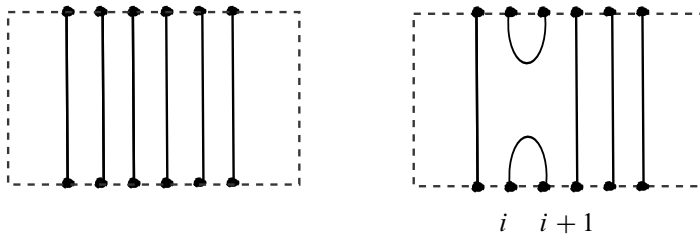


Figure 5: An example of the identity element $|_n$ and a generator e_n^i of TL_n for $n = 6$ and $i = 2$

We will use a shorthand notation which denotes n parallel strands, the identity 1_n, by $|_n$.

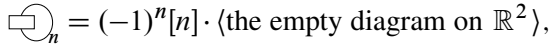
Suppose that q^2 is not a k^{th} root of unity for $k \leq n$. There is an element $\dot{+}_n$ in TL_n called the *Jones–Wenzl projector*, which is uniquely defined by the following properties. For the original reference where the projector was defined and studied, see [16].

(i) $\dot{+}_n \cdot e_n^i = e_n^i \cdot \dot{+}_n = 0$ for $1 \leq i \leq n-1$.

(ii) $\dot{+}_n - |_n$ belongs to the algebra generated by $\{e_n^1, e_n^2, \ldots, e_n^{n-1}\}$.

(iii) $\dot{+}_n \cdot \dot{+}_n = \dot{+}_n$.

(iv) Let $\mathcal{S}(S^1 \times I)$ be the linear skein of the annuli with no points marked on its boundaries. The image of $\dot{+}_n$ in $\mathcal{S}(\mathbb{R}^2)$ obtained from first sending it to $\mathcal{S}(S^1 \times I)$ by joining the n boundary points on the top with those at the bottom and then embedding $S^1 \times I$ in $\mathbb{R}^2$ is equal to



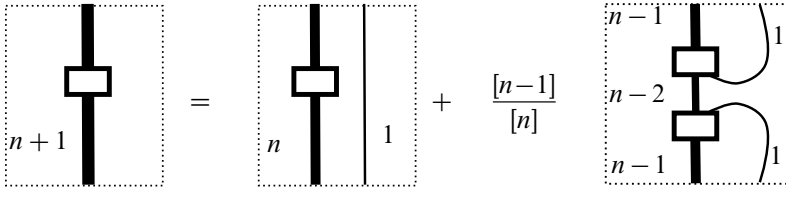
$$= (-1)^n [n] \cdot \langle \text{the empty diagram on } \mathbb{R}^2 \rangle,$$

where $[n]$ is the *quantum integer* defined by

$$[n] := \frac{q^{-(n+1)} - q^{n+1}}{q^{-1} - q}.$$

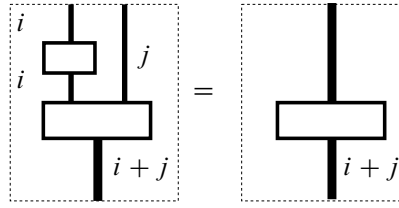
From the defining properties, the Jones–Wenzl projector also satisfies a recursion relation

(4)



and an identity

(5)



2.2 The colored Jones polynomial

Here we give a definition of the colored Jones polynomial in terms of skeins in the Temperley–Lieb algebra defined in Section 2.1. For the original construction of the invariant based on the representation theory of $U_q(\mathfrak{sl}_2)$ for generic q, see [13]. A discussion and proof of the equivalence of two definitions may be found in [5].

Definition 2.2 Let D be a diagram of a link $K \subset S^3$ with k components. For each component D_i for $i \in \{1, \dots, k\}$ of D, take an annulus A_i via the blackboard framing. Let

$$f_D\colon \underbrace{\mathcal{S}(S^1 \times I) \times \mathcal{S}(S^1 \times I) \times \cdots \times \mathcal{S}(S^1 \times I)}_{k \text{ times}} \to \mathcal{S}(\mathbb{R}^2)$$

be the map which sends a k–tuple of elements $(s_1, \dots, s_k)$ to $S(\mathbb{R}^2)$ by immersing the collection of annuli containing the skeins in the plane such that the over- and under-crossings of D are the over- and under-crossings of the annuli. The n^{th} *unreduced colored Jones polynomial* $J_K(q;n)$ may be defined as

$$J_K(q;n) := ((-1)^n q^{\frac{1}{2}(n^2+2n)})^{\omega(D)} \Big\langle f_D \underbrace{\Big(\boxed{n}, \boxed{n}, \dots, \boxed{n}\Big)}_{k \text{ times}} \Big\rangle.$$

See the right-hand side of Figure 6 for an example where D has 2 components. The Kauffman bracket here is extended by linearity and gives the polynomial multiplying the empty diagram after reducing the diagram via skein relations. Note that this gives $J_{\bigcirc}(q;n) = (-1)^n[n]$ as the normalization. To simplify notation we will denote the skein $f_D\big(\boxed{n}, \boxed{n}, \dots, \boxed{n}\big)$ from now on by $D^n_{\sqcup}$.

We now describe the categorification of the colored Jones polynomial, which is a homology theory whose Euler characteristic is equal to the polynomial. Here we summarize the description of Bar-Natan [3] and Rozansky [15]. The differences in conventions between those used in this paper and those authors are explained at the beginning of each section.

2.3 Khovanov bracket for tangles

We follow the approach of Bar-Natan [3] with a different grading convention: our graded category has grading in half-integers, and we do not consider complexes with bigrading shifted by the number of positive and negative crossings of the tangle. This results in a chain complex that is invariant up to a shift under a framing change which is specified in Section 2.6. For details, please consult [3].

We work in the additive category $\mathcal{TL}$ constructed from the category TL whose objects are finite sets of properly embedded arcs with no crossings and simple closed curves in $\mathcal{S}(D,n)$ for $n \in \mathbb{Z}_+$, considered up to boundary-preserving isotopy. For two such skeins in TL, a morphism between them is a cobordism which is a compact 2–manifold whose boundary is the disjoint union of the two skeins. The composition of morphisms is given by placing one cobordism on top of another in the obvious way. See [3] for the details of the construction from TL of $\mathcal{TL}$, where Bar-Natan shows that one can construct an additive category from any category. The process formally adds finite direct sums of $\mathbb{Z}$–linear combinations of objects and morphisms to TL in such a way that the composition of morphisms is bilinear.

From $\mathcal{TL}$ we consider the homotopy category of chain complexes over $\mathcal{TL}$ which are bounded from below, denoted by $\mathrm{Kom}^+(\mathcal{TL})$. Let $m \in \mathbb{Z}$ and let T_i be an object of $\mathcal{TL}$. An object of $\mathrm{Kom}^+(\mathcal{TL})$ is a chain complex $(\boldsymbol{T}, d_{\boldsymbol{T}})$,

$$\boldsymbol{T} = \cdots \to T_{i+1} \xrightarrow{d_{i+1}} T_i \xrightarrow{d_i} \cdots \to T_m, \tag{6}$$

considered up to homotopy equivalence $\sim$, and a morphism between two chain complexes is a chain map. We use the notation $\boldsymbol{T}_i := T_i$. We further mod out the set of morphisms $\mathrm{Kom}^+(\mathcal{TL})$ by the S, T and $4Tu$ relations, and consider as in [3] the *graded* category $\mathrm{Kob}(\mathcal{TL})$ where grading shifts of complexes are induced by cobordisms.

To an (n,n)–tangle T with crossings in $\mathcal{S}(D,n)$ is then associated a chain complex $(\boldsymbol{T}, d_{\boldsymbol{T}})$, where the $\boldsymbol{T}_i$ are smoothings of T and the boundary map $d_{\boldsymbol{T}}$ comes from the saddle cobordism

from the B–resolution of the crossing  to the A–resolution. For a tangle $T \in \mathrm{TL}$ we will often represent the associated complex $\boldsymbol{T}$ in $\mathrm{Kob}(\mathcal{TL})$ by a picture of the tangle itself. We may compose the complexes associated to tangles in the fashion of

planar algebras as also detailed in [3]. After the application of a TQFT this gives the Khovanov homology of tangles. We specify our grading conventions in Section 2.4.

2.4 Notation and grading for the Khovanov bracket

Our grading conventions are adapted from [15] with a minor change of variables. The difference with Rozansky's convention in [15] is that his q there is our q^{-1} and his h there is our h^{-1}.

