Tangent cones and local geometry
of the representation and character varieties of knot groups

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Let K be a knot in the 3–sphere S^3. We calculate explicitly the tangent cone to the representation variety at an abelian representation which corresponds to a double root of the Alexander polynomial. We also describe the local structure of the representation and character varieties.

1 Introduction

Given a knot K in S^3, we let $M = S^3 \setminus K$ denote its complement and $\Gamma := \pi_1(M)$ denote the fundamental group of M.

Gordon and Luecke [13] showed that knots are determined by their complements and Whitten [24] proved that the classification of prime knots reduces to that of their knot groups. So one way to the understanding of knots is the study of their groups.

Research has shown that many topological properties of a knot are encoded in the representation and character varieties of its group, so it is of interest to study their basic properties.

We denote by $R(\Gamma) := \mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{C}))$ the representation variety of Γ (see Section 2.1) and by $S(\Gamma)$ the set of abelian representations, ie those which factor through $\Phi \colon \Gamma \to \Gamma/\Gamma' = T$, where T is an infinite cyclic group. Note that $S(\Gamma) \subset R(\Gamma)$ is an irreducible algebraic component of $R(\Gamma)$ (see Klassen [18]).

A representation $\rho \colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$ is called *irreducible* if the only subspaces of $\mathbb{C}^2$ which are invariant under $\rho(\Gamma)$ are $\{0\}$ and $\mathbb{C}^2$. We denote $R^{\mathrm{irr}}(\Gamma) \subset R(\Gamma)$ the subset of irreducible representations and $R^{\mathrm{red}}(\Gamma)$ the subset of reducible ones. Note that $R^{\mathrm{irr}}(\Gamma)$ is Zariski open and $R^{\mathrm{red}}(\Gamma)$ is Zariski closed in $R(\Gamma)$ (see Culler and Shalen [11]).

The aim of this article is to study the local structure of the representation space of Γ in the Lie group $\mathrm{SL}_2(\mathbb{C})$ in the neighborhood of an abelian representation.

By a result of Thurston, under certain conditions, one can deform an irreducible representation nontrivially but there is no general theorem which allows the deformation of reducible representations. Many authors have studied this problem, for example, the author [2; 3], the author and Lines [4], Frohman and Klassen [12], Herald [14], Heusener and Kroll [15], Heusener, Porti and Suárez Peiró [17] and Shors [22]. The latter (in an unpublished work) has considered the case when the Alexander module has cyclic torsion. The present paper is also concerned with this case.

A first approximation to the local structure of the representation variety at a given representation is given by its Zariski tangent space and a much finer invariant is the tangent cone. In this article, we calculate explicitly the tangent cone to the representation variety at an abelian representation which corresponds to a root α of the Alexander polynomial Δ_1 of the knot K and such that the $(t-\alpha)$–torsion of the Alexander module $H_1(\widetilde{M}, \mathbb{C})$ is cyclic of order 2. Moreover, we give an approximation to the tangent cone in the case of cyclic $(t-\alpha)$–torsion of the Alexander module of order r, $r \geq 3$.

More precisely, let $\alpha = \lambda^2$ be a root of Δ_1 and suppose that the $(t-\alpha)$–torsion of the Alexander module is of the form

$$\frac{\mathbb{C}[t, t^{-1}]}{(t-\alpha)^r} \oplus \frac{\mathbb{C}[t, t^{-1}]}{(t-\alpha^{-1})^r}, \quad r \geq 2.$$

Let $\rho_\lambda \colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$ the abelian representation given by $\rho_\lambda(\mu) = \mathrm{diag}(\lambda, \lambda^{-1}) \in \mathrm{SL}_2(\mathbb{C})$.

As an approximation to the tangent cone $\mathrm{TC}_{\rho_\lambda}$, we have a descending sequence of cones included in the Zariski tangent space $Z^1(\Gamma, \mathfrak{sl}_{\rho_\lambda})$. The first of this cones is the quadratic cone $\mathrm{QC}_{\rho_\lambda}$ and the second one is the cubic cone $\mathrm{TC}^3_{\rho_\lambda}$.

We shall prove the following theorem:

Theorem 1.1 *Assume that the $(t-\alpha)$–torsion is cyclic of order $r \geq 2$.*

(1) *If $r = 2$ then $\mathrm{TC}_{\rho_\lambda} = \mathrm{TC}^3_{\rho_\lambda} \subsetneq \mathrm{QC}_{\rho_\lambda} = Z^1(\Gamma, \mathfrak{sl}_{\rho_\lambda})$.*

(2) *If $r \geq 3$ then $\mathrm{TC}_{\rho_\lambda} \subsetneq \mathrm{TC}^3_{\rho_\lambda} = \mathrm{QC}_{\rho_\lambda} = Z^1(\Gamma, \mathfrak{sl}_{\rho_\lambda})$.*

Let $\phi_\lambda \colon \Gamma \to \mathrm{SL}_2(\mathbb{C})$ be the reducible nonabelian representation with the same character as ρ_λ and given by $\phi_\lambda = \exp(V_+) \rho_\lambda$ (see Section 4.2). Note that ϕ_λ is unique up to conjugation. The study of the local structure of the representation variety near the abelian representation ρ_λ and near the reducible nonabelian representation ϕ_λ goes simultaneously.

Proposition 1.2 *Suppose that $r = 2$; then the representation ϕ_λ is scheme reduced. Moreover:*

(1) *If $\dim_{\mathbb{C}} Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) = 4$ then ϕ_λ is a smooth point of $R(\Gamma)$ which is contained in a 4–dimensional component $R_\lambda(\Gamma)$; so*

$$\mathrm{TC}_{\phi_\lambda} = \mathrm{QC}_{\phi_\lambda} = T^{\mathrm{Zar}}_{\phi_\lambda} R(\Gamma) = Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}).$$

Further, in this case, the abelian representation ρ_λ is contained in exactly two irreducible components of $R(\Gamma)$, namely $S(\Gamma)$ and $R_\lambda(\Gamma)$, and ρ_λ is a smooth point of both varieties.

(2) *If $\dim_{\mathbb{C}} Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) = 5$, then ϕ_λ is a singular point of $R(\Gamma)$ which is contained in, at least, two 4–dimensional components of $\overline{R^{\mathrm{irr}}(\Gamma)}$ and*

$$\mathrm{TC}_{\phi_\lambda} = \mathrm{QC}_{\phi_\lambda} \subsetneq T^{\mathrm{Zar}}_{\phi_\lambda} R(\Gamma) = Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}).$$

Further, in this case, the abelian representation ρ_λ is contained in at least 3 components of $R(\Gamma)$.

The variety of characters $X(\Gamma)$ of a finitely generated group Γ is the quotient in the algebraic category of the action of $\mathrm{SL}_2(\mathbb{C})$ by conjugation on the variety of representations $R(\Gamma)$. We denote the projection by $\pi\colon R(\Gamma) \to X(\Gamma)$. For a representation $\rho \in R(\Gamma)$, its projection onto $X(\Gamma)$, denoted by χ_ρ, is called *the character of ρ* and may be interpreted as a map $\chi_\rho\colon \Gamma \to \mathbb{C}$; $\gamma \mapsto \mathrm{tr}(\rho(\gamma))$ [11].

We denote by $Y(\Gamma)$ the projection of $S(\Gamma)$ onto the character variety $X(\Gamma)$ and by $X^{\mathrm{irr}}(\Gamma) := \pi(R^{\mathrm{irr}}(\Gamma))$ the set of irreducible characters of Γ. Note that $X^{\mathrm{irr}}(\Gamma)$ is Zariski open in $X(\Gamma)$ [11]. We also use the notation $\chi_\lambda := \chi_{\phi_\lambda} = \chi_{\rho_\lambda}$ and $X_\lambda(\Gamma) := \pi(R_\lambda(\Gamma))$.

As an application to the explicit description of the tangent cone, we give a description of the local geometry of the representation and character varieties in the case of a cyclic $(t - \alpha)$–torsion of the Alexander module of order 2.

Theorem 1.3 *Assume that the $(t - \alpha)$–torsion is cyclic of order 2.*

(1) *Suppose that $\dim_{\mathbb{C}} Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) = 4$. Then the curves $X_\lambda(\Gamma)$ and $Y(\Gamma)$ are the unique irreducible components of $X(\Gamma)$ that contain χ_λ. In addition, χ_λ is a smooth point of both curves, the two curves are tangent at χ_λ and $T^{\mathrm{Zar}}_{\chi_\lambda} \overline{X^{\mathrm{irr}}(\Gamma)} \cong H^1(\Gamma, \mathfrak{sl}_{\phi_\lambda})$.*

(2) *Suppose that $\dim_{\mathbb{C}} Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) = 5$. Then χ_λ is contained in at least 2 irreducible components of $\overline{X^{\mathrm{irr}}(\Gamma)}$ and is not a smooth point of the character variety of irreducible representations.*

The paper is organised as follows. In Section 2 the basic notation and facts are introduced. In Section 3, we recall some results about abelian representations. The proof of the Theorem 1.1 is presented in Section 4. Section 5 includes the proofs of Proposition 1.2 and Theorem 1.3 and contains examples. The last section, Section 6, is devoted to the proof of a technical result.

2 Representation spaces and deformations

In this section, we recall some notation and facts concerning the representation and character varieties and the formal deformations. We introduce also the notion of i–th tangent cone at a representation as well as the terminology of quadratic and strongly nonquadratic singularity of the representation variety.

2.1 Representation spaces

Given a finitely presented group $\Gamma = \langle S_1, \ldots, S_n \mid R_1, \ldots, R_m \rangle$ and a complex Lie group G, the space of all representations (ie homomorphisms) of the group Γ into G is denoted by $R(\Gamma, G)$. The space $R(\Gamma, G)$ is equipped with the compact open topology and can be identified with the following subspace of G^n:

$$R(\Gamma, G) := \{(g_1, \ldots, g_n) \mid R_j(g_1, \ldots, g_n) = E, \ j = 1, \ldots, m\},$$

where E is the unit in G. The relations R_j give a smooth map $R: G^n \to G^m$ and $R(\Gamma, G)$ can be identified with $R^{-1}(E, \ldots, E)$.

The space $R(\Gamma, G)$ inherits an algebraic structure if G is an algebraic group. It is easy to see that the map $R: G^n \to G^m$ is polynomial in the ambient coordinates. The set $R^{-1}(E, \ldots, E)$ is therefore an affine algebraic set and the induced algebraic structure on $R(\Gamma, G)$ does not depend on the presentation of Γ [19]. The space $R(\Gamma, G)$ carries two topologies, the Zariski and the complex or classical topology.

Let $\rho: \Gamma \to G$ be a representation. The Lie algebra $\mathcal{G}$ of G is turned via $\mathrm{Ad} \circ \rho$ into a Γ–module denoted by $\mathcal{G}_\rho$, ie for all $X \in \mathcal{G}$ and for $\gamma \in \Gamma$, we have $\gamma \cdot X = \rho(\gamma) X \rho(\gamma)^{-1}$. We denote by $C^n(\Gamma, \mathcal{G}_\rho) := \{u: \Gamma^n \to \mathcal{G}\}$ the space of n–cochains and the coboundary operator is denoted by $\partial: C^n(\Gamma, \mathcal{G}_\rho) \to C^{n+1}(\Gamma, \mathcal{G}_\rho)$. Let $B^*(\Gamma, \mathcal{G}_\rho)$

(resp. $Z^*(\Gamma, \mathcal{G}_\rho)$, resp. $H^*(\Gamma, \mathcal{G}_\rho)$) be the coboundaries (resp. cocycles, resp. coho-mology groups) of Γ with coefficients in $\mathcal{G}_\rho$.

We denote by $T_\rho^{\mathrm{Zar}} R(\Gamma, G)$ the Zariski tangent space of the variety $R(\Gamma, G)$ at the representation ρ. It was observed by Weil that there is a natural inclusion $T_\rho^{\mathrm{Zar}} R(\Gamma, G) \hookrightarrow Z^1(\Gamma, \mathcal{G}_\rho)$: for each analytic path $\rho_t: \Gamma \to G$ such that $\rho_0 = \rho$, the map $u: \Gamma \to \mathcal{G}$ given by

$$u(\gamma) := \left. \frac{d(\rho_t(\gamma))}{dt} \right|_{t=0} \rho^{-1}(\gamma)$$

is a cocycle [23]. It is also proved in [19] that the space $Z^1(\Gamma, \mathcal{G}_\rho)$ can be identified with the tangent space of the scheme of all representations of Γ to G at ρ. In general, the space of cocycles is larger then the Zariski tangent space. A representation $\rho: \Gamma \to G$ is called *scheme reduced* if the inclusion $T_\rho^{\mathrm{Zar}} R(\Gamma, G) \hookrightarrow Z^1(\Gamma, \mathcal{G}_\rho)$ is an isomorphism. Even for a scheme reduced representation, the space $Z^1(\Gamma, \mathcal{G}_\rho)$ gives only a first approximation to the local structure of the representation variety. A much finer invariant is the tangent cone $\mathrm{TC}_\rho R(\Gamma, G)$, that is the set of all cocycles $u \in Z^1(\Gamma, \mathcal{G}_\rho)$ which are tangent to an analytic path in $R(\Gamma, G)$.

