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Virtual amalgamation of relatively quasiconvex subgroups

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For relatively hyperbolic groups, we investigate conditions guaranteeing that the subgroup generated by two relatively quasiconvex subgroups Q_1 and Q_2 is relatively quasiconvex and isomorphic to $Q_1 *_{Q_1 \cap Q_2} Q_2$. The main theorem extends results for quasiconvex subgroups of word-hyperbolic groups, and results for discrete subgroups of isometries of hyperbolic spaces. An application on separability of double cosets of quasiconvex subgroups is included.

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1 Introduction

This paper continues the work started by the first author in [10] motivated by the following question:

Problem 1 Suppose G is a relatively hyperbolic group and Q_1 and Q_2 are relatively quasiconvex subgroups of G. Investigate conditions guaranteeing that the natural homomorphism

$$Q_1 *_{Q_1 \cap Q_2} Q_2 \longrightarrow G$$

is injective and that its image $\langle Q_1 \cup Q_2 \rangle$ is relatively quasiconvex.

Let G be a group hyperbolic relative to a finite collection of subgroups $\mathbb{P}$, and let dist be a proper left invariant metric on G.

Definition 1 Two subgroups Q and R of G have *compatible parabolic subgroups* if for any maximal parabolic subgroup P of G either $Q \cap P < R \cap P$ or $R \cap P < Q \cap P$.

Theorem 2 *For any pair of relatively quasiconvex subgroups Q and R of G with compatible parabolic subgroups, and any finite index subgroup H of $Q \cap R$, there is a constant $M = M(Q, R, H, \mathrm{dist}) \geq 0$ with the following property. Suppose that $Q' < Q$ and $R' < R$ are subgroups such that:*

(1) $H = Q' \cap R'$;

(2) $\mathrm{dist}(1, g) \geq M$ *for any g in $Q' \setminus Q' \cap R'$ or $R' \setminus Q' \cap R'$.*

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Then the subgroup $\langle Q' \cup R' \rangle$ *of* G *satisfies:*

(1) *The natural homomorphism*

$$Q' *_{Q' \cap R'} R' \longrightarrow \langle Q' \cup R' \rangle$$

is an isomorphism.

(2) *If* Q' *and* R' *are relatively quasiconvex, then so is* $\langle Q' \cup R' \rangle$ *.*

Theorem 2 extends results by Gitik [6, Theorem 1] for word-hyperbolic groups and by the first author [10, Theorem 1.1] for relatively hyperbolic groups. Yang recently obtained a similar combination results requiring stronger conditions [14]. His results include a combination result for HNN extensions and some applications to subgroup separability.

Definition 3 Two subgroups Q and R of a group G can be *virtually amalgamated* if there are finite index subgroups $Q' < Q$ and $R' < R$ such that the natural map $Q' *_{Q' \cap R'} R' \longrightarrow G$ is injective.

Let Q and R be relatively quasiconvex subgroups of G with compatible parabolic subgroups and let $M = M(Q, R, Q \cap R)$ be the constant provided by Theorem 2. If $Q \cap R$ is a separable subgroup of G, then there is a finite index subgroup G' of G containing $Q \cap R$ such that $\mathrm{dist}(1, g) > M$ for every $g \in G$ with $g \notin Q \cap R$. In this case, the subgroups $Q' = G' \cap Q$ and $R' = G' \cap R$ satisfy the hypothesis of Theorem 2; hence they have a quasiconvex virtual amalgam.

Corollary 4 (Virtual Quasiconvex Amalgam Theorem) *Let* Q *and* R *be quasiconvex subgroups of* G *with compatible parabolic subgroups, and suppose that* $Q \cap R$ *is separable. Then* Q *and* R *can be virtually amalgamated in* G *.*

It is known that many (relatively) hyperbolic groups have the property that all quasiconvex or all finitely generated subgroups are separable; see Agol, Long and Reid [2], Long and Reid [8; 9], Wise [12; 13], and Agol, Groves and Manning [1]. Still, it is a natural question to ask whether the corollary above holds under the hypothesis that G is residually finite.

A special case of the Virtual Quasiconvex Amalgam Theorem is the following by Baker and Cooper [3, Theorem 5.3].