Let A be a bigraded chain complex $(A_{i,j}, d_A)$, and let $A[1]$ the chain complex whose homological grading i is shifted by 1 with differential $-d_A$, so $A[1]_i = A_{i-1}$, while $A\{1\}$ has the j–grading shifted by 1. Following Rozansky's convention in using a nonstandard notation for the grading shift, we will also use the notation hA for $A[1]$, with

$$h^k A = A[k],$$

and qA for $A\{1\}$. This will completely specify the changes to the bigrading in what follows. Our grading convention is determined by the following presentation of the Khovanov bracket. Here $h^{\pm\frac{1}{2}}$ simply shifts the i–degree of the complex by $\pm\frac{1}{2}$.

(i) $\bigcirc = q\,\boxed{} + q^{-1}\,\boxed{}$, where $\boxed{}$ is the empty skein, and

(ii) $\times = \mathrm{Cone}\big(h^{-\frac{1}{2}})\,(\xrightarrow{s} h^{-\frac{1}{2}}\smile\big) = \boxed{h^{\frac{1}{2}})\,(\to h^{-\frac{1}{2}}\smile}$, where s is the saddle cobordism. Note that $\deg_h(s) = -1$.

The notation $\mathrm{Cone}(A \xrightarrow{f} B)$ for two chain complexes (A, d_A) and (B, d_B) indicates the complex of the mapping cone $hA \oplus B$ with differential

$$d_f = \begin{bmatrix} -d_A & 0 \\ f & d_B \end{bmatrix}.$$

We will also use the following notation for the mapping cone (the map f will be suppressed whenever it is clear):

$$\boxed{hA \to B} \;=\; \mathrm{Cone}(A \xrightarrow{f} B)$$

2.5 The chain complex for the Jones–Wenzl projector

Rozansky constructs a chain complex in $\mathrm{Kob}(\mathcal{TL})$, denoted here by $\overset{n}{\sqcap}$, which satisfies the following universal properties.

Definition 2.3 Given a chain complex $\boldsymbol{T}\colon \cdots \to T_{i+1} \to T_i \to \cdots \to T_m$ in $\mathrm{Kob}(\mathcal{TL})$, the *minimum homological degree* of $\boldsymbol{T}$, denoted by $|\boldsymbol{T}|$, is m.

Theorem 2.4 [14, Theorem 2.7] *Let e_n^i be a generator of TL_n. The complex $\dot{+}_n$ has the following properties:*

(i) *The complex of a composition of $\dot{+}_n$ with e_n^i for all $i = 1, \dots, n-1$ is contractible, ie*

$$e_n^i \circ \dot{+}_n \sim \dot{+}_n \circ e_n^i \sim 0.$$

(ii) *The complex $\dot{+}_n$ is idempotent with respect to tangle composition, ie*

$$\dot{+}_n \circ \dot{+}_n \sim \dot{+}_n.$$

(iii) *Let $\gamma \in \mathrm{TL}_n$ be an (n,n)–tangle in the disk D with n marked points above and below. A through strand is an arc joining a point on the upper edge to a point on the lower edge. Let $|\gamma|$ denote the number of through strands of γ. The complex $\dot{+}_n$ has a presentation*

$$\dot{+}_n = \mathrm{Cone}(\dot{+}_n \xrightarrow{f} |_n),$$

where

$$\dot{+}_n = \cdots \to \boldsymbol{h}^i \bigoplus_{0 \le j \le i, \gamma \in \mathrm{TL}_n, |\gamma| < n} \mu_{ij,\gamma} \boldsymbol{q}^j \gamma \to \cdots, \tag{7}$$

and where $\gamma \in \mathrm{TL}_n$ has $|\gamma| < n$ and $|\dot{+}_n| = 0$.

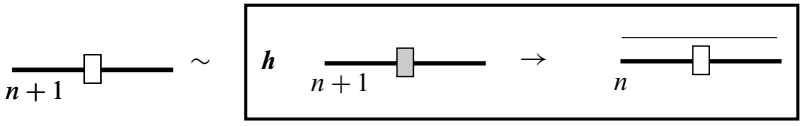
The reader may refer to [14] for the details of constructing $\dot{+}_n$ via the chain complexes for torus braids. We shall not use the construction in detail. We need only that the complex for the projector satisfies the universal properties listed in Theorem 2.4 . There are two mutually dual categorifications from this approach. For this paper we are using the complex whose homological grading is bounded from below.

We will also make use of the categorified versions of (4)–(5) proven in [15], which replaces tangles by their categorification complexes and equality by homotopy equivalence.

Lemma 2.5 [15, Proposition 3.6]



Lemma 2.6 [15, Theorem 3.8] *The $n+1$ strand categorified Jones–Wenzl projector has the mapping cone presentation*



2.6 Link homology, Euler characteristic and the colored Jones polynomial

The chain complex in $\mathrm{Kob}(\mathcal{TL})$, to which a 1+1 TQFT is applied to obtain the homology groups categorifying the n^{th} colored Jones polynomial, is obtained by composing each n–cabled link component from the complex of the tangle T^n, with the complex of the n^{th} Jones–Wenzl projector $\overset{n}{\sqcap\!\sqcup}$ as shown in Figure 6.

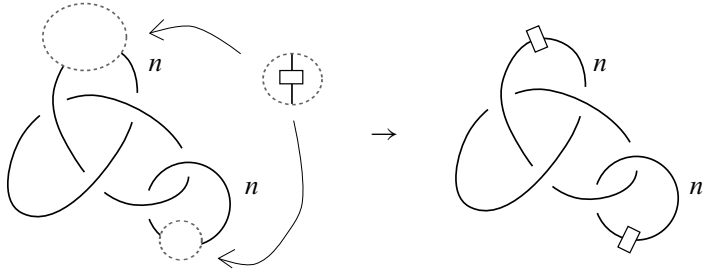
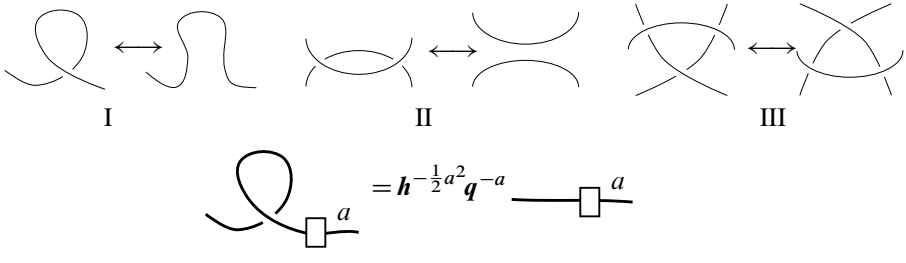


Figure 6

The homology thus defined is invariant under Reidemeister moves of type II and III, shown below, but has a degree shift under the Reidemeister move of type I:



We will not use the exact definition of a TQFT that will give the desired homology groups. The reader may consult [3, Definition 7.1] for the precise definition of such a TQFT. We denote the resulting complex by $[\![D^n_{\sqcap\!\sqcup}]\!]$, the underlying bigraded chain groups by $\{C^{\mathrm{Kh}}_{i,j}(D,n)\}$, and the resulting homology groups by $\{H^{\mathrm{Kh}}_{i,j}(D,n)\}$. From

Definition 2.2 of the colored Jones polynomial it is not hard to see that

$$J_K(q;n) = ((-1)^n q^{\frac{1}{2}(n^2+2n)})^{\omega(D)} \sum_{i,j} (-1)^j q^{i+j} \dim H^{\mathrm{Kh}}_{i,j}(D,n). \tag{9}$$

Note that the presence of the projector means that we are allowing infinite complexes bounded from below.

In general, for any skein $\mathcal{S}$ in $\mathcal{S}(\mathbb{R}^2)$ which may be decorated by the Jones–Wenzl projector, we denote the chain complex under the TQFT by $[\![\mathcal{S}]\!]$ with underlying chain groups $\{C^{\mathrm{Kh}}_{i,j}(\mathcal{S})\}$ and resulting homology groups $\{H^{\mathrm{Kh}}_{i,j}(\mathcal{S})\}$. In a figure we will often depict the complex $[\![\mathcal{S}]\!]$ by just the skein $\mathcal{S}$ without the double brackets.

3 The tail homology of the colored Jones polynomial

This section contains an exposition of Rozansky's work, where the results are taken directly, or summarized, from [15], with obvious adjustments for the change of variables. The goal is to give a precise definition of the tail homology which he has constructed.

3.1 Shifted homology

Let D be a link diagram in $\mathcal{S}(\mathbb{R}^2)$ decorated by Jones–Wenzl projectors; Rozansky defines a *shifted homology* in [15] which we write here with a different notation for the degree shifts.

