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Length functions of Hitchin representations

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Given a Hitchin representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$, we construct n continuous functions $\ell_i^\rho\colon \mathcal{C}^{\text{Höl}}(S) \to \mathbb{R}$ defined on the space of Hölder geodesic currents $\mathcal{C}^{\text{Höl}}(S)$ such that, for a closed, oriented curve γ in S, the i^{th} eigenvalue of the matrix $\rho(\gamma) \in \mathrm{PSL}_n(\mathbb{R})$ is of the form $\pm\exp \ell_i^\rho(\gamma)$: such functions generalize to higher rank Thurston's length function of Fuchsian representations. Identities and differentiability properties of these lengths ℓ_i^ρ, as well as applications to eigenvalue estimates, are also considered.

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Let S be a closed, connected, oriented surface S of genus $g \geq 2$. This article is concerned with homomorphisms $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ from the fundamental group $\pi_1(S)$ to the Lie group $\mathrm{PSL}_n(\mathbb{R})$ (equal to the special linear group $\mathrm{SL}_n(\mathbb{R})$ if n is odd, and to $\mathrm{SL}_n(\mathbb{R})/\{\pm\mathrm{Id}\}$ if n is even), and more precisely with elements lying in *Hitchin components* $\mathrm{Hit}_n(S)$ of the $\mathrm{PSL}_n(\mathbb{R})$–character variety

$$\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S) = \mathrm{Hom}\big(\pi_1(S), \mathrm{PSL}_n(\mathbb{R})\big)/\!\!/\mathrm{PSL}_n(\mathbb{R})$$

identified by N Hitchin [18]. Here, the "double bar" sign indicates that the precise definition of the character variety $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$ requires that the quotient be taken in the sense of geometric invariant theory (Mumford, Fogarty and Kirwan [21]); however, for the component $\mathrm{Hit}_n(S)$ that we are interested in, this quotient construction coincides with the usual topological quotient.

A *Hitchin component* $\mathrm{Hit}_n(S)$ is defined as a component of $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$ that contains some *n–Fuchsian representation*, namely some homomorphism $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ of the form

$$\rho = \iota \circ r,$$

where: $r\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ is a discrete, injective homomorphism, and

$$\iota\colon \mathrm{PSL}_2(\mathbb{R}) \to \mathrm{PSL}_n(\mathbb{R})$$

is the preferred homomorphism defined by the n–dimensional, irreducible representation of $\mathrm{SL}_2(\mathbb{R})$ into $\mathrm{SL}_n(\mathbb{R})$. These components $\mathrm{Hit}_n(S)$ were singled out by

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N Hitchin [18] who first suggested the interest in studying their elements. We shall refer to elements of $\mathrm{Hit}_n(S)$ as *Hitchin representations*.

Motivations for studying Hitchin representations find their origin in the case where $n=2$. Hitchin components $\mathrm{Hit}_2(S)$ then coincide with *Teichmüller components* $\mathcal{T}(S)$ of $\mathcal{R}_{\mathrm{PSL}_2(\mathbb{R})}(S)$, whose elements, known as *Fuchsian representations*, are of particular interest as they correspond to conjugacy classes of holonomies of marked hyperbolic structures on the surface S. In addition, every element of $\mathcal{T}(S)$ is a discrete, injective homomorphism, and reversely, any such homomorphism lies in some component $\mathcal{T}(S)$ (Weil [26] and Margulis [20]). It is a result due to W Goldman [13] that $\mathcal{R}_{\mathrm{PSL}_2(\mathbb{R})}(S)$ possesses exactly two Teichmüller components $\mathcal{T}(S)$; each of these components $\mathcal{T}(S)$ is known to be homeomorphic to $\mathbb{R}^{6g-6}$ (Thurston [25] and Fathi, Laudenbach and Poénaru [10]).

In his foundational paper, Hitchin [18] proved that, in the case where $n \geq 3$, there are one or two Hitchin components $\mathrm{Hit}_n(S)$ in $\mathcal{R}_{\mathrm{PSL}_n(\mathbb{R})}(S)$ according to whether n is odd or even, and a beautiful result of Hitchin is that each of these components $\mathrm{Hit}_n(S)$ is homeomorphic to $\mathbb{R}^{(2g-2)(n^2-1)}$. Hitchin's proof is based the theory of Higgs bundles, and as observed by Hitchin, this geometric analysis framework offers no information about the geometry of the elements of $\mathrm{Hit}_n(S)$. The first geometric result about Hitchin representations is to due to S Choi and W Goldman [8] who showed that, for $n=3$, the Hitchin component $\mathrm{Hit}_3(S)$ parametrizes the deformation space of *real convex projective structures* on the surface S. As a consequence of their work, they showed the faithfulness and the discreetness for the elements of $\mathrm{Hit}_3(S)$.

About a decade ago, F Labourie [19] (see also Guichard [16] and Guichard and Wienhard [17]) proved the following result.

Theorem 1 (Labourie [19]) *Let $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ be a Hitchin representation. Then ρ is discrete and injective. In addition, the image $\rho(\gamma) \in \mathrm{PSL}_n(\mathbb{R})$ of any nontrivial $\gamma \in \pi_1(S)$ is diagonalizable, its eigenvalues are all real with distinct absolute values.*

The above statement comes as a consequence (among others) of a remarkable Anosov property for Hitchin representations discovered by Labourie [19]. More precisely, let $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ be a Hitchin representation that lifts to $\rho\colon \pi_1(S) \to \mathrm{SL}_n(\mathbb{R})$; consider the flat, twisted $\mathbb{R}^n$–bundle $T^1S \times_\rho \mathbb{R}^n = T^1S \times \mathbb{R}^n/\pi_1(S) \to T^1S$, where T^1S is the unit tangent bundle of S; let $(G_t)_{t\in\mathbb{R}}$ on $T^1S \times_\rho \mathbb{R}^n$ be the flow that lifts the geodesic flow $(g_t)_{t\in\mathbb{R}}$ on T^1S via the flat connection. The total space $T^1S \times_\rho \mathbb{R}^n$ splits as a sum of line subbundles $V_1 \oplus \cdots \oplus V_n$ with the property that each line subbundle $V_i \to T^1S$ is invariant under the action of the flow $(G_t)_{t\in\mathbb{R}}$. In addition, the action of the flow $(G_t)_{t\in\mathbb{R}}$ is *Anosov* in the following sense: pick a Riemannian metric $\| \ \|$ on $T^1S \times_\rho \mathbb{R}^n \to T^1S$; there exist some constants $A \geq 0$ and $a > 0$ such that, for every $u \in T^1S$, for every unit vectors $X_i(u) \in V_i(u)$ and $X_j(u) \in V_j(u)$, for every $t > 0$,

$$Ae^{-at} \geq \begin{cases} \dfrac{\left\|G_t X_j(u)\right\|_{g_t(u)}}{\left\|G_t X_i(u)\right\|_{g_t(u)}} & \text{if } i > j; \\[2ex] \dfrac{\left\|G_{-t} X_j(u)\right\|_{g_{-t}(u)}}{\left\|G_{-t} X_i(u)\right\|_{g_{-t}(u)}} & \text{if } i < j. \end{cases}$$

Results

Given a Hitchin representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$, our main result uses Labourie's dynamical framework to define a family of n *length functions* ℓ_i^ρ associated to ρ; these length functions extend to Hitchin representations *Thurston's length function* of Fuchsian representations in the Teichmüller space $\mathcal{T}(S)$.