Corollary 5 *Let* G *be a geometrically finite subgroup of* $\mathrm{isom}(\mathbb{H}^n)$ *, and let* Q *and* R *be geometrically finite subgroups of* G *with compatible parabolic subgroups. Suppose* $Q \cap R$ *is separable in* G *. Then* Q *and* R *have a geometrically finite virtual amalgam.*

Separability of quasiconvex subgroups and double cosets of quasiconvex subgroups is of interest in the construction of actions on special cube complexes [13]. The machinery we use to prove the main result also gives the following.

Corollary 6 (Double cosets are separable) *Let G be a relatively hyperbolic group such that all its quasiconvex subgroups are separable. If Q and R are quasiconvex subgroups with compatible parabolic subgroups then the double coset QR is separable.*

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2 Preliminaries

2.1 Gromov-hyperbolic spaces

Let (X, dist) be a proper and geodesic δ–hyperbolic space. Recall that a (λ, μ)–quasigeodesic is a curve $\gamma\colon [a,b] \to X$ parameterized by arc length such that

$$|x-y|/\lambda - \mu \leq \mathrm{dist}(\gamma(x), \gamma(y)) \leq \lambda|x-y| + \mu$$

for all $x, y \in [a,b]$. The curve γ is a k–local (λ, μ)–quasigeodesic if the above condition is required only for $x, y \in [a,b]$ such that $|x-y| \leq k$.

Lemma 7 Coornaert, Delzant and Papadopoulos [5, Chapter 3, Theorem 1.2] (Morse Lemma) *For each λ, μ, δ there exists $k > 0$ with the following property. In a δ–hyperbolic geodesic space, any (λ, μ)–quasigeodesic at k–Hausdorff distance from the geodesic between its endpoints.*

Lemma 8 [5, Chapter 3, Theorem 1.4] *For each λ, μ, δ there exist k, λ', μ' so that any k–local (λ, μ)–quasigeodesic in a δ–hyperbolic geodesic space is a (λ', μ')–quasigeodesic.*

Fix a basepoint $x_0 \in X$. If G is a subgroup of $\mathrm{Isom}(X)$, we identify each element g of G with the point gx_0 of X. For $g_1, g_2 \in G$ denote by $\mathrm{dist}(g_1, g_2)$ the distance $\mathrm{dist}(g_1x_0, g_2x_0)$. Since X is a proper space, if G is a discrete subgroup of $\mathrm{Isom}(X)$, this is a proper and left invariant pseudometric on G.

Lemma 9 [10, Lemma 4.2] (Bounded Intersection) *Let G be a discrete subgroup of* $\mathrm{isom}(X)$, *let Q and R be subgroups of G, and let $\mu > 0$ be a real number. Then there is a constant $M = M(Q, R, \mu) \geq 0$ so that*

$$Q \cap \mathcal{N}_\mu(R) \subset \mathcal{N}_M(Q \cap R).$$

2.2 Relatively quasiconvex subgroups

We follow the approach to relatively hyperbolic groups as developed by Hruska [7].

Definition 10 (Relative Hyperbolicity) A group G is *relatively hyperbolic with respect to a finite collection of subgroups* $\mathbb{P}$ if G acts properly discontinuously and by isometries on a proper and geodesic δ–hyperbolic space X with the following property: X has a G–equivariant collection of pairwise disjoint horoballs whose union is an open set U, G acts cocompactly on $X \setminus U$, and $\mathbb{P}$ is a set of representatives of the conjugacy classes of parabolic subgroups of G.

Throughout the rest of the paper, G is a relatively hyperbolic group acting on a proper and geodesic δ–hyperbolic space X with a G–equivariant collection of horoballs satisfying all conditions of Definition 10. As before, we fix a basepoint $x_0 \in X \setminus U$, identify each element g of G with $gx_0 \in X$ and let $\mathrm{dist}(g_1, g_2)$ denote $\mathrm{dist}(g_1x_0, g_2x_0)$ for $g_1, g_2 \in G$.

