

Non-commutative multivariable Reidemeister torsion and the Thurston norm

SHELLY HARVEY
STEFAN FRIEDL

Given a 3–manifold the second author defined functions $\delta_n: H^1(M; \mathbb{Z}) \rightarrow \mathbb{N}$, generalizing McMullen’s Alexander norm, which give lower bounds on the Thurston norm. We reformulate these invariants in terms of Reidemeister torsion over a non-commutative multivariable Laurent polynomial ring. This allows us to show that these functions are semi-norms.

[57M27](#); [57N10](#)

1 Introduction

Let M be a 3–manifold. Throughout the paper we will assume that all 3–manifolds are compact, connected and orientable. Let $\phi \in H^1(M; \mathbb{Z})$. The *Thurston norm* of ϕ is defined as

$$\|\phi\|_T = \min\{\chi_-(S) \mid S \subset M \text{ properly embedded surface dual to } \phi\},$$

where for a surface S with connected components S_1, \dots, S_k we write $\chi_-(S) = \sum_{i=1}^k \max\{0, -\chi(S_i)\}$. We refer to Thurston [18] for details.

Generalizing work of Cochran [1], the second author introduced in [7] a function

$$\delta_n: H^1(M; \mathbb{Z}) \rightarrow \mathbb{N}_0 \cup \{-\infty\}$$

for every $n \in \mathbb{N}$ and showed that δ_n gives a lower bound on the Thurston norm. These functions are invariants of the 3–manifold and generalize the Alexander norm defined by C McMullen in [11]. We point out that the definition we use in this paper differs slightly from the original definition when $n = 0$ and a few other special cases. We refer to [Section 4.3](#) for details.

The relationship between the functions δ_n and the Thurston norm was further strengthened in Harvey [8] (cf also Cochran [1] and Friedl [4]) where it was shown that the

δ_n give a never decreasing series of lower bounds on the Thurston norm, ie for any $\phi \in H^1(M; \mathbb{Z})$ we have

$$\delta_0(\phi) \leq \delta_1(\phi) \leq \delta_2(\phi) \leq \dots \leq \|\phi\|_T.$$

Furthermore it was shown in Friedl–Kim [5] that under a mild assumption these inequalities are an equality modulo 2.

In his original paper [18], Thurston showed that $\|\cdot\|_T$ is a semi-norm. It is therefore a natural question to ask whether the invariants δ_n are semi-norms as well. In [7] this was shown to be the case for $n = 0$. The following theorem, which is a special case of the main theorem of this paper (cf Theorem 4.2), gives an affirmative answer to this question for $n \geq 1$.

Theorem 1.1 *Let M be a 3-manifold with empty or toroidal boundary. Assume that $\delta_n(\phi) \neq -\infty$ for some $\phi \in H^1(M; \mathbb{Z})$, then*

$$\delta_n: H^1(M; \mathbb{Z}) \rightarrow \mathbb{N}_0$$

is a semi-norm.

In particular, this allows us to show that the sequence $\{\delta_n\}$ is eventually constant. That is, there exists an $N \in \mathbb{N}$ such that $\delta_n = \delta_N$ for all $n \geq N$ (cf Proposition 4.4).

Before we address whether the δ_n are norms, we discuss a more algebraic problem. Recall that given a multivariable Laurent polynomial ring $\mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ over a commutative field \mathbb{F} we can associate to any non-zero $f = \sum_{\alpha \in \mathbb{Z}^m} a_\alpha t^\alpha \in \mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ a semi-norm on $\text{hom}(\mathbb{Z}^m, \mathbb{R})$ by

$$\|\phi\|_f := \sup\{\phi(\alpha) - \phi(\beta) \mid a_\alpha \neq 0, a_\beta \neq 0\}.$$

Thus, to any square matrix B over $\mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ with $\det(B) \neq 0$, we can associate a norm using $\det(B) \in \mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$.

Generalizing this idea to the non-commutative case, in Section 2.1 we introduce the notion of a *multivariable skew Laurent polynomial ring* $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ of rank m over a skew field \mathbb{K} . Given a square matrix B over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ we can study its Dieudonné determinant $\det(B)$ which is an element in the abelianization of the multiplicative group $\mathbb{K}(t_1, \dots, t_m) \setminus \{0\}$ where $\mathbb{K}(t_1, \dots, t_m)$ denotes the quotient field of $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. This determinant will in general not be represented by an element in $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. Our main technical result (Theorem 2.2) is that nonetheless there is a natural way to associate a norm to B that generalizes the commutative case.

Given a 3–manifold M and a ‘compatible’ representation

$$\pi_1(M) \rightarrow \text{GL}(\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d)$$

we will show in Section 3 that the corresponding Reidemeister torsion can be viewed as a matrix over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. Moreover, we will show in Section 4.3 that for appropriate representations the norm that we can associate to this matrix that agrees with ρ_n, δ_n . This implies Theorem 1.1. We conclude this paper with examples of links for which we compute the Thurston norm using the results in this paper.

As a final remark we point out that the results in this paper completely generalize the results in [6]. Furthermore, the results can easily be extended to studying 2–complexes together with the Turaev norm which is modeled on the definition of the Thurston norm of a 3–manifold. We refer to Thurston [21] for details.

Acknowledgments

The authors would like to thank Tim Cochran, John Hempel, Taehee Kim and Chris Rasmussen for helpful conversations.

2 The non-commutative Alexander norm

In this section we will introduce the notion of a multivariable skew Laurent polynomial ring and we will then show that matrices over such rings give rise to semi-norms.

2.1 Multivariable Laurent polynomials

Let \mathcal{R} be a (non-commutative) domain and $\gamma: \mathcal{R} \rightarrow \mathcal{R}$ a ring homomorphism. We denote by $\mathcal{R}[s^{\pm 1}]$ the *one-variable skew Laurent polynomial ring over \mathcal{R}* . Specifically, the elements in $\mathcal{R}[s^{\pm 1}]$ are formal sums $\sum_{i=m}^n a_i s^i$ ($m \leq n \in \mathbb{Z}$) with $a_i \in \mathcal{R}$. Addition is given by addition of the coefficients, and multiplication is defined using the rule $s^i a = \gamma^i(a) s^i$ for any $a \in \mathcal{R}$ (where $\gamma^i(a)$ stands for $(\gamma \circ \dots \circ \gamma)(a)$). We point out that any element $\sum_{i=m}^n a_i s^i \in \mathcal{R}[s^{\pm 1}]$ can also be written uniquely in the form $\sum_{i=m}^n s^i \tilde{a}_i$, indeed, $\tilde{a}_i = s^{-i} a_i s^i \in \mathcal{R}$.

In the following let \mathbb{K} be a skew field. We then define a *multivariable skew Laurent polynomial ring of rank m over \mathbb{K}* (in non-commuting variables) to be a ring R which is an algebra over \mathbb{K} with unit (ie we can view \mathbb{K} as a subring of R) together with a decomposition $R = \bigoplus_{\alpha \in \mathbb{Z}^m} V_{\alpha}$ such that the following hold:

- (1) V_{α} is a one-dimensional \mathbb{K} –vector space,

- (2) $V_\alpha \cdot V_\beta = V_{\alpha+\beta}$ and
- (3) $V_{(0,\dots,0)} = \mathbb{K}$.

In particular R is \mathbb{Z}^m -graded. Note that these properties imply that any V_α is invariant under left and right multiplication by \mathbb{K} , that any element in $V_\alpha \setminus \{0\}$ is a unit, and that R is a (non-commutative) domain. The example that the reader should keep in mind is a commutative Laurent polynomial ring $\mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. Indeed, let $t^\alpha := t_1^{\alpha_1} \cdots t_m^{\alpha_m}$ for $\alpha = (\alpha_1, \dots, \alpha_m)$, then $V_\alpha = \mathbb{F}t^\alpha$, $\alpha \in \mathbb{Z}^m$ has the required properties.

Let R be a multivariable skew Laurent polynomial ring of rank m over \mathbb{K} . To make our subsequent definitions and arguments easier to digest we will always pick $t^\alpha \in V_\alpha \setminus \{0\}$ for $\alpha \in \mathbb{Z}^m$. It is easy to see that we can in fact pick t^α , $\alpha \in \mathbb{Z}^m$ such that $t^{n\alpha} = (t^\alpha)^n$ for all $\alpha \in \mathbb{Z}^m$ and $n \in \mathbb{Z}$. Note that this choice implies that $t^{(0,\dots,0)} = 1$. Using the above choices, the set of t^α for $\alpha \in \mathbb{Z}$ satisfies the following properties:

- (1) $t^\alpha t^{\tilde{\alpha}} t^{-(\alpha+\tilde{\alpha})} \in \mathbb{K}^\times$ for all $\alpha, \tilde{\alpha} \in \mathbb{Z}^m$ and
- (2) $t^\alpha \mathbb{K} = \mathbb{K}t^\alpha$ for all α .

This shows that the notion of multivariable skew Laurent polynomial ring of rank m is a generalization of the notion of twisted group ring of \mathbb{Z}^m as defined in Passman [13, page 13]. If $m = 1$ then we have $t^{(n)} \in V_{(n)}$ such that $t^{(n)} = (t^{(1)})^n$ for any $n \in \mathbb{Z}$. We write $t^n = t^{(n)}$. In particular, when $m = 1$, R is a one-variable skew Laurent polynomial ring as above.

