The Lusternik–Schnirelmann category and the fundamental group

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We prove that

$$\mathrm{cat}_{\mathrm{LS}} X \le \mathrm{cd}(\pi_1(X)) + \left\lceil \frac{\dim X - 1}{2} \right\rceil$$

for every CW–complex X where $\mathrm{cd}(\pi_1(X))$ denotes the cohomological dimension of the fundamental group of X.

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1 Introduction

The *Lusternik–Schnirelmann category* $\mathrm{cat}_{\mathrm{LS}} X$ of a topological space X is the minimal number n such that there is an open cover $\{U_0, \dots, U_n\}$ of X by $n+1$ contractible in X sets (we note that sets U_i are not necessarily contractible). The Lusternik–Schnirelmann category has proven useful in different areas of mathematics. In particular, the classical theorem of Lusternik and Schnirelmann (see Cornea et al [3]) proven in the 30s states that $\mathrm{cat}_{\mathrm{LS}} M$ gives a lower bound for the number of critical points on M of any smooth not necessarily Morse function. For nice spaces, such as CW–complexes, it is an easy observation that $\mathrm{cat}_{\mathrm{LS}} X \le \dim X$. In the 40s Grossman [8] (and independently in the 50s G W Whitehead [16; 3]) proved that for simply connected CW–complexes $\mathrm{cat}_{\mathrm{LS}} X \le \dim X/2$. In the presence of a fundamental group as small as $\mathbb{Z}_2$ the Lusternik–Schnirelmann category can be equal to the dimension. An example is $\mathbb{R}P^n$.

Nevertheless, Yu Rudyak conjectured that in the case of free fundamental group there should be a Grossman–Whitehead-type inequality at least for closed manifolds. There were partial results towards Rudyak's conjecture (see Dranishnikov, Katz and Rudyak [6] and Strom [13]) until it was settled by the author [5]. Also it was shown in [5] that a Grossman–Whitehead-type estimate holds for complexes with fundamental group of cohomological dimension ≤ 2. We recall that free groups (and only them by Stallings [12] and Swan [15]) have cohomological dimension one. In this paper we prove an inequality for complexes with fundamental groups having finite cohomological dimension. Complexes of type $\mathbb{C}P^n \times B\pi$ show that our inequality is sharp when π is free.

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We conclude the introductory part with definitions and statements from [5] which are used in this paper. Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in A}$ be a family of sets in a topological space X. Formally, it is a function $U\colon A \to 2^X \setminus \{\varnothing\}$ from the index set to the set of nonempty subsets of X. The sets U_α in the family $\mathcal{U}$ will be called *elements of* $\mathcal{U}$. The *multiplicity* of $\mathcal{U}$ (or the *order*) at a point $x \in X$, denoted $\mathrm{Ord}_x \mathcal{U}$, is the number of elements of $\mathcal{U}$ that contain x. The *multiplicity* of $\mathcal{U}$ is defined as $\mathrm{Ord}\,\mathcal{U} = \sup_{x\in X} \mathrm{Ord}_x \mathcal{U}$. A family $\mathcal{U}$ is a cover of X if $\mathrm{Ord}_x \mathcal{U} \neq 0$ for all x. A cover $\mathcal{U}$ is a *refinement* of another cover $\mathcal{C}$ ($\mathcal{U}$ *refines* $\mathcal{C}$) if for every $U \in \mathcal{U}$ there exists $C \in \mathcal{C}$ such that $U \subset C$. We recall that the *covering dimension* of a topological space X does not exceed n, $\dim X \le n$, if for every open cover $\mathcal{C}$ of X there is an open refinement $\mathcal{U}$ with $\mathrm{Ord}\,\mathcal{U} \le n+1$.

Definition 1.1 A family $\mathcal{U}$ of subsets of X is called a k*–cover*, $k \in N$, if every subfamily of k elements forms a cover of X.

The following is obvious (see Dranishnikov [5]).