Thurston [25] considers the space of *measured laminations* $\mathcal{ML}(S)$ of S, that is a certain completion of the set of all isotopy classes of simple, closed, unoriented curves in S. He then associates to a Fuchsian representation $r\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ in $\mathcal{T}(S)$ a continuous, homogeneous function $\ell^r : \mathcal{ML}(S) \to \mathbb{R}$ such that, for every simple, closed, unoriented curve $\gamma \subset S$,

$$\ell^r(\gamma) = \tfrac{1}{2}\log\left|\lambda_1^r(\gamma)\right|,$$

where $|\lambda_1^r(\gamma)|$ is the largest absolute value of the eigenvalues of $r(\gamma) \in \mathrm{PSL}_2(\mathbb{R})$. Geometrically, $r\colon \pi_1(S) \to \mathrm{PSL}_2(\mathbb{R})$ is the holonomy of a marked hyperbolic structure m on the surface S; the number $\ell^r(\gamma)$ is then the length of the unique, simple, closed, unoriented m–geodesic in S that is freely homotopic to the simple, closed, unoriented curve γ. This length function $\ell^r\colon \mathcal{ML}(S) \to \mathbb{R}$ has proved to be a fundamental tool in the study of 2 and 3–dimensional hyperbolic manifolds.

Thurston's length function ℓ^r was later extended by F Bonahon [1; 2] to the larger space of *measure geodesic currents* $\mathcal{C}(S)$ of S, which is a certain completion of the set of all isotopy classes of closed, oriented curves in S. Later, Bonahon [3] also developed a differential calculus for measured laminations, that is based on *Hölder geodesic currents*. In particular, he obtains differentiability properties for Thurston's original function $\ell^r\colon \mathcal{ML}(S) \to \mathbb{R}$ by continuously extending $\ell^r\colon \mathcal{ML}(S) \to \mathbb{R}$ to the space of Hölder geodesic currents $\mathcal{C}^{\mathrm{Höl}}(S)$ of S.

We generalize these constructions in the case where $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ is a Hitchin representation. Let $\gamma \in \pi_1(S)$ be nontrivial element; by Theorem 1, the eigenvalues

$\lambda_i^\rho(\gamma)$ of the matrix $\rho(\gamma) \in \mathrm{PSL}_n(\mathbb{R})$ can be indexed so that

$$|\lambda_1^\rho(\gamma)| > |\lambda_2^\rho(\gamma)| > \cdots > |\lambda_n^\rho(\gamma)|.$$

Theorem 2 (Length functions) *Let $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ be a Hitchin representation, and let $\mathcal{C}^{\text{Höl}}(S)$ be the vector space of Hölder geodesic currents. For every $i = 1, 2, \dots, n$, there exists a continuous, linear function*

$$\ell_i^\rho\colon \mathcal{C}^{\text{Höl}}(S) \to \mathbb{R}$$

such that, for every closed, oriented curve $\gamma \subset S$, $\ell_i^\rho(\gamma) = \log|\lambda_i^\rho(\gamma)|$. This continuous extension is unique on the space of measure geodesic currents $\mathcal{C}(S) \subset \mathcal{C}^{\text{Höl}}(S)$.

In addition, let $\mathfrak{R}\colon T^1S \to T^1S$ be the *orientation reversing involution*, namely $\mathfrak{R}$ is the map defined by $\mathfrak{R}(u) = -u$, where $u \in T_x^1S$. For every Hölder geodesic current $\alpha \in \mathcal{C}^{\text{Höl}}(S)$, $\mathfrak{R}^*\alpha$ is the *pullback current* of α under the involution $\mathfrak{R}$.

Theorem 3 (Identities) *For every Hölder geodesic current $\alpha \in \mathcal{C}^{\text{Höl}}(S)$:*

(i) $\displaystyle\sum_{i=1}^n \ell_i^\rho(\alpha) = 0$

(ii) $\ell_i^\rho(\mathfrak{R}^*\alpha) = -\ell_{n-i+1}^\rho(\alpha)$

The above two identities are suggested by the case where $\alpha \in \mathcal{C}^{\text{Höl}}(S)$ is a closed, oriented curve $\gamma \in \pi_1(S)$. Indeed, since $\rho(\gamma) \in \mathrm{PSL}_n(\mathbb{R})$, $\sum_{i=1}^n \log|\lambda_i^\rho(\gamma)| = 0$. Moreover, as a consequence of our indexing conventions, $\lambda_i^\rho(\gamma^{-1}) = 1/\lambda_{n-i+1}^\rho(\gamma)$, and thus $\log|\lambda_i^\rho(\gamma^{-1})| = -\log|\lambda_{n-i+1}^\rho(\gamma)|$.

The continuity property of Theorem 2 is the fundamental feature of the length functions $\ell_i^\rho : \mathcal{C}^{\text{Höl}}(S) \to \mathbb{R}$. As an application of this continuity, we prove the two following results; the proofs use the full force of Hölder geodesic currents.

Theorem 4 (Tangentiability) *The functions $\ell_i^\rho\colon \mathcal{C}^{\text{Höl}}(S) \to \mathbb{R}$ restrict to functions $\ell^\rho_{i\,|\mathcal{ML}(S)}\colon \mathcal{ML}(S) \to \mathbb{R}$ that are tangentiable, namely, if $(\alpha_t)_{t\geq 0} \subset \mathcal{ML}(S)$ is a smooth 1–parameter family of measured laminations with tangent vector $\dot{\alpha_0} = \frac{d}{dt^+}\alpha_t|_{t=0}$ at α_0, then*

$$\frac{d}{dt^+}\ell_i^\rho(\alpha_t)\Big|_{t=0} = \ell_i^\rho(\dot{\alpha}_0).$$

Finally, we prove the following asymptotic estimate for the eigenvalues of a Hitchin representation.

Theorem 5 (Eigenvalue estimate) *Let $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ be a Hitchin representation, and let α, $\beta \in \pi_1(S)$. For every $i = 1, \dots, n$, the ratio*

$$\frac{\lambda_i^\rho(\alpha^m \beta)}{\lambda_i^\rho(\alpha)^m}$$

has a finite limit as m tends to ∞. This limit is equal to $e^{\ell_i^\rho(\dot\alpha)}$ for a certain Hölder geodesic current $\dot\alpha \in \mathcal{C}^{\mathrm{Höl}}(S)$.

Remarks

Dreyer [9] extends to Hitchin representations *Thurston's cataclysm deformations*, which themselves generalize (left) earthquake deformations of hyperbolic structures on surfaces (Thurston [24; 25]). Given a Hitchin representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$, we study various geometric aspects of these cataclysms, and prove a variational formula for the associated length functions ℓ_i^ρ.

Another motivation for introducing length functions associated to a Hitchin representation is part of the development of a new system of coordinates on Hitchin components $\mathrm{Hit}_n(S)$. In [18], Hitchin showed that $\mathrm{Hit}_n(S)$ is diffeomorphic to $\mathbb{R}^{(2g-2)(n^2-1)}$; his parametrization is based on Higgs bundle techniques, and in particular requires the initial choice of a complex structure on S. In joint work with F Bonahon [5; 6], we construct a geometric, real analytic parametrization of Hitchin components $\mathrm{Hit}_n(\mathbb{R}^n)$. One feature of this parametrization is that it is based on topological data only. In essence, our coordinates are an extension of Thurston's shearing coordinates (Thurston [24] and Bonahon [1]) on the Teichmüller space $\mathcal{T}(S)$, combined with Fock–Goncharov coordinates [11] on moduli spaces of positive, framed, local systems on a punctured surface. In particular, the length functions ℓ_i^ρ play a crucial role in analyzing the image of this parametrization.

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1 The eigenbundles of a Hitchin representation

Our construction makes great use of the machinery developed in Labourie [19], we thus begin with reviewing some of Labourie's framework. It is convenient to endow the surface S with an arbitrary hyperbolic metric m_0. It induces a m_0–geodesic flow $(g_t)_{t\in\mathbb{R}}$ on the unit tangent bundle T^1S of S; we refer to the associated orbit space as the m_0–geodesic foliation $\mathcal{F}$ of T^1S.