Definition 2.1 A representation $\rho: \Gamma \to G$ is called *regular* if the tangent cone is equal to $Z^1(\Gamma, \mathcal{G}_\rho)$. Otherwise, ρ is called *singular*.

We are interested in the case where Γ is a knot group and G is the Lie group $\mathrm{SL}_2(\mathbb{C})$ which we denote SL for simplicity.

2.2 Formal deformations and cones

Let M_1 and M_2 be two Γ–modules. The cup product of two cochains $u \in C^p(\Gamma, M_1)$ and $v \in C^q(\Gamma, M_2)$ is the cochain $u \smile v \in C^{p+q}(\Gamma, M_1 \otimes M_2)$ defined by

$$u \smile v(\gamma_1, \ldots, \gamma_{p+q}) := u(\gamma_1, \ldots, \gamma_p) \otimes \gamma_1 \cdots \gamma_p \cdot v(\gamma_{p+1}, \ldots, \gamma_{p+q}).$$

It is possible to combine the cup product with any $\mathbb{C}[\Gamma]$–bilinear map $b: M_1 \otimes M_2 \to M_3$. So we obtain a cup product $C^p(\Gamma, M_1) \otimes C^q(\Gamma, M_2) \to C^{p+q}(\Gamma, M_3)$ [8].

We are mainly interested in the case where the Γ–module $\mathcal{G} = M_1 = M_2$ is also a Lie algebra. Then by using the Lie brackets of $\mathcal{G}$ as a bilinear map we obtain the cup bracket of two cochains which will be denoted by $u \smile v$. Note that the cup bracket is not associative on the cochain level.

Definition 2.2 A *formal deformation* of $\rho: \Gamma \to G$ is a representation $\rho_t: \Gamma \to G(\mathbb{C}[\![t]\!])$ such that $\rho_0 = \rho$. Here $\mathbb{C}[\![t]\!]$ denotes the ring of formal power series and $\rho_0: \Gamma \to G$ is the evaluation of ρ_t at $t = 0$.

Every formal deformation ρ_t of $\rho\colon \Gamma \to G$ can be written in the form $\rho_t(\gamma) = \exp(tv_1(\gamma)+t^2v_2(\gamma)+\cdots)\rho(\gamma)$ where $v_i\colon \Gamma \to \mathcal{G}$ are maps, ie elements of $C^1(\Gamma, \mathcal{G}_\rho)$ and Weil's construction gives that $v_1 \in Z^1(\Gamma, \mathcal{G}_\rho)$ is a cocycle.

More generally, we have the following:

Lemma 2.3 *Let $\rho\colon \Gamma \to G$ be a representation. Then $\rho_t\colon \Gamma \to G(\mathbb{C}[\![t]\!])$ given by $\rho_t = \exp(\sum_{i\geq 1} t^i v_i)\rho$ is a homomorphism if and only if*

$$\begin{cases} -\partial v_1 = 0 \\ -\partial v_n = \dfrac{1}{2}\displaystyle\sum_{k=1}^{n-1} v_k \smile v_{n-k} + \sum_{j=3}^{n} A_j(v_1,\ldots,v_{n-2}), \ \text{for } n \geq 2, \end{cases}$$

where, for $\gamma_1, \gamma_2 \in \Gamma$, $A_j(v_1,\ldots,v_{n-2})(\gamma_1,\gamma_2)$ is a sum of monomials which are multiple brackets of exactly j–terms $v_i(\gamma_1)$ or $\gamma_1 \cdot v_l(\gamma_2)$, $1 \leq i, l \leq n-2$ [2].

For all $n \geq 2$, denote by $\zeta_n(v_1,\ldots,v_{n-1})$ the 2–cochain on Γ such that

$$\zeta_n(v_1,\ldots,v_{n-1}) = \frac{1}{2}\sum_{k=1}^{n-1} v_k \smile v_{n-k} + \sum_{j=3}^{n} A_j(v_1,\ldots,v_{n-2}).$$

It is easy to see that $\zeta_n(v_1,\ldots,v_{n-1}) \in Z^2(\Gamma, \mathcal{G}_\rho)$ (see [4, proof of Proposition 4.10]). Note that to construct a formal deformation of ρ it suffices to solve the infinite sequence of obstructions

$$\begin{cases} -\partial v_1 = 0 \\ -\partial v_n = \zeta_n(v_1,\ldots,v_{n-1}), \ n \geq 2. \end{cases}$$

The first of these obstructions is the cup product $\{v_1 \smile v_1\}$ and each obstruction is defined in terms of the preceding solution on the cochain level and each taking values in $H^2(\Gamma, \mathcal{G}_\rho)$ [20].

Definition 2.4 We call a cocycle $v_1 \in Z^1(\Gamma, \mathcal{G}_\rho)$ *formally integrable* if there exists a formal deformation ρ_t with first order term v_1.

By a theorem of Artin [1], the tangent cone to $R(\Gamma, G)$ at ρ,

$$\mathrm{TC}_\rho\, R(\Gamma, G) = \left\{ \frac{d\rho_t\rho^{-1}}{dt}\bigg|_{t=0} \,\bigg|\, \rho_t \text{ is an analytic curve in } R(\Gamma, G) \text{ with endpoint } \rho \right\}$$

is exactly the set of formally integrable cocycles.

As an approximation to the tangent cone, we have a descending sequence of cones and the first of them is the quadratic cone.

Definition 2.5 Let $\rho\colon \Gamma \to G$ be a representation. The *quadratic cone* at ρ to $R(\Gamma, G)$ is the subset of $Z^1(\Gamma, \mathcal{G}_\rho)$ given by

$$\mathrm{QC}_\rho\, R(\Gamma, G) = \{u \in Z^1(\Gamma, \mathcal{G}_\rho) \mid \{u \smile u\} = 0 \in H^2(\Gamma, \mathcal{G}_\rho)\}.$$

We say $R(\Gamma, G)$ has a *quadratic singularity* at ρ if ρ is singular and $\mathrm{TC}_\rho\, R(\Gamma, G) = \mathrm{QC}_\rho\, R(\Gamma, G)$, ie the obstructions are at most quadratic.

Now let $u_1 \in Z^1(\Gamma, \mathcal{G}_\rho)$ be a cocycle in the quadratic cone, ie $\{u_1 \smile u_1\} = 0$ in $H^2(\Gamma, \mathcal{G}_\rho)$. Then there exists a cochain $u_2\colon \Gamma \to \mathcal{G}_\rho$ such that $-\partial u_2 = \frac{1}{2} u_1 \smile u_1$. The cochain u_2 is only unique up to addition of a cocycle $Z \in Z^1(\Gamma, \mathcal{G}_\rho)$.

The third obstruction is given by the cocycle

$$\zeta_3(u_1, u_2)(\gamma_1, \gamma_2) = \frac{1}{2} u_1 \smile u_2(\gamma_1, \gamma_2) + \frac{1}{2} u_2 \smile u_1(\gamma_1, \gamma_2)$$

$$+ \frac{1}{12}[[u_1(\gamma_1), \gamma_1 \cdot u_1(\gamma_2)], \gamma_1 \cdot u_1(\gamma_2) - u_1(\gamma_1)].$$

Its existence depends on that of u_2 and is unique up to adding elements of the form $\frac{1}{2} Z \smile u_1 + \frac{1}{2} u_1 \smile Z$ where $Z \in Z^1(\Gamma, \mathcal{G}_\rho)$.

The cubic cone at ρ to $R(\Gamma, G)$ is by definition the set

$$\mathrm{TC}^3_\rho\, R(\Gamma, G) := \{u_1 \in \mathrm{QC}_\rho(\Gamma) \mid \exists\, u_2 \in C^1(\Gamma, \mathcal{G}_\rho) \text{ such that}$$
$$\zeta_3(u_1, u_2) \text{ exists and } \{\zeta_3(u_1, u_2)\} = 0 \in H^2(\Gamma, \mathcal{G}_\rho)\}.$$

More generally, for $i \geq 3$, we define *the i–th tangent cone* at ρ to $R(\Gamma, G)$ as

$$\mathrm{TC}^i_\rho\, R(\Gamma, G) := \{u_1 \in Z^1(\Gamma, \mathcal{G}_\rho) \mid \exists\, u_2, \ldots, u_{i-1} \in C^1(\Gamma, \mathcal{G}_\rho) \text{ such that}$$
$$\zeta_j(u_1, \ldots, u_{j-1}) \text{ exists and } \{\zeta_j(u_1, \ldots, u_{j-1})\} = 0\ \forall\ 2 \leq j \leq i\}.$$

So we have the inclusions

$$\mathrm{TC}_\rho\, R(\Gamma, G) \subset \cdots \subset \mathrm{TC}^{i+1}_\rho\, R(\Gamma, G)$$
$$\subset \mathrm{TC}^i_\rho\, R(\Gamma, G) \subset \cdots \subset \mathrm{QC}_\rho\, R(\Gamma, G) \subset Z^1(\Gamma, \mathcal{G}_\rho)$$

and by Artin's theorem, $\mathrm{TC}_\rho\, R(\Gamma, G) = \bigcap_{i \geq 1} \mathrm{TC}^i_\rho\, R(\Gamma, G)$.

Definition 2.6 For a given representation $\rho \in R(\Gamma, G)$, we say that ρ is a *strongly nonquadratic singularity* of $R(\Gamma, G)$ if there exists a cocycle $u_1 \in \mathrm{QC}_\rho\, R(\Gamma, G)$ in the quadratic cone for which the third obstruction does not vanish. In other words, $\mathrm{TC}^3_\rho\, R(\Gamma, G) \subsetneq \mathrm{QC}_\rho\, R(\Gamma, G)$.

Notation If $G = \mathrm{SL}$ we will adopt the notation $\mathrm{QC}_\rho := \mathrm{QC}_\rho R(\Gamma, \mathrm{SL})$, $\mathrm{TC}_\rho^i :=$ $\mathrm{TC}_\rho^i R(\Gamma, \mathrm{SL})$ for $i \geq 3$ and $\mathrm{TC}_\rho := \mathrm{TC}_\rho R(\Gamma, \mathrm{SL})$.

3 Abelian representations of knot groups and the quadratic cone

From now and on we suppose that $G = \mathrm{SL}$ and Γ is a knot group.

In this section we study mainly the Zariski tangent space and the quadratic cone at an abelian representation which corresponds to a root of the Alexander polynomial. In particular, we recall a result of [4] which allows us to calculate explicitly the quadratic cone.

3.1 Alexander invariants

Recall that if Γ is a knot group then we have a canonical surjection $\Phi\colon \Gamma \to T$ where $T := \Gamma/\Gamma'$ is an infinite cyclic group. We denote by $\widetilde{M}$ the infinite cyclic covering determined by the epimorphism $\Phi\colon \Gamma \to T \cong \mathbb{Z}$. Let t be a generator of T and let $\Gamma = \langle S_1, \ldots, S_n | R_1, \ldots, R_{n-1} \rangle$ be a presentation of Γ such that $\Phi(S_i) = t$. We choose then $\mu := S_1$ as a meridian of K.

The vector space $H_1(\widetilde{M}, \mathbb{Q})$ is a torsion $\mathbb{Q}T$–module and the group algebra $\Lambda_0 = \mathbb{Q}T$ can be then identified with the ring of Laurent polynomials $\mathbb{Q}[t, t^{-1}]$. The module $H_1(\widetilde{M}, \mathbb{Q})$ decomposes as a sum of cyclic Λ_0–modules

$$H_1(\widetilde{M}, \mathbb{Q}) = \bigoplus_{i=1}^{k} \frac{\Lambda_0}{(\Pi_i)} \quad \text{where } \Pi_i \in \Lambda_0.$$

The ideals (Π_i) satisfy $(\Pi_i) \subset (\Pi_{i+1})$, ie $\Pi_k \mid \Pi_{k-1} \cdots \mid \Pi_1$. The i–th Alexander polynomial Δ_i is by definition $\Delta_i := \Pi_k \cdots \Pi_i$, so $\Pi_i = \Delta_i/\Delta_{i+1}$.

The first Alexander polynomial Δ_1 is the generator of the order ideal of the Alexander module $H_1(\widetilde{M}, \mathbb{Q})$. By the Blanchfield duality theorem [5], the Alexander polynomials are symmetric and unique up to multiplication with elements of the form $\{\alpha t^n \mid \alpha \in \mathbb{Q} \setminus \{0\}, \ n \in \mathbb{Z}\}$. Note that $\Delta_i(1) \neq 0$.

3.2 Abelian representations

For a given $\lambda \in \mathbb{C}^*$, we denote by ρ_λ the abelian representation which is given by $\mu \mapsto \mathrm{diag}(\lambda, \lambda^{-1}) \in \mathrm{SL}$. The Lie algebra $\mathfrak{sl} := \mathfrak{sl}_2(\mathbb{C})$ splits as a Γ–module

$$\mathfrak{sl}_{\rho_\lambda} = \mathcal{H} \oplus \mathcal{G}_+ \oplus \mathcal{G}_-,$$

where $\mathcal{H}$ is generated by

$$g_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$\mathcal{G}_+$ is generated by

$$g_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and $\mathcal{G}_-$ is generated by

$$g_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Here Γ acts on $\mathcal{G}_\pm$ via $\gamma \cdot z_\pm = \Phi(\gamma)(\lambda^{\pm 2})z_\pm$ and $\mathcal{H}$ is the trivial Γ–module. If $f\colon \Gamma \to \mathfrak{sl}$ is a cochain, we denote f^0, f^+, f^- its coordinates in g_0, g_+ and g_-.