Proposition 1.2 *A family $\mathcal{U}$ that consists of m subsets of X is an $(n+1)$–cover of X if and only if $\mathrm{Ord}_x \mathcal{U} \ge m-n$ for all $x \in X$.*

The following theorem can be found in Ostrand [10].

Theorem 1.3 (Kolmogorov–Ostrand) *A metric space X is of dimension $\le n$ if and only if for each open cover $\mathcal{C}$ of X and each integer $m \ge n$, there exist m disjoint families of open sets $\mathcal{V}_0, \dots, \mathcal{V}_m$ such that their unions $\bigcup \mathcal{V}_i$ is an $(n+1)$–cover of X and it refines $\mathcal{C}$.*

Let $f\colon X \to Y$ be a map and let $X' \subset X$. A set $U \subset X$ is *fiberwise contractible to* X' if there is a homotopy $H\colon U \times [0,1] \to X$ such that $H(x,0) = x$, $H(U \times \{1\}) \subset X'$, and $f(H(x,t)) = f(x)$ for all $x \in U$.

We refer to [5] for the proof of the following:

Theorem 1.4 *Let $\mathcal{U} = \{U_0, \dots, U_k\}$ be an open cover of a normal topological space X. Then for any $m = 0, 1, 2, \dots, \infty$ there is an open $(k+1)$–cover $\mathcal{U}_m = \{U_0, \dots, U_{k+m}\}$ of X extending $\mathcal{U}$ such that for $n > k$, $U_n = \bigcup_{i=0}^k V_i$ is a disjoint union with $V_i \subset U_i$.*

Corollary 1.5 *Let $f\colon X \to Y$ be a continuous map of a normal topological space and let $\mathcal{U} = \{U_0, \dots, U_k\}$ be an open cover of X by sets fiberwise contractible to $X' \subset X$. Then for any $m = 0, 1, 2, \dots, \infty$ there is an open $(k+1)$–cover $\mathcal{U}_m = \{U_0, \dots, U_{k+m}\}$ of X by sets fiberwise contractible to X'.*

2 Generalization of Ganea's fibrations

Let $A \subset Z$ be a closed subset of a path-connected space and let F denote the homotopy fiber of the inclusion. By A_Z we denote the space of paths in Z issued from A, ie the space of continuous maps $\phi\colon [0,1] \to Z$ with $\phi(0) \in A$ and we define a map $p_A\colon A_Z \to Z$ by the formula $p(\phi) = \phi(1)$. Note that A_Z deforms to A and p_A is a Hurewicz fibration. Then by the definition F is the fiber of p_A.

Proposition 2.1 *There is a Hurewicz fibration $\pi\colon F \to A$ with fiber ΩZ, the loop space on Z.*

Proof The map $q'\colon A_Z \to A \times Z$ that sends a path to the end points is a Hurewicz fibration as a pullback of the Hurewicz fibration $q\colon Z^{[0,1]} \to Z \times Z$ [11]. The fiber of q is the loop space ΩZ. Since $p_A = \mathrm{pr}_2 \circ q'$, the fiber $F = p_A^{-1}(x) = (q')^{-1}\,\mathrm{pr}_2^{-1}(x) = q^{-1}(A)$ is the total space of a Hurewicz fibration q over A with the fiber ΩZ. □

We define the k–th *generalized Ganea's fibration* $p_k\colon E_k(Z,A) \to Z$ over a path connected space Z with a fixed closed subset A as the fiberwise join product of $k+1$ copies of the fibrations $p_A\colon A_Z \to Z$. Since p_A is a Hurewicz fibration and the fiberwise join of Hurewicz fibrations is a Hurewicz fibration, so are all p_k by Švarc [14]. Note that the fiber of p_k is the join product $*^{k+1}F$ of $k+1$ copies of F (see Cornea et al [3] for more details). Also we note that for $A = \{z_0\}$ the fibration p_k is the standard Ganea fibration. The following is a generalization of the Ganea–Švarc theorem.