Let $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ be a Hitchin representation. Since ρ lies in the same component as some n–Fuchsian representation, it lifts to a representation valued in $\mathrm{SL}_n(\mathbb{R})$, that we still denote by $\rho\colon \pi_1(S) \to \mathrm{SL}_n(\mathbb{R})$; see Goldman [13] for details. Consider the flat twisted $\mathbb{R}^n$–bundle

$$T^1S \times_\rho \mathbb{R}^n = T^1\widetilde{S} \times \mathbb{R}^n/\pi_1(S),$$

where $\widetilde{S}$ is the universal cover of S, and where the action of $\pi_1(S)$ is defined by the property that, for every $\gamma \in \pi_1(S)$, for every $(u,X) \in T^1\widetilde{S} \times \mathbb{R}^n$,

$$\gamma(u,X) = (\gamma u, \rho(\gamma)X).$$

Let $(G_t)_{t\in\mathbb{R}}$ be the flow on the total space $T^1S \times_\rho \mathbb{R}^n$ that lifts the geodesic flow $(g_t)_{t\in\mathbb{R}}$ on T^1S via the flat connection; here, the "flatness" condition means that, if one looks at the situation in the universal cover, the lift $(\widetilde{G}_t)_{t\in\mathbb{R}}$ acts on $T^1\widetilde{S} \times \mathbb{R}^n$ as the geodesic flow $(\widetilde{g})_{t\in\mathbb{R}}$ on the first factor, and trivially on the second factor. We shall refer to $T^1S \times_\rho \mathbb{R}^n \to T^1S$ as the *associated, flat* $\mathbb{R}^n$*–bundle of the Hitchin representation* $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$.

For every nontrivial $\gamma \in \pi_1(S)$, index the eigenvalues $\lambda_i^\rho(\gamma)$ of $\rho(\gamma) \in \mathrm{PSL}_n(\mathbb{R})$ as in Theorem 2 so that

$$|\lambda_1^\rho(\gamma)| > |\lambda_2^\rho(\gamma)| > \cdots > |\lambda_n^\rho(\gamma)|.$$

The key tool underlying Labourie's analysis is the following decomposition.

Theorem 6 (Labourie [19] eigenbundle decomposition) *The associated, flat* $\mathbb{R}^n$*–bundle* $p\colon T^1S \times_\rho \mathbb{R}^n \to T^1S$ *splits as a sum of n line subbundles* $V_1 \oplus \cdots \oplus V_n$ *that satisfy the following properties:*

(i) *Each line subbundle* $V_i \to T^1S$ *is invariant under the flow* $(G_t)_{t\in\mathbb{R}}$.

(ii) *If* $u \in T^1S$ *is fixed by* $g_{t_0}\colon T^1S \to T^1S$ *for some* $t_0 > 0$, *and if* $\gamma \in \pi_1(S)$ *represents the corresponding closed orbit of the geodesic flow, then the lift* G_{t_0} *acts on the fibre* $p^{-1}(u) = V_1(u) \oplus \cdots \oplus V_n(u)$ *by multiplication by* $1/\lambda_i^\rho(\gamma)$ *on the line* $V_i(u)$.

(iii) *Each line* $V_i(u)$ *depends smoothly on* $u \in T^1S$ *along the leaves of the geodesic foliation* $\mathcal{F}$, *and is transversally Hölder continuous.*

The terminology *eigenbundle decomposition* is motivated by the property (ii) that we can make more precise as follows. For every $i = 1, \dots, n$, let $\widetilde{V}_i \to T^1\widetilde{S}$ that lifts the line subbundle $V_i \to T^1 S$. Let $(\widetilde{g}_t)_{t\in\mathbb{R}}$ be the lift on the universal cover $T^1\widetilde{S}$ of the geodesic flow $(g_t)_{t\in\mathbb{R}}$ on $T^1 S$. Let $u \in T^1 S$ that is fixed by $g_{t_0}\colon T^1 S \to T^1 S$ for some $t_0 > 0$. Let $\widetilde{u} \in T^1\widetilde{S}$ that lifts $u \in T^1 S$, and let $\gamma \in \pi_1(S)$ be the (unique) nontrivial element such that $\widetilde{g}_{t_0}(\widetilde{u}) = \gamma\widetilde{u}$. Because of the flat connection, and the invariance of the line subbundle $V_i \to T^1 S$ under the flow $(G_t)_{t\in\mathbb{R}}$, we have that $\widetilde{V}_i(\widetilde{g}_{t_0}(\widetilde{u})) = \widetilde{V}_i(\widetilde{u})$ as lines of $\mathbb{R}^n$. Also, $\widetilde{V}_i(\widetilde{g}_{t_0}(\widetilde{u})) = \widetilde{V}_i(\gamma\widetilde{u}) = \rho(\gamma)\widetilde{V}_i(\widetilde{u})$ (it is the equivariance property for the lift $\widetilde{V}_i \to T^1\widetilde{S}$). Hence $\widetilde{V}_i(\widetilde{u}) \subset \mathbb{R}^n$ is an eigenspace for $\rho(\gamma) \in \mathrm{PSL}_n(\mathbb{R})$, and $\rho(\gamma)$ is diagonalizable. Finally, note that, for every $X \in \mathbb{R}^n$, $(\widetilde{g}_{t_0}(\widetilde{u}), \widetilde{G}_{t_0} X) = (\gamma\widetilde{u}, X)$ identifies in the quotient with $(\widetilde{u}, \rho(\gamma)^{-1}X)$. Therefore, the lift G_{t_0} acts on the line $V_i(u)$ by multiplication by $1/\lambda_i^\rho(\gamma)$.

As a consequence of the above discussion, we make the following observation, that we state as a lemma for future reference.

Lemma 7 *Let $\widetilde{V}_i \to T^1\widetilde{S}$ and $\widetilde{V}_{n-i+1} \to T^1\widetilde{S}$ that lift the line subbundles $V_i \to T^1 S$ and $V_{n-i+1} \to T^1 S$, respectively. For every $u \in T^1 S$ that lifts to $\widetilde{u} \in T^1\widetilde{S}$, the fibres $\widetilde{V}_i(\widetilde{u})$ and $\widetilde{V}_{n-i+1}(-\widetilde{u})$ coincide as lines of $\mathbb{R}^n$.*

Proof When u lies in a closed leaf of the geodesic foliation $\mathcal{F}$, the assertion immediately comes as a consequence of the property (ii) of Theorem 6, and from our indexing conventions for the eigenvalues of $\rho(\gamma) \in \mathrm{PSL}_n(\mathbb{R})$ when $\gamma \in \pi_1(S)$. The general case then follows from the latter by density of closed leaves in $T^1 S$. $\square$

The existence of an eigenbundle decomposition for the associated, flat $\mathbb{R}$–bundle $T^1 S \times_\rho \mathbb{R}^n \to T^1 S$ of a Hitchin representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ as in Theorem 6 is a consequence of *Labourie's Anosov Property* for Hitchin representations; see Labourie [19], Guichard [16], Guichard and Wienhard [17] and Dreyer [9] for additional details.

2 The length functions of a Hitchin representation

2.1 Hölder geodesic currents

Before tackling the construction of the length functions $\ell_i^\rho\colon \mathcal{C}^{\mathrm{Höl}}(S) \to \mathbb{R}$, we need to remind the reader of the definition of Hölder geodesic currents; see Bonahon [2; 3; 4] for details.

Let (X, d) be metric space. A *Hölder distribution* α is a continuous, linear functional on the space of compactly supported, Hölder continuous functions $\phi\colon X \to \mathbb{R}$. A special case of Hölder distributions are *positive Radon measures*, which are linear functionals on the space of compactly supported, continuous functions, and associate to a nonnegative function a nonnegative number.

The unit tangent bundle T^1S is a 3–dimensional manifold, and the orbits of the m_0–geodesic flow $(g_t)_{t\in\mathbb{R}}$ define a 1–dimensional foliation $\mathcal{F}$ of T^1S called its m_0*–geodesic foliation*. It turns out that, whereas the geodesic flow depends of the auxiliary metric m_0 that we have chosen on S, the geodesic foliation does not. Indeed, if another negatively curved metric m' defines a geodesic foliation $\mathcal{F}'$, there is a homeomorphism of T^1S that sends $\mathcal{F}$ to $\mathcal{F}'$. In addition, this homeomorphism can be chosen to be isotopic to the identity, and Hölder bicontinuous; see Bridson and Haefliger [7], Ghys [12] and Gromov [14; 15] for details.