Lemma 11 Bowditch [4, Lemma 6.4] (Cocompact actions of parabolic subgroups on thick horospheres) *Let B be a horoball of X with G–stabilizer P. For any $M > 0$, P acts cocompactly on $\mathcal{N}_M(B) \cap (X \setminus U)$.*

Lemma 12 (Parabolic approximation) *Let Q be a subgroup of G and let $\mu > 0$ be a real number. There is a constant $M = M(Q, \mu)$ with the following property. If P is a maximal parabolic subgroup of G stabilizing a horoball B, and $\{1, q\} \subset Q \cap \mathcal{N}_\mu(B)$ then there is $p \in Q \cap P$ such that* $\mathrm{dist}(p, q) < M$.

Proof By Lemma 11, $\mathrm{dist}(q, P) < M_1$ for some constant $M_1 = M_1(Q, P)$. Then Lemma 9 implies that $\mathrm{dist}(q, Q \cap P) < M_2$ where $M_2 = N(Q, P, M_1)$. Since B is a horoball at distance less than μ from 1, there are only finitely many possibilities for B and hence for the subgroup P. Let M the maximum of all $N(Q, P, \mu)$ among the possible P. □

Definition 13 (Relatively quasiconvex subgroup) A subgroup Q of G is *relatively quasiconvex* if there is $\mu \geq 0$ such that for any geodesic c in X with endpoints in Q, $c \cap (X \setminus U) \subset N_\mu(Q)$.

The choice of horoballs turns out not to make a difference.

Proposition 14 [7] *If Q is relatively quasiconvex in G then for any $L \geq 0$ there is $\mu \geq 0$ such that for any geodesic c in X with endpoints in Q, $c \cap \mathcal{N}_L(X \setminus U) \subset N_\mu(Q)$.*

3 A lemma on Gromov's inner product

Let Q and R be relatively quasiconvex subgroups with compatible parabolic subgroups, and let H be a finite index subgroup of $Q \cap R$.

Let Q' and R' be subgroups of Q and R respectively such that $Q' \cap R' = H$. Let $g \in Q'R'$ (or $g \in R'Q'$) such that $g \notin H$. Suppose $g = qr$ (or $g = rq$) with $q \in Q'$, $r \in R'$ and such that $\mathrm{dist}(1,q) + \mathrm{dist}(1,r)$ is minimal among all such products.

Lemma 15 *Suppose that there exists $a \in H$ and a point p at distance at most A from the geodesic segment $[1,g]$ so that $\mathrm{dist}(p,qa) \leq B$. Then*

$$\mathrm{dist}(1,q) + \mathrm{dist}(1,r) \leq \mathrm{dist}(1,g) + 2A + 2B.$$

Proof Let $p' \in [1,g]$ be such that $\mathrm{dist}(p,p') < A$. Then

$$\begin{aligned}\mathrm{dist}(1,qa) + \mathrm{dist}(1,a^{-1}r) &\leq \mathrm{dist}(1,p') + \mathrm{dist}(p',qa) + \mathrm{dist}(qa,p') + \mathrm{dist}(p',g)\\ &\leq \mathrm{dist}(1,g) + 2A + 2B.\end{aligned}$$

Since g can be written as $(qa)(a^{-1}r)$, the minimality assumption implies $\mathrm{dist}(1,q) + \mathrm{dist}(1,r) \leq \mathrm{dist}(1,g) + 2A + 2B$. □

Lemma 16 (Gromov's inner product is bounded) *There is a constant $K = K(Q,R,H)$ with the following property:*

$$\mathrm{dist}(1,q) + \mathrm{dist}(1,r) \leq \mathrm{dist}(1,g) + K.$$

Proof Constants which depend only on Q, R, H and δ are denoted by M_i, the index counts positive increments of the constant during the proof. Suppose $g = qr$, the other case being symmetric. The constant K of the statement corresponds to M_{13}.

Consider a triangle Δ with vertices $1, q, g$. Let $p \in [1,q]$ be a center of Δ, ie the δ–neighborhood of p intersects all sides of Δ.