Definition 3.1 $$\widetilde{H}^{\mathrm{Kh}}(D,n) := \boldsymbol{h}^{\frac{1}{2}n^2c(D)}\boldsymbol{q}^{n|s_A(D)|}H^{\mathrm{Kh}}(D,n).$$

This is identical to his definition in [15],

$$\widetilde{H}^{\mathrm{Kh}}(D_n) := \boldsymbol{h}^{\frac{1}{2}n_\times(D_n)}\boldsymbol{q}^{n_\circ(D_n)}H^{\mathrm{Kh}}(D_n),$$

where for him D_n is the n–blackboard cable of D decorated by the Jones–Wenzl projector. The number $n_\times(D_n)$, which is the total number of single line crossings in D_n (as he defines in [15]), is the same as $n^2c(D)$. The quantity $n_\circ(D_n)$, which he defines as the total number of circles resulting from replacing the Jones–Wenzl projectors with identity braids and performing the B–splicings on all crossings, is also the same as $n|s_B(D)|$. Adjusting for our convention for the variable changes of $\boldsymbol{q}$ to $\boldsymbol{q}^{-1}$ and $\boldsymbol{h}$ to $\boldsymbol{h}^{-1}$ gives the corresponding quantities concerning the all-A Kauffman state.

3.2 Tail homology

Rozansky's main result in [15] is the following:

Theorem 3.2 [15, Theorem 2.13] *For any link diagram D there is a sequence of degree-preserving maps*

$$\widetilde{H}^{\mathrm{Kh}}(D,n) \xrightarrow{f_n} \widetilde{H}^{\mathrm{Kh}}(D,n+1), \tag{10}$$

which are isomorphisms on $\widetilde{H}^{\mathrm{Kh}}_{i,*}$ for $i \le n-1$.

This implies that the directed system formed by the maps f_n has a limit.

Definition 3.3 The *tail homology* $H^\infty(D)$ of a link diagram D is the limit of the direct system $\{\widetilde{H}^{\mathrm{Kh}}(D,n), f_n\}$ formed by the maps f_n of Theorem 3.2:

$$H^\infty(D) = \varinjlim \widetilde{H}^{\mathrm{Kh}}(D,n).$$

Corollary 3.4 *The i^{th} homology group of $H^\infty(D)$ is isomorphic to $\widetilde{H}^{\mathrm{Kh}}_{i,*}(D,i+1)$.*

In addition, Rozansky proves the following bounds on shifted homology. Note again that what he means by a "uni-colored diagram D_n" is the same as $D^n_{\Yup}$ considered in this paper.

Theorem 3.5 [15, Theorem 2.12] *The shifted homology has a bound: $\widetilde{H}^{\mathrm{Kh}}_{i,j}(D,n)=0$ if*

$$j < -\tfrac{1}{2}i - \tfrac{1}{2}c(D) - \tfrac{3}{2}c(D)^\ell \quad \text{or} \quad j < -i - c(D)^\ell,$$

where $c(D)^\ell$ is the number of crossings in D whose corresponding segments in the all-A state are one-edged loops. Moreover, if D is A–adequate, then

$$\dim \widetilde{H}^{\mathrm{Kh}}_{i,-i} = \begin{cases} 0 & \text{if } i>0, \\ 1 & \text{if } i=0. \end{cases}$$

The graded Euler characteristic of the tail homology is

$$J_{D,\infty}(q) = \sum_{i,j} (-1)^j q^{i+j} \dim H^\infty_{i,j}(D).$$

Because of the homological bounds of Theorem 3.5, $J_{D,\infty}(q)$ is well-defined.

Suppose K is A–adequate. Then $H^{\infty}(D)$ is independent of the A–adequate diagram chosen, and so is $J_{D,\infty}(q)$. Let $c(K)$ be the minimal crossing number among A–adequate diagrams representing K, then the inequalities of Theorem 3.5 translate to

$$j < -\tfrac{1}{2}i - \tfrac{1}{2}c(K) \quad \text{and} \quad j < -i.$$

In particular, let D be an A–adequate diagram of K. The homology group $\widetilde{H}^{\mathrm{Kh}}_{i,j}(D,n)$ equals 0 if

$$i + j < \tfrac{1}{2}i - \tfrac{1}{2}c(K).$$

In view of (9), let $n \gg c(K)$. If $i \geq n$, then $\widetilde{H}^{\mathrm{Kh}}_{i,j}(D,n) = 0$ for D if $i + j < \frac{1}{2}n - \frac{1}{2}c(K)$. Therefore, the terms of the graded Euler characteristic,

$$\sum_{i,j} (-1)^j q^{i+j} \dim \widetilde{H}^{\mathrm{Kh}}_{i,j}(D,n),$$

where $i + j < \frac{1}{2}n - \frac{1}{2}c(K)$ are determined by $J_{D,\infty}(q)$ up to multiplication by a power of q. This is the following corollary from [15], which we adjust for unframed links by multiplying by a power of q.

Theorem 3.6 [15, Theorem 2.6] *The graded Euler characteristic of the tail homology determines the lower powers of q in the colored Jones polynomial of an A–adequate link:*

$$J_K(q;n) = \big((-1)^n q^{\frac{1}{2}(n^2+2n)}\big)^{\omega(D)} q^{-\frac{1}{2}n^2 c(D) - n|s_A(D)|} \big(J_{D,\infty}(q) + O(q^{\frac{1}{2}n - \frac{1}{2}c(K)})\big).$$

It is in this sense that the tail homology categorifies the tail of the colored Jones polynomial when the link is A–adequate.

4 Proof of Conjecture 1.4

A substantial portion of our proof relies on interpreting and adapting the content of Rozansky's arguments [15], which we will indicate and give original references whenever appropriate.

We consider the following combinatorial data of a skein $\mathcal{S} \in \mathrm{TL}$ with crossings which may or may not be decorated by Jones–Wenzl projectors. We also consider a more general Kauffman state σ, which is a choice of A– or B–resolution at a crossing as before, but restricted to a specified subset of the crossings of $\mathcal{S}$. It should be assumed that a Kauffman state is applied to all the crossings of $\mathcal{S}$ unless otherwise indicated.

- $C(\mathcal{S})$ is the set of crossings in $\mathcal{S}$, and let $c(\mathcal{S}) = |C(\mathcal{S})|$ be the number of crossings in $\mathcal{S}$.
- $\operatorname{sgn}(\sigma) := \operatorname{sgn}_{\boldsymbol{B}}(\sigma) - \operatorname{sgn}_{\boldsymbol{A}}(\sigma)$, where

$$\operatorname{sgn}_{\boldsymbol{A}}(\sigma) := \tfrac{1}{2}(\#\text{ of crossings where the } A\text{–resolution is chosen in } \sigma),$$
$$\operatorname{sgn}_{\boldsymbol{B}}(\sigma) := \tfrac{1}{2}(\#\text{ of crossings where the } B\text{–resolution is chosen in } \sigma).$$

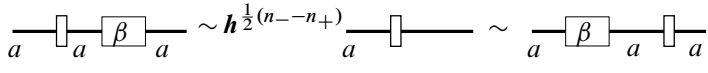
4.1 Tools for working with the chain complex

We will use the following lemmas. Recall from Definition 2.3 that $|\boldsymbol{T}|$ is the minimum homological degree of the complex $(\boldsymbol{T}, d_{\boldsymbol{T}})$.

Lemma 4.1 [15, Theorem 2.10] *Let $\mathcal{S}$ be a skein with crossings, which may include disjoint circles and may be decorated by Jones–Wenzl projectors. Let $c(\mathcal{S})$ be the total number of crossings of $\mathcal{S}$. The minimum homological degree of the complex $[\![\mathcal{S}]\!]$ is bounded below by $-\frac{1}{2}c(\mathcal{S})$, ie*

$$|[\![\mathcal{S}]\!]| \geq -\tfrac{1}{2}c(\mathcal{S}).$$

Theorem 4.2 [15, Theorem 3.3] *Let β be a braid as shown below, with the braid strands going from left to right. Let a crossing of β be positive or negative depending on whether it is $\times$ or $\times$, respectively. We have the homotopy equivalence*



where n_+ (resp. n_-) is the number of positive (resp. negative) crossings of the braid.

Lemma 4.3 *If a chain complex $\boldsymbol{T}$ has the mapping cone presentation*

$$\boldsymbol{T} \sim \operatorname{Cone}(\boldsymbol{A} \xrightarrow{f} \operatorname{Cone}(\boldsymbol{B} \xrightarrow{g} \boldsymbol{C})),$$

then it also admits the mapping cone presentation

$$\boldsymbol{T} \sim \operatorname{Cone}\big(\operatorname{Cone}(\boldsymbol{A}[-1] \xrightarrow{f_{\boldsymbol{B}}} \boldsymbol{B}) \xrightarrow{f_{\boldsymbol{C}} \oplus g} \boldsymbol{C}\big),$$

where $f_{\boldsymbol{B}}$ is the component of f in $\boldsymbol{B}$ and $f_{\boldsymbol{C}}$ is the component in $\boldsymbol{C}$.