A *Hölder geodesic current* α on S is a transverse Hölder distribution for the geodesic foliation $\mathcal{F}$, namely α assigns a Hölder distribution α_D on every surface $D\subset T^1S$ transverse to $\mathcal{F}$. This assignment is invariant under restriction: for any subsurface $D'\subset D$, $\alpha_D|_{D'}=\alpha_{D'}$, and is homotopy invariant: for any (Hölder) homotopy $h\colon D\to D''$ from D to another transverse surface D'' that preserves $\mathcal{F}$, $\alpha_D=h^*\alpha_{D''}$; $h^*\alpha_{D''}$ is the pullback of $\alpha_{D''}$ by h.

When the transverse Hölder distribution α is actually a measure α_D for every surface $D\subset T^1S$ transverse to $\mathcal{F}$, the corresponding Hölder geodesic current is a *measure geodesic current* of S. Let $\mathcal{C}^{\text{Höl}}(S)$ and $\mathcal{C}(S)$ be respectively the space of Hölder geodesic currents, and the space of measure geodesic currents. Note that $\mathcal{C}^{\text{Höl}}(S)$ is a (real) vector space, and $\mathcal{C}(S)$ is stable under positive scalar multiplication.

The space of Hölder geodesic currents $\mathcal{C}^{\text{Höl}}(S)$ is endowed with the *weak-$*$ topology*, namely the weakest topology for which, for every surface $D\subset T^1S$ transverse to $\mathcal{F}$, the linear function $\varphi_D\mapsto\alpha_D(\varphi_D)$ is continuous, where φ_D ranges over all compactly supported, Hölder continuous functions on the surface D.

A typical example of measure geodesic current is provided by the free homotopy class of a closed, oriented curve $\gamma\subset S$. Let $k\geq 0$ be the largest integer such that γ is homotopic to a k–multiple γ_1^k of a closed curve γ_1. The homotopically primitive curve γ_1 is freely homotopic to a unique closed, oriented geodesic, which itself corresponds to a closed leaf γ_1^* of the geodesic foliation $\mathcal{F}$. In particular, we associate to γ_1 the transverse 1–weighted Dirac measure for $\mathcal{F}$ defined by the closed orbit γ_1^*: for every surface D transverse to $\mathcal{F}$, the measure γ_{1D} is the counting measure at the intersection points $D\cap\gamma_1^*$. As a result, we associate to $\gamma=\gamma_1^k$ k–times the transverse 1–weighted Dirac measure associated to γ_1. Hence the following embedding

$$\{\text{closed, oriented curves in } S\}/\text{homotopy}\subset\mathcal{C}(S)\subset\mathcal{C}^{\text{Höl}}(S).$$

In addition, the set of positive real, linear combinations of multiples of homotopy classes of closed, oriented curves are dense in $\mathcal{C}(S)$.

Finally, note that to a closed, *unoriented* curve $\overline{\gamma}$ in S corresponds two closed leaves γ^* and $(\mathfrak{R}(\gamma))^*$ of the geodesic foliation $\mathcal{F}$, and thus two measure geodesic currents γ and $\mathfrak{R}^*\gamma \in \mathcal{C}(S)$ (as in Theorem 3, $\mathfrak{R}\colon T^1S \to T^1S$ denotes the orientation reversing involution, and $\mathfrak{R}^*\colon \mathcal{C}^{\text{Höl}}(S) \to \mathcal{C}^{\text{Höl}}(S)$ is the pullback involution induced by $\mathfrak{R}$). Therefore, the set of closed, *unoriented* curves in S can be formally embedded in $\mathcal{C}^{\text{Höl}}(S)$ as follows: for every closed, unoriented curve $\overline{\gamma} \subset S$,

$$\overline{\gamma} = \tfrac{1}{2}\gamma + \tfrac{1}{2}\mathfrak{R}^*\gamma \in \mathcal{C}^{\text{Höl}}(S). \tag{1}$$

In particular, the space of measured laminations $\mathcal{ML}(S)$, that is defined as the closure in $\mathcal{C}(S)$ of the set of positive real multiples of homotopy classes of *simple, closed, unoriented* curves in S (Bonahon [1; 2]), corresponds to the closure in $\mathcal{C}(S)$ of the set of positive real, linear combinations of elements of the above form (1).

2.2 Lengths of closed, oriented curves

Let $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ be a Hitchin representation. For a nontrivial $\gamma \in \pi_1(S)$, Theorem 1 shows that the eigenvalues $\lambda_i^\rho(\gamma)$ of the matrix $\rho(\gamma) \in \mathrm{PSL}_n(\mathbb{R})$ are all real, and can be indexed so that

$$\left|\lambda_1^\rho(\gamma)\right| > \left|\lambda_2^\rho(\gamma)\right| > \cdots > \left|\lambda_n^\rho(\gamma)\right|.$$

Set $\ell_i^\rho(\gamma) = \log|\lambda_i^\rho(\gamma)|$. Note that $\ell_i^\rho(\gamma)$ depends only on the conjugacy class of $\gamma \in \pi_1(S)$, and thus depends only on the free homotopy class of the closed, oriented curve $\gamma \subset S$. We have n maps

$$\ell_i^\rho\colon \{\text{closed, oriented curves in } S\}/\text{homotopy} \to \mathbb{R}.$$

2.3 1–forms along the geodesic foliation

We now construct, for every $i = 1, \dots, n$, a 1–form ω_i along the leaves of the geodesic foliation $\mathcal{F}$ of T^1S.

Given a Hitchin representation $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$, consider its associated flat $\mathbb{R}^n$–bundle $T^1S \times_\rho \mathbb{R}^n \to T^1S$ as in Section 1. Let $(G_t)_{t\in\mathbb{R}}$ the flow on $T^1S \times_\rho \mathbb{R}^n$ that lifts the geodesic flow on T^1S via the flat connection. Let $V_1, V_2, \dots, V_n \to T^1S$ be the line subbundles of the eigenbundle decomposition of Theorem 6; a fundamental property for each of the line subbundles $V_i \to T^1S$ is to be invariant under the action of the flow $(G_t)_{t\in\mathbb{R}}$. Finally, pick a Riemannian metric $\|\ \|$ on $T^1S \times_\rho \mathbb{R}^n \to T^1S$.

Let $\mathcal{L}$ be a leaf of the geodesic foliation $\mathcal{F}$. Pick a point $u_0 \in \mathcal{L} \subset T^1 S$ and a vector $X_i(u_0)$ in the fibre $V_i(u_0)$ of the line subbundle $V_i \to T^1 S$. For every t on a neighborhood of 0 in $\mathbb{R}$, set

$$f_{X_i(u_0)}\big(g_t(u_0)\big) = \log \|G_t X_i(u_0)\|_{g_t(u_0)}.$$

The above expression defines a function $f_{X_i(u_0)}\colon I \to \mathbb{R}$, where I is a neighborhood of u_0 in the leaf $\mathcal{L}$. In addition, the fibre $V_i(u)$ depending smoothly on the point $u \in T^1 S$ along the leaves of the geodesic foliation $\mathcal{F}$, the function $f_{X_i(u_0)}$ is smooth along the leaf $\mathcal{L}$. For every u on the same neighborhood I of u_0 in the leaf $\mathcal{L}$, set

$$\omega_i(u) = -d_u f_{X_i(u_0)},$$

where the differential $d_u f_{X_i(u_0)}$ is taken along the leaf $\mathcal{L}$.

Lemma 8 *Defined as above, ω_i is a well-defined 1–form along the leaf $\mathcal{L} \subset \mathcal{F}$.*

Proof We must verify that ω_i does not depend on the choices of the point $u_0 \in I$, and of the vector $X_i(u_0) \in V_i(u_0)$.