Theorem 2.2 *Let $A \subset X$ be a subcomplex contractible in X. Then $\mathrm{cat}_{\mathrm{LS}}(X) \le k$ if and only if the generalized Ganea fibration*

$$p_k\colon E_k(Z,A) \to Z$$

admits a section.

Proof When A is a point this statements turns into the classical Ganea–Švarc theorem [3; 14]. Since for $z_0 \in A$, the above fibration $p_k\colon E_k(Z,z_0) \to Z$ is contained in $p_k\colon E_k(Z,A) \to Z$, the classical Ganea–Švarc theorem implies the only if direction.

The barycentric coordinates of a section to p_k define an open cover $U_0, \dots U_k$ of U_i with each U_i contractible to A. Since A is contractible in Z, all sets U_i are contractible in Z. □

We call a map $f\colon X \to Y$ a *stratified locally trivial bundle* (with two strata) with fiber (Z, A) if there $X' \subset X$, such that $(f^{-1}(y), g^{-1}(y)) \cong (Z, A)$ for all $y \in Y$, where $g = f|_{X'}$, and there is an open cover $\mathcal{U} = \{U\}$ of Y such that $(f^{-1}(U), g^{-1}(U))$ is homeomorphic as a pair to $(Z \times U, A \times U)$ by means of a fiber preserving homeomorphism. Such a bundle is called a *trivial stratified bundle* if one cant take $\mathcal{U}$ consisting of one element $U = Y$.

Now let $f\colon X \to Y$ be a stratified locally trivial bundle with a subbundle $g\colon X' \to Y$ and a fiber (Z, A). We define a space

$$E_0 = \{\phi \in C(I, X) \mid f\phi(I) = f\phi(0),\ \phi(0) \in g^{-1}(f\phi(0))\}$$

to be the space of all paths ϕ in $f^{-1}(y)$ for all $y \in Y$ with the initial point in $g^{-1}(y)$. The topology in E_0 is inherited from $C(I, X)$. We define a map $\xi_0\colon E_0 \to X$ by the formula $\xi_0(\phi) = \phi(1)$. Then $\xi_k\colon E_k \to X$ is defined as the fiberwise join of $k+1$ copies of ξ_0. Formally, we define inductively E_k as a subspace of the join $E_0 * E_{k-1}$:

$$E_k = \bigcup \{\phi * \psi \in E_0 * E_{k-1} \mid \xi_0(\phi) = \xi_{k-1}(\psi)\},$$

which is the union of all intervals $[\phi, \psi] = \phi * \psi$ with the endpoints $\phi \in E_0$ and $\psi \in E_{k-1}$ such that $\xi_0(\phi) = \xi_{k-1}(\psi)$. There is a natural projection $\xi_k\colon E_k \to X$ that takes all points of each interval $[\phi, \psi]$ to $\phi(0)$.

Note that when $f\colon X = Z \times Y \to Y$ is a trivial stratified bundle with the subbundle $g\colon A \times Y \to Y$, $A \subset Z$, then $E_k = E_k(Z, A) \times Y$ and $\xi_k = p_k \times 1_Y$ where $p_k\colon (E_k, A) \to Z$ is the generalized Ganea fibration.

Lemma 2.3 *Let $f\colon X \to Y$ be a stratified locally trivial bundle between paracompact spaces with a fiber (Z, A) in which A is contractible in Z. Then:*

(i) *For each k the map $\xi_k\colon E_k \to X$ is a Hurewicz fibration.*

(ii) *The fiber of ξ_k is the join of $k+1$ copies of the fiber F of $p_A\colon A_Z \to Z$.*

(iii) *If the projection ξ_k has a section, then X has an open cover $\mathcal{U} = \{U_0, \dots, U_k\}$ by sets each of which admits a fiberwise deformation into X' where $g\colon X' \to Y$ is the subbundle.*

Proof (i) First, we note that this statement holds true for trivial stratified bundles. By the assumption there is a cover $\mathcal{U}$ of Y such that $f|_{f^{-1}U}\colon f^{-1}(U) \to U$ is a trivial stratified bundle and hence ξ_k is a Hurewicz fibration over $f^{-1}(U)$ for all $U \in \mathcal{U}$. Then by Hurewicz [9] (see also Dold [4]) we conclude that ξ_k is a Hurewicz fibration over X.