By [18], $H^1(\Gamma, \mathcal{G}_\pm) = 0$ if and only if $\Delta_1(\alpha^{\pm 1}) \neq 0$, where $\alpha := \lambda^2$. It turns out that in this case $\dim_{\mathbb{C}} Z^1(\Gamma, \mathfrak{sl}_{\rho_\lambda}) = 3$, ρ_λ is a smooth point of the representation variety $R(\Gamma)$ and $S(\Gamma)$ is the unique component through ρ_λ (see Klassen [18]).

Remark The representation ρ_λ is a singular point of the representation variety $R(\Gamma)$ if $\Delta_1(\alpha) = 0$. Moreover, it was observed in [3, Corollary 4.4] that the representation ρ_λ is scheme reduced, so $T_{\rho_\lambda}^{\mathrm{Zar}} R(\Gamma) = Z^1(\Gamma, \mathfrak{sl}_{\rho_\lambda})$.

Lemma 3.1 *If $u_1 = W + u_+ + u_- \in Z^1(\Gamma, \mathcal{H}) \oplus Z^1(\Gamma, \mathcal{G}_+) \oplus Z^1(\Gamma, \mathcal{G}_-)$ is formally integrable, then so are the cocycles cu_1, $c \in \mathbb{C}$ and $\tilde{u}_1 = W + \beta u_+ + \beta^{-1} u_-$, $\beta \in \mathbb{C}^*$.*

Moreover, if $u_1 = u_+ + u_-$ is formally integrable then $\hat{u}_1 = \beta_+ u_+ + \beta_- u_-$ is also formally integrable for all $\beta_\pm \in \mathbb{C}$.

Proof Let $\rho_t = \exp(\sum_{i \geq 1} t^i u_i)\rho_\lambda$ be a formal deformation of ρ_λ.

It is clear that $\rho_t = \exp(\sum_{i \geq 1} c^i t^i u_i)\rho_\lambda$ is also a formal deformation of ρ_λ for any $c \in \mathbb{C}$.

On the other hand, the stabilizer of ρ_λ under conjugation of SL are the diagonal matrices. We have for all $\beta \in \mathbb{C}^*$,

$$\begin{pmatrix} \beta^{1/2} & 0 \\ 0 & \beta^{-1/2} \end{pmatrix} \begin{pmatrix} x & y \\ z & -x \end{pmatrix} \begin{pmatrix} \beta^{-1/2} & 0 \\ 0 & \beta^{1/2} \end{pmatrix} = \begin{pmatrix} x & \beta y \\ \beta^{-1} z & -x \end{pmatrix},$$

so the lemma follows for $\tilde{u}_1$.

If $\beta_+ \beta_- = 0$, then as $\exp(tu_\pm)\rho_\lambda$ is a formal deformation of ρ_λ, $\hat{u}_1$ is formally integrable.

If $\beta_+\beta_- \neq 0$, then the properties above and the equality

$$(\beta_+, \beta_-) = (\beta_+\beta_-)^{1/2}(\beta_+^{1/2}\beta_-^{-1/2}, \beta_+^{-1/2}\beta_-^{1/2})$$

show that $\hat{u}_1$ is formally integrable. $\square$

For $\alpha \in \mathbb{C}$ and a polynomial $\Pi \in \Lambda_0$, we denote by $\nu_\alpha(\Pi) \in \mathbb{Z}$ the multiplicity of Π at α. So α is a root of Π if and only if $\nu_\alpha(\Pi) > 0$.

Lemma 3.2 *Let $\alpha \in \mathbb{C}^*$ be a root of Δ_1 and set $l = \max\{i \mid \Pi_i(\alpha) = 0\}$ where $\Pi_i = \Delta_i/\Delta_{i+1}$.*

Then $\dim_\mathbb{C} H^1(\Gamma, \mathcal{G}_\pm) = l$ and $\nu_\alpha(\Delta_1) = \sum_{i=1}^l \nu_\alpha(\Pi_i)$.

Proof This is a result of Burde [9] and de Rham [21] and is a consequence of the fact that

$$H^1(\Gamma, \mathcal{G}_\pm) \cong \mathrm{Hom}\left(H_1(\widetilde{M}, \mathbb{C}), \frac{\mathbb{C}[t, t^{-1}]}{(t - \alpha^{\pm 1})}\right) \quad \text{and} \quad H_1(\widetilde{M}, \mathbb{C}) \cong \bigoplus_{i=1}^k \frac{\Lambda_0}{(\Pi_i)} \otimes \mathbb{C}. \quad \square$$

3.3 The quadratic cone

Let $u_1 = W + u_+ + u_- \in Z^1(\Gamma, \mathfrak{sl}_{\rho_\lambda})$ be a cocycle where $W \in Z^1(\Gamma, \mathcal{H})$ and $u_\pm \in Z^1(\Gamma, \mathcal{G}_\pm)$. The distributive property for the cup product, together with the fact that $[\mathcal{H}, \mathcal{H}] = 0 = [\mathcal{G}_+, \mathcal{G}_+] = [\mathcal{G}_-, \mathcal{G}_-]$, give

$$u_1 \smile u_1 = (u_+ \smile u_- + u_- \smile u_+) + (W \smile u_+ + u_+ \smile W) + (W \smile u_- + u_- \smile W).$$

Since $H^2(\Gamma, \mathcal{H}) = 0$, $\{u_1 \smile u_1\} = \{(W \smile u_+ + u_+ \smile W) + (W \smile u_- + u_- \smile W)\} = 2\{W \smile u_+\} + 2\{W \smile u_-\}$.

Notice that the cup product is commutative in cohomology because the Lie bracket is anticommutative. It follows that the quadratic cone $\mathrm{QC}_{\rho_\lambda}$ contains the space $D^+ := Z^1(\Gamma, \mathcal{H}) \oplus B^1(\Gamma, \mathcal{G}_+) \oplus B^1(\Gamma, \mathcal{G}_-)$ of cocycles which are cohomologous to diagonal cocycles, and contains the space of antidiagonal cocycles $D^- := Z^1(\Gamma, \mathcal{G}_+) \oplus Z^1(\Gamma, \mathcal{G}_-)$. It may be worth to notice that D^+ is tangent to the abelian deformations obtained by perturbing λ.

In fact, the following result of [4, Theorem 3.2] allows us to determine the quadratic cone.

Proposition 3.3 *Let $W \in Z^1(\Gamma, \mathcal{H})$ be a nonzero cocycle. The homomorphism*

$$W \smile \cdot : H^1(\Gamma, \mathcal{G}_\pm) \to H^2(\Gamma, \mathcal{G}_\pm)$$

is an isomorphism if and only if the $(t - \alpha^{\pm 1})$–torsion of the Alexander module $H_1(\widetilde{M}, \mathbb{C})$ is semisimple, that is of the form

$$\frac{\mathbb{C}[t, t^{-1}]}{(t - \alpha^{\pm 1})} \oplus \cdots \oplus \frac{\mathbb{C}[t, t^{-1}]}{(t - \alpha^{\pm 1})}.$$

Note that the latter is equivalent to the equality $\dim_{\mathbb{C}} H^1(\Gamma, \mathcal{G}_\pm) = \nu_{\alpha^{\pm 1}}(\Delta_1)$.

Corollary 3.4 *Let $k = \max\{i \mid \Pi_i(\alpha) = 0\}$.*
(1) *If $\dim_{\mathbb{C}} H^1(\Gamma, \mathcal{G}_\pm) = \nu_{\alpha^{\pm 1}}(\Delta_1)$, then every cocycle $u \in \mathrm{QC}_{\rho_\lambda}$ is an antidiagonal cocycle or cohomologous to a diagonal cocycle, ie $\mathrm{QC}_{\rho_\lambda} = D^+ \cup D^-$.*
(2) *If $\nu_{\alpha^\pm}(\Pi_i) > 1$ for all $i = 1, \ldots, k$, then the quadratic cone and the Zariski tangent space coincide, ie $Z^1(\Gamma, \mathfrak{sl}_{\rho_\lambda}) = \mathrm{QC}_{\rho_\lambda}$.*

Proof Let $u_1 = W + u_+ + u_- \in \mathrm{QC}_{\rho_\lambda}$ be a cocycle where $W \in Z^1(\Gamma, \mathcal{H})$, $u_+ \in Z^1(\Gamma, \mathcal{G}_+)$ and $u_- \in Z^1(\Gamma, \mathcal{G}_-)$. We have $u_1 \smile u_1$ is homologous to zero if and only if $W \smile u_+$ and $W \smile u_-$ are homologous to zero.

If $\dim_{\mathbb{C}} H^1(\Gamma, \mathcal{G}_\pm) = \nu_{\alpha^{\pm 1}}(\Delta_1)$ then, by [Proposition 3.3](), we obtain $W = 0$ or $u_\pm \in B^1(\Gamma, \mathcal{G}_\pm)$.

If $\nu_{\alpha^\pm}(\Pi_i) > 1$ for all $i = 1, \ldots, k$, then $W \smile u_+$ and $W \smile u_-$ are homologous to zero (see [4], proof of Theorem 3.2]) and hence $Z^1(\Gamma, \mathfrak{sl}_{\rho_\lambda}) = \mathrm{QC}_{\rho_\lambda}$. $\square$

Now, Theorem 5.2 of [3] can be rephrased as follows:

Corollary 3.5 *Let K be a knot in S^3 with group Γ. Let $\lambda \in \mathbb{C}^*$ such that λ^2 is a simple root of the Alexander polynomial and ρ_λ the abelian representation which is given by $\mu \mapsto \mathrm{diag}(\lambda, \lambda^{-1}) \in \mathrm{SL}$. Then the representation variety $R(\Gamma)$ has a quadratic singularity at ρ_λ, ie*

$$\mathrm{TC}_{\rho_\lambda} = \mathrm{QC}_{\rho_\lambda} \subsetneqq Z^1(\Gamma, \mathfrak{sl}_{\rho_\lambda}).$$

The following result asserts the existence of knots with nonquadratic singularities.

Theorem 3.6 *Let $K \subset S^3$ be a knot and let $\lambda \in \mathbb{C}^*$ such that the $(t - \alpha)$–torsion of the Alexander module of K with complex coefficients is cyclic of the form*

$$\frac{\mathbb{C}[t, t^{-1}]}{(t - \lambda^2)^r} \oplus \frac{\mathbb{C}[t, t^{-1}]}{(t - \lambda^{-2})^r}, \quad r \geq 2.$$

Then the representation variety $R(\Gamma)$ has a nonquadratic singularity at ρ_λ, where ρ_λ is the abelian representation which is given by $\mu \mapsto \mathrm{diag}(\lambda, \lambda^{-1}) \in \mathrm{SL}$.

Proof The tangent space $Z^1(\Gamma, \mathfrak{sl}_{\rho_\lambda})$ and the quadratic cone $\mathrm{QC}_{\rho_\lambda}$ coincide by Corollary 3.4. Since the $(t - \alpha)$–torsion of the Alexander module of K is cyclic, we have $\dim_{\mathbb{C}} H^1(\Gamma, \mathcal{G}_\pm) = 1$ and hence $\dim_{\mathbb{C}} Z^1(\Gamma, \mathfrak{sl}_{\rho_\lambda}) = 5$.

On the other hand, it is proved in [17, Lemma 6.1] that there are no 5–dimensional components going through ρ_λ. Hence there are cocycles in the quadratic cone which do not appear in the tangent cone. □

Remarks (1) Theorem 1.1 allows us to determine a family of knots such that the corresponding representation variety $R(\Gamma)$ admits a strongly nonquadratic singularity. Namely, every knot which has Alexander module of the form $\Lambda/(t - \alpha)^2 \oplus \Lambda/(t - \alpha^{-1})^2$ admits a strongly nonquadratic singularity.

(2) One way of constructing a family of knots of S^3 admitting strongly nonquadratic singularities (and thus nonquadratic singularities) was communicated to me by M Heusener. It corresponds to the product knots $K \sharp K$ where K is a knot in S^3 such that α is a simple root of the Alexander polynomial Δ_1 of K.

(3) It is to note that knots having Alexander module of the form $\Lambda/(t - \alpha)^r \oplus \Lambda/(t - \alpha^{-1})^r$, $r \geq 3$, admit nonquadratic singularities but no strongly nonquadratic singularities.

4 The third obstruction

This section is devoted to the proof of the Theorem 1.1 and is divided into four subsections. In the first one we recall the construction of the rationalized group and the involution on the knot group. In the second one we construct an appropriate basis of the first and second homological groups corresponding to the abelian representation ρ_λ. At the end of this subsection, we calculate the space of cocycles corresponding to a reducible nonabelian representation with the same character as ρ_λ. The third and fourth subsections contain the proof of Theorem 1.1.