Since the fibre $V_i(u_0)$ is a line, any other choice $X_i'(u_0)$ for $X_i(u_0)$ is of the form $X_i'(u_0) = c X_i(u_0)$ for some $c \in \mathbb{R}$. Then

$$f_{X_i'(u_0)} = f_{X_i(u_0)} + \log |c|$$

and $df_{X_i'(u_0)} = df_{X_i(u_0)}$. Hence ω_i is independent of the choice of $X_i(u_0) \in V_i(u_0)$.

Let $u_0' = g_{t_0}(u_0) \in I$ be another point, and let $X_i'(u_0') = G_{t_0} X_i(u_0) \in V_i(u_0')$. Then the functions

$$f_{u_0', X_i'(u_0')} = f_{u_0, X_i(u_0)}$$

coincide on I since $G_t X_i'(u_0') = G_{t+t_0} X_i(u_0)$, and thus have the same differential along the leaf $\mathcal{L}$. □

It follows from Lemma 8 that ω_i is a well-defined 1–form along the leaves of $\mathcal{F}$. Moreover, let us also make the following observation regarding the global regularity of the 1–form ω_i.

Lemma 9 *The 1–form ω_i is smooth along the leaves of the geodesic foliation $\mathcal{F}$, and is transversally Hölder continuous.*

Proof This is an immediate consequence of the regularity property (iii) of Theorem 6. □

2.4 Lengths of Hölder geodesic currents

We now make use of the 1–forms ω_i to define the lengths $\ell_i^\rho(\alpha)$ of a Hölder geodesic current $\alpha \in \mathcal{C}^{\text{Höl}}(S)$.

Let $\{\mathcal{U}_j\}_{j=1,\dots,m}$ be a finite family of *flow boxes* $\{\mathcal{U}_j\}_{j=1,\dots,m}$ that covers the compact, foliated 3–manifold T^1S. By flow box, we mean an open subset $\mathcal{U}_j \subset T^1S$ such that there exists a diffeomorphism $\mathcal{U}_j \cong D_j \times (0,1)$, where D_j is an open subset of $\mathbb{R}^2$, and where, for every $x \in D_j$, the interval $\{x\} \times (0,1)$ corresponds to an arc in a leaf of $\mathcal{F}$. Let $\{\xi_j\}_{j=1,\dots,m}$ be a partition of unity subordinate to the open covering $\{\mathcal{U}_j\}_{j=1,\dots,m}$. By integrating the 1–form $\xi_j\omega_i$ along the arcs of leaves in $\mathcal{U}_j$, we define a function $\phi_j\colon D_j \to \mathbb{R}$ by

$$\phi_j(x) = \int_{\{x\}\times(0,1)} \xi_j\omega_i.$$

Note that, by Lemma 9, the above function $\phi_j\colon D_j \to \mathbb{R}$ is Hölder continuous with compact support. The Hölder geodesic current α induces a Hölder distribution α_{D_j} on D_j, that we shall still denote by α to alleviate notations. Let us denote the evaluation of α at the function ϕ_j by

$$\alpha(\phi_j) = \int_{\mathcal{U}_j} \xi_j\omega_i \, d\alpha,$$

where the integral notation is suggested by the case where α is a transverse measure for $\mathcal{F}$. Finally, set

$$\ell_i^\rho(\alpha) = \int_{T^1S} \omega_i \, d\alpha = \sum_{j=1}^{m} \int_{\mathcal{U}_j} \xi_j\omega_i \, d\alpha.$$

By the usual linearity arguments, $\ell_i^\rho(\alpha)$ is independent of the choice of the open covering $\{\mathcal{U}_j\}_{j=1,\dots,m}$ and of the partition of unity $\{\xi_j\}_{j=1,\dots,m}$.

Theorem 10 *Defined as above, for every $i = 1,\dots,n$,*

$$\ell_i^\rho\colon \mathcal{C}^{\text{Höl}}(S) \to \mathbb{R}$$

is a continuous, linear function on the vector space of Hölder geodesic currents $\mathcal{C}^{\text{Höl}}(S)$ that extends the length ℓ_i^ρ of closed, oriented curves of Section 2.2. This continuous extension is unique on the space of measure geodesic currents $\mathcal{C}(S) \subset \mathcal{C}^{\text{Höl}}(S)$. In addition, the length ℓ_i^ρ does not depend on the choice of the Riemannian metric $\|\ \|$ on the associated, flat $\mathbb{R}^n$–bundle $T^1S \times_\rho \mathbb{R}^n \to T^1S$ that defines the 1–form ω_i of Section 2.3.

Proof of Theorem 10

We organize the proof into several steps.

Lemma 11 *For every closed, oriented curve* $\gamma \subset S$,

$$\ell_i^\rho(\gamma) = \log \left|\lambda_i^\rho(\gamma)\right|,$$

where $\ell_i^\rho(\gamma)$ *is the image of the Hölder geodesic current* $\gamma \in \mathcal{C}^{\text{Höl}}(S)$ *under the function* $\ell_i^\rho\colon \mathcal{C}^{\text{Höl}}(S) \to \mathbb{R}$*, and where* $\lambda_i^\rho(\gamma)$ *is the* i^{th} *eigenvalue of* $\rho(\gamma)$.

Proof We need to return to the definition of the Hölder geodesic current $\gamma \in \mathcal{C}^{\text{Höl}}(S)$.

By homogeneity of the function ℓ_i^ρ, we can focus attention to the case where the closed, oriented curve $\gamma \subset S$ is homotopically primitive, namely γ is not homotopic to a multiple γ_1^k of a closed, oriented curve γ_1 with $k \geq 2$. Thus γ determines a closed, oriented m_0–geodesic of S, and a closed leaf γ^* of the geodesic foliation $\mathcal{F}$. Identify the closed, oriented curve $\gamma \subset S$ with the Hölder geodesic current $\gamma \in \mathcal{C}^{\text{Höl}}(S)$, which is the transverse 1–weighted Dirac measure defined by the associated closed leaf γ^* (see Section 2.1).

By definition of the function $\ell_i^\rho\colon \mathcal{C}^{\text{Höl}}(S) \to \mathbb{R}$,

$$\ell_i^\rho(\gamma) = \int_{T^1 S} \omega_i \, d\gamma = \int_{\gamma^*} \omega_i.$$

To compute this integral, pick a point $u_0 \in \gamma^* \subset T^1 S$, and a nonzero vector $X_i(u_0) \in V_i(u_0)$ in the fibre of the line subbundle $V_i \to T^1 S$. By definition of the 1–form ω_i (see Section 2.3),

$$\begin{aligned}
\int_{\gamma^*} \omega_i &= \int_0^{t_\gamma} \frac{d}{ds} \left(-\log \|G_s X_i(u_0)\|_{g_s(u_0)}\right) ds \\
&= \log \|G_0 X_i(u_0)\|_{u_0} - \log \|G_{t_\gamma} X_i(u_0)\|_{g_{t_\gamma}(u_0)},
\end{aligned}$$

where t_γ is the necessary time to go around the closed leaf $\gamma^* \subset \mathcal{F}$ by the geodesic flow $(g_t)_{t \in \mathbb{R}}$, namely t_γ is the smallest $t > 0$ such that $g_t(u_0) = u_0$. By the property (ii) of Theorem 6, $G_{t_\gamma} X_i(u_0) = 1/\lambda_i^\rho(\gamma) X_i(u_0)$, and $G_0 X_i(u_0) = X_i(u_0)$ since $(G_t)_{t \in \mathbb{R}}$ is a flow, which proves the assertion. □