The argument of Dodziuk et al [3, Corollary 6.3] can be used to show that any such Laurent polynomial ring is a (left and right) Ore domain and in particular has a (skew) quotient field. We normally denote a multivariable skew Laurent polynomial ring of rank m over \mathbb{K} suggestively by $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and we denote the quotient field of $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ by $\mathbb{K}(t_1, \dots, t_m)$.

2.2 The Dieudonné determinant

In this section we recall several well-known definitions and facts about the Dieudonné determinant. Let \mathcal{K} be a skew field; in our applications \mathcal{K} will be the quotient field of a multivariable skew Laurent polynomial ring. First define $\text{GL}(\mathcal{K}) := \varinjlim \text{GL}(\mathcal{K}, n)$, where we have the maps $\text{GL}(\mathcal{K}, n) \rightarrow \text{GL}(\mathcal{K}, n+1)$ in the direct system, given by $A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$, then define $K_1(\mathcal{K}) = \text{GL}(\mathcal{K})/[\text{GL}(\mathcal{K}), \text{GL}(\mathcal{K})]$. For details we refer the reader to Milnor [12] or Turaev [19].

Let A a square matrix over \mathcal{K} . After elementary row operations and destabilization we can arrange that in $K_1(\mathcal{K})$ the matrix A is represented by a 1×1 -matrix (d) . Then

the Dieudonné determinant $\det(A) \in \mathcal{K}_{ab}^\times := \mathcal{K}^\times / [\mathcal{K}^\times, \mathcal{K}^\times]$ (where $\mathcal{K}^\times := \mathcal{K} \setminus \{0\}$) is defined to be d . It is well-known that the Dieudonné determinant induces an isomorphism $\det: K_1(\mathcal{K}) \rightarrow \mathcal{K}_{ab}^\times$. We refer to Rosenberg [14, Theorem 2.2.5 and Corollary 2.2.6] for more details.

2.3 Multivariable skew Laurent polynomial rings and semi-norms

In this section we show that matrices defined over a multivariable skew Laurent polynomial ring give rise to a semi-norm. We also relate this norm to degrees of one-variable polynomials.

Let $\mathbb{K}[s^{\pm 1}]$ be a one-variable skew Laurent polynomial ring and let $f \in \mathbb{K}[s^{\pm 1}]$. If $f = 0$ then we write $\deg(f) = -\infty$, otherwise, for $f = \sum_{i=m}^n a_i s^i \in \mathbb{K}[s^{\pm 1}]$ with $a_m \neq 0, a_n \neq 0$ we define $\deg(f) := n - m$. This extends to a homomorphism $\deg: \mathbb{K}(t) \setminus \{0\} \rightarrow \mathbb{Z}$ via $\deg(fg^{-1}) = \deg(f) - \deg(g)$. Since \deg is a homomorphism to an abelian group this induces a homomorphism $\deg: \mathbb{K}(t)_{ab}^\times \rightarrow \mathbb{Z}$. Note that throughout this paper we will apply the convention that $-\infty < a$ for any $a \in \mathbb{Z}$.

For the remainder of this section let $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ be a multivariable skew Laurent polynomial ring of rank m together with a choice of $t^\alpha, \alpha \in \mathbb{Z}^m$ as above. Let $f \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. We can write $f = \sum_{\alpha \in \mathbb{Z}^m} a_\alpha t^\alpha$ for some $a_\alpha \in \mathbb{K}$. We associate a semi-norm $\|-\|_f$ on $\text{hom}(\mathbb{R}^m, \mathbb{R})$ to f as follows. If $f = 0$, then we set $\|-\|_f := 0$. Otherwise we set

$$\|\phi\|_f := \sup\{\phi(\alpha) - \phi(\beta) \mid a_\alpha \neq 0, a_\beta \neq 0\}.$$

Clearly $\|-\|_f$ is a semi-norm and does not depend on the choice of t^α . This semi-norm should be viewed as a generalization of the degree function.

Now let $\tau \in K_1(\mathbb{K}(t_1, \dots, t_m))$ and let $f_n, f_d \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \setminus \{0\}$ such that $\det(\tau) = f_n f_d^{-1} \in \mathbb{K}(t_1, \dots, t_m)_{ab}^\times$. Then define

$$\|\phi\|_\tau := \max\{0, \|\phi\|_{f_n} - \|\phi\|_{f_d}\}$$

for any $\phi \in \text{hom}(\mathbb{R}^m, \mathbb{R})$. By the following proposition this function is well-defined.

Proposition 2.1 *Let $\tau \in K_1(\mathbb{K}(t_1, \dots, t_m))$. Let $f_n, f_d, g_n, g_d \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \setminus \{0\}$ such that $\det(\tau) = f_n f_d^{-1} = g_n g_d^{-1} \in \mathbb{K}(t_1, \dots, t_m)_{ab}^\times$. Then*

$$\|-\|_{f_n} - \|-\|_{f_d} = \|-\|_{g_n} - \|-\|_{g_d}.$$

We postpone the proof to [Section 2.4](#).

Let B be a matrix defined over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. In general, it is not the case that $\det(B)$ can be represented by an element in $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. But we still have the following result which is the main technical result of this paper.

Theorem 2.2 *If $\tau \in K_1(\mathbb{K}(t_1, \dots, t_m))$ can be represented by a matrix defined over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$, then $\|-\|_{\tau}$ defines a semi-norm on $\text{hom}(\mathbb{R}^m, \mathbb{R})$.*

We postpone the proof to [Section 2.5](#).

Now let $\phi: \mathbb{Z}^m \rightarrow \mathbb{Z}$ be a non-trivial homomorphism. We will show that $\|\phi\|_B$ can also be viewed as the degree of a polynomial associated to B and ϕ . We begin with some definitions. Consider

$$\mathbb{K}[\text{Ker}(\phi)] := \bigoplus_{\alpha \in \text{Ker}(\phi)} \mathbb{K}t^{\alpha} \subset \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}].$$

This clearly defines a subring of $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and the argument of Dodziuk et al [\[3, Corollary 6.3\]](#) shows that $\mathbb{K}[\text{Ker}(\phi)]$ is an Ore domain with skew field which we denote by $\mathbb{K}(\text{Ker}(\phi))$.

Let $d \in \mathbb{Z}$ such that $\text{Im}(\phi) = d\mathbb{Z}$ and pick $\beta = (\beta_1, \dots, \beta_m) \in \mathbb{Z}^m$ such that $\phi(\beta) = d$. Let $\mu := t^{\beta}$. Then we can form one-variable Laurent polynomial rings $(\mathbb{K}[\text{Ker}(\phi)])[s^{\pm 1}]$ and $\mathbb{K}(\text{Ker}(\phi))[s^{\pm 1}]$ where $sk := \mu k \mu^{-1} s$ for all $k \in \mathbb{K}[\text{Ker}(\phi)]$ respectively for all $k \in \mathbb{K}(\text{Ker}(\phi))$. We get an isomorphism

$$\begin{aligned} \gamma_{\phi}: \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] &\xrightarrow{\cong} (\mathbb{K}[\text{Ker}(\phi)])[s^{\pm 1}] \\ \sum_{\alpha \in \mathbb{Z}^m} k_{\alpha} t^{\alpha} &\mapsto \sum_{\alpha \in \mathbb{Z}^m} k_{\alpha} t^{\alpha} \mu^{-\phi(\alpha)/d} s^{\phi(\alpha)/d}, \end{aligned}$$

where $k_{\alpha} \in \mathbb{K}$ for all $\alpha \in \mathbb{Z}^m$. Note that $k_{\alpha} t^{\alpha} \mu^{-\phi(\alpha)/d} \in \mathbb{K}[\text{Ker}(\phi)]$. An easy computation shows that γ_{ϕ} is an isomorphism of rings. We also get an induced isomorphism $\mathbb{K}(t_1, \dots, t_m) \xrightarrow{\cong} (\mathbb{K}(\text{Ker}(\phi)))(s)$.

Let B be a matrix over $\mathbb{K}(t_1, \dots, t_m)$. Define $\deg_{\phi}(B) := \deg(\det(\gamma_{\phi}(B)))$ where we view $\gamma(B)$ as a matrix over $(\mathbb{K}(\text{Ker}(\phi)))(s)$.

Theorem 2.3 *Let B be a matrix over $\mathbb{K}(t_1, \dots, t_m)$. Let $\phi \in \text{hom}(\mathbb{Z}^m, \mathbb{Z})$ be non-trivial and let $d \in \mathbb{N}$ such that $\text{Im}(\phi) = d\mathbb{Z}$. Then*

$$\|\phi\|_B = d \max\{0, \deg_{\phi}(B)\}.$$

In particular, this shows that $\deg_{\phi}(B)$ is independent of the choice of β . The above theorem is a generalization of [\[7, Proposition 5.12\]](#) to the non-commutative case.