In this section we will assume that $\alpha := \lambda^2$ is a root of the Alexander polynomial and that the $(t - \alpha)$–torsion of the Alexander module is cyclic of the form $\Lambda/(t - \alpha)^r \oplus \Lambda/(t - \alpha^{-1})^r$ where $\Lambda := \mathbb{C}[t, t^{-1}]$ and r is a positive integer.

By Corollary 3.5, if $r = 1$ then $\mathrm{TC}_{\rho_\lambda} = \mathrm{QC}_{\rho_\lambda} \subsetneq Z^1(\Gamma, \mathfrak{sl}_{\rho_\lambda})$. In particular, $\mathrm{TC}_{\rho_\lambda} = \mathrm{TC}^3_{\rho_\lambda} = \mathrm{QC}_{\rho_\lambda}$.

Our aim is to determine the cubic tangent cone at ρ_λ to $R(\Gamma)$ as an approach to the tangent cone $\mathrm{TC}_{\rho_\lambda}$.

As a consequence of Theorem 1.1, we observe that the third obstruction vanishes if $r \geq 3$. So our studies suggest the following conjecture:

Conjecture If α is a root of the Alexander polynomial such that the $(t - \alpha)$−torsion of the Alexander module is of the form $\Lambda/(t - \alpha)^r \oplus \Lambda/(t - \alpha^{-1})^r$, $r \geq 1$

then
$$\mathrm{TC}_{\rho_\lambda} = \mathrm{TC}^{r+1}_{\rho_\lambda} \subsetneqq \mathrm{TC}^r_{\rho_\lambda} = \cdots = Z^1(\Gamma, \mathfrak{sl}_{\rho_\lambda}).$$

This conjecture is verified for $r = 1$ and $r = 2$. For $r \geq 3$, we just proved that $\mathrm{TC}^3_{\rho_\lambda} = \mathrm{QC}_{\rho_\lambda} = Z^1(\Gamma, \mathfrak{sl}_{\rho_\lambda})$. Moreover, by Proposition 4.1, $W + V_+ \in \mathrm{TC}^r_{\rho_\lambda}$. So, we project to prove that $W + V_+ \notin \mathrm{TC}^{r+1}_{\rho_\lambda}$ as a first step in proving the conjecture.

The proof of Theorem 1.1 will be divided into two parts. The first step consists in the choice of an antidiagonal, nontrivial cocycle $u_1 \in Z^1(\Gamma, \mathcal{G}_- \oplus \mathcal{G}_+)$. Using the techniques of deformation of [4], we will construct in the case where $r = 2$ and u_1 is a cocycle which verifies the third obstruction, a formal deformation tangent to u_1. In the case where the third obstruction does not vanish for u_1 and $r \geq 2$ we will use a result of [16] to conclude.

4.1 The rationalized group and involution

Our basic reference in this paragraph is [4], all proofs and details can be found there. However, for the sake of completeness, we recall the following constructions in order to solve the obstruction equations.

Let K be a knot in S^3. Since the Alexander polynomials are reciprocal, the Λ_0−module $H_1(\widetilde{M}, \mathbb{Q})$ decomposes as a direct sum of modules of the form $\Lambda_0/p^m(t)$ and $\Lambda_0/q^n(t) \oplus \Lambda_0/q^n(t^{-1})$, where $p(t)$ is an irreducible reciprocal polynomial, $q(t)$ is an irreducible nonreciprocal polynomial and m, n are nonnegative integers.

Construct an anti-involution j on $H_1(\widetilde{M}, \mathbb{Q})$ by setting

$$\begin{cases} j([R(t)]) = R([t^{-1}]) & \text{on } \dfrac{\Lambda_0}{p^m(t)} \\ j([S^1(t)], [S^2(t)]) = ([S^2(t^{-1})], [S^1(t^{-1})]) & \text{on } \dfrac{\Lambda_0}{q^n(t)} \oplus \dfrac{\Lambda_0}{q^n(t^{-1})}. \end{cases}$$

This anti-involution extends to $H_1(\widetilde{M}, \mathbb{C})$ to an anti-involution still denoted j. In particular, if we suppose that the $(t - \alpha)$−torsion of the Λ−module $H_1(\widetilde{M}, \mathbb{C})$ is isomorphic to $\Lambda/(t - \alpha)^r \oplus \Lambda/(t - \alpha^{-1})^r$ then $j([Y_1(t)], [Y_2(t)]) = ([Y_2(t^{-1})], [Y_1(t^{-1})])$.

Let $N = N_1 = \Gamma'$ be the commutator subgroup of $\Gamma = \pi_1(M)$, $N_{i+1} = [N, N_i]$ and let $\widehat{\Gamma} = \widehat{N/N_i} \rtimes T$ be the rationalized associated group as constructed in [4]. Then the anti-involution j extends to an involution on $\widehat{\Gamma}$ and induce involutions in cochains and in cohomology which we simply denote j^*.

By restricting ourselves to a subcomplex of the standard cochain complex $C^*(\widehat{\Gamma}, \mathfrak{sl}_{\rho_\lambda})$, we obtain the cochain complex $\overline{C}^*(\widehat{\Gamma}, \mathfrak{sl}_{\rho_\lambda})$ which is isomorphic to the direct limit $\varinjlim C^*((\widehat{N/N_i}) \rtimes T, \mathfrak{sl}_{\rho_\lambda})$ and whose cohomology groups are the same as those of Γ with coefficients in $\mathfrak{sl}_{\rho_\lambda}$. In particular, if q denotes the natural map $\Gamma = N \rtimes T \to \widehat{\Gamma}$ then q induces isomorphisms $q^* \colon \overline{H}^*(\widehat{\Gamma}, \mathcal{G}_\pm) \to H^*(\Gamma, \mathcal{G}_\pm)$ for $i = 1, 2$, (for details see [4, Section 3, proof of Proposition 3.1]).

Remark It is to notice that $\partial \circ j^* = j^* \circ \partial$ and that, for all $f, g \in \overline{Z}^1(\widehat{\Gamma}, \mathfrak{sl}_{\rho_\lambda})$, $j^*(f \smile g) = -j^*f \smile j^*g$ [2, Chapter III].

4.2 Construction of an appropriate basis of $H^1(\Gamma, \mathcal{G}_+ \oplus \mathcal{G}_-)$ and $H^2(\Gamma, \mathcal{G}_+ \oplus \mathcal{G}_-)$

As α is a root of the Alexander polynomial, we have an isomorphism [4]

$$\bar{\psi} \colon \mathrm{Hom}_\Lambda(H_1(\widetilde{M}, \mathbb{C}), \mathcal{G}_+ \oplus \mathcal{G}_-) \to \overline{H}^1(\widehat{\Gamma}, \mathcal{G}_+ \oplus \mathcal{G}_-) \cong H^1(\Gamma, \mathcal{G}_+ \oplus \mathcal{G}_-).$$

Let y_1 be a Λ–basis of the $(t - \alpha)$–torsion of $H_1(\widetilde{M}, \mathbb{C})$, $\Lambda/(t - \alpha)^r$, and set $y_2 = j(y_1)$. Let (V_+, V_-) be cocycles in $\overline{Z}^1(\widehat{\Gamma}, \mathcal{G}_+ \oplus \mathcal{G}_-)$ such that

$$\begin{cases} \bar{\psi}^{-1}(\{V_+\})(y_1) = g_+ \\ \bar{\psi}^{-1}(\{V_-\})(y_2) = g_-. \end{cases}$$

Then $\{V_+\}$ (resp. $\{V_-\}$) is a basis of $\overline{H}^1(\widehat{\Gamma}, \mathcal{G}_+)$ (resp. $\overline{H}^1(\widehat{\Gamma}, \mathcal{G}_-)$) such that $j^*(V_+) = V_-$.

We will use later the following result whose proof will be postponed to Section 6. One can show that this result, which provides us a basis of $H^2(\Gamma, \mathcal{G}_+)$, may be generalized to the case where the torsion of $H_1(\widetilde{M}, \mathbb{C})$ is arbitrary, but we will restrict ourselves to the special case where the torsion is cyclic.

Proposition 4.1 *If $H_1(\widetilde{M}, \mathbb{C})$ is a cyclic module with $(t - \alpha)$–torsion $\Lambda/(t - \alpha)^r$, $r \geq 2$, and $W \in Z^1(\Gamma, \mathcal{H})$ a nonzero cocycle, then there exist cochains $h_i^+ \in C^1(\Gamma, \mathcal{G}_+)$, $2 \leq i \leq r$ such that $\{\zeta_{r+1}(W + V_+, h_2^+, \ldots, h_r^+)\}$ is a basis of $H^2(\Gamma, \mathcal{G}_+)$.*

The proof shows that the cochains h_i^+ factor through N_2 and define cochains from $\widehat{\Gamma}$ to $\mathcal{G}_+$ still denoted by h_i^+.

Corollary 4.2 *Let the cochains $h_i^- := j^*h_i^+ \in \overline{C}^1(\widehat{\Gamma}, \mathcal{G}_-)$, $2 \leq i \leq r$. Then $\{\zeta_{r+1}(W + V_-, h_2^-, \ldots, h_r^-)\}$ is a basis of $\overline{H}^2(\widehat{\Gamma}, \mathcal{G}_-)$.*

In fact the cochains h_i^- are defined on Γ.

The cocycle $U_1 = V_+ + V_-$ satisfies $j^*U_1 = U_1$.

Consider the second obstruction equation

(E_2)
$$-\partial U_2 = \zeta(U_1) = \frac{1}{2}U_1 \smile U_1.$$

Then there exists a cochain $U_2 \in \overline{C}^1(\widehat{\Gamma}, \mathcal{H})$ which solves (E_2) and verifies $U_2(\mu) = 0$.

Applying the involution j^* to the obstruction equation (E_2) [3, Section 4.1], one obtains

$$-\partial(j^*U_2) = -\frac{1}{2}U_1 \smile U_1 = \partial U_2.$$

So $j^*U_2 = -U_2 + Z$ where $Z \in \overline{Z}^1(\widehat{\Gamma}, \mathcal{H})$. As $(j^*U_2)(\mu) = U_2(\mu^{-1}) = 0 = U_2(\mu)$, $Z(\mu) = 0$ and $Z = 0$. Thus $j^*U_2 = -U_2$.

Consider now the 2–cocycle $\zeta_3(U_1, U_2) \in \overline{Z}^2(\widehat{\Gamma}, \mathcal{G}_+ \oplus \mathcal{G}_-)$; then $\{\zeta_3(U_1, U_2 + Z)\} = \{\zeta_3(U_1, U_2)\} + \{W \cup Z\}$ for all $Z \in \overline{Z}^1(\widehat{\Gamma}, \mathfrak{sl}_{\rho_\lambda})$.

By Corollary 3.4 (2), as $r \geq 2$,

$\{\zeta_3(U_1, U_2)\} = 0$ in $\overline{H}^2(\widehat{\Gamma}, \mathfrak{sl}_{\rho_\lambda})$ if and only if

$$\{\zeta_3(U_1, U_2 + Z)\} = 0 \text{ in } \overline{H}^2(\widehat{\Gamma}, \mathfrak{sl}_{\rho_\lambda}) \text{ for all } Z \in \overline{Z}^1(\widehat{\Gamma}, \mathfrak{sl}_{\rho_\lambda}).$$

Notation In the sequel we let $\phi_\lambda \colon \Gamma \to \mathrm{SL}$ be the reducible nonabelian representation given by $\phi_\lambda = \exp(V_+)\rho_\lambda$.