(ii) We note that ξ_k over $f^{-1}(y)$ coincides with the generalized Ganea fibration p_k for (Z, A). Therefore, the fiber of ξ_k coincides with the fiber of p_k. Then we apply Proposition 2.1

(iii) Suppose ξ_k has a section $\sigma\colon X \to E_k$. For each $x \in X$ the element $\sigma(x)$ of $*^{k+1}\Omega F$ can be presented as the $(k+1)$–tuple

$$\sigma(x) = ((\phi_0, t_0), \dots, (\phi_k, t_k)) \text{ where } \sum t_i = 1 \text{ and } t_i \geq 0.$$

Here we use the notation $t_i = t_i(x)$ and $\phi_i = \phi_i^x$. Clearly, $t_i(x)$ and ϕ_i^x are continuous functions of x.

A section $\sigma\colon X \to E_k$ defines a cover $\mathcal{U} = \{U_0, \dots, U_k\}$ of X as follows:

$$U_i = \{x \in X \mid t_i(x) > 0\}.$$

By the construction of U_i for $i \leq n$ for every $x \in U_i$ there is a canonical path connecting x with $g^{-1}f(x)$. These paths define a fiberwise deformation $H\colon U_i \times [0,1] \to X'$ of U_i into $g^{-1}f(U_i) \subset X'$ by the formula $H(x,t) = \phi_i^x(1-t)$. $\square$

3 The main result

We recall that the *homotopical dimension* of a space X, $\operatorname{hd}(X)$, is the minimal dimension of a CW–complex homotopy equivalent to X [3].

Proposition 3.1 *Let $p\colon E \to X$ be a fibration with $(n-1)$–connected fiber where $n = \operatorname{hd}(X)$. Then p admits a section.*

Proof Let $h\colon Y \to X$ be a homotopy equivalence with the homotopy inverse $g\colon X \to Y$ where Y is a CW–complex of dimension n. Since the fiber of p is $(n-1)$–connected, the map h admits a lift $h'\colon Y \to E$. Let H be a homotopy connecting $h \circ g$ with 1_X. By the homotopy lifting property there is a lift $H'\colon X \times I \to E$ of H with $H|_{X\times\{0\}} = h' \circ g$. Then the restriction $H|_{X\times\{1\}}$ is a section. $\square$

We recall that $\lceil x \rceil$ denotes the smallest integer n such that $x \leq n$.

Lemma 3.2 *Suppose that a stratified locally trivial bundle $f\colon X \to Y$ with a fiber (Z, A) is such that Z is r–connected, A is $(r-1)$–connected, A is contractible in Z, and Y is locally contractible. Then*

$$\operatorname{cat}_{\mathrm{LS}} X \leq \dim Y + \left\lceil \frac{\operatorname{hd}(X) - r}{r+1} \right\rceil.$$

Proof Let $\dim Y = m$ and $\mathrm{hd}(X) = n$.

By Lemma 2.3 the fiber K of the fibration $\xi_k\colon E_k \to X$ is the join product $*^{k+1}F$ of $k+1$ copies of the fiber F of the map $p_A\colon A_Z \to Z$. By Proposition 2.1, F admits a fibration $\phi\colon F \to A$ with fibers homotopy equivalent to the loop space ΩZ. Since the base A and the fibers are $(r-1)$–connected, F is $(r-1)$–connected. Thus, K is $(k+(k+1)r-1)$–connected. By Proposition 3.1 there is a section $\sigma\colon X \to E_k$ to the fibration $\xi_k\colon E_k \to X$, whenever $k(r+1)+r \geq n$. Let k be the smallest integer satisfying this condition. Thus, $k = \lceil (n-r)/(r+1) \rceil$.