Proof Since γ and deg are homomorphisms it is clearly enough to show that for any $g \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \setminus \{0\}$ we have

$$\|\phi\|_g = d \text{deg}(\gamma_\phi(g)).$$

Write $g = \sum_{\alpha \in \mathbb{Z}^m} a_\alpha t^\alpha$ with $a_\alpha \in \mathbb{K}$. Let d, β, μ and $\gamma: \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \xrightarrow{\cong} (\mathbb{K}[\text{Ker}(\phi)])[s^{\pm 1}]$ as above. Note that $\text{Ker}(\phi) \oplus \mathbb{Z}\beta = \mathbb{Z}^m$, hence

$$\begin{aligned} g &= \sum_{i \in \mathbb{Z}} \sum_{\alpha \in \text{Ker}(\phi)} a_{\alpha+i\beta} t^{\alpha+i\beta}, \\ \gamma_\phi(g) &= \sum_{i \in \mathbb{Z}} \left(\sum_{\alpha \in \text{Ker}(\phi)} a_{\alpha+i\beta} t^{\alpha+i\beta} \mu^{-i} \right) s^i. \end{aligned}$$

Note that $a_{\alpha+i\beta} t^{\alpha+i\beta} \mu^{-i} \in \mathbb{K}t^\alpha$. Since $\mathbb{K}[\text{Ker}(\phi)] = \bigoplus_{\alpha \in \text{Ker}(\phi)} \mathbb{K}t^\alpha$ we get the following equivalences:

$$\begin{aligned} \sum_{\alpha \in \text{Ker}(\phi)} a_{\alpha+i\beta} t^{\alpha+i\beta} \mu^{-i} &= 0 \\ \Leftrightarrow a_{\alpha+i\beta} t^{\alpha+i\beta} \mu^{-i} &= 0 \text{ for all } \alpha \in \text{Ker}(\phi) \\ \Leftrightarrow a_{\alpha+i\beta} &= 0 \text{ for all } \alpha \in \text{Ker}(\phi). \end{aligned}$$

Therefore

$$\begin{aligned} \|\phi\|_g &= d \max_{i \in \mathbb{Z}} \{ \text{there exists } \alpha \in \text{Ker}(\phi) \text{ such that } a_{\alpha+i\beta} \neq 0 \} \\ &\quad - d \min_{i \in \mathbb{Z}} \{ \text{there exists } \alpha \in \text{Ker}(\phi) \text{ such that } a_{\alpha+i\beta} \neq 0 \} \\ &= d \max_{i \in \mathbb{Z}} \{ \sum_{\alpha \in \text{Ker}(\phi)} a_{\alpha+i\beta} t^{\alpha+i\beta} \mu^{-i} \neq 0 \} \\ &\quad - d \min_{i \in \mathbb{Z}} \{ \sum_{\alpha \in \text{Ker}(\phi)} a_{\alpha+i\beta} t^{\alpha+i\beta} \mu^{-i} \neq 0 \} \\ &= d \text{deg}(\gamma_\phi(g)). \end{aligned}$$

Thus the theorem is proved. □

2.4 Proof of Proposition 2.1

We start out with the following three basic lemmas.

Lemma 2.4 Let $f, g \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \setminus \{0\}$, then $\|-\|_{fg} = \|-\|_f + \|-\|_g$.

This lemma is well-known. It follows from the fact that the Newton polytope of non-commutative multivariable polynomials fg is the Minkowski sum of the Newton polytopes of f and g . We refer to Sturmfels [16, page 31] for details.

Lemma 2.5 Let $d \in \mathbb{K}(t_1, \dots, t_m)$ and let $f_n, f_d, g_n, g_d \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ such that $d = f_n f_d^{-1} = g_n g_d^{-1} \in \mathbb{K}(t_1, \dots, t_m)$. Then

$$\|-\|_{f_n} - \|-\|_{f_d} = \|-\|_{g_n} - \|-\|_{g_d}.$$

In particular

$$\|-\|_d := \|-\|_{f_n} - \|-\|_{f_d}$$

is well-defined.

Proof Recall that by the definition of the Ore localization $f_n f_d^{-1} = g_n g_d^{-1} \in \mathbb{K}(t_1, \dots, t_m)$ is equivalent to the existence of $u, v \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \setminus \{0\}$ such that $f_n u = g_n v$ and $f_d u = g_d v$. The lemma now follows immediately from [Lemma 2.4](#). \square

Lemma 2.6 Let $d, e \in \mathbb{K}(t_1, \dots, t_m)$, then

$$\|-\|_{de} = \|-\|_d + \|-\|_e.$$

Proof Pick $f_n, f_d, g_n, g_d \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ such that $f_n f_d^{-1} = d$ and $g_n g_d^{-1} = e$. By the Ore property there exist $u, v \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \setminus \{0\}$ such that $g_n u = f_d v$. It follows that

$$f_n f_d^{-1} g_n g_d^{-1} = f_n v u^{-1} g_d^{-1} = (f_n v)(g_d u)^{-1}.$$

The lemma now follows immediately from [Lemma 2.4](#). \square

We can now give the proof of [Proposition 2.1](#).

Proof of Proposition 2.1 Let B be a matrix defining an element $K_1(\mathbb{K}(t_1, \dots, t_m))$. Assume that we have $f_n, f_d, g_n, g_d \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ such that $\det(B) = f_n f_d^{-1} = g_n g_d^{-1} \in \mathbb{K}(t_1, \dots, t_m)_{ab}^\times$. We can lift the equality $f_n f_d^{-1} = g_n g_d^{-1} \in \mathbb{K}(t_1, \dots, t_m)_{ab}^\times$ to an equality

$$(1) \quad f_n f_d^{-1} = \prod_{i=1}^r [a_i, b_i] g_n g_d^{-1} \in \mathbb{K}(t_1, \dots, t_m)^\times$$

for some $a_i, b_i \in \mathbb{K}(t_1, \dots, t_m)$. It follows from [Lemma 2.6](#) that $\|-\|_{[a_i, b_i]} = 0$. It then follows from [Lemma 2.6](#) that $\|-\|_{f_n f_d^{-1}} = \|-\|_{g_n g_d^{-1}}$. \square

2.5 Proof of Theorem 2.2

Let $\tau \in K_1(\mathbb{K}(t_1, \dots, t_m))$ that can be represented by a matrix B defined over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. We will show that $\|-\|_\tau = \|-\|_B$ defines a semi-norm on $\text{hom}(\mathbb{R}^m, \mathbb{R})$. Because of the continuity and the \mathbb{N} -linearity of $\|-\|_B$ it is enough to show that for any two non-trivial homomorphisms $\phi, \tilde{\phi}: \mathbb{Z}^m \rightarrow \mathbb{Z}$ we have

$$\|\phi + \tilde{\phi}\|_B \leq \|\phi\|_B + \|\tilde{\phi}\|_B.$$

Let $\phi, \tilde{\phi}: \mathbb{Z}^m \rightarrow \mathbb{Z}$ be non-trivial homomorphisms. Let $d \in \mathbb{Z}$ such that $\text{Im}(\phi) = d\mathbb{Z}$ and pick β with $\phi(\beta) = d$. We write $\mu = t^\beta$. As in Section 2.3 we can form $\mathbb{K}[\text{Ker}(\phi)]$ and we also have an isomorphism $\gamma_\phi: \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \xrightarrow{\cong} (\mathbb{K}[\text{Ker}(\phi)])[s^{\pm 1}]$.

Consider $\gamma_\phi(B)$, it is defined over the PID $\mathbb{K}[\text{Ker}(\phi)][s^{\pm 1}]$. Therefore we can use elementary row operations to turn $\gamma_\phi(B)$ into a diagonal matrix with entries in $\mathbb{K}[\text{Ker}(\phi)][s^{\pm 1}]$. In particular we can find $a_i, b_i \in \mathbb{K}[\text{Ker}(\phi)]$ such that

$$\det(\gamma_\phi(B)) = \sum_{i=r_1}^{r_2} s^i a_i b_i^{-1}.$$

Since $\mathbb{K}[\text{Ker}(\phi)]$ is an Ore domain we can in fact find a common denominator for $a_i b_i^{-1}, i = r_1, \dots, r_2$. More precisely, we can find $c_{r_1}, \dots, c_{r_2} \in \mathbb{K}[\text{Ker}(\phi)]$ and $d \in \mathbb{K}[\text{Ker}(\phi)]$ such that $a_i b_i^{-1} = c_i d^{-1}$ for $i = r_1, \dots, r_2$. Now let $c = \sum_{i=r_1}^{r_2} s^i c_i$. Then

$$\det(\gamma_\phi(B)) = cd^{-1} \in \mathbb{K}(\text{Ker}(\phi))(s)_{ab}^\times$$

where $c \in \mathbb{K}[\text{Ker}(\phi)][s^{\pm 1}]$ and $d \in \mathbb{K}[\text{Ker}(\phi)]$. Now let $f = \gamma_\phi^{-1}(c) \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and $g = \gamma_\phi^{-1}(d) \in \mathbb{K}[\text{Ker}(\phi)]$. Then $\det(B) = fg^{-1}$ and by Proposition 2.1 we have

$$\|-\|_B = \|-\|_f - \|-\|_g.$$

We now observe that $\|\phi\|_g = 0$ and $\|\phi + \tilde{\phi}\|_g = \|\tilde{\phi}\|_g$ since $g \in \mathbb{K}[\text{Ker}(\phi)]$. Therefore it follows that

$$\begin{aligned} \|\phi + \tilde{\phi}\|_B &= \|\phi + \tilde{\phi}\|_f - \|\phi + \tilde{\phi}\|_g \\ &= \|\phi + \tilde{\phi}\|_f - \|\tilde{\phi}\|_g \\ &\leq \|\phi\|_f + \|\tilde{\phi}\|_f - \|\tilde{\phi}\|_g \\ &= (\|\phi\|_f - \|\phi\|_g) + (\|\tilde{\phi}\|_f - \|\tilde{\phi}\|_g) \\ &= \|\phi\|_B + \|\tilde{\phi}\|_B. \end{aligned}$$

This concludes the proof of Theorem 2.2.