Lemma 4.3 *Suppose that $r = 2$. Then*

$$\dim_{\mathbb{C}} Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) = \begin{cases} 4 & \text{if } \{\zeta_3(U_1, U_2)\} \neq 0 \text{ in } H^2(\Gamma, \mathfrak{sl}_{\rho_\lambda}) \\ 5 & \text{if } \{\zeta_3(U_1, U_2)\} = 0 \text{ in } H^2(\Gamma, \mathfrak{sl}_{\rho_\lambda}). \end{cases}$$

More precisely, let $W \in Z^1(\Gamma, \mathbb{C})$ be a cocycle such that

$$W(\mu) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, with the same notation as above we obtain:

(1) *if $\{\zeta_3(U_1, U_2)\} \neq 0$ in $H^2(\Gamma, \mathfrak{sl}_{\rho_\lambda})$ then*

$$Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) = B^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) \oplus \left\langle \left(W + h_2^+ - \frac{1}{2}[W, V_+] \right) \right\rangle.$$

(2) if $\{\zeta_3(U_1, U_2)\} = 0$ in $H^2(\Gamma, \mathfrak{sl}_{\rho_\lambda})$ and $U_3 \in C^1(\Gamma, \mathcal{G}_+ \oplus \mathcal{G}_-)$ is a cochain such that $-\partial U_3 = \zeta_3(U_1, U_2)$ then

$$Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) = B^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) \oplus \left\langle \left(W + h_2^+ - \frac{1}{2}[W, V_+] \right) \right\rangle$$

$$\oplus \left\langle \left(V_- + U_2 + \frac{1}{2}[V_+, V_-] + U_3^+ + \frac{1}{2}[V_+, U_2] - \frac{1}{6}[V_+, [V_+, V_-]] \right) \right\rangle.$$

Proof Let $\mathcal{U} \in Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda})$ be a cocycle; then for all $\gamma_1, \gamma_2 \in \Gamma$

$$(*) \qquad \mathcal{U}(\gamma_1 \gamma_2) = \mathcal{U}(\gamma_1) + \phi_\lambda(\gamma_1) \mathcal{U}(\gamma_2) \phi_\lambda^{-1}(\gamma_1).$$

Now, using the decomposition $\mathcal{U} = \mathcal{U}^0 + \mathcal{U}^+ + \mathcal{U}^-$, where $\mathcal{U}^\pm$ takes values in $\mathcal{G}^\pm$ and $\mathcal{U}^0$ in $\mathcal{H}$, one obtains that, for all $\gamma_1, \gamma_2 \in \Gamma$, $(*)$ is equivalent to the system

$$\begin{cases} \mathcal{U}^-(\gamma_1 \gamma_2) = \mathcal{U}^-(\gamma_1) + \gamma_1 \cdot \mathcal{U}^-(\gamma_2) \\ \mathcal{U}^0(\gamma_1 \gamma_2) = \mathcal{U}^0(\gamma_1) + \mathcal{U}^0(\gamma_2) + [V_+(\gamma_1), \gamma_1 \cdot \mathcal{U}^-(\gamma_2)] \\ \mathcal{U}^+(\gamma_1 \gamma_2) = \mathcal{U}^+(\gamma_1) + \gamma_1 \cdot \mathcal{U}^+(\gamma_2) + [V_+(\gamma_1), \mathcal{U}^0(\gamma_2)] \\ \qquad\qquad + \frac{1}{2}[V_+(\gamma_1), [V_+(\gamma_1), \gamma_1 \cdot \mathcal{U}^-(\gamma_2)]], \end{cases}$$

which is equivalent to

$$\begin{cases} \qquad\qquad \mathcal{U}^- \in Z^1(\Gamma, \mathcal{G}_-) \\ \qquad -\partial \mathcal{U}^0 = V_+ \smile \mathcal{U}^- \\ -\partial \mathcal{U}^+(\gamma_1, \gamma_2) = V_+ \smile \mathcal{U}^0(\gamma_1, \gamma_2) + \frac{1}{2}[V_+(\gamma_1), V_+ \smile \mathcal{U}_-(\gamma_1, \gamma_2)]. \end{cases}$$

Note that if $\mathcal{U}^- = 0$ then $\mathcal{U}^0 \in Z^1(\Gamma, \mathcal{H})$ and if $\mathcal{U}^- = 0$ and $\mathcal{U}^0 = 0$ then $\mathcal{U}^+ \in Z^1(\Gamma, \mathcal{G}_+)$. On the other hand, as $\dim_{\mathbb{C}} Z^1(\Gamma, \mathcal{G}_\pm) = 2$ and $\dim_{\mathbb{C}} Z^1(\Gamma, \mathcal{H}) = 1$, one obtains $\dim_{\mathbb{C}} Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) \leq 5$. Observe further that the space of coboundaries, $B^1(\Gamma, \mathfrak{sl}_{\phi_\lambda})$, is three dimensional and is spanned by the following three elements $v_i = (\mathcal{U}_i^-, \mathcal{U}_i^0, \mathcal{U}_i^+) \colon \Gamma \to \mathfrak{sl}$:

$$v_1 = (0, 0, \partial B_+), \quad v_2 = (0, 0, V_+), \quad v_3 = (\partial B_-, *, *), \quad B_\pm \in \mathcal{G}_\pm.$$

A simple cohomological calculation enables us to show that, when U_3 exists, the cochains $v_4, v_5 \colon \Gamma \to \mathfrak{sl}$ given, respectively, by

$$\begin{cases} \mathcal{U}^- = 0 \\ \mathcal{U}^0 = W \\ \mathcal{U}^+ = h_2^+ - \frac{1}{2}[W, V_+] \end{cases} \qquad \begin{cases} \mathcal{U}^- = V_- \\ \mathcal{U}^0 = U_2 + \frac{1}{2}[V_+, V_-] \\ \mathcal{U}^+ = U_3^+ + \frac{1}{2}[V_+, U_2] - \frac{1}{6}[V_+, [V_+, V_-]] \end{cases}$$

are linearly independent solutions of the preceding system which are not coboundaries. When $\{\zeta_3(U_1, U_2)\} \neq 0$ in $H^2(\Gamma, \mathfrak{sl}_{\rho_\lambda})$, v_4 is the unique solution of the system which is not a coboundary. $\qquad\square$

4.3 The case where $r = 2$ and $\{\zeta_3(U_1, U_2)\} = 0$ in $H^2(\Gamma, \mathfrak{sl}_{\rho_\lambda}))$

In this case we will construct explicitly a formal deformation of the abelian representation ρ_λ by resolving all the obstruction equations (for more details see Ben Abdelghani and Lines [4]).

As $\{\zeta_3(U_1, U_2)\} = 0$ in $\overline{H^2}(\widehat{\Gamma}, \mathfrak{sl}_{\rho_\lambda})$, we choose a cochain $U_3 \in \overline{C}^1(\widehat{\Gamma}, \mathcal{G}_+ \oplus \mathcal{G}_-)$ such that

$$\begin{cases} -\partial U_3 = \zeta_3(U_1, U_2) \\ j^* U_3 = U_3. \end{cases}$$

This is always possible. Indeed, if $j^* U_3 = U_3$ we are done, if not we choose U_3^0 any cochain such that $-\partial U_3^0 = \zeta_3(U_1, U_2)$ and we set $U_3 = \frac{1}{2}(U_3^0 + j^*(U_3^0))$.

Now, $\{\zeta_4(U_1, U_2, U_3)\} = 0$ in $\overline{H^2}(\widehat{\Gamma}, \mathcal{H})$ and there exists a cochain $U_4 \in \overline{C}^1(\widehat{\Gamma}, \mathcal{H})$ such that

$$\begin{cases} -\partial U_4 = \zeta_4(U_1, U_2, U_3) \\ U_4(\mu) = 0. \end{cases}$$

Using the properties of the involution as previously, it is simple to see that $j^* U_4 = -U_4$. The 2–cocycle $\zeta_5(U_1, U_2, U_3, U_4)$ may not be cohomologous to zero in $\overline{H^2}(\widehat{\Gamma}, \mathfrak{sl}_{\rho_\lambda})$, so we will perturb the solution U_2 by a trivial cocycle and study the changes on the related cochains.

Let $\beta \in \mathbb{C}$; then for each cocycle $W \in Z^1(\Gamma, \mathcal{H})$ such that

$$W(\mu) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

the cochain $U_2 + \beta W$ solves (E_2) and verifies $j^*(U_2 + \beta W) = -(U_2 + \beta W)$. The cochain $U_3 + \beta(h_2^+ + h_2^-)$ is then a solution of (E_3) such that $j^* \left(U_3 + \beta(h_2^+ + h_2^-) \right) = U_3 + \beta(h_2^+ + h_2^-)$ and

$$\zeta_4 \left(U_1, U_2 + \beta W, U_3 + \beta(h_2^+ + h_2^-) \right) = \zeta_4(U_1, U_2, U_3) + \beta \eta_4$$

where, for all $\gamma_1, \gamma_2 \in \widehat{\Gamma}$,

$$\eta_4(\gamma_1, \gamma_2) = \frac{1}{2} U_1 \smile (h_2^+ + h_2^-)(\gamma_1, \gamma_2) + \frac{1}{2}(h_2^+ + h_2^-) \smile U_1(\gamma_1, \gamma_2)$$

$$+ \frac{1}{12} [[U_1(\gamma_1), \gamma_1 \cdot W(\gamma_2)] + [W(\gamma_1), \gamma_1 \cdot U_1(\gamma_2)], \gamma_1 \cdot U_1(\gamma_2) - U_1(\gamma_1)].$$

As $\eta_4 \in \overline{Z}^2(\widehat{\Gamma}, \mathcal{H})$, there exists a cochain $Z_4 \in \overline{C}^1(\widehat{\Gamma}, \mathcal{H})$ such that

$$\begin{cases} -\partial Z_4 = \eta_4 \\ Z_4(\mu) = 0. \end{cases}$$

Using again the properties of the involution, one can prove that $j^* Z_4 = -Z_4$.

Then a simple calculation gives

$$\begin{aligned} \zeta_5 \left(U_1, U_2 + \beta W, U_3 + \beta(h_2^+ + h_2^-), U_4 + \beta Z_4 \right) \\ = \zeta_5(U_1, U_2, U_3, U_4) + \beta \eta_5 + \beta^2 \left(\zeta_3(W + V_+, h_2^+) + \zeta_3(W + V_-, h_2^-) \right), \end{aligned}$$

where $\eta_5 \in \overline{Z}^2(\widehat{\Gamma}, \mathcal{G}_+ \oplus \mathcal{G}_-)$ and is such that $j^* \eta_5 = \eta_5$.

By [Proposition 4.1](#), there exist two tuples (δ, δ'), $(\lambda_5, \lambda_5') \in \mathbb{C}^2$ such that

$$\{\eta_5\} = \delta\{\zeta_3(W + V_1, h_2^+)\} + \delta'\{\zeta_3(W + V_2, h_2^-)\}$$

and $\quad \{\zeta_5(U_1, U_2, U_3, U_4)\} = \lambda_5\{\zeta_3(W + V_1, h_2^+)\} + \lambda_5'\{\zeta_3(W + V_2, h_2^-)\}.$

Using the techniques of [3, Proposition 4.7], one can show $\delta = \delta'$ and $\lambda_5 = \lambda_5'$. So

$$\{\eta_5\} = \delta\{\zeta_3(W + V_1, h_2^+) + \zeta_3(W + V_2, h_2^-)\}$$

and $\quad \{\zeta_5(U_1, U_2, U_3, U_4)\} = \lambda_5\{\zeta_3(W + V_1, h_2^+) + \zeta_3(W + V_2, h_2^-)\}.$

On the other hand,

$$\begin{aligned} \{\zeta_5\left(U_1, U_2 + \beta W, U_3 + \beta(h_2^+ + h_2^-), U_4 + \beta Z_4\right)\} \\ = \{\zeta_5(U_1, U_2, U_3, U_4)\} + (\beta\delta + \beta^2)\{\zeta_3(W + V_+, h_2^+) + \zeta_3(W + V_-, h_2^-)\} \\ = (\beta^2 + \beta\delta + \lambda_5)\{\zeta_3(W + V_+, h_2^+) + \zeta_3(W + V_-, h_2^-)\}. \end{aligned}$$

By these equalities and as $\{\zeta_3(W + V_+, h_2^\pm)\} \neq 0$ in $\overline{H}^2(\widehat{\Gamma}, \mathcal{G}_\pm)$, we can see that there exist cochains, which we will denote U_2^0, U_3^0, U_4^0, such that $\{\zeta_5(U_1, U_2^0, U_3^0, U_4^0)\} \neq 0$ in $\overline{H}^2(\widehat{\Gamma}, \mathfrak{sl}_{\rho_\lambda})$. Then, we can assert that there exists $\lambda_5 \in \mathbb{C}^*$ such that

$$\{\zeta_5(U_1, U_2^0, U_3^0, U_4^0)\} = \lambda_5(\{\zeta_3(W + V_+, h_2^+) + \zeta_3(W + V_-, h_2^-)\}).$$

Note that, if β is a solution of the equation $\beta^2 + \beta\delta + \lambda_5 = 0$ then $\beta \neq 0$. So, by the equalities above, we can always manage to have two distinct possible choices $\beta_0 \in \mathbb{C}^*$ for which

$$\{\zeta_5(U_1, U_2^0 + \beta_0 W, U_3^0 + \beta_0(h_2^+ + h_2^-), U_4^0 + \beta_0 Z_4)\} = 0 \quad \text{in } \overline{H}^2(\widehat{\Gamma}, \mathfrak{sl}_{\rho_\lambda}).$$

We denote $U_2 := U_2^0 + \beta_0 W$, $U_3 := U_3^0 + \beta_0(h_2^+ + h_2^-)$, $U_4 := U_4^0 + \beta_0 Z_4$, so

$$\{\zeta_5(U_1, U_2, U_3, U_4)\} = 0 \quad \text{in } \overline{H}^2(\widehat{\Gamma}, \mathfrak{sl}_{\rho_\lambda})$$

and there exists a cochain U_5 in $\overline{C}^1(\widehat{\Gamma}, \mathcal{G}_+ \oplus \mathcal{G}_-)$ such that

$$\begin{cases} -\partial U_5 = \zeta_5(U_1, U_2, U_3, U_4) \\ j^* U_5 = U_5. \end{cases}$$