Suppose that $p \in X \setminus U$. Then $\mathrm{dist}(p,Q), \mathrm{dist}(p,qR) \leq M_1$ by relative quasiconvexity of Q and R. By Lemma 9, there exists $a \in Q \cap R$ so that $\mathrm{dist}(p,qa) \leq M_2$. Since H

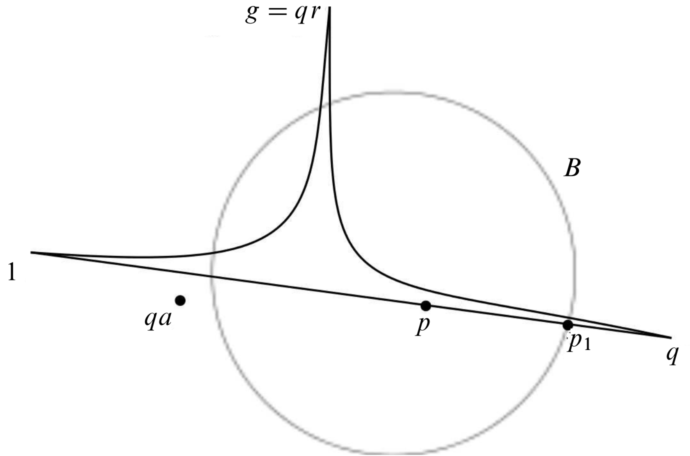

Figure 1

is a finite index subgroup of $Q \cap R$, there is $b \in H$ such that $\mathrm{dist}(p, qb) \leq M_3$. By Lemma 15, $\mathrm{dist}(1, q) + \mathrm{dist}(1, r) \leq \mathrm{dist}(1, g) + 2M_3 + 2\delta$.

Suppose instead that p is in a horoball B, whose stabilizer is P. We can assume $\mathrm{dist}(q, B) \leq M_8$. Indeed, let p_1 be the entrance point of the geodesic $[q, 1]$ in B; then $\mathrm{dist}(p_1, Q) < M_4$ by quasiconvexity of Q. Notice that $\mathrm{dist}(p_1, [q, g])$ is at most 2δ since p is a center of Δ and $p_1 \in [q, p]$ (consider a triangle with vertices p, q, p' for $p' \in [q, g]$ so that $d(p, p') \leq \delta$). By quasiconvexity of R, there is $p_2 \in [q, g]$ such that $\mathrm{dist}(p_1, p_2), \mathrm{dist}(p_2, qR) < M_5$. Lemma 9 implies there is $a \in Q \cap R$ such that $\mathrm{dist}(qa, p_1), \mathrm{dist}(qa, p_2) < M_6$. Since H is a finite index subgroup of $Q \cap R$, there is $b \in H$ such that $\mathrm{dist}(qb, p_1), \mathrm{dist}(qb, p_2) < M_7$. Since g can be written as $(qb)(b^{-1}r)$, by minimality we have

$$\begin{aligned}\mathrm{dist}(1, p_1) + \mathrm{dist}(p_1, q) + {}&\mathrm{dist}(q, p_2) + \mathrm{dist}(p_2, g)\\ &= \mathrm{dist}(1, q) + \mathrm{dist}(1, g)\\ &\leq \mathrm{dist}(1, qb) + \mathrm{dist}(1, b^{-1}r)\\ &= \mathrm{dist}(1, p_1) + \mathrm{dist}(p_1, qb) + \mathrm{dist}(qb, p_2) + \mathrm{dist}(p_2, g),\end{aligned}$$

and therefore

$$\begin{aligned}2\,\mathrm{dist}(q, B) &= 2\,\mathrm{dist}(p_1, q)\\ &\leq \mathrm{dist}(p_1, q) + \mathrm{dist}(q, p_2) + \mathrm{dist}(p_1, p_2)\\ &\leq \mathrm{dist}(p_1, qb) + \mathrm{dist}(qb, p_2) + \mathrm{dist}(p_1, p_2)\\ &\leq 2M_8.\end{aligned}$$