3 Applications to the Thurston norm

In this section we will show that the Reidemeister torsion corresponding to ‘compatible’ representations over a multivariable skew Laurent polynomial ring give rise to semi-norms that give lower bounds on the Thurston norm.

3.1 Reidemeister torsion

Let X be a finite connected CW-complex. Denote the universal cover of X by \tilde{X} . We view $C_*(\tilde{X})$ as a right $\mathbb{Z}[\pi_1(X)]$ -module via deck transformations. Let R be a ring and let $\varphi: \pi_1(X) \rightarrow \mathrm{GL}(R, d)$ be a representation. This equips R^d with a left $\mathbb{Z}[\pi_1(X)]$ -module structure. We can therefore consider the right R -module chain complex $C_*^\varphi(X; R^d) := C_*(\tilde{X}) \otimes_{\mathbb{Z}[\pi_1(X)]} R^d$. We denote its homology by $H_i^\varphi(X; R^d)$. If $H_*^\varphi(X; R^d) = 0$, then we define the Reidemeister torsion $\tau(X, \varphi) \in K_1(R) / \pm \varphi(\pi_1(X))$ otherwise we write $\tau(X, \varphi) := 0$. If the homomorphism φ is clear we may also write $\tau(X, R^d)$.

Let M be a manifold. Since Reidemeister torsion only depends on the homeomorphism type of the space we can define $\tau(M, \varphi)$ by picking any CW-structure for M . We refer to the excellent book of Turaev [19] for the details.

3.2 Compatible homomorphisms and the higher order Alexander norm

In the following let M be a 3-manifold with empty or toroidal boundary. Let $\psi: H_1(M) \rightarrow \mathbb{Z}^m$ be an epimorphism. Let $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ be a multivariable skew Laurent polynomial ring of rank m as in Section 2.1.

A representation $\varphi: \pi_1(M) \rightarrow \mathrm{GL}(\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d)$ is called ψ -compatible if for any $g \in \pi_1(M)$ we have $\varphi(g) = At^{\psi(g)}$ for some $A \in \mathrm{GL}(\mathbb{K}, d)$. This generalizes definitions in Turaev [20] and Friedl [4]. We denote the induced representation $\pi_1(M) \rightarrow \mathrm{GL}(\mathbb{K}(t_1, \dots, t_m), d)$ by φ as well and consider the corresponding Reidemeister torsion $\tau(M, \varphi) \in K_1(\mathbb{K}(t_1, \dots, t_m)) / \pm \varphi(\pi_1(M)) \cup \{0\}$.

We say φ is a *commutative representation* if there exists a commutative subfield \mathbb{F} of \mathbb{K} such that for all g we have $\varphi(g) = At^{\psi(g)}$ with A defined over \mathbb{F} and if $t^\alpha, t^{\tilde{\alpha}}$ commute for any $\alpha, \tilde{\alpha} \in \mathbb{Z}^m$. The following result is our main application of the purely algebraic results of Section 2.

Theorem 3.1 *Let M be a 3-manifold with an empty or toroidal boundary, let $\psi: H_1(M) \rightarrow \mathbb{Z}^m$ be an epimorphism and let $\varphi: \pi_1(M) \rightarrow \mathrm{GL}(\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d)$ be a ψ -compatible representation such that $\tau(M, \varphi) \neq 0$. If one of the following holds:*

- (1) φ is commutative or
- (2) there exists $g \in \mathrm{Ker}\{\pi_1(M) \rightarrow \mathbb{Z}^m\}$ such that $\varphi(g) - id$ is invertible over \mathbb{K} ,

then $\|-\|_{\tau(M, \varphi)}$ is a semi-norm on $\mathrm{hom}(\mathbb{R}^m, \mathbb{R})$ and for any $\phi: \mathbb{R}^m \rightarrow \mathbb{R}$ we have

$$\|\phi \circ \psi\|_T \geq \|\phi\|_{\tau(M, \varphi)}.$$

We point out that if $g \in \text{Ker}\{\pi_1(M) \rightarrow \mathbb{Z}^m\}$, then $\varphi(g) - \text{id}$ is defined over \mathbb{K} since φ is ψ -compatible. We refer to $\|-\|_{\tau(M,\varphi)}$ as the *higher-order Alexander norm*.

In the case that $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ equals $\mathbb{Q}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$, the usual commutative Laurent polynomial ring, we recover McMullen’s Alexander norm $\|-\|_A$ (cf McMullen [11]). The general commutative case is the main result in Friedl–Kim [6]. The proof we give here is different in its nature from the proofs in [11] and [6].

Proof In the case that $m = 1$ it is clear that $\|-\|_{\tau(M,\varphi)}$ is a semi-norm. The fact that it gives a lower bound on the Thurston norm was shown in [1; 7; 20; 4]. We therefore assume now that $m > 1$.

We first show that $\|\phi \circ \psi\|_T \geq \|\phi\|_{\tau(M,\varphi)}$ for any $\phi: \mathbb{R}^m \rightarrow \mathbb{R}$. Since both sides are \mathbb{N} -linear and continuous we only have to show that $\|\phi \circ \psi\|_T \geq \|\phi\|_{\tau(M,\varphi)}$ for all epimorphisms $\phi: \mathbb{Z}^m \rightarrow \mathbb{Z}$. So from now on, we will assume that $\phi: \mathbb{Z}^m \rightarrow \mathbb{Z}$ is an epimorphism.

Pick $\mu \in \mathbb{Z}^m$ with $\phi(\mu) = 1$ as in the definition of $\text{deg}_\phi(\tau(M, \varphi))$. We can then form the rings $\mathbb{K}[\text{Ker}(\phi)][s^{\pm 1}]$ and $\mathbb{K}(\text{Ker}(\phi))(s)$. First note that by Theorem 2.3

$$\|\phi\|_{\tau(M,\varphi)} = \text{deg}_\phi(\tau(M, \varphi))$$

since ϕ is surjective. The representation

$$\pi_1(M) \rightarrow \text{GL}(\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d) \rightarrow \text{GL}(\mathbb{K}(\text{Ker}(\phi))[s^{\pm 1}], d)$$

is ϕ -compatible since $\pi_1(M) \rightarrow \text{GL}(\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d)$ is ψ -compatible. It now follows from Friedl [4, Theorem 1.2] that $\|\phi \circ \psi\|_T \geq \text{deg}(\tau(M, \mathbb{K}(\text{Ker}(\phi))(s))) = \text{deg}_\phi(\tau(M, \varphi))$ (cf also Turaev [20]).

In the remainder of the proof we will show that if $m > 1$ then the Reidemeister torsion $\tau(M, \varphi) \in K_1(\mathbb{K}(t_1, \dots, t_m)) / \pm \varphi(\pi_1(M))$ can be represented by a matrix defined over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. It then follows from Theorem 2.2 that $\|-\|_{\tau(M,\varphi)}$ is a semi-norm.

Consider the case that φ is a commutative representation and let \mathbb{F} be the commutative subfield \mathbb{F} in the definition of a commutative representation. Denote by $\mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ the ordinary Laurent polynomial ring. Then we have ψ -compatible representations $\pi_1(M) \rightarrow \text{GL}(\mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d) \hookrightarrow \text{GL}(\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}], d)$. By [19, Proposition 3.6] we have

$$\tau(M, \mathbb{F}(t_1, \dots, t_m)) = \tau(M, \mathbb{K}(t_1, \dots, t_m)) \in K_1(\mathbb{K}(t_1, \dots, t_m)) / \pm \varphi(\pi_1(M)).$$

Since $m > 1$ it follows from [19, Theorem 4.7] combined with [6, Lemmas 6.2 and 6.5] that $\det(\tau(M, \mathbb{F}(t_1, \dots, t_m))) \in \mathbb{F}(t_1, \dots, t_m)$ equals the twisted multivariable

Alexander polynomial, in particular it is defined over $\mathbb{F}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. This concludes the proof in the commutative case.

It therefore remains to consider the case when there exists $g \in \text{Ker}\{\pi_1(M) \rightarrow \mathbb{Z}^m\}$ such that $\varphi(g) - \text{id}$ is invertible. We first consider the case that M is a closed 3-manifold. The proof will use the special CW-structure from the next claim.

Claim There exists a CW-structure for M with one 0-cell and one 3-cell and such the closure of a 1-cell and the cocore of a 2-cell represent g .¹

In order to prove the claim pick a Heegaard decomposition $M = G_0 \cup H_0$. We can add a handle g_0 (respectively h_0) to G_0 (respectively H_0) in H_0 (respectively G_0) so that the core of g_0 (respectively h_0) represents g . Adding further handles h_1, \dots, h_r (respectively g_1, \dots, g_r) in $M \setminus G_0$ (respectively $M \setminus H_0$) we can assume that complement $H_0 \setminus (\bigcup_{i=0}^r g_i)$ (respectively $G_0 \setminus (\bigcup_{i=0}^s h_i)$) is again a handlebody. It follows that $G := (G_0 \cup \bigcup_{i=0}^r g_i) \setminus (\bigcup_{i=0}^s h_i)$ and $H := (H_0 \cup \bigcup_{i=0}^s h_i) \setminus (\bigcup_{i=0}^r g_i)$ are handlebodies and hence $M = G \cup H$ is a handlebody decomposition of M .