Proof This is immediate from writing out the chain complexes of the two presentations as direct sums and comparing the differentials. □

4.2 Decomposition over Kauffman states

Let D be a non-A–adequate link diagram, which means that its all-A state has a loop corresponding to a crossing ℓ. Let ℓ^{n+1} denote the $n+1$–cabled crossings corresponding to ℓ in D^{n+1}. Without loss of generality, we assume that the one-edged loop in $s_A(D)$ corresponding to ℓ lies in the bounded component of the state circle to which it is attached. Let $C(D)$ be the set of crossings of a diagram D; we consider Kauffman states on D^{n+1} restricted to the set of crossings $C(D^{n+1}) \setminus \ell^{n+1}$.

We have the skein exact sequence on homology groups from choosing the $A-$ or B–resolution at each crossing via the map

$$\boxed{[\![\times]\!] = \boldsymbol{h}^{\frac{1}{2}}[\!]\,(\,)[\!] \xrightarrow{s} \boldsymbol{h}^{-\frac{1}{2}}[\!]\,\asymp\,[\!]}.$$

Pick an order on the crossings in $C(D^{n+1})\setminus\ell^{n+1}$; we paste together the exact sequence for every crossing to obtain the presentation of $[\![D^{n+1}_{\sqcup}]\!]$ as an iterated mapping cone. Let $\mathcal{S}_\sigma$ be a skein with crossings and decorated by Jones–Wenzl projectors resulting from a state σ. The chain groups of $[\![D^{n+1}_{\sqcup}]\!]$ is a direct sum of the chain group of $[\![\mathcal{S}_\sigma]\!]$,

$$[\![D^{n+1}_{\sqcup}]\!] = \bigoplus_{\sigma \text{ on } C(D^{n+1})\setminus\ell^{n+1}} [\![\mathcal{S}_\sigma]\!].$$

Lemma 4.4 *The unshifted homology group $H^{\mathrm{Kh}}_{i,*}(D, n+1) = 0$ if $H^{\mathrm{Kh}}_{i-\mathrm{sgn}(\sigma),*}(\mathcal{S}_\sigma) = 0$ for all σ.*

Proof We have the long exact sequence

$$\cdots \to H^{\mathrm{Kh}}_{i-\frac{1}{2},j}(\,)\,(\,) \to H^{\mathrm{Kh}}_{i,j}(\times) \to H^{\mathrm{Kh}}_{i+\frac{1}{2},j}(\asymp) \to \cdots,$$

where $H^{\mathrm{Kh}}_{i,j}(D, n+1) = H^{\mathrm{Kh}}_{i,j}(\times)$ for a choice of a crossing in $C(D^{n+1}) \setminus \ell^{n+1}$. We have another long exact sequence with the remaining crossings with each of $H^{\mathrm{Kh}}_{i-\frac{1}{2},j}(\,)\,(\,)$ and $H^{\mathrm{Kh}}_{i+\frac{1}{2},j}(\asymp)$. We repeat this process, obtaining an exact sequence for every choice of $A-$ or B–resolution, until all the crossings in $C(D^{n+1}) \setminus \ell^{n+1}$ are resolved.

Each time a resolution is chosen, the homological degree of the constituent homology group in the exact sequence is shifted by opposite the sign of the choice of the resolution. If each of these homology groups is trivial, then the original homology group is trivial.

□

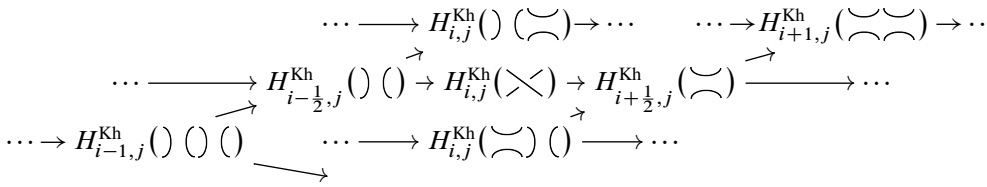


Figure 7: The pasted exact sequences from resolving two crossings

Lemma 4.5 *Let $D^{n+1}_{\sqcup}$ be the $n+1$–cable of a non-A–adequate link diagram D decorated by a Jones–Wenzl projector and let σ be a Kauffman state on $C(D^{n+1})\backslash \ell^{n+1}$ with ℓ^{n+1} the set of $n+1$–cabled crossings corresponding to the loop crossing ℓ of $s_A(D)$. The skein $\mathcal{S}_\sigma$ obtained by applying σ to $D^n_{\sqcup}$ contains a cap or a cup composed with a Jones–Wenzl projector unless it is isotopic to a disjoint union of a skein $\mathcal{S}^k_\sigma$ with circles, where $\mathcal{S}^k_\sigma$ has the form shown in Figure 8.*

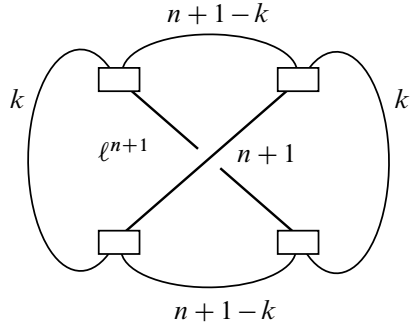
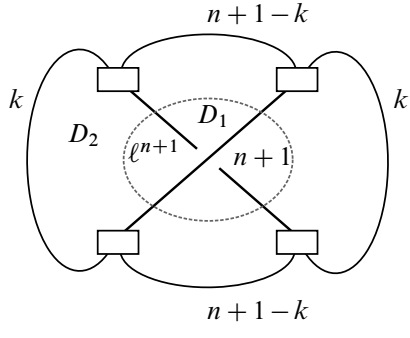


Figure 8: The skein $\mathcal{S}^k_\sigma$ with $0 \le k \le n+1$

Proof The skein $\mathcal{S}_\sigma$ may be written as a composition of the skeins of two disks $\mathcal{S}(D_1, 2(n+1))$ and $\mathcal{S}(D_2, 2(n+1))$, one containing the $n+1$–cabled crossing ℓ^{n+1} and the other containing nonintersecting strands between the projectors:



Configurations of these nonintersecting strands in D_2 are determined by the presence of projectors. A configuration containing a cap/cup composed with a projector will result in a homotopically trivial complex. It is straightforward to see that a nontrivial configuration has to have the form indicated in Figure 8 as follows. Here we flatten out the disk D_2 and denote the four projectors by tl (top-left), tr (top-right), ll (lower-left), and lr (lower-right) as shown in Figure 9.

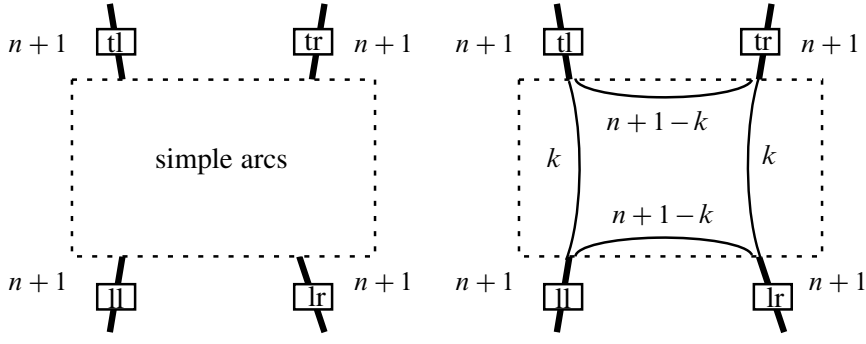


Figure 9

We first argue that, from the perspective shown for D_2 in Figure 9, we cannot have arcs connecting the tl projector to the lr projector, since that will necessarily result in a cap/cup composed with the tr or ll projector by comparing the number of strands to the available endpoints. This also uses the fact that the strands cannot intersect. There cannot be arcs connecting the tr projector to the ll projector for the same reason. Thus we can only have arcs between the tr and lr projectors, and arcs between the tl and ll projectors. The number of strands between each such pair is denoted by k and k', respectively. If $k \neq k'$ then again there is a cap/cup composed with a projector. Therefore, $k = k'$, and we come to the form in the lemma. □

4.3 Long exact sequences of homology from local transformations

A *local replacement*, as it is called in [15], is a triple $M = (\tau_i, \tau_c, \tau_f)$ of tangle diagrams decorated by the Jones–Wenzl projector. Let a triple of diagrams (D_i, D_c, D_f) differ locally in the skeins (τ_i, τ_c, τ_f).