Since Q and R have compatible parabolic subgroups, assume $Q \cap q^{-1}Pq \leq R \cap q^{-1}Pq$, the other case being symmetric. By quasiconvexity of Q, there is $q_1 \in Q$ at distance M_9 from the entrance point of $[1, q]$ in B. In particular, the distance from q_1 to $[1, g]$ is at most M_{10}. Applying the parabolic approximation lemma to $\{1, q^{-1}q_1\} \subset Q \cap \mathcal{N}_{M_{10}}(q^{-1}B)$, there is an element $a \in Q \cap q^{-1}Pq$ such that $\mathrm{dist}(qa, q_1) \leq M_{11}$. Since $Q \cap q^{-1}Pq \leq R \cap q^{-1}Pq$ it follows that $a \in Q \cap R$. Since H is finite index in $Q \cap R$, by increasing the constant we can assume that $a \in H$ and $\mathrm{dist}(qa, q_1) \leq M_{12}$. Then Lemma 15 implies

$$\mathrm{dist}(1, q) + \mathrm{dist}(1, r) \leq \mathrm{dist}(1, g) + M_{13}. \qquad \square$$

4 Proof of Theorem 2

Let Q and R be relatively quasiconvex subgroups with compatible parabolic subgroups, and let H be a finite index subgroups of $Q \cap R$.

Let $K = K(Q, R, H)$ be the constant of Lemma 16. Let M be large enough so that $M > k, \lambda'\mu'$ where k, λ' and μ' are as in Lemma 8 for $\lambda = 1, \mu = K$.

Let Q' and R' be subgroups satisfying the hypothesis of the theorem, in particular $Q' \cap R' = H$. Consider $1 \neq g \in Q' *_{Q' \cap R'} R'$ and suppose that $g \notin Q' \cap R'$. Then $g = g_1 \dots g_n$ where the g_i's are alternatively elements of $Q' \setminus Q' \cap R'$ and $R' \setminus Q' \cap R'$. Moreover, assume that this product is *minimal* in the sense that $\sum \mathrm{dist}(1, g_i)$ is minimal among all such products describing g.

Lemma 17 *For each i, let $h_i = g_1 \dots g_i$. Then the concatenation $\alpha = \alpha_1 \cdots \alpha_{n-1}$ of geodesics α_i from h_i to h_{i+1} is an M–local $(1, K)$–quasigeodesic.*

Proof By the choice of Q' and R' each segment α_i has length at least M. Let $x \in [h_{i-1}, h_i]$ and $y \in [h_i, h_{i+1}]$. By Lemma 16, we have

$$\begin{aligned}
\mathrm{dist}(h_{i-1}, x) + \mathrm{dist}(x, y) + \mathrm{dist}(y, h_{i+1}) &\geq \mathrm{dist}(h_{i-1}, h_{i+1}) \\
&\geq \mathrm{dist}(h_{i-1}, h_i) + \mathrm{dist}(h_i, h_{i+1}) - K \\
&= \mathrm{dist}(h_{i-1}, x) + \mathrm{dist}(x, h_i) + \mathrm{dist}(h_i, y) + \mathrm{dist}(y, h_{i+1}) - K.
\end{aligned}$$

Therefore $\mathrm{dist}(x, y) + K \geq \mathrm{dist}(x, h_i) + \mathrm{dist}(h_i, y)$. $\square$

Since $M > k$, Lemma 8 implies that α is a (λ', μ')–quasigeodesic. Since $M > \lambda'\mu'$, it follows that α has different endpoints. Therefore we have shown that the map $Q' *_{Q' \cap R'} R' \to G$ is injective.

It is left to prove that if Q' and R' are relatively quasiconvex, then $\langle Q', R'\rangle$ is relatively quasiconvex. Let $g \in \langle Q \cap R\rangle$ and let γ be a geodesic from 1 to g. Since H is quasiconvex, if $g \in H$ then $\gamma \cap (X \setminus U)$ is uniformly close to H and hence to $\langle Q \cap R\rangle$. Suppose that $g \notin H$. By Lemma 7 (Morse Lemma), any (λ', μ')–quasigeodesic is at Hausdorff distance at most L from any geodesic between its endpoints. In particular, $\gamma \cap (X \setminus U) \subseteq \mathcal{N}_L(\alpha) \cap (X \setminus U)$ where α is the quasigeodesic constructed above. It is enough to show that $\alpha \cap \mathcal{N}_L(X \setminus U)$ is contained in $\mathcal{N}_\mu(\langle Q' \cup R'\rangle)$. Let $p \in \alpha \cap \mathcal{N}_L(X \setminus U)$ and let i be so that $p \in [h_i, h_{i+1}] \cap N_L(X \setminus U)$. Assume $g_{i+1} \in Q'$, the other case being symmetric. As Q' is relatively quasiconvex and in view of Proposition 14, there is a constant μ so that $p \in \mathcal{N}_\mu(h_i Q') \subseteq \mathcal{N}_\mu(\langle Q' \cup R'\rangle)$ (as $h_i \in \langle Q' \cup R'\rangle$).