Now give M the CW structure as follows: take one 0-cell, attach 1-cells along a choice of cores of G such that g is represented by the closure of a 1-cell. Attach 2-cells along cocores of H such that one cocore represents g . Finally attach one 3-cell. This CW-structure clearly has the required properties to complete the claim.

Denote the number of 1-cells by n . Consider the chain complex of the universal cover \tilde{M} :

$$0 \rightarrow C_3(\tilde{M})^1 \xrightarrow{\partial_3} C_2(\tilde{M})^n \xrightarrow{\partial_2} C_1(\tilde{M})^n \xrightarrow{\partial_1} C_0(\tilde{M})^1 \rightarrow 0,$$

where the superscript indicates the rank over $\mathbb{Z}[\pi_1(M)]$. Picking appropriate lifts of the cells of M to cells of \tilde{M} and picking an appropriate order we get bases for the $\mathbb{Z}[\pi_1(M)]$ -modules $C_i(\tilde{M})$, such that if A_i denotes the matrix corresponding to ∂_i , then A_1 and A_3 are of the form

$$\begin{aligned} A_3 &= (1 - g, 1 - b_1, \dots, 1 - b_{n-1})^t, \\ A_1 &= (1 - g, 1 - a_1, \dots, 1 - a_{n-1}), \end{aligned}$$

for some $a_i, b_i \in \pi_1(M), i = 1, \dots, n-1$. By assumption $\text{id} - \varphi(g)$ is invertible over \mathbb{K} . Denote by B_2 the result of deleting the first column and the first row of A_2 . Let $\tau := (\text{id} - \varphi(g))^{-1} \varphi(B_2) (\text{id} - \varphi(g))^{-1}$. Note that τ is defined over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. Since we assume that $\tau(M, \varphi) \neq 0$ it follows that $\varphi(B_2)$ is invertible over $\mathbb{K}(t_1, \dots, t_m)$ and $\tau(M, \varphi) = \tau \in K_1(\mathbb{K}(t_1, \dots, t_m)) / \pm \varphi(\pi_1(M))$ (we refer to [19, Theorem 2.2]

¹By cocore, we mean the element in π_1 given by taking a point on the 2-cell and connecting the two push-offs through an arc in the 3-cell.

for details). Therefore $\tau(M, \varphi) \in K_1(\mathbb{K}(t_1, \dots, t_m)) / \pm \varphi(\pi_1(M))$ can be represented by a matrix defined over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. This concludes the proof in the case that M is a closed 3-manifold.

In the case that M is a 3-manifold with non-empty toroidal boundary we can find a (simple) homotopy equivalence to a 2-complex X with $\chi(X) = 0$. We can assume that the CW-structure has one 0-cell, n 1-cells and $n - 1$ 2-cells, furthermore we can assume that the closure of a 1-cell represents an element $g \in \text{Ker}\{\psi: \pi_1(X) \rightarrow \mathbb{Z}^m\}$ such that $\text{id} - \varphi(g)$ is invertible. We get a chain complex

$$0 \rightarrow C_2(\tilde{X})^{n-1} \xrightarrow{\partial_2} C_1(\tilde{X})^n \xrightarrow{\partial_1} C_0(\tilde{X})^1 \rightarrow 0.$$

Picking appropriate lifts of the cells of X to cells of \tilde{X} we get bases for the $\mathbb{Z}[\pi_1(X)]$ -modules $C_i(\tilde{X})$, such that if A_i denotes the matrix corresponding to ∂_i , then A_1 is of the form

$$A_1 = (1 - g, 1 - a_1, \dots, 1 - a_{n-1}),$$

for some $a_i \in \pi_1(M)$. Now denote by B_2 the result of deleting the first row of A_2 . Then $\tau := \varphi(B_2)(\text{id} - \varphi(g))^{-1}$ is again defined over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and the proof continues as in the case of a closed 3-manifold. \square

Remark It follows from [4] that if M is closed, or if M has toroidal boundary, then $\tau(M, \varphi) \neq 0$ is equivalent to $H_1(M; \mathbb{K}(t_1, \dots, t_m)) = 0$, or equivalently, that $H_1(M; \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}])$ has rank zero over $\mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$.

Remark The computation of polynomials

$$f_d \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}] \text{ and } f_n \in \mathbb{K}[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$$

such that $\det(\tau(M, \varphi)) = f_n f_d^{-1}$ is computationally equivalent to the computation of $\text{deg}_\phi(\tau(M, \varphi))$ for some $\phi: H_1(M) \rightarrow \mathbb{Z}$. Put differently we get the perhaps surprising fact that computing the higher-order Alexander norm does not take longer than computing a single higher-order, one-variable Alexander polynomial.

4 Examples of ψ -compatible homomorphisms

Following [1] and [7] we will use the rational derived series to give examples of ψ -compatible homomorphisms. For a given a 3-manifold, we will show that these give rise to a never decreasing, eventually constant sequence of semi-norms all of which give lower bounds on the Thurston norm.

4.1 Skew fields of group rings

A group G is called *locally indicable* if for every finitely generated non-trivial subgroup $U \subset G$ there exists a non-trivial homomorphism $U \rightarrow \mathbb{Z}$. We recall the following well-known theorem.

Theorem 4.1 *Let G be a locally indicable and amenable group and let R be a subring of \mathbb{C} . Then $R[G]$ is an Ore domain, in particular it embeds in its classical right ring of quotients $\mathbb{K}(G)$.*

Higman [9] showed that $R[G]$ has no zero divisors. The theorem now follows from [17] or [3, Corollary 6.3].

We recall that a group G is called *poly-torsion-free-abelian* (PTFA) if there exists a filtration

$$1 = G_0 \subset G_1 \subset \dots \subset G_{n-1} \subset G_n = G$$

such that G_i/G_{i-1} is torsion free abelian. It is well-known that PTFA groups are amenable and locally indicable (cf [15]). The group rings of PTFA groups played an important role in Cochran–Orr–Teichner [2], Cochran [1] and Harvey [7].

4.2 Admissible pairs and multivariable skew Laurent polynomial rings

We slightly generalize a definition from Harvey [8].

Definition Let π be a group and let $\psi: \pi \rightarrow \mathbb{Z}^m$ be an epimorphism and let $\varphi: \pi \rightarrow G$ be an epimorphism to a locally indicable and amenable group G such that there exists a map $G \rightarrow \mathbb{Z}^m$ (which we also denote by ψ) such that

$$\begin{array}{ccc} \pi & \xrightarrow{\varphi} & G \\ & \searrow \psi & \downarrow \psi \\ & & \mathbb{Z}^m \end{array}$$

commutes. Following [8, Definition 1.4] we call (φ, ψ) an *admissible pair* for π .

Clearly $G_\psi := \text{Ker}\{G \rightarrow \mathbb{Z}^m\}$ is locally indicable and amenable. It follows now from Passman [13, Lemma 3.5 (ii)] that $(\mathbb{Z}[G], \mathbb{Z}[G_\psi] \setminus \{0\})$ satisfies the Ore property. Now pick elements $t^\alpha \in G, \alpha \in \mathbb{Z}^m$ such that $\psi(t^\alpha) = \alpha$ and $t^{n\alpha} = (t^\alpha)^n$ for any $\alpha \in \mathbb{Z}^m, n \in \mathbb{Z}$.

Clearly $\mathbb{Z}[G](\mathbb{Z}[G_\psi] \setminus \{0\})^{-1} = \sum_{\alpha \in \mathbb{Z}^m} \mathbb{K}(G_\psi)t^\alpha$ is a multivariable skew Laurent polynomial ring of rank m over the field $\mathbb{K}(G_\psi)$ as defined in Section 2.1. We denote

this ring by $\mathbb{K}(G_\psi)[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$. Note that $\mathbb{Z}[\pi] \rightarrow \mathbb{Z}[G] \rightarrow \mathbb{K}(G_\psi)[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ is a ψ -compatible homomorphism and that $\mathbb{K}(G_\psi)(t_1, \dots, t_m)$ is canonically isomorphic to $\mathbb{K}(G)$.

A family of examples of admissible pairs is provided by the rational derived series of a group π introduced by the second author (cf [7, Section 3]). Let $\pi_r^{(0)} := \pi$ and define inductively

$$\pi_r^{(n)} := \{g \in \pi_r^{(n-1)} \mid g^d \in [\pi_r^{(n-1)}, \pi_r^{(n-1)}] \text{ for some } d \in \mathbb{Z} \setminus \{0\}\}.$$

Note that $\pi_r^{(n-1)} / \pi_r^{(n)} \cong (\pi_r^{(n-1)} / [\pi_r^{(n-1)}, \pi_r^{(n-1)}]) / \mathbb{Z}$ -torsion. By [7, Corollary 3.6] the quotients $\pi / \pi_r^{(n)}$ are PTFA groups for any π and any n . If $\psi: \pi \rightarrow \mathbb{Z}^m$ is an epimorphism, then $(\pi \rightarrow \pi / \pi_r^{(n)}, \psi)$ is an admissible pair for π for any $n > 0$.