Let $n_f = \frac{1}{2}c(D_f) - \frac{1}{2}c(D_i)$ and $m_f = |s_A(D_f)| - |s_A(D_i)|$. The quantity $|s_A(D_f)|$ is the number of circles obtained from taking the A–resolution on all of the crossings of D_f and replacing each projector by $|_n$. Similarly, we have $|s_A(D_i)|$. Let $\tau_f' =$

$\boldsymbol{h}^{n_f}\boldsymbol{q}^{m_f}\tau_f$. If there is a mapping cone presentation of τ_i

$$[\![\tau_i]\!] \sim \boxed{[\![\tau_c]\!] \xrightarrow{f} [\![\tau'_f]\!]} \tag{11}$$

then we also have the mapping cone presentation

$$[\![\tau_c]\!] = \boxed{\boldsymbol{h}[\![\tau'_f]\!] \xrightarrow{g} [\![\tau_i]\!]} \tag{12}$$

from the natural map $\tau'_f \xrightarrow{g} \tau_i$. Here we indicate explicitly the map f in the mapping cone presentation for $[\![\tau_i]\!]$. Otherwise, the notation agrees with that of [15, (3.1), page 554]. A *local transformation* is a local replacement (τ_i, τ_c, τ_f) along with a map $g\colon \tau'_f \to \tau_i$ specifying the mapping cone presentation (12).

With the natural map $\tau_i \to \tau_c$, this induces a long exact sequence of homology groups

$$\boldsymbol{h}^{-1}H^{\mathrm{Kh}}(D'_c) \xrightarrow{f} \widetilde{H}^{\mathrm{Kh}}(D_f) \xrightarrow{g} \widetilde{H}^{\mathrm{Kh}}(D_i) \to H^{\mathrm{Kh}}(D'_c), \tag{13}$$

where $H^{\mathrm{Kh}}(D'_c) = \boldsymbol{h}^{\frac{1}{2}c(D_i)}\boldsymbol{q}^{|s_A(D_i)|}H^{\mathrm{Kh}}(D_c)$. The long exact sequence implies the following:

Theorem 4.6 [15, Proposition 3.1] *If $H^{\mathrm{Kh}}_{i,*}(D_c) = 0$ for $i \le m-1$, then the degree-preserving map*

$$\widetilde{H}^{\mathrm{Kh}}(D_f) \xrightarrow{g} \widetilde{H}^{\mathrm{Kh}}(D_i)$$

is an isomorphism on $\widetilde{H}^{\mathrm{Kh}}_{i,*}(D_f)$ for

$$i \le m + \tfrac{1}{2}c(D_i) - 2. \tag{14}$$

4.4 Local transformations

We exhibit three local transformations M_{I}, M_{II} and M_{III}, shown in Figures 11, 13 and 15, and we apply Theorem 4.6 to triples of diagrams corresponding to transformations. The local transformations are the same as those from [15] with obvious adjustments for the variable change in our conventions. However, we prove the statements for a different set of triples of diagrams. In particular, we consider triples of diagrams from the skein $\mathcal{S}^k_\sigma$ differing locally in transformations M_{I}, M_{II} and M_{III}. Since the state σ is not relevant for this part of the argument, we will suppress the notation indicating σ and just refer to $\mathcal{S}^k$ for Sections 4.4–4.8.

To specify the location where we apply the local transformation, we will label the projectors of $\mathcal{S}^k$ as shown in Figure 10.

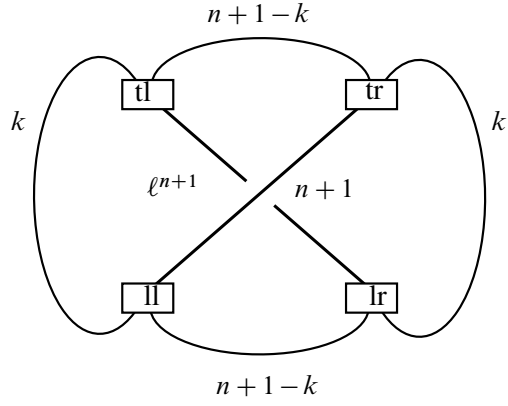


Figure 10: The top-left, top-right, lower-left and lower-right projectors

4.5 Local transformation I

Proposition 4.7 [15, Proposition 4.1] *Let $\mathcal{S}^k$ be a skein of the form in Figure 8. Consider the local transformation $M_{\mathrm{I}} = (\tau_i, \tau_c, \tau_f)$ on the $n+1$–cabled crossing in $\mathcal{S}^k$, with (D_i, D_c, D_f) as shown in Figure 12.*

Then:

(a) *There is a map f_{I} such that we have a mapping cone presentation*

$$\tau_i = \boxed{\tau_c \xrightarrow{f_{\mathrm{I}}} \tau_f'},$$

where $\tau_f' = \boldsymbol{h}^{-n-\frac{1}{2}}\tau_f$.

(b) *The induced degree-preserving map*

$$\widetilde{H}^{\mathrm{Kh}}(D_f) \xrightarrow{g_{\mathrm{I}}} \widetilde{H}^{\mathrm{Kh}}(D_i)$$

is an isomorphism on $\widetilde{H}^{\mathrm{Kh}}_{i,*}$ for $i \le 2n-1$.

Proof (a) The map inducing the mapping cone presentation is the saddle cobordism

$$\times = \mathrm{Cone}\Big(\boldsymbol{h}^{-\frac{1}{2}}\ )\ (\ \xrightarrow{s}\ \boldsymbol{h}^{-\frac{1}{2}}\ \smile\Big).$$

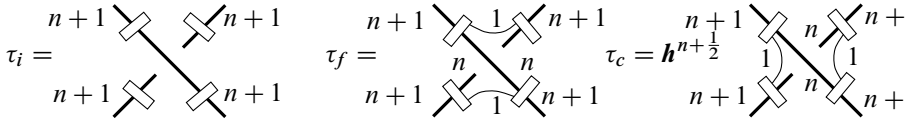


Figure 11: Local transformation M_{I}. This is a version of the skein relation in Definition 2.1; see also [17].

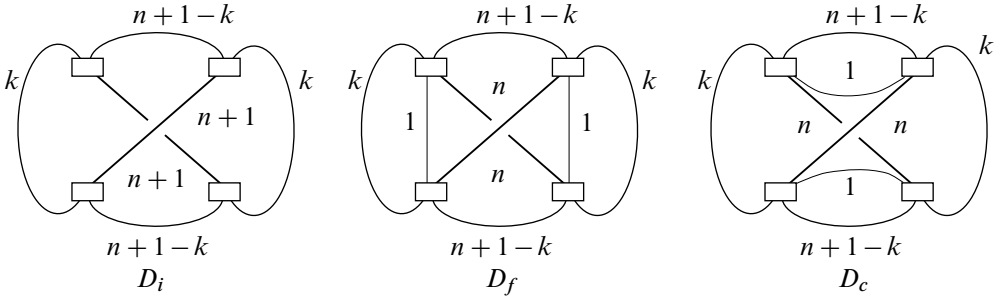


Figure 12

The argument follows from that of [15, Proposition 4.1] without further adjustment except for the sign of the crossing, since Rozansky's arguments are completely local.

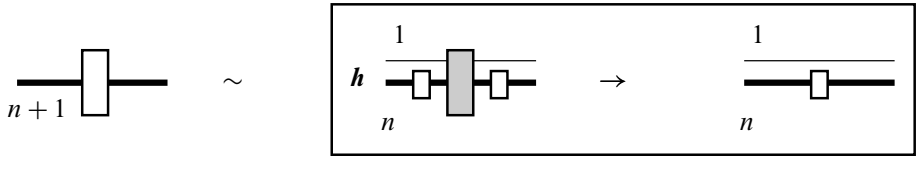
(b) By Theorem 4.6, we need to show that $H^{\mathrm{Kh}}_{i,*}(D_c) = 0$ for $i \le 2n - \frac{1}{2}c(D_i)$. Note that $c(D_c) = c(D_i) - 2n - 1$. By Lemma 4.1, $H^{\mathrm{Kh}}_{i,*}(D_c) = 0$ for

$$i < -\tfrac{1}{2}c(D_c) + \underbrace{n + \tfrac{1}{2}}_{\substack{\text{the } \boldsymbol{h}^{n+1/2} \\ \text{shifting on } D_c}} = -\tfrac{1}{2}c(D_i) + 2n + 1. \qquad \square$$

4.6 Local transformation II

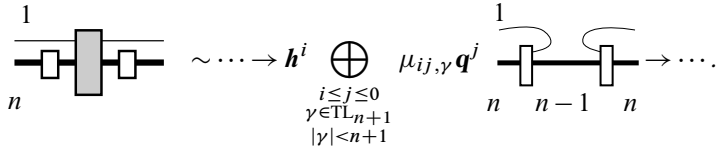
Before showing the second local transformation we will use yet another mapping cone presentation for the Jones–Wenzl projector. The arguments in [15] are again completely local, so we will refer the reader to [15] for the proofs.