Suppose that for all $1 \le i \le n-1$, there exists a cochain $U_{2i+1} \in \overline{C}^1(\widehat{\Gamma}, \mathcal{G}_+ \oplus \mathcal{G}_-)$ and a cochain $U_{2i+2} \in \overline{C}^1(\widehat{\Gamma}, \mathcal{H})$ such that

$$\begin{cases} -\partial U_{2i+1} = \zeta_{2i+1}(U_1, \ldots, U_{2i}) \\ j^* U_{2i+1} = U_{2i+1} \end{cases} \qquad \begin{cases} -\partial U_{2i+2} = \zeta_{2i+1}(U_1, \ldots, U_{2i+1}) \\ j^* U_{2i+2} = -U_{2i+2} \end{cases}$$

and consider the 2–cocycle $\zeta_{2n+1}(U_1, \ldots, U_{2n}) \in \overline{Z}^2(\widehat{\Gamma}, \mathcal{G}_+ \oplus \mathcal{G}_-)$. If the latter is cohomologous to zero in $\overline{H}^2(\widehat{\Gamma}, \mathcal{G}_+ \oplus \mathcal{G}_-)$ we are done, if not, there exists $\lambda_{2n+1} \in \mathbb{C}^*$ such that

$$\{\zeta_{2n+1}(U_1, \ldots, U_{2n})\} = \lambda_{2n+1}(\{\zeta_3(W + V_+, h_2^+) + \zeta_3(W + V_-, h_2^-)\}).$$

Now for $\beta \in \mathbb{C}$,

$$\{\zeta_{2n+1}(U_1, \ldots, U_{2n-2} + \beta W, U_{2n-1} + \beta(h_2^+ + h_2^-), U_{2n} + \beta Z_4)\}$$
$$= \{\zeta_{2n+1}(U_1, \ldots, U_{2n})\} + \beta\{\eta_4\} + \beta\beta_0(\{\zeta_3(W + V_+, h_2^+) + \zeta_3(W + V_-, h_2^-)\})$$
$$= (\lambda_{2n+1} + \beta\delta + \beta\beta_0)(\{\zeta_3(W + V_+, h_2^+) + \zeta_3(W + V_-, h_2^-)\}).$$

As $\beta_0 + \delta \neq 0$ (by our assumption on λ_5), for $\beta = -(\lambda_{2n+1})/(\beta_0 + \delta)$ we have

$$\{\zeta_{2n+1}(U_1, \ldots, U_{2n-2} + \beta W, U_{2n-1} + \beta(h_2^+ + h_2^-), U_{2n} + \beta Z_4)\} = 0$$
$$\text{in } \overline{H}^2(\widehat{\Gamma}, \mathfrak{sl}_{\rho_\lambda}).$$

Let then U_{2n+1} be a cochain in $\overline{C}^1(\widehat{\Gamma}, \mathcal{G}_+ \oplus \mathcal{G}_-)$ which solves (E_{2n+1}) such that $j^* U_{2n+1} = U_{2n+1}$, (such a cochain exists, cf [4] for the details).

This finishes the resolution of the infinite sequence of obstruction equations.

Now let $q: \Gamma \to \widehat{\Gamma}$ denote the natural map and $U_t = \sum_{i \ge 1} U_i t^i$ be the formal curve constructed above. Then by a theorem of M Artin [1] there exists a convergent curve $\widetilde{U}_t$ such that $\widetilde{U}_t = U_t \circ q \mod t^{n_0} \mathfrak{sl}[\![t]\!]$ (where $n_0 \ge 2$ is a fixed integer) and $\rho_t = \exp \widetilde{U}_t \rho_\lambda$ is an arc of representations of Γ in SL for $t > 0$ (see Ben Abdelghani and Lines [4] for more details).

This proves that $U_1 = V_+ + V_- \in \mathrm{TC}_{\rho_\lambda}$ and, by Lemma 3.1, $Z^1(\Gamma, \mathcal{G}_+) \oplus Z^1(\Gamma, \mathcal{G}_-) \subset \mathrm{TC}_{\rho_\lambda}$.

Now, for all $x \in \mathbb{C}^*$, we have

$$\{\zeta_3(xW + U_1, x(h_2^+ + h_2^-) + U_2)\}$$
$$= x^2\{\zeta_3(W + V_+, h_2^+) + \zeta_3(W + V_-, h_2^-)\} + \{\zeta_3(U_1, U_2)\}.$$

As $r = 2$, and by Proposition 4.1, $\{\zeta_3(W + V_+, h_2^+) + \zeta_3(W + V_-, h_2^-)\} \neq 0$. Moreover, $\{\zeta_3(U_1, U_2)\} = 0$, so for all $x \in \mathbb{C}^*$, $xW + U_1 \notin \mathrm{TC}_{\rho_\lambda}^3$. Thus

$$\mathrm{TC}_{\rho_\lambda} = \mathrm{TC}_{\rho_\lambda}^3 = Z^1(\Gamma, \mathcal{H}) \oplus B^1(\Gamma, \mathcal{G}_+) \oplus B^1(\Gamma, \mathcal{G}_-) \cup Z^1(\Gamma, \mathcal{G}_+) \oplus Z^1(\Gamma, \mathcal{G}_-)$$
$$\subsetneq CQ_{\rho_\lambda} = Z^1(\Gamma, \mathfrak{sl}_{\rho_\lambda}).$$

This finishes the proof when $r = 2$ and $\{\zeta_3(U_1, U_2)\} = 0$.

4.4 The case where $r \geq 2$ and $\{\zeta_3(U_1, U_2)\} \neq 0$ in $H^2(\Gamma, \mathfrak{sl}_{\rho_\lambda})$

To proceed with the proof in this case we need the following proposition which is based on a result of [16]:

Proposition 4.4 If $r \geq 2$ and $\{\zeta_3(U_1, U_2)\} \neq 0$ in $H^2(\Gamma, \mathfrak{sl}_{\rho_\lambda})$ then the tangent cone $\mathrm{TC}_{\rho_\lambda}$ contains an irreducible component of dimension 4.

Proof Suppose that $r \geq 2$ and $\{\zeta_3(U_1, U_2)\} \neq 0$ in $H^2(\Gamma, \mathfrak{sl}_{\rho_\lambda})$, then we obtain $\dim_{\mathbb{C}} Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) = 4$ (see Lemma 4.3). As ϕ_λ is nonabelian, $\dim_{\mathbb{C}} H^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) = 1$. By [16, Lemma 7.5], every cocycle in $Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda})$ is integrable. So the representation ϕ_λ is a smooth point of $R(\Gamma)$ with local dimension 4. As ρ_λ is in the adherence of the orbit of ϕ_λ, it is contained in the same irreducible component $R_\lambda(\Gamma)$ of dimension 4. Thus, we conclude the tangent cone of $R(\Gamma)$ at ρ_λ contains an irreducible component of dimension 4. $\square$

Now if $r = 2$ and $\{\zeta_3(U_1, U_2)\} \neq 0$ in $H^2(\Gamma, \mathfrak{sl}_{\rho_\lambda})$, there exists $\beta \in \mathbb{C}^*$ such that

$$\{\zeta_3(U_1, U_2)\} = \beta\{\zeta_3(W + V_+, h_2^+) + \zeta_3(W + V_-, h_2^-)\},$$

and for $\beta_0 = \pm\sqrt{-\beta}$, we obtain

$$\{\zeta_3(\beta_0 W + U_1, U_2 + \beta_0(h_2^+ + h_2^-))\} = 0 \text{ in } H^2(\Gamma, \mathfrak{sl}_{\rho_\lambda}).$$

Using Lemma 3.1, we deduce that the cubic tangent cone is given by

$$\mathrm{TC}^3_{\rho_\lambda}(\Gamma) = \{\delta(\beta_0 W + x V_+ + x^{-1} V_-) + B_+ + B_- \mid \delta \in \mathbb{C}, x \in \mathbb{C}^*, B_\pm \in B^1(\Gamma, \mathcal{G}_\pm)\}$$
$$\cup \{\delta W + B_+ + B_- \mid \delta \in \mathbb{C}, B_\pm \in B^1(\Gamma, \mathcal{G}_\pm)\}.$$

Hence, it has two components: one of dimension 4 and one of dimension 3; so it is of dimension 4. As $\mathrm{TC}_{\rho_\lambda} \subset \mathrm{TC}^3_{\rho_\lambda}$ and $\mathrm{TC}_{\rho_\lambda}$ has an irreducible component of dimension 4, we conclude that $\mathrm{TC}_{\rho_\lambda} = \mathrm{TC}^3_{\rho_\lambda}$ and $R(\Gamma)$ is of dimension 4 locally at ρ_λ.

Now, for the second part of the Theorem 1.1, we use an argument by contradiction. Suppose that $r \geq 3$ and $\{\zeta_3(U_1, U_2)\} \neq 0$ in $H^2(\Gamma, \mathfrak{sl}_{\rho_\lambda})$. Then, by Proposition 4.1, $\{\zeta_3(W + V_\pm, h_2^\pm)\} = 0$ and for all $\delta \in \mathbb{C}$, $\{\zeta_3(\delta W + U_1, U_2 + \delta(h_2^+ + h_2^-))\} \neq 0$ in $H^2(\Gamma, \mathfrak{sl}_{\rho_\lambda})$. So, we will have

$$\mathrm{TC}^3_{\rho_\lambda} = Z^1(\Gamma, \mathcal{H}) \oplus B^1(\Gamma, \mathcal{G}_+) \oplus B^1(\Gamma, \mathcal{G}_-),$$

and $\mathrm{TC}_{\rho_\lambda} = \mathrm{TC}^3_{\rho_\lambda}$ will be of dimension 3. This is impossible by Proposition 4.4.

Thus, we conclude that if $r \geq 3$ then necessarily $\{\zeta_3(U_1, U_2)\} = 0$ in $H^2(\Gamma, \mathfrak{sl}_{\rho_\lambda})$ and for all $\delta \in \mathbb{C}$, $\delta W + U_1 \in \mathrm{TC}^3_{\rho_\lambda}$. This implies that $\mathrm{TC}^3_{\rho_\lambda} = Z^1(\Gamma, \mathfrak{sl}_{\rho_\lambda})$ is of dimension 5 and, in particular, $\mathrm{TC}_{\rho_\lambda} \subsetneq \mathrm{TC}^3_{\rho_\lambda}$ because there are no 5–dimensional components going through ρ_λ.

5 Applications and examples

As an application to Theorem 1.1, we prove Proposition 1.2 which describes the local geometry of the representation variety at an abelian representation and at a reducible nonabelian representation with the same character χ_λ as the abelian one. Further, we prove Theorem 1.3, that describes the local geometry of the character variety at χ_λ and we illustrate our discussion with the examples of the two-bridge knot $b(49, 17)$ and the knot 8_{20}.

Let $\alpha := \lambda^2$ be a root of Δ_1 and suppose that the $(t - \alpha)$–torsion of the Alexander module $H_1(\pi_1(\widetilde{M}), \mathbb{C})$ is cyclic of the form $\Lambda/(t - \alpha)^2 \oplus \Lambda/(t - \alpha^{-1})^2$, $r \geq 2$. Let $\rho_\lambda \colon \Gamma \to \mathrm{SL}$ be the abelian representation given by $\rho_\lambda(\mu) = \mathrm{diag}(\lambda, \lambda^{-1})$ and $\phi_\lambda \colon \Gamma \to \mathrm{SL}$ be the reducible nonabelian representation given by $\phi_\lambda = \exp(V_+)\,\rho_\lambda$.

Proof of Proposition 1.2 (1) Suppose that $\dim_{\mathbb{C}} Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) = 4$, then by Section 4.4, ϕ_λ is a smooth point of $R(\Gamma)$ which is contained in a unique 4–dimensional component $R_\lambda(\Gamma)$. In particular, ϕ_λ is scheme reduced and $\mathrm{TC}_{\phi_\lambda} = Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) = \langle (W + h_2^+ - 1/2[W, V_+]) \rangle \oplus B^1(\Gamma, \mathfrak{sl}_{\Phi_\lambda})$.

Now to prove the fact that $S(\Gamma)$ and $R_\lambda(\Gamma)$ are the only irreducible components of $R(\Gamma)$ that contain ρ_λ and that ρ_λ is a smooth point of both varieties, it suffices to proceed exactly as in [4, Corollary 5.3; 17, Corollary 1.3].

(2)   If $\dim_{\mathbb{C}} Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) = 5$, then by Section 4.3, ρ_λ is contained in at least one 4–dimensional component which contains irreducible representations. By [4, Theorem 5.2, ii)], the nonabelian representation ϕ_λ is contained in each 4–dimensional component of $R(\Gamma)$ containing ρ_λ and containing irreducible representations.

By the analysis of the tangent cone in Section 4.3, we proved the existence of an arc of irreducible representations with endpoint ρ_λ of the form

$$\rho_t = \exp\left(tU_1 + t^2(U_2^0 + \beta_0 W) + t^3(U_3^0 + \beta_0(h_2^+ + h_2^-)) + o(t^4)\right)\rho_\lambda.$$

Set, for $t > 0$, $\Phi_t = A_t^{-1}\rho_t A_t$ with

$$A_t = \begin{pmatrix} t^{1/2} & 0 \\ 0 & t^{-1/2} \end{pmatrix}.$$

Then $\Phi_t = \exp\left(V_+ + t^2(V_- + U_2^0 + \beta_0 W + (U_3^0)^+ + \beta_0 h_2^+) + o(t^3)\right)\rho_\lambda,$

and Φ_t is an arc of irreducible representations with endpoint ϕ_λ. Using the Campbell–Hausdorff formula [6, Chapter II, Section 6.5], one can see that Φ_t is tangent to the cocycle

$$Z = \beta_0\left(W + h_2^+ - \frac{1}{2}[W, V_+]\right) + V_- + U_2^0 + \frac{1}{2}[V_+, V_-]$$
$$+ (U_3^0)^+ + \frac{1}{2}[V_+, U_2^0] - \frac{1}{6}[V_+, [V_+, V_-]].$$

This proves that $Z \in \mathrm{TC}_{\phi_\lambda}$. As we have two possible choices for β_0, which will be denoted β_1, β_2, we have two possible choices for the cocycle Z which we denote by Z_1 and Z_2.