4.3 Admissible pairs and semi-norms

Let M be a 3-manifold with empty or toroidal boundary. Let

$$(\varphi: \pi_1(M) \rightarrow G, \psi: \pi_1(M) \rightarrow \mathbb{Z}^m)$$

be an admissible pair for $\pi_1(M)$. We denote the induced map

$$\mathbb{Z}[\pi_1(M)] \rightarrow \mathbb{K}(G_\psi)(t_1, \dots, t_m)$$

by φ as well.

Let $\phi: \mathbb{Z}^m \rightarrow \mathbb{Z}$ be a non-trivial homomorphism. We denote the induced homomorphism $G \rightarrow \mathbb{Z}^m \rightarrow \mathbb{Z}$ by ϕ as well. We write $G_\phi := \text{Ker}\{G \rightarrow \mathbb{Z}\}$. Pick $\mu \in G$ such that $\phi(\mu)\mathbb{Z} = \text{Im}(\phi)$. We define $\mathbb{Z}[G_\phi][u^{\pm 1}]$ via $uf = \mu f \mu^{-1}u$. Note that we get an isomorphism $\mathbb{K}(G_\phi)(u) \cong \mathbb{K}(G)$. If $\tau(M, \varphi) \neq 0$, then we define

$$\delta_G(\phi) := \max\{0, \text{deg}(\tau(M, \mathbb{K}(G_\phi)(u)))\},$$

otherwise we write $\delta_G(\phi) = -\infty$. We will adopt the convention that $-\infty < a$ for any $a \in \mathbb{Z}$. By [4] this agrees with the definition in [8, Definition 1.6] if $\delta_G(\phi) \neq -\infty$ and if $\varphi: G \rightarrow \mathbb{Z}^m$ is not an isomorphism or if $m > 1$. In the case that $\varphi: G \rightarrow \mathbb{Z}$ is an isomorphism and $M \neq S^1 \times D^2, S^1 \times S^2$, this definition differs from [8, Definition 1.6] by the term $1 + b_3(M)$. In the case that $\varphi: \pi \rightarrow \pi / \pi_r^{(n+1)}$ then we also write $\delta_n(\phi) = \delta_{\pi / \pi_r^{(n+1)}}(\phi)$.

The following theorem implies [Theorem 1.1](#).

Theorem 4.2 *Let M be a 3-manifold with an empty or toroidal boundary. Let $(\varphi: \pi_1(M) \rightarrow G, \psi: \pi_1(M) \rightarrow \mathbb{Z}^m)$ be an admissible pair for $\pi_1(M)$ such that*

$\tau(M, \varphi) \neq 0$. Then for any $\phi: \mathbb{Z}^m \rightarrow \mathbb{Z}$ we have $\|\phi\|_{\tau(M, \varphi)} = \delta_G(\phi)$ and $\phi \mapsto \max\{0, \delta_G(\phi)\}$ defines a semi-norm which is a lower bound on the Thurston norm.

Proof Let $\phi: \mathbb{Z}^m \rightarrow \mathbb{Z}$ be a non-trivial homomorphism. As in Section 2.1 we can form $\mathbb{K}(G_\phi)[s^{\pm 1}]$ and $\mathbb{K}(G_\psi)(\text{Ker}(\phi))[s^{\pm 1}]$. Note that these rings are canonically isomorphic Laurent polynomial rings. If $\psi: G \rightarrow \mathbb{Z}^m$ is an isomorphism, then φ is commutative. Otherwise we can find a non-trivial $g \in \text{Ker}(\psi)$, so clearly $1 - \varphi(g) \neq 0 \in \mathbb{K}(G)$. This shows that we can apply Theorem 3.1 which then concludes the proof. \square

In the case that $\varphi: \pi \rightarrow \pi/\pi_r^{(n+1)}$ we denote the semi-norm $\phi \mapsto \max\{0, \delta_n(\phi)\}$ by $\|-\|_n$. Note that in the case $n = 0$ this was shown by the second author [7, Proposition 5.12] to be equal to McMullen's Alexander norm [11].

4.4 Admissible triple

We now slightly extend a definition from [8].

Definition Let π be a group and $\psi: \pi \rightarrow \mathbb{Z}^m$ an epimorphism. Furthermore let $\varphi_1: \pi \rightarrow G_1$ and $\varphi_2: \pi \rightarrow G_2$ be epimorphisms to locally indicable and amenable groups G_1 and G_2 . We call $(\varphi_1, \varphi_2, \psi)$ an *admissible triple* for π if there exist epimorphisms $\Phi: G_1 \rightarrow G_2$ and $\psi_2: G_2 \rightarrow \mathbb{Z}^m$ such that $\varphi_2 = \Phi \circ \varphi_1$, and $\psi = \psi_2 \circ \varphi_2$.

Note that $(\varphi_i, \psi), i = 1, 2$ are admissible pairs for π . Combining Theorem 4.2 with [4, Theorem 1.3] (cf also [8]) we get the following result.

Theorem 4.3 *Let M be a 3-manifold with empty or toroidal boundary. If $(\varphi_1, \varphi_2, \psi)$ is an admissible triple for $\pi_1(M)$ such that $\tau(M, \varphi_2) \neq 0$, then we have the following inequalities of semi-norms:*

$$\|-\|_{\tau(M, \varphi_2)} \leq \|-\|_{\tau(M, \varphi_1)} \leq \|-\|_T.$$

In particular we have

$$\|-\|_0 \leq \|-\|_1 \leq \dots \leq \|-\|_T.$$

Let M be a 3-manifold with empty or toroidal boundary and let $\phi \in H^1(M; \mathbb{Z})$. Since $\delta_n(\phi) \in \mathbb{N}$ for all n it follows immediately from Theorem 4.3 that there exists $N \in \mathbb{N}$ such that $\delta_n(\phi) = \delta_N(\phi)$ for all $n \geq N$. But we can in fact prove a slightly stronger statement, namely that there exists such an N independent of the choice of $\phi \in H^1(M; \mathbb{Z})$.

Proposition 4.4 *Let M be a 3-manifold with empty or toroidal boundary. There exists $N \in \mathbb{N}$ such that $\delta_n(\phi) = \delta_N(\phi)$ for all $n \geq N$ and all $\phi \in H^1(M; \mathbb{R})$.*

Proof Write $\pi = \pi_1(M)$, $\pi_n = \pi/\pi_r^{(n+1)}$ and $m = b_1(M)$. Let $\psi: \pi \rightarrow \mathbb{Z}^m$ be an epimorphism. Write $(\pi_n)_\psi = \text{Ker}\{\psi: \pi_n \rightarrow \mathbb{Z}^m\}$. Now pick elements $t^\alpha \in \pi_n, \alpha \in \mathbb{Z}^m$ such that $\psi(t^\alpha) = \alpha$ and $t^{k\alpha} = (t^\alpha)^k$ for any $\alpha \in \mathbb{Z}^m, k \in \mathbb{Z}$. Consider the map $\mathbb{Z}[\pi] \rightarrow \mathbb{Z}[\pi_n] \rightarrow \mathbb{K}((\pi_n)_\psi)(t_1, \dots, t_m)$. We write $\tau_n = \tau(M, \mathbb{K}((\pi_n)_\psi)(t_1, \dots, t_m))$. We can find $f_n, g_n \in \mathbb{K}((\pi_n)_\psi) \in [t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ such that $\tau_n = f_n g_n^{-1}$.

Now let H be a real vector space and $C \subset H$ a convex subset. Then C defines a dual convex subset $d(C) \subset H^* = \text{hom}(H, \mathbb{R})$. Under the canonical identification $(H^*)^* = H$ we have $d(d(C)) = C$. We use ψ to identify $H_1(M; \mathbb{R})$ with \mathbb{R}^m . Let $f = \sum_{\alpha \in \mathbb{Z}^m} a_\alpha t^\alpha \in \mathbb{K}((\pi_n)_\psi) \in [t_1^{\pm 1}, \dots, t_m^{\pm 1}]$ and denote by $N(f)$ its Newton polytope, ie $N(f)$ is the convex hull of $\{\alpha | a_\alpha \neq 0\} \subset H_1(M; \mathbb{R})$. Clearly $d(N(f)) \subset (H_1(M; \mathbb{R}))^* = H^1(M; \mathbb{R})$ equals the norm ball of $\|-\|_f$. By the above discussion we see that $d(\|-\|_f) = N(f)$, in particular $d(\|-\|_f)$ has only integral vertices.

By the definition of $\delta_n = \|-\|_{\tau_n} = \|-\|_{f_n g_n^{-1}}$ it follows that

$$d(\delta_n) + d(g_n) = d(\tau_n) + d(g_n) = d(f_n)$$

where “+” denotes the Minkowski sum of convex sets. It is easy to see that this implies that $d(\delta_n)$ has only integral vertices.