Theorem 4.8 [15, Theorem 3.8] *The Jones–Wenzl projector has the mapping cone presentation*



where the first complex of the mapping cone is homotopic to a complex in which each individual term has the form indicated in the right-hand side of

(15)



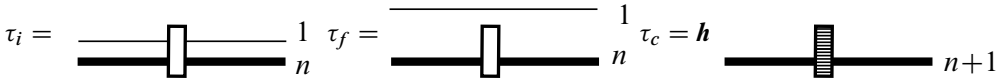


Figure 13: Local transformation M_{II}. The complex $[\![\tau_c]\!]$ comes from the first complex of the mapping cone presentation in Theorem 4.8.

We would just like to remark that in Rozansky's proof, the map giving the mapping cone presentation comes from Lemmas 2.5 and 2.6.

Let



be a shorthand notation for the complex of the right-hand side of the homotopy equivalence (15); we can now present local transformation II.

Proposition 4.9 [15, Proposition 4.2] *Let $\mathcal{S}_{\mathrm{I}}^k$ be the skein obtained from $\mathcal{S}^k$ by an application of the local transformation I to the $n+1$–cabled crossing as in Proposition 4.7. Pick the top-right projector* (recall the labeling in the caption of Figure 10) *and consider the triple of local transformation $M_{\mathrm{II}} = (\tau_i, \tau_c, \tau_f)$ applied to the chosen projector of $\mathcal{S}_{\mathrm{I}}^k$, with (D_i, D_c, D_f) as shown in Figure 14.*

Then:

(a) *There is a map f_{II} such that we have a mapping cone presentation*

$$\tau_i = \boxed{\tau_c \xrightarrow{f_{\mathrm{II}}} \tau_f'},$$

where $\tau_f' = \tau_f$.

(b) *The induced degree-preserving map*

$$\widetilde{H}^{\mathrm{Kh}}(D_f) \xrightarrow{g_{\mathrm{II}}} \widetilde{H}^{\mathrm{Kh}}(D_i)$$

is an isomorphism on $\widetilde{H}^{\mathrm{Kh}}_{i,*}$ for $i \leq n-1$.

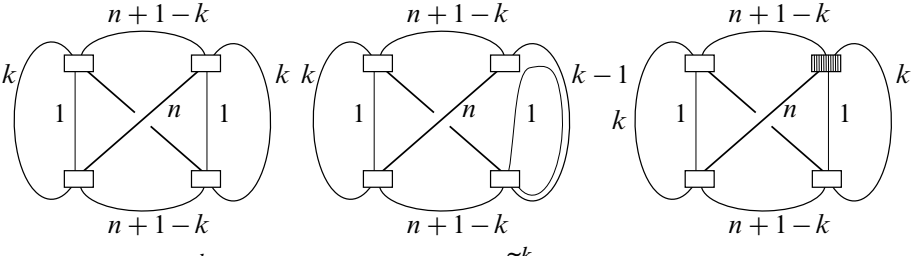
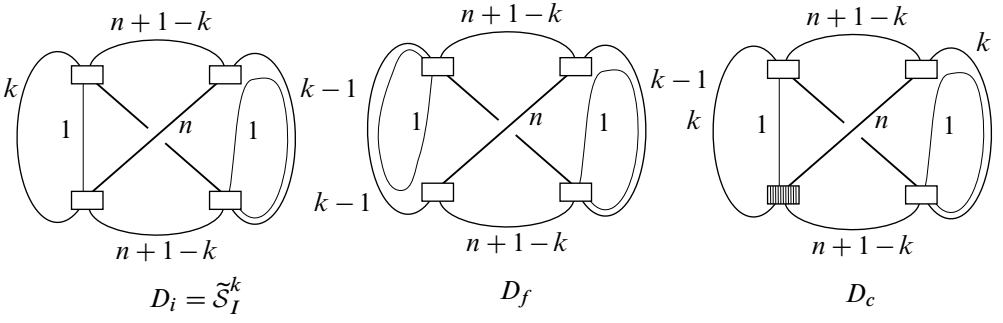


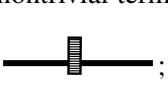
Figure 14

Let $\widetilde{\mathcal{S}}^k_{\mathrm{I}}$ be the skein D_f, and now let $D_i = \widetilde{\mathcal{S}}^k_{\mathrm{I}}$; we have the same statement for the triple $M_{\mathrm{II}} = (\tau_i, \tau_c, \tau_f)$ applied to the lower-left projector of $\widetilde{\mathcal{S}}^k_{\mathrm{I}}$:

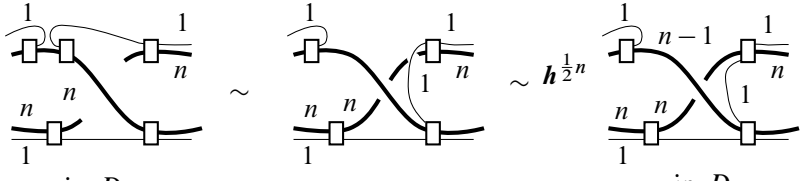


Proof (a) We have the desired mapping cone presentation by Theorem 4.8. Note that $c(D_f) = c(D_i)$ and $|s_A(D_f)| = |s_A(D_i)|$.

(b) As in Rozansky's proof, there is the following equivalence by sliding a single strand along the projector for each nontrivial term in the complex



the last step comes from Theorem 4.2:



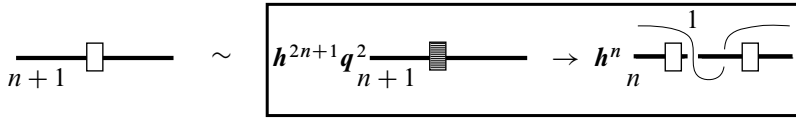
The number of crossings is reduced in the final chain complex, say D_e, by n, therefore $c(D_e) = c(D_i) - n$. Lemma 4.1 says that $|[\![D_e]\!]| \geq -\frac{1}{2}c(D_e)$. Therefore, $H^{\mathrm{Kh}}_{i,*}(D_c) = 0$ for $i \leq -\frac{1}{2}c(D_e) - 1 + 1 = -\frac{1}{2}c(D_i) + n$, where we also account for the degree shift of $\boldsymbol{h}^{\frac{1}{2}n}$ on D_e and the shift of $\boldsymbol{h}$ on D_c. This implies by Theorem 4.6, that $\widetilde{H}^{\mathrm{Kh}}(D_f) \xrightarrow{g_{\mathrm{II}}} \widetilde{H}^{\mathrm{Kh}}(D_i)$ is an isomorphism for $i < n+1-2 = n-1$. Note that we do not need to consider other possible intersections of the single strand with the rest of the diagram as Rozansky does in his proof, since in the skein $\mathcal{S}^k$ there is only one $n+1$–cabled crossing.

For the last statement of the lemma, concerning $\widetilde{\mathcal{S}}^k_{\mathrm{I}}$, we apply the exact same steps to the lower-left projector of $\widetilde{\mathcal{S}}^k_{\mathrm{I}}$. □

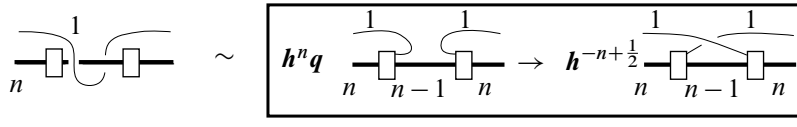
4.7 Local transformation III

We first expand upon the mapping cone presentations of the Jones–Wenzl projector. The next few statements are taken directly from [15] with minor adjustments for our convention in crossing changes and reorganized. The proofs again are completely local; therefore, the reader can consult [15] for details.

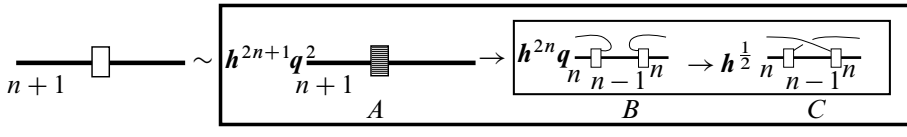
Theorem 4.10 [15, Theorem 3.9] *There is a homotopy equivalence*

(16) 

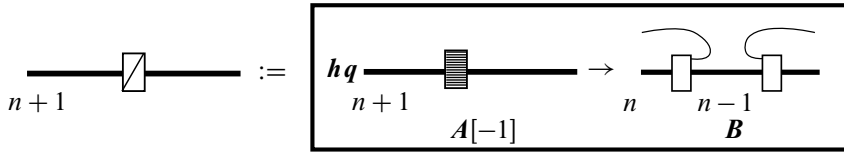
Lemma 4.11 [15, Lemma 3.12] *There is a homotopy equivalence*



Combining Theorem 4.10 and Lemma 4.11 gives the homotopy equivalence

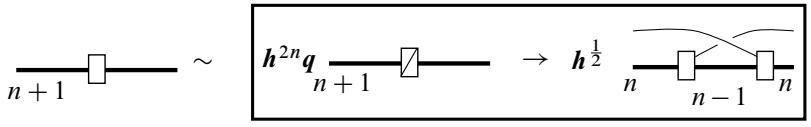
(17) 

Let $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ denote the chain complexes in the above mapping cone presentation. By Lemma 4.3, if we let



be the complex $\mathrm{Cone}(A[-1]\to B)$ resulting from writing the mapping cone presentation $\mathrm{Cone}(A\to \mathrm{Cone}(B\to C))$ of (17) as $\mathrm{Cone}(\mathrm{Cone}(A[-1]\to B)\to C)$, then we have the following theorem.