Lemma 12 *The function* $\ell_i^\rho\colon \mathcal{C}^{\text{Höl}}(S) \to \mathbb{R}$ *is linear and continuous. Its restriction* $\ell_i^\rho{}_{|\mathcal{C}(S)}\colon \mathcal{C}(S) \to \mathbb{R}$ *is positively homogeneous, and is the unique, continuous extension to the space of measure geodesic currents* $\mathcal{C}(S) \subset \mathcal{C}^{\text{Höl}}(S)$ *for the length* ℓ_i^ρ *of closed, oriented curves of Section 2.2.*

Proof By construction, for every Hölder geodesic current $\alpha \in \mathcal{C}^{\text{Höl}}(S)$,

$$\ell_i^{\rho}(\alpha) = \sum_{j=1}^{m} \alpha(\phi_j).$$

The linearity and homogeneity are immediate. The continuity follows from the definition of the weak-$*$ topology of $\mathcal{C}^{\text{Höl}}(S)$. Finally, since the set of positive real, linear combinations of multiples of closed, oriented curves are dense in the space of measure geodesic currents $\mathcal{C}(S)$, the restriction of this continuous extension to $\mathcal{C}(S)$ is unique. □

Lemma 13 *The length $\ell_i^{\rho}(\alpha)$ is independent of the choice of the Riemannian metric $\| \; \|$ on the associated, flat $\mathbb{R}^n$–bundle $T^1S \times_{\rho} \mathbb{R}^n \to T^1S$.*

Proof Let $\| \; \|'$ be another Riemannian metric on the associated, flat $\mathbb{R}^n$–bundle $T^1S \times_{\rho} \mathbb{R}^n$; it induces another 1–form ω_i' along the leaves of the geodesic foliation $\mathcal{F}$. Since the line $V_i(u)$ depends smoothly on $u \in T^1S$ along the leaves of $\mathcal{F}$, there exists a positive function $f\colon T^1S \to \mathbb{R}$, smooth along of the leaves of $\mathcal{F}$, such that, for every $X_i(u) \in V_i(u)$, $\|X_i(u)\|_u = f(u)\,\|X_i(u)\|_u'$. As a result, $\omega_i' = \omega_i - d\log f$, which implies that

$$\int_{T^1S} \omega_i' \, d\alpha = \int_{T^1S} \omega_i \, d\alpha - \int_{T^1S} d\log f \, d\alpha.$$

Since $\sum_{j=1}^{m} \xi_j = 1$,

$$\int_{T^1S} d\log f \, d\alpha = \sum_{j=1}^{m} \int_{\mathcal{U}_j} d(\xi_j \log f) d\alpha.$$

For our notation conventions, for every $j = 1, \dots, m$,

$$\int_{\mathcal{U}_j} d(\xi_j \log f) d\alpha = \alpha_{D_j} \int_{\{x\}\times(0,1)} d(\xi_j \log f),$$

where $\mathcal{U}_j \cong D_j \times (0,1)$. By Stokes's, $\int_{\{x\}\times(0,1)} d(\xi_j \log f) = 0$, which proves the assertion. □

This achieves the proof of Theorem 10. □

3 Properties of the length functions

3.1 Symmetries of the lengths

We now prove the two properties of Theorem 3.

Proposition 14 *For every Hölder geodesic current $\alpha \in \mathcal{C}^{\text{Höl}}(S)$,*

$$\sum_{i=1}^{n} \ell_i^\rho(\alpha) = 0.$$

Proof Endow each fibre $\{\widetilde{u}\} \times \mathbb{R}^n$ of the trivial bundle $T^1\widetilde{S} \times \mathbb{R}^n \to T^1\widetilde{S}$ with the canonical volume form $\sigma = dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$ of $\mathbb{R}^n$. Recall that $\pi_1(S)$ acts on $T^1\widetilde{S} \times \mathbb{R}^n$ via the diagonal action. Since each $\rho(\gamma)$ is in $\mathrm{SL}_n(\mathbb{R})$, the form σ is invariant under the action of $\pi_1(S)$. In addition, because of the flat connection, the lift $(\widetilde{G}_t)_{t\in\mathbb{R}}$ on $T^1\widetilde{S} \times \mathbb{R}^n$ of the geodesic flow $(\widetilde{g}_t)_{t\in\mathbb{R}}$ on $T^1\widetilde{S}$ acts trivially on the factor $\mathbb{R}^n$ of $T^1\widetilde{S} \times \mathbb{R}^n$, and consequently preserves σ. As a result, σ descends to a well-defined G_t–invariant volume form on the fibres of the bundle $T^1S \times_\rho \mathbb{R}^n$.

Recall that the length functions $\ell_i^\rho \colon \mathcal{C}^{\text{Höl}}(S) \to \mathbb{R}$ are independent of the choice of the Riemannian metric $\| \ \|$ on the bundle $T^1S \times_\rho \mathbb{R}^n$. Without loss of generality, we can arrange that the line subbundles V_i are orthogonal for $\| \ \|$, and that the volume form defined by $\| \ \|$ coincides with the volume form σ.

By definition of the 1–form ω_i, for every $u \in T^1S$, for every vector $X_i(u) \in V_i(u)$,

$$\begin{aligned}\sum_{i=1}^{n} \omega_i(u) &= \sum_{i=1}^{n} \frac{d}{dt}\big(-\log \|G_t X_i(u)\|_{g_t(u)}\big) dt\Big|_{t=0} \\ &= -\frac{d}{dt}\log\Big(\prod_{i=1}^{n} \|G_t X_i(u)\|_{g_t(u)}\Big) dt\Big|_{t=0} \\ &= -\frac{d}{dt}\log\Big(\sigma_{g_t(u)}\big(G_t X_1(u), G_t X_2(u), \dots, G_t X_n(u)\big)\Big) dt\Big|_{t=0} \\ &= -\frac{d}{dt}\log\Big(\sigma_u\big(X_1(u), X_2(u), \dots, X_n(u)\big)\Big) dt\Big|_{t=0} = 0.\end{aligned}$$

By integrating, it follows that, for every Hölder geodesic current $\alpha \in \mathcal{C}^{\text{Höl}}(S)$,

$$\sum_{i=1}^{n} \ell_i^\rho(\alpha) = 0. \qquad \square$$

The unit tangent space T^1S comes endowed with a natural, fibrewise involution $\mathfrak{R}\colon T^1S \to T^1S$, which to $u \in T^1_xS$ associates $\mathfrak{R}(u) = -u$: it is the *orientation reversing involution*. In particular, $\mathfrak{R}$ respects the geodesic foliation $\mathcal{F}$, and thus induces an involution $\mathfrak{R}^*\colon \mathcal{C}^{\text{Höl}}(S) \to \mathcal{C}^{\text{Höl}}(S)$ defined as follows: for every Hölder geodesic current $\alpha \in \mathcal{C}^{\text{Höl}}(S)$, $\mathfrak{R}^*\alpha$ is the *pullback current* of α under the involution $\mathfrak{R}$, namely, if $\varphi\colon D \to \mathbb{R}$ is Hölder continuous with compact support defined on a transverse surface $D \subset T^1S$, then $\mathfrak{R}^*\alpha(\varphi) = \alpha(\varphi \circ \mathfrak{R})$. Note that the restriction to each oriented leaf of the geodesic foliation $\mathcal{F}$ of the involution $\mathfrak{R}$ is orientation reversing.

Proposition 15 *For every Hölder geodesic current $\alpha \in \mathcal{C}^{\text{Höl}}(S)$,*

$$\ell_i^\rho(\mathfrak{R}^*\alpha) = -\ell_{n-i+1}^\rho(\alpha).$$

Proof The involution $\mathfrak{R}$ acts freely on the unit tangent bundle T^1S with quotient $\widehat{T}^1S = T^1S/\mathfrak{R}$. Thus, it also acts freely on the total space $T^1S \times_\rho \mathbb{R}^n$ with quotient a bundle $\widehat{T}^1S \times_\rho \mathbb{R}^n$. Consider a Riemannian metric $\|\ \|$ on $T^1S \times_\rho \mathbb{R}^n$ obtained by lifting a Riemannian metric on $\widehat{T}^1S \times_\rho \mathbb{R}^n$. By construction, the Riemannian metric $\|\ \|$ is invariant under the involution $\mathfrak{R}$.