Theorem 4.3 implies that there is a sequence of inclusions

$$d(\delta_0) \subset d(\delta_1) \subset \dots \subset d(\|-\|_T).$$

Since $d(\|-\|_T)$ is compact and since $d(\delta_n)$ has integral vertices for all n it follows immediately that there exists $N \in \mathbb{N}$ such that $d(\delta_n) = d(\delta_N)$ for all $n \geq N$. This completes the proof of the proposition. \square

5 Examples

Before we discuss the Thurston norm of a family of links we first need to introduce some notation for knots. Let K be a knot. We denote the knot complement by $X(K)$. Let $\phi: H_1(X(K)) \rightarrow \mathbb{Z}$ be an isomorphism. We write $\delta_n(K) := \delta_n(\phi)$. This agrees with the original definition of Cochran [1] for $n > 0$ and if $\Delta_K(t) = 1$, and it is one less than Cochran’s definition otherwise.

In the following let $L = L_1 \cup \dots \cup L_m$ be any ordered oriented m -component link. Let $i \in \{1, \dots, m\}$. Let K be an oriented knot with $\Delta_K(t) \neq 1$ which is separated

from L by a sphere S . We pick a path from a point on K to a point on L_i and denote by $L\#_i K$ the link given by performing the connected sum of L_i with K (cf Figure 1). Note that this connected sum is well-defined, ie independent of the choice of the path. We will study the Thurston norm of $L\#_i K$.

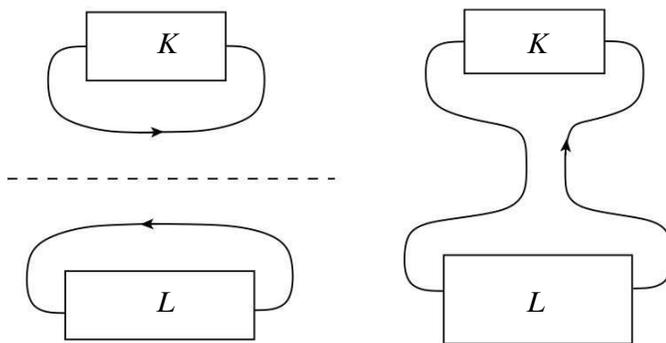


Figure 1: The link $L\#_i K$.

Now assume that L is a non-split link with at least two components and such that $\|-\|_0 = \|-\|_T$. Many examples of such links are known (cf [11]). For the link $L\#_i K$ denote its meridians by $\mu_i, i = 1, \dots, m$. Let $\psi: H_1(X(L\#_i K)) \rightarrow \mathbb{Z}^m$ be the isomorphism given by $\psi(\mu_i) = e_i$, where e_i is the i th vector of the standard basis of \mathbb{Z}^m .

We write $\pi := \pi_1(X(L\#_i K))$. For all $\alpha \in \mathbb{Z}^m$ we pick $t^\alpha \in \pi/\pi_r^{(n+1)}$ with $\psi(t^\alpha) = \alpha$ and such that $t^{l\alpha} = (t^\alpha)^l$ for all $\alpha \in \mathbb{Z}^m$ and $l \in \mathbb{Z}$. Furthermore write $t_i := t^{e_i}$.

Proposition 5.1 Consider the natural map

$$\varphi: \pi \rightarrow \mathbb{K}(\pi/\pi_r^{(n+1)}) = \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})(t_1, \dots, t_m).$$

where π is as defined above. There exists

$$f(t_i) \in \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})[t_i^{\pm 1}] \subset \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})[t_1^{\pm 1}, \dots, t_m^{\pm 1}]$$

such that $\deg(f(t_i)) = \delta_n(K) + 1$, and there exists a $d = d(t_1, \dots, t_m) \in \mathbb{K}(t_1, \dots, t_m)$ with $\|-\|_d = \|-\|_0$, such that

$$(2) \quad \tau(X(L\#_i K), \varphi) = d(t_1, \dots, t_m) f(t_i) \in K_1(\mathbb{K}(\pi_\psi/\pi_r^{(n+1)})(t_1, \dots, t_m)) / \pm \varphi(\pi).$$

Furthermore, if $\delta_n(K) = 2\text{genus}(K) - 1$, then

$$\|-\|_{\tau(X(L\#_i K), \varphi)} = \|-\|_T.$$

Proof Let S be the embedded sphere in S^3 coming from the definition of the connected sum operation (cf Figure 1). Let D be the annulus $S \cap X(L\#_i K)$ and we denote by P the closure of the component of $X(L\#_i K) \setminus D$ corresponding to K . We denote the closure of the other component by P' (see Figure 2 below). Note that P is homeomorphic to $X(K)$ and P' is homeomorphic to $X(L)$. Denote the induced

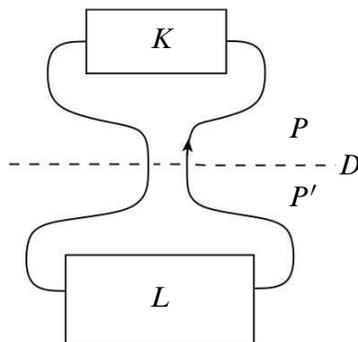


Figure 2: The link complement of $L\#_i K$ cut along the annulus D .

maps to $(K) := \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})(t_1, \dots, t_m)$ by φ as well. We get an exact sequence

$$0 \rightarrow C_*^\varphi(D; (K)) \rightarrow C_*^\varphi(P; (K)) \oplus C_*^\varphi(P'; (K)) \rightarrow C_*^\varphi(X(L\#_i K); (K)) \rightarrow 0$$

of chain complexes. It follows from [19, Theorem 3.4] that

$$(3) \quad \tau(P, \varphi)\tau(P', \varphi) = \tau(D, \varphi)\tau(X(L\#_i K), \varphi) \in (K_1((K))/\pm\varphi(\pi)) \cup \{0\}.$$

First note that D is homotopy equivalent to a circle and that $\text{Im}\{\psi: \pi_1(D) \rightarrow \mathbb{Z}^m\} = \mathbb{Z}e_i$. It is now easy to see that $\tau(D, \varphi) = (1-at_i)^{-1}$ for some $a \in \mathbb{K}(\pi_\psi/\pi_r^{(n+1)}) \setminus \{0\}$.

Next note that $\text{Im}\{\psi: \pi_1(P) \rightarrow \mathbb{Z}^m\} = \mathbb{Z}e_i$. In particular $\tau(P, \varphi)$ is defined over the one-variable Laurent polynomial ring $\mathbb{K}(\pi_\psi/\pi_r^{(n+1)})[t_i^{\pm 1}]$ which is a PID. Recall that we can therefore assume that its Dieudonné determinant $f(t_i)$ lies in $\mathbb{K}(\pi_\psi/\pi_r^{(n+1)})[t_i^{\pm 1}]$ as well.

Claim

$$\deg(\tau(P, \varphi: \pi_1(P) \rightarrow \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})(t_i)) = \delta_n(K).$$

Proof First recall that there exists a homeomorphism $P \cong X(K)$. We also have an inclusion $X(L\#_i K) \rightarrow X(L\#K)$. Combining with the degree one map $X(L\#_i K) \rightarrow X(K)$ we get a factorization of an automorphism of $\pi_1(X(K))$ as follows:

$$\pi_1(X(K)) \cong \pi_1(P) \rightarrow \pi_1(X(L\#_i K)) \rightarrow \pi_1(X(L\#K)) \rightarrow \pi_1(X(K)).$$

Since the rational derived series is functorial (cf [7]) we in fact get that

$$\begin{aligned} \pi_1(X(K))/\pi_1(X(K))_r^{(n+1)} &\cong \pi_1(P)/\pi_1(P)_r^{(n+1)} \\ &\rightarrow \pi_1(X(L\#K))/\pi_1(X(L\#K))_r^{(n+1)} \\ &\rightarrow \pi_1(X(K))/\pi_1(X(K))_r^{(n+1)} \end{aligned}$$

is an isomorphism. In particular

$$\pi_1(X(K))/\pi_1(X(K))_r^{(n+1)} \rightarrow \pi_1(X(L\#_i K))/\pi_1(X(L\#_i K))_r^{(n+1)}$$

is injective, and the induced map on Ore localizations is injective as well. Finally note that $\text{Ker}\{\pi_1(X(K)) \rightarrow \pi_1(P) \xrightarrow{\psi} \mathbb{Z}^m\} = \text{Ker}(\phi)$ where $\phi: \pi_1(X(K)) \rightarrow \mathbb{Z}$ is the abelianization map. It now follows that

$$\begin{aligned} \delta_n(K) &= \text{deg}(\tau(X(K), \pi_1(X(K)) \rightarrow \mathbb{K}(\pi_1(X(K))_\phi/\pi_1(X(K))_r^{(n+1)})(t_i)) \\ &= \text{deg}(\tau(X(K), \pi_1(X(K)) \rightarrow \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})(t_i)) \\ &= \text{deg}(\tau(P, \pi_1(P) \rightarrow \mathbb{K}(\pi_\psi/\pi_r^{(n+1)})(t_i)). \end{aligned}$$

Note that the second equality follows from the functoriality of torsion (cf [19, Proposition 3.6]) and the fact that going to a supfield does not change the degree of a rational function. This concludes the proof of the claim. \square

Claim We have the following equality of norms on $H^1(X(L); \mathbb{Z})$:

$$\|-\|_{\tau(P', \varphi)} = \|-\|_T.$$

Proof First recall that P' is homeomorphic to $X(L)$. The claim now follows immediately from [Theorem 4.3](#) applied to φ and to the abelianization map of $\pi_1(P')$, and from the assumption that $\|-\|_0 = \|-\|_T$ on $H^1(X(L); \mathbb{Z})$. \square

Putting these computations together and using [Equation \(3\)](#) we now get a proof of [Equation \(2\)](#).