Theorem 4.12 [15, Theorem 3.11] *The Jones–Wenzl projector has the mapping cone presentation*

(18) 

Now we can finally specify local transformation III.

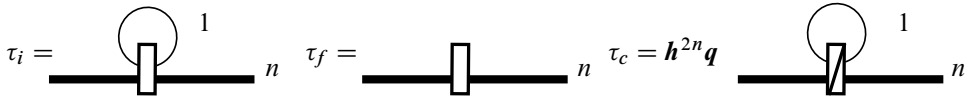


Figure 15: Local transformation M_{III}

Proposition 4.13 [15, Proposition 4.5] *Let $\mathcal{S}^k_{\mathrm{II}}$ be the skein obtained from $\mathcal{S}_{\mathrm{I}}$ by two applications of the local transformation M_{II} as in Proposition 4.9, where there is a single line circle attached to the lower-right projector and a single line circle attached to the upper-left projector, with (D_i, D_c, D_f) as shown in Figure 16.*

Then:

(a) *There is a map f_{III} such that we have a mapping cone presentation*

$$\tau_i = \boxed{\tau_c \xrightarrow{f_{\mathrm{III}}} \tau'_f},$$

where $\tau'_f \sim \boldsymbol{q}^{-1}\tau_f$.

(b) *The induced degree-preserving map*

$$\widetilde{H}^{\mathrm{Kh}}(D_f) \xrightarrow{g_{\mathrm{III}}} \widetilde{H}^{\mathrm{Kh}}(D_i)$$

is an isomorphism on $\widetilde{H}^{\mathrm{Kh}}_{i,*}$ for $i \le 2n-2$.

Let $\widetilde{\mathcal{S}}^k_{\mathrm{II}}$ be the skein D_f, and now let $D_i = \widetilde{\mathcal{S}}^k_{\mathrm{II}}$; we have the same statement for the triple $M_{\mathrm{III}} = (\tau_i, \tau_f, \tau_c)$ applied to the upper-left projector with the remaining single

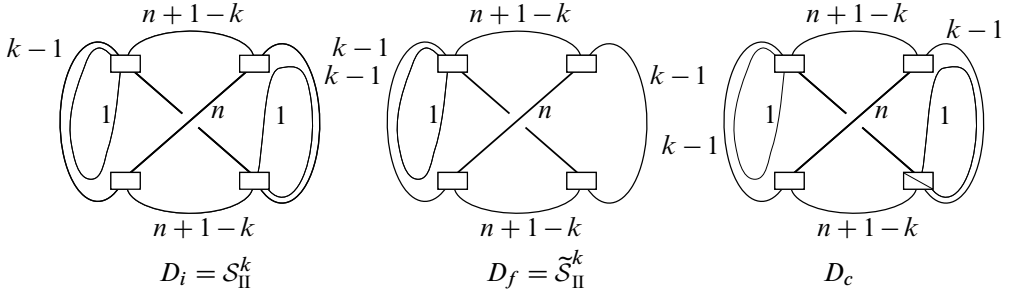
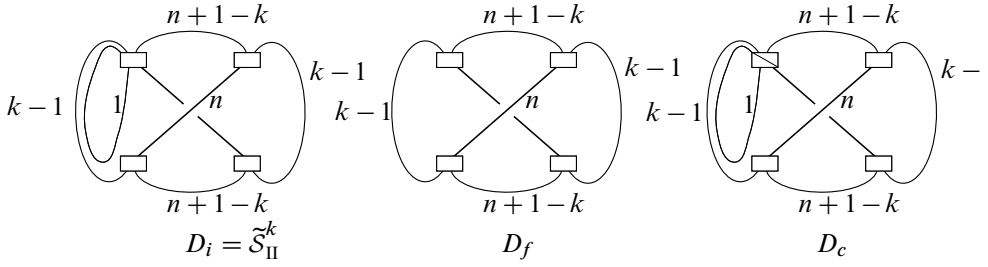
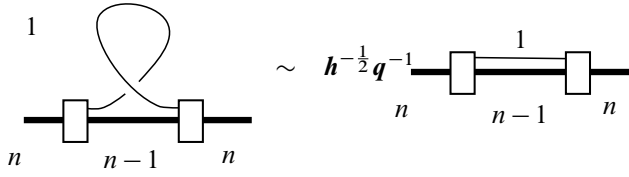


Figure 16

line circle attached:



Proof (a) From Theorem 4.12 we perform a type I Reidemeister move to complete the proof of part (a):



Note that $c(D_i) = c(D_f)$ and $|s_A(D_i)| - 1 = |s_A(D_f)|$.

(b) It is clear that D_c has the same number of crossings as D_i, so

$$|[\![D_c]\!]| \geq 2n - \tfrac{1}{2}c(D_c) = 2n - \tfrac{1}{2}c(D_i),$$

taking into account the degree shift $\boldsymbol{h}^{2n}\boldsymbol{q}$. Then, since $H^{\mathrm{Kh}}_{i,*}(D_c) = 0$ for $i < 2n - \frac{1}{2}c(D_i)$, Theorem 4.6 says that the map g_{III} is an isomorphism on $\widetilde{H}^{\mathrm{Kh}}_{i,*}(D_f)$ for $i \leq 2n - \frac{1}{2}c(D_i) + \frac{1}{2}c(D_i) - 2 = 2n - 2$. □

4.8 Simplification of $\mathcal{S}^k$ via local transformations

We apply the machinery set up in the previous section to understand the first few homology groups of the complex $[\![\mathcal{S}^k]\!]$. The goal is to repeatedly apply local transformations M_{I}, M_{II} and M_{III} until we get to the cabled unknot.

Let $\mathcal{S} \in \mathcal{S}(\mathbb{R}^2)$ be a skein decorated by Jones–Wenzl projectors, and let $\bar{\mathcal{S}}$ be the skein obtained by replacing the Jones–Wenzl projectors appearing in $\mathcal{S}$ by the identity. The shifted homology of $\mathcal{S}$ is defined as

$$\widetilde{H}^{\mathrm{Kh}}(\mathcal{S}) = \boldsymbol{h}^{\frac{1}{2}c(\mathcal{S})}\boldsymbol{q}^{|s_A(\bar{\mathcal{S}})|}H^{\mathrm{Kh}}(\mathcal{S}).$$

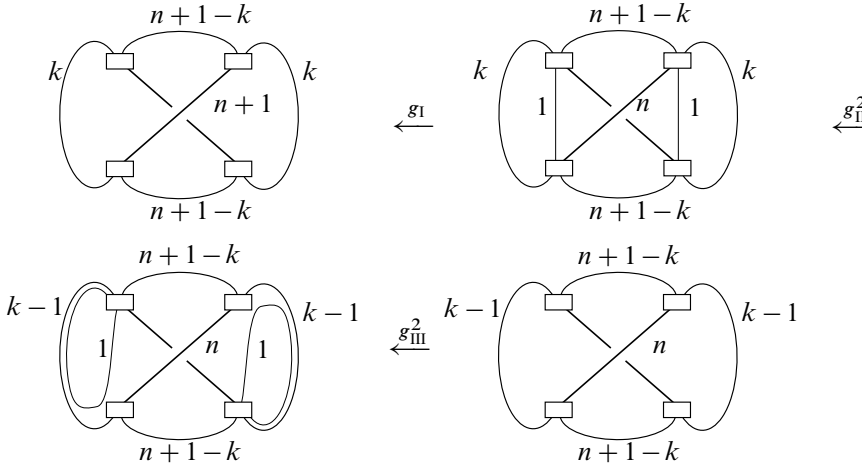


Figure 17: The sequence of local transformations reducing k

Theorem 4.14 *Let U^{n+1-k} be the $(n+1-k)$–cabled diagram of the standard diagram of the unknot with a left-hand twist, denoted by U, decorated by a Jones–Wenzl projector; see Figure 18. There is a degree-preserving map*

$$\widetilde{[\![\mathcal{S}^k]\!]} \to \widetilde{[\![U^{n+1-k}]\!]},$$

which is an isomorphism for $0 \le i \le n-k$.