As $\beta_0 \neq 0$ and by Lemma 4.3, $\langle Z_1, Z_2 \rangle \oplus B^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) = Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda})$ and $Z_1, Z_2 \in \mathrm{TC}_{\phi_\lambda} \subset T_{\phi_\lambda}^{\mathrm{Zar}} R(\Gamma)$, so $Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) \subset T_{\phi_\lambda}^{\mathrm{Zar}} R(\Gamma)$ and $Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) = T_{\phi_\lambda}^{\mathrm{Zar}} R(\Gamma)$. This proves that ϕ_λ is scheme reduced.

On the other hand, it is easy to see that there is no 5–component containing ϕ_λ because there is no 5–component containing ρ_λ. This implies that $\dim_{\mathbb{C}} \mathrm{TC}_{\phi_\lambda} = 4$ and has at least two components:

$$\langle Z_1 \rangle \oplus B^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) \cup \langle Z_2 \rangle \oplus B^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) \subset \mathrm{TC}_{\phi_\lambda}.$$

It remains to prove that $QC_{\phi_\lambda} \subsetneq Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda})$ and that

$$TC_{\phi_\lambda} = QC_{\phi_\lambda} = \langle Z_1 \rangle \oplus B^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) \cup \langle Z_2 \rangle \oplus B^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}).$$

Let $\mathcal{U} = \alpha_1 Z_1 + \alpha_2 Z_2 \in QC_{\phi_\lambda}$. Our goal is to prove that either $\alpha_1 = 0$ or $\alpha_2 = 0$. Let ϕ_t be the formal deformation modulo order 2 of ϕ_λ given by

$$\phi_t = \exp\left(t\mathcal{U} + o(t^2)\right) \exp(V_+) \rho_\lambda.$$

Set $\rho_t = B_t \phi_t B_t^{-1}$ with

$$B_t = \begin{pmatrix} t^{1/4} & 0 \\ 0 & t^{-1/4} \end{pmatrix}.$$

Then

$$\begin{aligned}
\rho_t &= B_t \exp\left(t(\alpha_1 Z_1 + \alpha_2 Z_2) + o(t^2)\right) \exp(V_+) B_t^{-1} \rho_\lambda \\
&= \exp\left(t(\alpha_1(Z_1^0 + t^{1/2} Z_1^+ + t^{-1/2} Z_1^-) + \alpha_2(Z_2^0 + t^{1/2} Z_2^+ + t^{-1/2} Z_2^-) + o(t^2)\right) \\
&\quad \times \exp(t^{1/2} V_+) \rho_\lambda \\
&= \exp\left(t^{1/2}(\alpha_1 Z_1^- + \alpha_2 Z_2^-) + t(\alpha_1 Z_1^0 + \alpha_2 Z_2^0) + o(t^{3/2})\right) \exp(t^{1/2} V_+) \rho_\lambda \\
&= \exp\left(t^{1/2}(V_+ + \alpha_1 Z_1^- + \alpha_2 Z_2^-) + t(\alpha_1 Z_1^0 + \alpha_2 Z_2^0 + \tfrac{1}{2}[\alpha_1 Z_1^- + \alpha_2 Z_2^-, V_+])\right) \\
&\quad + o(t^{3/2}))\rho_\lambda.
\end{aligned}$$

As

$$\begin{cases} V_+ + \alpha_1 Z_1^- + \alpha_2 Z_2^- = V_+ + (\alpha_1 + \alpha_2) V_- \\ \alpha_1 Z_1^0 + \alpha_2 Z_2^0 + \tfrac{1}{2}[\alpha_1 Z_1^- + \alpha_2 Z_2^-, V_+] = (\alpha_1 \beta_1 + \alpha_2 \beta_2) W + (\alpha_1 + \alpha_2) U_2^0, \end{cases}$$

we obtain

$$\rho_t = \exp\left(t^{1/2}\left(V_+ + (\alpha_1 + \alpha_2) V_-\right) + t\left((\alpha_1 \beta_1 + \alpha_2 \beta_2) W + (\alpha_1 + \alpha_2) U_2^0\right) + o(t^{3/2})\right)\rho_\lambda.$$

The analysis of the tangent cone at ρ_λ in shows that we have necessarily

$$\begin{cases} \alpha_1 + \alpha_2 = x \\ \alpha_1 \beta_1 + \alpha_2 \beta_2 = \beta_0 x \end{cases} \quad x \in \mathbb{C} \text{ and } \beta_0 = \beta_1 \text{ or } \beta_2.$$

If $\beta_0 = \beta_1$ we obtain $x = \alpha_1$ and if $\beta_0 = \beta_2$ we obtain $x = \alpha_2$. Thus $\alpha_2 = 0$ or $\alpha_1 = 0$. This implies that $\mathcal{U} = x Z_1$ or $\mathcal{U} = x Z_2$; $x \in \mathbb{C}$. So we conclude that

$$QC_{\phi_\lambda} = \langle Z_1 \rangle \oplus B^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) \cup \langle Z_2 \rangle \oplus B^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) \subset TC_{\phi_\lambda}.$$

As $TC_{\phi_\lambda} \subset QC_{\phi_\lambda}$, we are done. $\qquad \square$

Remarks (1) If $r = 2$ and $\dim_{\mathbb{C}} Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) = 5$, then ϕ_λ is a quadratic singularity of $R(\Gamma)$.

(2) If $r \geq 3$ then $\dim_{\mathbb{C}} Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) = 5$ and ϕ_λ is necessarily a singular point of $R(\Gamma)$.

Proof of Theorem 1.3 (1) By [17, Lemma 5.4], the projection $\pi: R(\Gamma) \to X(\Gamma)$ is open at the abelian representation ρ_λ; ie if U is a classical neighborhood of ρ_λ then $\pi(U)$ is a classical neighborhood of χ_λ. Moreover, by Proposition 1.2, the analytic germ of $R(\Gamma)$ at ρ_λ has only two irreducible components which are precisely the germ of $R_\lambda(\Gamma)$ and the germ of $S(\Gamma)$. These two results imply that $X_\lambda(\Gamma)$ and $Y(\Gamma)$ are the unique irreducible components of $X(\Gamma)$ that contain χ_λ.

Now, Proposition 5.6 of [17] asserts that χ_λ is a smooth point of the curve $Y(\Gamma)$. So it remains to prove that χ_λ is a smooth point of the curve $X_\lambda(\Gamma)$ and that $X_\lambda(\Gamma)$ and $Y(\Gamma)$ are tangent curves. As the set of irreducible representations of $R_\lambda(\Gamma)$ is dense in $R_\lambda(\Gamma)$, $X_\lambda(\Gamma) \subset \overline{X^{\mathrm{irr}}(\Gamma)}$. By Theorem A of [7], χ_λ is a smooth point of $\overline{X^{\mathrm{irr}}(\Gamma)}$ and therefore of the irreducible component $X_\lambda(\Gamma)$. By [11, Corollary 1.5.3], $\dim X_\lambda(\Gamma) = \dim R_\lambda(\Gamma) - 3 = 1$ and $X_\lambda(\Gamma)$ is a curve in $X(\Gamma)$.

We recall that the function algebra $\mathbb{C}[X(\Gamma)]$ is finitely generated by the evaluation functions $I_\gamma: X(\Gamma) \to \mathbb{C}$; $\chi_\rho \mapsto I_\gamma(\chi_\rho) = \mathrm{tr}(\rho(\gamma))$, where $\gamma \in \Gamma$, which are regular maps [11]. Let $W \in Z^1(\Gamma, \mathbb{C})$ be an abelian cocycle such that

$$W(S_1) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then $\rho_s(\gamma) = \exp(sW(\gamma))\rho_\lambda(\gamma)$, $\gamma \in \Gamma$, is a path of abelian representations with endpoint ρ_λ and tangent to W at $s = 0$. For $\gamma \in \Gamma$, we write

$$W(\gamma) = \begin{pmatrix} |\gamma| & 0 \\ 0 & -|\gamma| \end{pmatrix}$$

with $|\gamma| = \Phi(\gamma)$ where $\Phi: \Gamma \to \Gamma/\Gamma' \cong \mathbb{Z}$ designs the canonical projection. Since, for $\gamma \in \Gamma$,

$$I_\gamma(\pi(\rho_s)) = \mathrm{tr}\,\rho_s(\gamma) = \lambda^{|\gamma|} + \frac{1}{\lambda^{|\gamma|}} + s|\gamma|\left(\lambda^{|\gamma|} - \frac{1}{\lambda^{|\gamma|}}\right) + o(s^2),$$

it follows that $d_{\chi_\lambda} I_{S_i}(d_{\rho_\lambda}\pi(W)) = \lambda - 1/\lambda \neq 0$. Thus, $d_{\rho_\lambda}\pi(W)$ is a basis for $T^{\mathrm{Zar}}_{\chi_\lambda} Y(\Gamma) \cong \mathbb{C}$.

On the other hand, we know by the proof of Proposition 1.2 that

$$\phi_s(\gamma) = \exp\left(s\big(W(\gamma) + h_2^+(\gamma) - 1/2[W(\gamma), V_+(\gamma)]\big) + o(s^2)\right)\phi_\lambda(\gamma)$$

is a path of irreducible representations in $R_\lambda(\Gamma)$ with endpoint ϕ_λ and tangent to $\mathcal{U} = W + h_2^+ - 1/2[W, V_+]$ at $s = 0$. As adding elements of $\mathcal{G}_+$ to a deformation of ρ_λ or ϕ_λ do not change the trace at the first order, we obtain for $\gamma \in \Gamma$

$$I_\gamma \left(\pi(\phi_s) \right) = \operatorname{tr} \phi_s(\gamma) = \lambda^{|\gamma|} + \frac{1}{\lambda^{|\gamma|}} + s|\gamma| \left(\lambda^{|\gamma|} - \frac{1}{\lambda^{|\gamma|}} \right) + o(s^2).$$

It follows that for all $\gamma \in \Gamma$

$$d_{\chi_\lambda} I_\gamma \left(d_{\phi_\lambda} \pi(\mathcal{U}) \right) = |\gamma| \left(\lambda^{|\gamma|} - \frac{1}{\lambda^{|\gamma|}} \right) = d_{\chi_\lambda} I_\gamma \left(d_{\rho_\lambda} \pi(W) \right).$$

Therefore $d_{\phi_\lambda} \pi(\mathcal{U})$ is a basis of $T_{\chi_\lambda}^{\mathrm{Zar}} X_\lambda(\Gamma) \cong \mathbb{C}$ and $d_{\phi_\lambda} \pi(\mathcal{U}) = d_{\rho_\lambda} \pi(W)$. Thus, $T_{\chi_\lambda}^{\mathrm{Zar}} Y(\Gamma) = T_{\chi_\lambda}^{\mathrm{Zar}} X_\lambda(\Gamma)$ and $Y(\Gamma)$ and $X_\lambda(\Gamma)$ are tangent curves.

The last assertion is a consequence of [7, Theorem A].

(2) By the second assertion of Proposition 1.2, the representation ϕ_λ is a singular point of $R(\Gamma)$ which is contained in at least two 4–dimensional local components of $R(\Gamma)$ which contain irreducible representations. Thus, as ρ_λ is contained in the adherence of the orbit of ϕ_λ, the abelian representation ρ_λ is itself contained in these 4–dimensional components of $\overline{R^{\mathrm{irr}}(\Gamma)}$. Using again the fact that the projection $\pi\colon R(\Gamma) \to X(\Gamma)$ is open at the abelian representation ρ_λ, we can assert that χ_λ is contained in at least 2 irreducible components of $\overline{X^{\mathrm{irr}}(\Gamma)}$. This implies, in particular, that χ_λ is not a smooth point of $\overline{X^{\mathrm{irr}}(\Gamma)}$. $\qquad \square$

5.1 The knot $b(49, 17)$

Let $K \subset S^3$ be the 2–bridge knot $b(49, 17)$ [10]. This is an alternating knot with 12 crossings in a minimal projection and with Alexander polynomial $\Delta_1(t) = (2t^2 - 3t + 2)^2$. Let λ be a complex number on the unit circle such that $\Delta_1(\lambda^2) = 0$. Then $\alpha = \lambda^2$ is a double root of the Alexander polynomial and the Alexander module with complex coefficients of $b(49, 17)$ is given by $\Lambda/(t - \alpha)^2 \oplus \Lambda/(t - \alpha^{-1})^2$ [10].

A computer supported calculation [17] shows that $\dim_{\mathbb{C}} Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) = 4$. Section 4.4 shows that the 2–bridge knot $b(49, 17)$ admits a strongly nonquadratic singularity in the SL representation variety. Moreover, by Theorem 1.3, in this case every reducible but nonabelian representation ϕ_λ such that $\chi_{\phi_\lambda} = \chi_\lambda$ is a smooth point of the representation variety and the components $X_\lambda(\Gamma))$ and $Y(\Gamma)$ are tangent at χ_λ.