On the other hand, let $\widetilde{V}_i \to T^1\widetilde{S}$ and $\widetilde{V}_{n-i+1} \to T^1\widetilde{S}$ that lift the subbundles $V_i \to T^1S$ and $V_{n-i+1} \to T^1S$, respectively. By Lemma 7, for every $\tilde{u} \in T^1S$, for every $t \in \mathbb{R}$, the fibres $\widetilde{V}_i(\tilde{g}_t(\mathfrak{R}(\tilde{u})))$ and $\widetilde{V}_{n-i+1}(\tilde{g}_{-t}(\tilde{u}))$ coincide as lines of $\mathbb{R}^n$. Therefore, $\mathfrak{R}^*\omega_i = \omega_{n-i+1}$. By integrating, it follows that, for every Hölder geodesic current $\alpha \in \mathcal{C}^{\text{Höl}}(S)$, $\ell_i^\rho(\mathfrak{R}^*\alpha) = -\ell_{n-i+1}^\rho(\alpha)$; note that a minus sign pops up due to the orientation reversing property of the involution $\mathfrak{R}$. □

3.2 Differentiability of the lengths

We discuss some regularity properties for the length functions $\ell_i^\rho\colon \mathcal{C}^{\text{Höl}}(S) \to \mathbb{R}$.

Thurston [25] considers the space of measured laminations $\mathcal{ML}(S)$ of S, that is a certain completion of the set of all isotopy classes of simple, closed, unoriented curves in S. A fundamental feature of the space $\mathcal{ML}(S)$ is that it is a *piecewise linear* manifold, which is homeomorphic to $\mathbb{R}^{6g-6}$. Therefore, every measured lamination admits tangent vectors, which allows to tackle *tangentiality* properties for functions that are defined on $\mathcal{ML}(S)$. We refer the reader to Penner and Harer [22] and Bonahon [3] for additional details about the PL–structure of $\mathcal{ML}(S)$.

The definition of tangent vectors to $\mathcal{ML}(S)$ is rather abstract and not very convenient in practice. In [3], Bonahon gives an analytical interpretation for the tangent vectors to

$\mathcal{ML}(S)$ as certain Hölder geodesic currents. In particular, a consequence of this work is the following simple criterion.

Recall that the space of measured laminations $\mathcal{ML}(S)$ can be viewed as the closure in $\mathcal{C}^{\text{Höl}}(S)$ of the set of positive real, linear combinations elements of the form

$$\bar{\gamma} = \tfrac{1}{2}\gamma + \tfrac{1}{2}\mathfrak{R}^*\gamma,$$

where $\gamma \in \mathcal{C}^{\text{Höl}}(S)$ is a closed, oriented curve, and $\mathfrak{R}\colon T^1S \to T^1S$ is the orientation reversing involution; see Section 2.1.

Theorem 16 (Bonahon [3]) *Let $f\colon \mathcal{ML}(S) \to \mathbb{R}$ be a homogeneous function defined on the space of measured lamination $\mathcal{ML}(S)$. If f admits a continuous, linear extension $f\colon \mathcal{C}^{\text{Höl}}(S) \to \mathbb{R}$ to the vector space of Hölder geodesic currents $\mathcal{C}^{\text{Höl}}(S)$, then $f\colon \mathcal{ML}(S) \to \mathbb{R}$ is* tangentiable, *namely it is differentiable with respect to directions of tangent vectors to $\mathcal{ML}(S)$.*

For every $i = 1, \dots, n$, let $\ell_i^\rho|_{\mathcal{ML}(S)}\colon \mathcal{ML}(S) \to \mathbb{R}$ be the functions obtained by restricting the length functions $\ell_i^\rho\colon \mathcal{C}^{\text{Höl}}(S) \to \mathbb{R}$ to the space of measured laminations $\mathcal{ML}(S)$. The following corollary is a straightforward consequence of the criterion of Theorem 16.

Corollary 17 *The functions $\ell_i^\rho|_{\mathcal{ML}(S)}\colon \mathcal{ML}(S) \to \mathbb{R}$ are tangentiable, namely, if $(\alpha_t)_{t\geq 0} \subset \mathcal{ML}(S)$ is a smooth 1–parameter family of measured laminations with tangent vector $\dot{\alpha_0} = \frac{d}{dt^+}\alpha_t|_{t=0}$ at α_0, then*

$$\frac{d}{dt^+}\ell_i^\rho(\alpha_t)\Big|_{t=0} = \ell_i^\rho(\dot{\alpha}_0).$$

4 An asymptotic estimate for the eigenvalues

As another application of the continuity property of the lengths ℓ_i^ρ, we now prove the following estimate.

Theorem 18 *Let $\rho\colon \pi_1(S) \to \mathrm{PSL}_n(\mathbb{R})$ be a Hitchin representation, and let $\alpha, \beta \in \pi_1(S)$. For every $i = 1, \dots, n$, the ratio*

$$\frac{\lambda_i^\rho(\alpha^m\beta)}{\lambda_i^\rho(\alpha)^m}$$

has a finite limit as m tends to ∞. This limit is equal to $e^{\ell_i^\rho(\dot{\alpha})}$, where $\dot{\alpha}$ is the Hölder geodesic current $\dot{\alpha} = \lim_{m\to\infty} \alpha^m\beta - m\alpha \in \mathcal{C}^{\text{Höl}}(S)$.

Proof Without loss of generality, we can assume that α is primitive in $\pi_1(S)$. As in Section 2.1, identify the closed, oriented curves α and $\alpha^m\beta$ with the corresponding closed leaves α^* and $\alpha^m\beta^*$ of the geodesic foliation $\mathcal{F}$ of T^1S. Endowing these closed leaves with the transverse Dirac measures that they define, we can regard α and $\alpha^m\beta$ as Hölder geodesic currents.

For m large enough, the closed leaf $\alpha^m\beta$ is made up of one piece of uniformly bounded length, and of another piece that wraps m times around α. As m tends to ∞, this closed leaf converges to the union of the closed orbit α and of an infinite leaf $\alpha^\infty\beta$ of the geodesic foliation whose two ends spiral around α; Figure 1 shows the situation in the surface S.

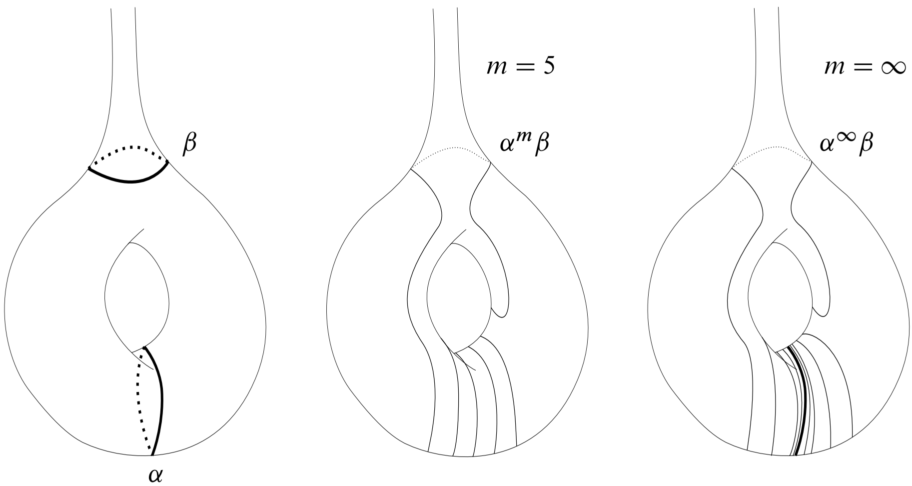

Figure 1: Projection of the leaf $\alpha\beta^m$ of the geodesic foliation $\mathcal{F}$ in the surface S

More precisely, let $D \subset T^1S$ be a small surface transverse to the geodesic foliation $\mathcal{F}$ that intersects the closed leaf α^* in one point x_∞^∞, as shown on Figure 2. The infinite leaf $\alpha^\infty\beta^*$ intersects D in two sequences of points $x_1^\infty, x_2^\infty, \dots$ and $y_1^\infty, y_2^\infty, \dots$ in such a way that $x_1^\infty, x_2^\infty, \dots$ converges in this order to one end of $\alpha^\infty\beta$, and $y_1^\infty, y_2^\infty, \dots$ converges in this order to the other end. Moreover, the two sequences $x_1^\infty, x_2^\infty, \dots$ and $y_1^\infty, y_2^\infty, \dots$ both converge to the point x_∞^∞.