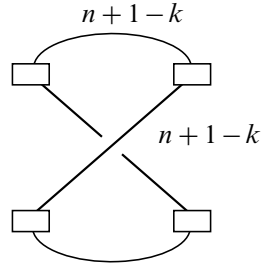


Figure 18: U^{n+1-k}

Proof There is nothing to prove if $k=0$, since then $\mathcal{S}^0 = U^{n+1}$. For $k \ge 1$, as shown in Figure 17, we may apply one M_{I} move, then two M_{II} moves, followed by two M_{III} moves to $\mathcal{S}^k$, to obtain a skein with k reduced to $k-1$ and an n–cabled crossing. The composition of the induced degree-preserving map $g_n = g_{\mathrm{I}} \circ g_{\mathrm{II}}^2 \circ g_{\mathrm{III}}^2$ is then an isomorphism for $i \le n-1$ by Theorem 4.6. We apply the same sequence of steps to the resulting skein in order to reduce k. We similarly obtain a degree-preserving map

g_{n-1} by composition, except that this is now an isomorphism for $i < n-2$. Therefore, if we apply the sequence of steps k times, we end up with a degree-preserving map g_{n-k} which is an isomorphism for $0 \le i \le n-k$. □

4.9 Trivial homology of the unknot

Lemma 4.15 *Let $0 \le k < n+1$. The homology group $\widetilde{H}^{\mathrm{Kh}}_{i,*}(U^{n+1-k}) = 0$ whenever $0 \le i \le n-k$.*

Proof Since we just have U^{n+1-k} with a full left-hand twist, we can remove the twist with a degree shift of $\boldsymbol{h}^{\frac{1}{2}a^2}\boldsymbol{q}^a$ for $a = n+1-k$. Note that $\frac{1}{2}a^2 > a-1$ for $a > 0$. □

4.10 Putting everything together

By Corollary 3.4, it suffices to show that the homology group $\widetilde{H}^{\mathrm{Kh}}_{n,*}(D, n+1)$ is trivial. We will show that $\widetilde{H}^{\mathrm{Kh}}_{i,*}(D, n+1)$ is trivial for $0 \le i \le n$.

Theorem 4.16 *The homology groups $\widetilde{H}^{\mathrm{Kh}}_{i,*}(D, n+1)$ are trivial for $i \le n$.*

Proof Recall that D is a non-A–adequate diagram and ℓ is a crossing in D whose corresponding segment in the all-A state is a one-edged loop. We may decompose the complex $[\![D^{n+1}_{\boxplus}]\!]$ over Kauffman states restricted to the set of crossings $C(D^{n+1}) \setminus \ell^{n+1}$, where ℓ^{n+1} is the set of $(n+1)^2$ crossings corresponding to ℓ in D^{n+1}. On the level of chain groups, $C^{\mathrm{Kh}}(D, n+1)$ may be written as a direct sum of the chain groups $C^{\mathrm{Kh}}(\mathcal{S}_\sigma)$ with the appropriate grading shifts:

$$C^{\mathrm{Kh}}(D, n+1) = \bigoplus_{\sigma \text{ a Kauffman state}} \boldsymbol{h}^{\mathrm{sgn}(\sigma)} C^{\mathrm{Kh}}(\mathcal{S}_\sigma),$$

and by Lemma 4.5, $\mathcal{S}_\sigma$ is a disjoint union of $\mathcal{S}^k_\sigma$ with circles, so

$$C^{\mathrm{Kh}}(D, n+1) = \bigoplus_{\sigma \text{ a Kauffman state}} \boldsymbol{h}^{\mathrm{sgn}(\sigma)} C^{\mathrm{Kh}}(\mathcal{S}^k_\sigma \sqcup \text{disjoint circles}).$$

Let σ_A be the all-A state which chooses the A–resolution on every crossing in $C(D^{n+1}) \setminus \ell^{n+1}$, and let $c = \mathrm{sgn}(\sigma) - \mathrm{sgn}(\sigma_A)$. Because $\mathcal{S}_\sigma = \mathcal{S}^k_\sigma \sqcup$ disjoint circles, if $\widetilde{H}^{\mathrm{Kh}}_{i-c,*}(\mathcal{S}^k_\sigma) = 0$, then $\widetilde{H}^{\mathrm{Kh}}(\mathcal{S}_\sigma) = 0$, which implies $\widetilde{H}^{\mathrm{Kh}}_{i,*}(D, n+1) = 0$ by Lemma 4.4. Thus it suffices to show for any σ that

$$\widetilde{H}^{\mathrm{Kh}}_{i-c,*}(\mathcal{S}^k_\sigma) = 0 \quad \text{for } i \le n.$$

For a fixed σ, if $i-c<0$, then there is nothing to prove since the homology groups with negative i–grading all vanish in the shifted homology. So the remaining case is where $i-c\geq 0$. Theorem 4.16 will immediately follow from Theorem 4.14 and Lemma 4.15 as long as we can show $c\geq k$ for every σ.

That is the content of the following lemma, where we use the fact that ℓ is a crossing corresponding to a one-edged loop in the all-A state of D.

Lemma 4.17 *Let ℓ be a crossing in the non-A–adequate diagram D whose corresponding segment in $s_A(D)$ is a one-edged loop. For a skein $\mathcal{S}^k_\sigma$ containing ℓ^{n+1} as in Figure 8 from a Kauffman state σ on $C(D^{n+1})\backslash\ell^{n+1}$, we have $c=\operatorname{sgn}(\sigma)-\operatorname{sgn}(\sigma_A)\geq k$.*

Proof There are two cases. A state $\sigma\neq\sigma_A$ with $k=0$ certainly has $c\geq k$. If σ is a state for which $k>0$, then we argue that it must choose the B–resolution on at least k crossings.

Since the segment corresponding to ℓ in the all-A state of D is a one-edged loop, the skein $\mathcal{S}^k_A$ from the state σ_A must have $k=0$. In other words, $\mathcal{S}^k_A=\mathcal{S}^0_A=U^{n+1}$, as in Figure 18. Let B_σ be the set of crossings in $C(D^{n+1})\setminus\ell^{n+1}$ on which σ chooses the B–resolution. We can obtain $\mathcal{S}_\sigma$ from $\mathcal{S}_{\sigma_A}$ by locally removing two arcs of the A–resolution and replacing them by the two arcs of the B–resolution at each segment in $\mathcal{S}_{\sigma_A}$ corresponding to a crossing in B_σ. See Figure 19 for the local replacement. A state has a local replacement on a segment if it chooses the B–resolution at the crossing corresponding to the segment.



Figure 19: We change the state circles locally.

Number the outermost k strands joining the top-right and the top-left projectors in S^0_A, outermost first, from $s_1,s_2,\dots$ to s_k. Suppose that $|B_\sigma|<k$. There must be some $i\in 1,\dots,k-1$ such that σ has no local replacement from A to B on any segment between s_i and s_{i+1}, or σ has no local replacement from A to B on any segment attached to s_1 on the outside; see Figure 20 for the convention of what is meant by the

"outside" of the strand s_1 and "between" the strands s_i and s_{i+1}. Either of these cases results in fewer than $2k$ strands joining the top and bottom pairs of projectors for $\mathcal{S}^k_\sigma$, a contradiction. This means that $|B_\sigma| \geq k$, and we have the statement of the lemma. □

This concludes the proof of Theorem 4.16. □

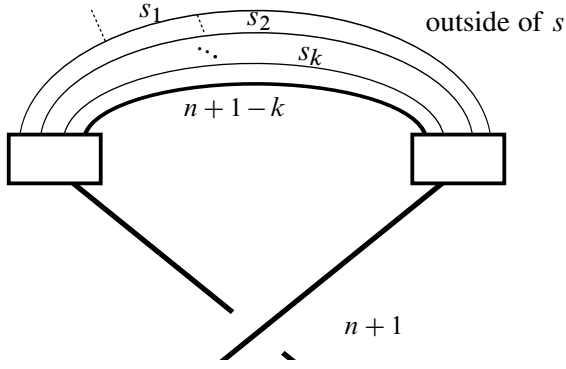


Figure 20: The strands $s_1, \ldots, s_k$ are marked here for $\mathcal{S}^0_A$. Two dashed segments are shown. One is attached to s_1 from the outside, and another is between s_1 and s_2. Segments which are not shown may attach to the strands in a complicated fashion. However, if σ has no local replacement on any segment attached to s_1 from the outside, then $k = 0$ for $\mathcal{S}^k_\sigma$. If σ has no local replacement on any segment between s_i and s_{i+1}, then there cannot be more than $2i$ strands which end up connecting the top and bottom pairs of projectors in $\mathcal{S}^k_\sigma$.

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