5.2 The knot 8_{20}

Let K be the knot 8_{20} in Rolfsen's list. We have that $\Delta_1(t) = (t^2 - t + 1)^2$ and we denote $\lambda := \exp(i\pi/6)$. Then this knot has an Alexander module with

complex coefficients of the form $\Lambda/(t-\alpha)^2 \oplus \Lambda/(t-\alpha^{-1})^2$ [10]. But this time $\dim_{\mathbb{C}} Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda}) = 5$. So by Section 4.3, we have again a strongly nonquadratic singularity. Moreover, by Theorem 1.3, the character χ_λ is not a smooth point of the character variety of irreducible representations. More precisely, there are at least two irreducible components of $\overline{X^{\mathrm{irr}}(\Gamma)}$ passing through X_λ.

Remarks (1) The examples of the two-bridge knot $b(49, 17)$ and the knot 8_{20} were treated in [17], but our methods differ from theirs.

(2) Each one of the knots $b(49, 17)$ and 8_{20} has a cyclic Alexander module of order 2. But our study of the local structure of their representation and character varieties proves that one cannot decide of this local structure unless one calculates the dimension of the space of cocycles $Z^1(\Gamma, \mathfrak{sl}_{\phi_\lambda})$. So this remains an open question: can we guess the local structure without cohomological computations?

6 Proof of Proposition 4.1

We will construct inductively the cochains $h_i^+ \colon \Gamma \to \mathcal{G}_+$, $2 \le i \le r$, such that the cochain $\zeta_j(W + V_+, h_2^+, \dots, h_{j-1}^+)$ is defined for every $3 \le j \le r$ and the class $\{\zeta_{r+1}(W + V_+, h_2^+, \dots, h_r^+)\} \ne 0$ in $H^2(\Gamma, \mathcal{G}_+)$.

Consider the second obstruction equation

$$(E_2) \qquad -\partial h_2^+ = \zeta_2(W + V_+) = \frac{1}{2}W \smile V_+ + \frac{1}{2}V_+ \smile W.$$

Observe first that we may suppose $h_2^+(\mu) = 0$. Indeed, if $h_2^+(\mu) = g \in \mathcal{G}_+$, we can replace h_2^+ by $h_2^+ + \partial C$ where C is the 0–cochain with value $(1/(1-\alpha))g$. Then using (E_2) and induction on k, we see that $h_2^+(\mu^k) = 0$.

Further, for all $\gamma_1, \gamma_2 \in N$, $h_2^+(\gamma_1\gamma_2) = h_2^+(\gamma_1) + h_2^+(\gamma_2)$. So we deduce that $h_2^+|_{N_2} = 0$ and h_2^+ factors through N/N_2.

Now, for all $(x, t^k), (y, t^l) \in N/N_2 \rtimes T$, Equation (E_2) is equivalent to

$$h_2^+(x + t^k y, t^{k+l}) = h_2^+(x, t^k) + \alpha^k h_2^+(y, t^l) + \zeta_2(W + V_+)\big((x, t^k), (y, t^l)\big).$$

As $(x, t^k) = (x, 1)(0, t^k)$, we obtain the equivalent system

$$\begin{cases} h_2^+(x + y, 1) = h_2^+(x, 1) + h_2^+(y, 1) \\ \qquad h_2^+(x, t^k) = h_2^+(x, 1) + \zeta_2(W + V_+)\big((x, 1), (0, t^k)\big) \\ h_2^+(t^k y, 1) - \alpha^k h_2^+(y, 1) = -\zeta_2(W + V_+)\big((t^k y, 1), (0, t^k)\big) \\ \qquad\qquad\qquad\qquad + \zeta_2(W + V_+)\big((0, t^k), (y, 1)\big). \end{cases}$$

The first equation of the above system and the fact that h_2^+ factors through N/N_2 imply that we can extend the map $N/N_2 \to \mathcal{G}_+$; $x \mapsto h_2^+(x, 1)$ to a $\mathbb{C}$–linear map denoted by φ_2^+. So the solution of (E_2) is equivalent to the existence of a $\mathbb{C}$–linear map $\varphi_2^+ \colon N/N_2 \otimes \mathbb{C} \to \mathcal{G}_+$ such that for all $y \in N/N_2 \otimes \mathbb{C}$ and $k \in \mathbb{Z}$:

$$(\tilde{E}_2) \quad \begin{aligned} \varphi_2^+&((t^k - \alpha^k)y) \\ &= -\zeta_2(W + V_+)((t^k y, 1), (0, t^k)) + \zeta_2(W + V_+)\big((0, t^k), (y, 1)\big). \end{aligned}$$

On the other hand, let $(e_0 = [1], e_1 = [t - \alpha], \dots, e_{r-1} = [(t - \alpha)^{r-1}])$ be a $\mathbb{C}$–basis for $\Lambda/(t - \alpha)^r$, then for all $0 \le i \le r - 1$:

$$(t^k - \alpha^k)e_i = a_1 e_{i+1} + \cdots + a_{r-1-i} e_{r-1} \quad \text{with } a_1 = k\alpha^{k-1}.$$

In particular

$$\begin{cases} (t^k - \alpha^k)e_{r-2} = a_1 e_{r-1} \\ (t^k - \alpha^k)e_{r-3} = a_1 e_{r-2} + a_2 e_{r-1} \\ \quad \vdots \\ (t^k - \alpha^k)e_{i+1} = a_1 e_{i+2} + \cdots + a_{r-i-2} e_{r-1} \\ (t^k - \alpha^k)e_i = a_1 e_{i+1} + \cdots + a_{r-i-1} e_{r-1} \\ (t^k - \alpha^k)e_{i-1} = a_1 e_i + \cdots + a_{r-i} e_{r-1} \\ \quad \vdots \\ (t^k - \alpha^k)e_0 = a_1 e_1 + \cdots + a_{r-1} e_{r-1}. \end{cases}$$

Letting $y = e_i$, $0 \le i \le r - 2$ in Equation $(\tilde{E}_2)$ and using the fact that $V_+(e_i) = 0$ for all $1 \le i \le r - 1$, one can see that $(\tilde{E}_2)$ is equivalent to the linear system with complex coefficients

$$\begin{pmatrix} 0 & \cdots & 0 & a_1 \\ 0 & \cdots & a_1 & a_2 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_{r-1} \end{pmatrix} \begin{pmatrix} \varphi_2^+(e_1) \\ \vdots \\ \vdots \\ \vdots \\ \varphi_2^+(e_{r-1}) \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ \vdots \\ 2k\alpha^k W(\mu) V_+(e_0) \end{pmatrix}.$$

As $a_1 \ne 0$, this system has a unique solution defined by

$$\begin{cases} \varphi_2^+(e_1) = 2\alpha W(\mu) V_+(e_0) \\ \varphi_2^+(e_i) = 0 \text{ for all } 2 \le i \le r - 1. \end{cases}$$

Note that this solution does not depend on k, as required.

Proceeding in the same manner, we can construct for all $2 \leq j \leq r$ cochains $h_j^+ \colon \Gamma \to \mathcal{G}_+$ resolving

$$(E_j) \qquad\qquad -\partial h_j^+ = \zeta_j(W + V_+, h_2^+, \dots, h_{j-1}^+)$$

and such that $h_j^+(\mu) = 0$, by constructing $\mathbb{C}$–linear maps $\varphi_j^+ \colon N/N_2 \otimes \mathbb{C} \to \mathcal{G}_+$ which satisfy the condition that $(\varphi_j^+(e_1), \dots, \varphi_j^+(e_{r-1}))$ is a solution of the system

$$(S_j) \qquad \begin{pmatrix} 0 & \cdots & 0 & a_1 \\ 0 & \cdots & a_1 & a_2 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_{r-1} \end{pmatrix} \begin{pmatrix} \varphi_j^+(e_1) \\ \vdots \\ \vdots \\ \vdots \\ \varphi_j^+(e_{r-1}) \end{pmatrix} = \begin{pmatrix} b_1^+ \\ \vdots \\ \vdots \\ \vdots \\ b_{r-1}^+ \end{pmatrix}$$

and $\quad b_i^+ = -\zeta_j(W + V_+, \varphi_2^+, \dots, \varphi_{j-1}^+)\big((t^k e_i, 1), (0, t^k)\big)$

$$+ \zeta_j(W + V_+, \varphi_2^+, \dots, \varphi_{j-1}^+)\big((0, t^k), (e_i, 1)\big)$$

for all $1 \leq i \leq r$.

It is easy to see that for all $2 \leq j \leq r$,

$$\begin{cases} \varphi_j^+(e_j) = \cdots = \varphi_j^+(e_{r-1}) = 0 \\ \varphi_j^+(e_{j-1}) = (2\alpha W(\mu))^{j-1} V_+(e_0) \end{cases}$$

and that for all $2 \leq j \leq r$ and for all $(x, t^k) \in \Gamma = N \rtimes T$,

$$h_j^+(x, t^k) = \varphi_j^+(x) + \zeta_j(W + V_+, h_2^+, \dots, h_{j-1}^+)\big((x, 1), (0, t^k)\big).$$

Now to prove that $\{\zeta_{r+1}(W + V_+, h_2^+, \dots, h_r^+)\} \neq 0$ in $H^2(\Gamma, \mathcal{G}_+)$ we use an argument by contradiction. Suppose that there exists a cochain $h_{r+1}^+ \colon \Gamma \to \mathcal{G}_+$ such that

$$-\partial h_{r+1}^+ = \zeta_{r+1}(W + V_+, h_2^+, \dots, h_r^+).$$

Then h_{r+1}^+ factors through N/N_2 and defines a $\mathbb{C}$–linear map φ_{r+1}^+ such that for all $y \in N/N_2 \otimes \mathbb{C}$

$$(\tilde{E}_{r+1}) \qquad \begin{aligned} \varphi_{r+1}^+((t^k - \alpha^k)y) &= -\zeta_{r+1}(W + V_+, \varphi_2^+, \dots, \varphi_r^+)\big((t^k y, 1), (0, t^k)\big) \\ &\quad + \zeta_{r+1}(W + V_+, \varphi_2^+, \dots, \varphi_r^+)\big((0, t^k), (y, 1)\big) \end{aligned}$$

On the other hand,

$$\begin{cases} (t^r - \alpha^r)e_{r-2} = r\alpha^{r-1}e_{r-1} \\ (t^{r-1} - \alpha^{r-1})e_{r-2} = (r-1)\alpha^{r-2}e_{r-1}. \end{cases}$$

So, for $k = r$ and $y = e_{r-2}$, Equation ($\widetilde{E}_{r+1}$) implies that

$$\begin{aligned} \varphi_{r+1}^+(r\alpha^{r-1}e_{r-1}) &= -\zeta_{r+1}(W+V_+, \varphi_2^+, \ldots, \varphi_r^+)\big((\alpha^r e_{r-2}+r\alpha^{r-1}e_{r-1}, 1), (0, t^r)\big) \\ &\quad + \zeta_{r+1}(W+V_+, \varphi_2^+, \ldots, \varphi_r^+)\big((0, t^r), (e_{r-2}, 1)\big) \\ &= 2r\alpha^r W(\mu)\varphi_r^+(e_{r-2}) + r^2\alpha^{r-1}W(\mu)\varphi_r^+(e_{r-1}). \end{aligned}$$

For $k = r - 1$ and $y = e_{r-2}$, Equation ($\widetilde{E}_{r+1}$) implies

$$\begin{aligned} \varphi_{r+1}^+\big((r-1)\alpha^{r-2}e_{r-1}\big) \\ = 2(r-1)\alpha^{r-1}W(\mu)\varphi_r^+(e_{r-2}) + (r-1)^2 W(\mu)\alpha^{r-2}\varphi_r^+(e_{r-1}). \end{aligned}$$

So we obtain the system

$$\begin{cases} r\alpha^{r-1}\varphi_{r+1}^+(e_{r-1}) = 2r\alpha^r \varphi_r^+(e_{r-2}) + r^2\alpha^{r-1}\varphi_r^+(e_{r-1}) \\ (r-1)\alpha^{r-2}\varphi_{r+1}^+(e_{r-1}) = 2(r-1)\alpha^{r-1}\varphi_r^+(e_{r-2}) + (r-1)^2\alpha^{r-2}\varphi_r^+(e_{r-1}), \end{cases}$$

which is equivalent to

$$\begin{cases} \varphi_{r+1}^+(e_{r-1}) = 2\alpha\varphi_r^+(e_{r-2}) + r\varphi_r^+(e_{r-1}) \\ \qquad\qquad = 2\alpha\varphi_r^+(e_{r-2}) + (r-1)\varphi_r^+(e_{r-1}). \end{cases}$$

So $\varphi_r^+(e_{r-1}) = 0 = (2\alpha)^{r-1}V_+(e_0)$ and $V_+(e_0) = 0$ ie $V_+ = 0$ which is absurd.

We conclude that $\{\zeta_{r+1}(W + V_+, h_2^+, \ldots, h_r^+)\} \neq 0$ in $H^2(\Gamma, \mathcal{G}_+)$. $\qquad\qquad\square$

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