Likewise, the closed leaf $\alpha^m\beta^*$ intersects D in points $x_1^m, x_2^m, \dots, x_{k_m}^m, y_1^m, y_2^m, \dots, y_{l_m}^m$ (see Figure 2), in such a way that, as m tends to ∞, each x_k^m converges to x_k^∞, and each y_l^m converges to y_l^∞. Besides, the total number $k_m + l_m$ of points is of the order of m, and more precisely, the difference $m - (k_m + l_m)$ is equal to a constant

c_D for m large enough. As a result, if φ is a continuous function defined on D,

$$\lim_{m\to\infty}\frac{1}{m}\alpha^m\beta(\varphi)-\alpha(\varphi)$$
$$=\lim_{m\to\infty}\frac{1}{m}\Bigg(\sum_{i=1}^{k_m}\varphi(x_i^m)+\sum_{j=1}^{l_m}\varphi(y_j^m)-(k_m+l_m+c_D)\varphi(x_\infty^\infty)\Bigg)$$
$$=0.$$

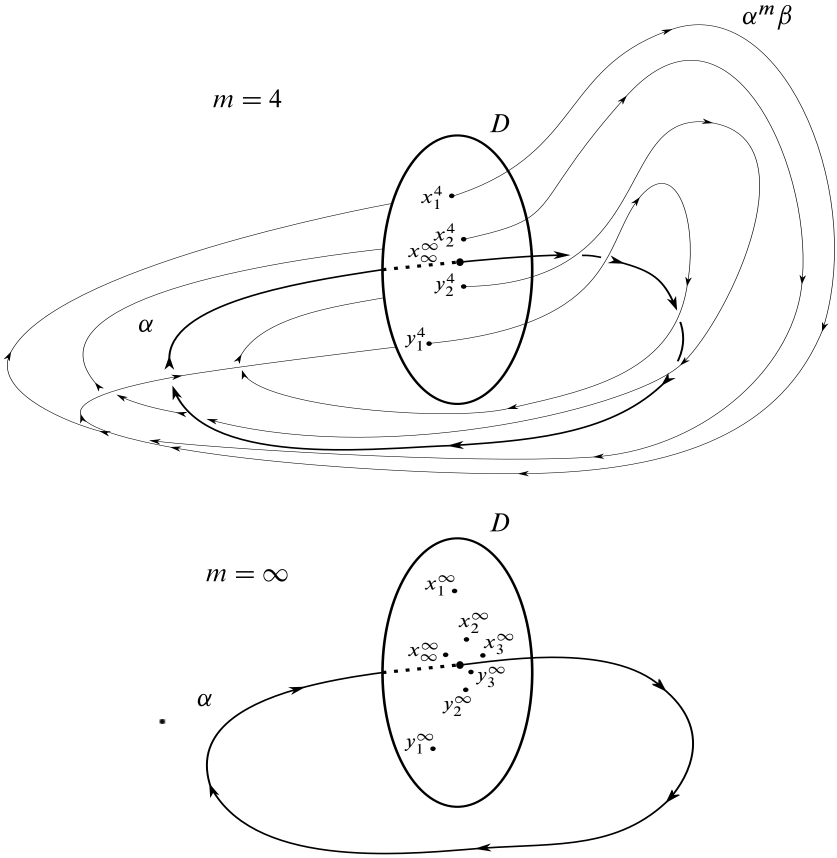

Figure 2: A transverse surface D to the geodesic foliation $\mathcal{F}$ that intersects both leaves α and $\alpha\beta^m$

If, in addition, we assume φ to be Hölder continuous, we can refine the above estimate. Because of the compacity of S, classical hyperbolic estimates guarantee that there exists some bound $M>0$ such that, for every $m\geq M$, both sequences x_i^m and y_i^m converge uniformly to x_∞^∞ as i tends to ∞, and this convergence is of exponential order. Therefore

$$\lim_{m\to+\infty} \big(\alpha^m\beta - m\alpha\big)(\varphi)$$
$$= \lim_{m\to+\infty} \sum_{i=1}^{k_m}\big(\varphi(x_i^m)-\varphi(x_\infty^\infty)\big) + \sum_{j=1}^{l_m}\big(\varphi(y_j^m)-\varphi(x_\infty^\infty)\big) - c_D\varphi(x_\infty^\infty)$$
$$= \sum_{i=1}^{\infty}\big(\varphi(x_i^\infty)-\varphi(x_\infty^\infty)\big) + \sum_{j=1}^{\infty}\big(\varphi(y_j^\infty)-\varphi(x_\infty^\infty)\big) - c_D\varphi(x_\infty^\infty)$$

exists and is finite.

In other words, the above calculation shows that the limit

$$\lim_{m\to\infty} \alpha^m\beta - m\alpha = \dot\alpha$$

exists in the space of Hölder geodesic currents $\mathcal{C}^{\text{Höl}}(S)$. The limit Hölder geodesic current $\dot\alpha$ is supported in the union of the closed leaf α^*, and of the infinite leaf $\alpha^\infty\beta^*$ whose two ends spiral around α^*.

Thus, by linearity and continuity of the length functions ℓ_i^ρ,

$$\lim_{m\to\infty} \ell_i^\rho(\alpha^m\beta) - m\ell_i^\rho(\alpha)$$

exists and is equal to $\ell_i^\rho(\dot\alpha)$. Taking the exponential on both sides, we conclude that

$$\frac{\lambda_i^\rho(\alpha^m\beta)}{\lambda_i^\rho(\alpha)^m}$$

converges to $\mathrm{e}^{\ell_i^\rho(\dot\alpha)}$, which proves the required result. □

We conclude this section with one last observation:

$$\dot\alpha = \lim_{m\to\infty} \alpha^m\beta - m\alpha = \lim_{m\to\infty} \frac{\frac{1}{m}\alpha^m\beta - \alpha}{\frac{1}{m}}$$

is very reminiscent of the expression of a derivative. In fact, there exists a 1–parameter family of Hölder geodesic currents $\alpha_t \in \mathcal{C}^{\text{Höl}}(S)$, $t\in[0,\varepsilon]$, such that $\alpha_0=\alpha$, $\alpha_{1/m} = \frac{1}{m}\alpha^m\beta$, and $\dot\alpha = \frac{d}{dt^+}\alpha_t|_{t=0}$. As a result, the distribution $\dot\alpha$ should be understood as a tangent vector at the point α, and the above estimate comes as a consequence of the first order approximation

$$\ell_i^\rho(\alpha_t) \approx \ell_i^\rho(\alpha_0) + t\frac{d}{dt^+}\ell_i^\rho(\alpha_t)\Big|_{t=0},$$

where $\frac{d}{dt^+}\ell_i^\rho(\alpha_t)|_{t=0} = \ell_i^\rho(\dot\alpha)$ by the previous facts. This should be compared with the differentiability property of the length functions ℓ_i^ρ of Corollary 17.

Remark In a preprint following the publication of an earlier version of the present paper, M Pollicott and R Sharp [23] proposed an alternative proof of the estimate of Theorem 18 that is based on symbolic dynamics. Their method enables to improve the result. They also prove asymptotic growth estimates for the length functions ℓ_i^ρ. In particular, their approach makes crucially use of the 1–forms ω_i introduced in Section 2.3.

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