5 Separability of double cosets

We now show Corollary 6. Suppose that all quasiconvex subgroups of G are separable. Let Q and R be quasiconvex subgroups with compatible parabolic subgroups. Let $g \in G$ and suppose that $g \notin QR$. We follow an argument described in Minasyan [11] and Yang [14].

Let $K = K(Q, R, Q \cap R)$ be the constant of Lemma 16. As in the proof of Theorem 2, let M be large enough so that $M > k, \lambda'\mu'$ where k, λ' and μ' are as in Lemma 8 for $\lambda = 1, \mu = K$. In addition, assume that

$$M > \lambda' \operatorname{dist}(1, g) + \lambda'\mu'. \tag{1}$$

Lemma 18 *There are finite index subgroups Q' and R' of Q and R respectively such that $g \notin Q\langle Q', R'\rangle R$.*

Proof Since $Q \cap R$ is separable, there are finite index subgroups Q' and R' of Q and R respectively, such that $Q' \cap R' = Q \cap R$ and $\operatorname{dist}(1, f) \geq 2M$ for any f in $Q' \setminus Q' \cap R'$ or $R' \setminus Q' \cap R'$. By Theorem 2 $\langle Q' \cup R'\rangle$ is a quasiconvex subgroup of G isomorphic to $Q' *_{Q \cap R} R'$.

Suppose that $g \in Q\langle Q', R'\rangle R$. Since $g \notin QR$ it follows that $g = g_1 \dots g_{2n}$ where $g_1 \in Q$, $g_{2n} \in R$, $g_{2i+1} \in Q' \setminus Q \cap R$, $g_{2i} \in R' \setminus Q \cap R$, and $n \geq 2$. Assume that this product is minimal in the sense that $\sum \operatorname{dist}(1, g_i)$ is minimal among all such products describing g.

For each i, let $h_i = g_1 \dots g_i$; let α_i be a geodesic from h_i to h_{i+1}. By the choice of Q' and R' each segment α_i has length at least $2M$ except α_1 and α_{2n-1}.

Notice that $g_2 \cdots g_{2n-1}$ represents an element of $Q' *_{Q\cap R} R'$ and such product is minimal in the sense of the previous section, so that by Lemma 17 the concatenation $\alpha_2 \cdots \alpha_{2n-1}$ is an M–local $(1, K)$–quasigeodesic. Minimality of $g_1 \dots g_{2n}$ and Lemma 16 imply that the concatenations $\alpha_1\alpha_2$ and $\alpha_{2n-1}\alpha_{2n}$ are M–local $(1, K)$–quasigeodesics. Since α_2 and α_{2n-1} have both length at least $2M$, it follows that the concatenation $\alpha = \alpha_1 \cdots \alpha_{2n}$ an M–local $(1, K)$–quasigeodesic.

By Lemma 8, it follows that α is a (λ', μ')–quasigeodesic between 1 and g. It follows that $\mathrm{dist}(1, g) \geq 4M/\lambda' - \mu'$; this is a contradiction with Equation (1) above. □

Since Q' and R' are of finite index, there are $q_1, \dots, q_k \in Q$ and $r_1, \dots, r_m \in R$ such that

$$Q\langle Q', R'\rangle R = \bigcup_{q_i, r_j} q_i \langle Q', R'\rangle r_j.$$

Since $\langle Q', R'\rangle$ is quasiconvex, it is closed in the profinite topology. It follows that $Q\langle Q', R'\rangle R$ is a finite union of closed sets. Therefore $Q\langle Q', R'\rangle R$ is a closed set in the profinite topology containing QR and such that $g \notin Q\langle Q', R'\rangle R$. Since g was an arbitrary element of $g \in G$ not in QR, it follows that QR is closed in the profinite topology of G.

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