Now assume that $\delta_n(K) = 2\text{genus}(K) - 1$. Let S_i be a Seifert surface of K with minimal genus. Let $\phi: \mathbb{Z}^m \rightarrow \mathbb{Z}$ be an epimorphism and let $l = \phi(\mu_i) \in \mathbb{Z}$. We first view ϕ as an element in $\text{hom}(H_1(X(L); \mathbb{Z}))$. A standard argument shows that ϕ is dual to a (possibly disconnected) surface S which intersects the tubular neighborhood of L_i in exactly l disjoint curves. Then the connected sum S' of S with l copies of S_i gives a surface in $X(L\#_i K)$ which is dual to ϕ viewed as an element in $\text{hom}(H_1(X(L\#_i K); \mathbb{Z}))$. A standard argument shows that S' is Thurston norm minimizing (cf eg [10, page 18]).

Clearly $\chi(S') = \chi(S) + l(\chi(S_i) - 1)$. A straightforward argument shows that furthermore $\chi_-(S') = \chi_-(S) + l(\chi_-(S_i) + 1)$ since L is not a split link and since K is non-trivial.

We now compute

$$\begin{aligned}
 \|\phi\|_T &= \chi_-(S') \\
 &= \chi_-(S) - n(\chi(S_i) - 1) \\
 &= \|\phi\|_T + 2l\text{genus}(\mathbf{K}) \\
 &= \|\phi\|_d + 2(\delta_n(\mathbf{K}) + 1) \\
 &= \|\phi\|_d + 2 \deg(f(t_i)) \\
 &= \|\phi\|_{\tau(X(L\#_i K), \varphi)}.
 \end{aligned}$$

By the \mathbb{R} -linearity and the continuity of the norms it follows that

$$\|\phi\|_{\tau(X(L\#_i K), \varphi)} = \|\phi\|_T$$

for all $\phi: \mathbb{Z}^m \rightarrow \mathbb{R}$. □

We now combine Proposition 5.1 with results of [1] to give explicit examples of the sequence of semi-norms $\|\cdot\|_n$.

Denote by $\diamond(n, m)$ the convex polytope given by the vertices $(\pm\frac{1}{n}, 0)$ and $(0, \pm\frac{1}{m})$. Let $(n_i)_{i \in \mathbb{N}}$ and $(m_i)_{i \in \mathbb{N}}$ be never decreasing sequences of odd positive numbers which are eventually constant, ie there exists an N such that $n_i = n_N$ for all $i \geq N$ and $m_i = m_N$ for all $i \geq N$. According to [1] we can find knots K_1 and K_2 such that $\delta_i(K_1) = n_i$ for any i , $\delta_N(K_1) = 2 \text{genus}(\mathbf{K}_1) - 1$ and $\delta_i(K_2) = m_i$ for any i and $\delta_N(K_2) = 2 \text{genus}(\mathbf{K}_2) - 1$.

Let $H(K_1, K_2)$ be the link formed by adding the two knots K_1 and K_2 from above to the Hopf link (cf Figure 3). Recall that the Thurston norm ball of the Hopf link is given by $\diamond(1, 1)$. Let $\pi := \pi_1(X(L))$. It follows immediately from applying Proposition 5.1

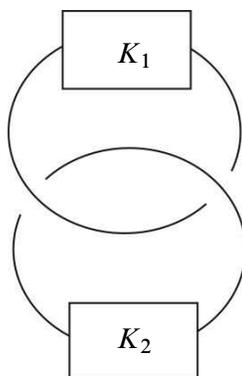


Figure 3: $H(K_1, K_2)$ is obtained by tying K_1 and K_2 into the Hopf link

twice that the norm ball of $\|-\|_i$ equals $\diamond(n_i + 1, m_i + 1)$ and that $\|-\|_N = \|-\|_T$. The following result is now an immediate consequence of [Proposition 5.1](#).

Corollary 5.2 *We have the following sequence of inequalities of semi-norms*

$$\|-\|_A = \|-\|_0 \leq \|-\|_1 \leq \|-\|_2 \leq \cdots \leq \|-\|_N = \|-\|_T.$$

In [\[7\]](#) the second author gave examples of 3-manifolds M such that

$$\|-\|_A = \|-\|_0 \leq \|-\|_1 \leq \|-\|_2 \leq \cdots$$

but in that case it was not known whether the sequence of norms $\|-\|_i$ eventually agrees with $\|-\|_T$.

It is an interesting question to determine which 3-manifolds satisfy $\|-\|_T = \|-\|_n$ for large enough n . We conclude this paper with the following conjecture.

Conjecture 5.3 *If $\pi_1(M)_r^{(\omega)} := \bigcap_{n \in \mathbb{N}} \pi_1(M)_r^{(n)} = \{1\}$, then there exists $n \in \mathbb{N}$ such that $\|-\|_T = \|-\|_n$.*

References

- [1] **T D Cochran**, *Noncommutative knot theory*, *Algebr. Geom. Topol.* 4 (2004) 347–398 [MR2077670](#)
- [2] **T D Cochran**, **K E Orr**, **P Teichner**, *Knot concordance, Whitney towers and L^2 -signatures*, *Ann. of Math.* (2) 157 (2003) 433–519 [MR1973052](#)
- [3] **J Dodziuk**, **P Linnell**, **V Mathai**, **T Schick**, **S Yates**, *Approximating L^2 -invariants and the Atiyah conjecture*, *Comm. Pure Appl. Math.* 56 (2003) 839–873 [MR1990479](#) Dedicated to the memory of Jürgen K Moser
- [4] **S Friedl**, *Reidemeister torsion, the Thurston norm and Harvey’s invariants*, *Pacific J. Math.* 230 (2007) 271–206
- [5] **S Friedl**, **T Kim**, *The parity of the Cochran–Harvey invariants of 3-manifolds*, *Trans. Amer. Math. Soc.* (2005) In press
- [6] **S Friedl**, **T Kim**, *Twisted Alexander norms give lower bounds on the Thurston norm*, *Trans. Amer. Math. Soc.* (2005) In press
- [7] **S L Harvey**, *Higher-order polynomial invariants of 3-manifolds giving lower bounds for the Thurston norm*, *Topology* 44 (2005) 895–945 [MR2153977](#)
- [8] **S L Harvey**, *Monotonicity of degrees of generalized Alexander polynomials of groups and 3-manifolds*, *Math. Proc. Cambridge Philos. Soc.* 140 (2006) 431–450 [MR2225642](#)

- [9] **G Higman**, *The units of group-rings*, Proc. London Math. Soc. (2) 46 (1940) 231–248 [MR0002137](#)
- [10] **W B R Lickorish**, *An introduction to knot theory*, Graduate Texts in Mathematics 175, Springer, New York (1997) [MR1472978](#)
- [11] **C T McMullen**, *The Alexander polynomial of a 3–manifold and the Thurston norm on cohomology*, Ann. Sci. École Norm. Sup. (4) 35 (2002) 153–171 [MR1914929](#)
- [12] **J Milnor**, *Whitehead torsion*, Bull. Amer. Math. Soc. 72 (1966) 358–426 [MR0196736](#)
- [13] **D S Passman**, *The algebraic structure of group rings*, Robert E. Krieger Publishing Co., Melbourne, FL (1985) [MR798076](#) Reprint of the 1977 original
- [14] **J Rosenberg**, *Algebraic K–theory and its applications*, Graduate Texts in Mathematics 147, Springer, New York (1994) [MR1282290](#)
- [15] **R Strebel**, *Homological methods applied to the derived series of groups*, Comment. Math. Helv. 49 (1974) 302–332 [MR0354896](#)
- [16] **B Sturmfels**, *Solving systems of polynomial equations*, CBMS Regional Conference Series in Mathematics 97, Published for the Conference Board of the Mathematical Sciences, Washington, DC (2002) [MR1925796](#)
- [17] **D Tamari**, *A refined classification of semi-groups leading to generalized polynomial rings with a generalized degree concept*, from: “Proceedings of the International Congress of Mathematicians, Amsterdam, 1954”, volume 3 (1957) 439–440
- [18] **W P Thurston**, *A norm for the homology of 3–manifolds*, Mem. Amer. Math. Soc. 59 (1986) i–vi and 99–130 [MR823443](#)
- [19] **V Turaev**, *Introduction to combinatorial torsions*, Lectures in Mathematics ETH Zürich, Birkhäuser Verlag, Basel (2001) [MR1809561](#) Notes taken by Felix Schlenk
- [20] **V Turaev**, *A homological estimate for the Thurston norm* (2002) [arXiv: math.GT/0207267](#)
- [21] **V Turaev**, *A norm for the cohomology of 2–complexes*, Algebr. Geom. Topol. 2 (2002) 137–155 [MR1885218](#)

Department of Mathematics, Rice University
6100 Main Street, MS 136, Houston TX 77005, USA

Département de Mathématiques, UQAM
C P 8888, Succursale Centre-ville, Montréal, Qc H3C 3P8, Canada

shelly@math.rice.edu, friedl@alumni.brandeis.edu

Received: 18 August 2006