By Lemma 2.3 a section $\sigma\colon X \to E_k$ defines a cover $\mathcal{U} = \{U_0, \dots, U_k\}$ by sets fiberwise contractible to X' where $X' \subset X$ is the first stratum. Let $\mathcal{U}_m = \{U_0, \dots, U_{k+m}\}$ be an extension of $\mathcal{U}$ to a $(k+1)$–cover of X from Corollary 1.5.

Let $\mathcal{O}$ be an open cover of Y such that f is trivial stratified bundle over each $O \in \mathcal{O}$. Let $\mathcal{C}$ be an open cover of Y such that for every $C \in \mathcal{C}$ there is $O \in \mathcal{O}$ such that $C \subset O$ and C is contractible in O. Such a cover exists since Y is locally contractible. By Theorem 1.3 there are $m+k+1$ families of open sets $\mathcal{V}_0, \dots, \mathcal{V}_{m+k}$ such that their union forms an $(m+1)$–cover of Y refining $\mathcal{C}$. We define $V_i = \bigcup_\alpha V_i^\alpha$ to be the unions of all sets from $\mathcal{V}_i = \{V_i^\alpha\}$. Then $\mathcal{V} = \{V_0, \dots, V_{m+k}\}$ is an open $(m+1)$–cover of Y such that for every i, $V_i = \bigcup_\alpha V_i^\alpha$ is a disjoint union of open sets V_i^α contractible to a point in $O_i^\alpha \in \mathcal{O}$.

We show that for all $i \in \{0, 1, \dots, m+k\}$, the sets $W_i = f^{-1}(V_i) \cap U_i$ are contractible in X. Since
$$W_i = \bigcup_\alpha f^{-1}(V_i^\alpha) \cap U_i$$
is a disjoint union, it suffices to show that the sets $f^{-1}(V_i^\alpha) \cap U_i$ are contractible in X for all α. By Corollary 1.5 the set U_i is fiberwise contractible into X' for $i \leq m+k$. Hence we can contract $f^{-1}(V_i^\alpha) \cap U_i$ to $f^{-1}(V_i^\alpha) \cap X' \cong V_i^\alpha \times A$ in X. Then we apply a contraction to a point of V_i^α in O_i^α and A in F to obtain a contraction to a point of $f^{-1}(V_i^\alpha) \cap X' \cong V_i^\alpha \times A$ in $f^{-1}(O_i^\alpha) \cong O_i^\alpha \times F$.

Next we show that $\{W_i\}_{i=0}^{m+k}$ is a cover of X. Since $\mathcal{V}$ is an $(m+1)$–cover, by Proposition 1.2 every $y \in Y$ is covered by at least $k+1$ elements $V_{i_0}, \dots, V_{i_k}$ of $\mathcal{V}$. Since $\mathcal{U}_m$ is a $(k+1)$–cover, $U_{i_0}, \dots, U_{i_k}$ is a cover of X. Hence $W_{i_0}, \dots, W_{i_k}$ covers $f^{-1}(y)$. □

Theorem 3.3 *For every CW–complex X with the following inequality holds true:*
$$\mathrm{cat}_{\mathrm{LS}} X \leq \mathrm{cd}(\pi_1(X)) + \left\lceil \frac{\mathrm{hd}(X)-1}{2} \right\rceil.$$

Proof Let $\pi = \pi_1(X)$ and let $\widetilde{X}$ denote the universal cover of X. We consider Borel's construction:

$$\begin{tikzcd} \widetilde{X} \arrow[d] & \widetilde{X} \times E\pi \arrow[l] \arrow[r] \arrow[d] & E\pi \arrow[d] \\ X & \widetilde{X} \times_\pi E\pi \arrow[l, "g"'] \arrow[r, "f"] & B\pi. \end{tikzcd}$$

We refer for the properties of Borel's construction also known as the twisted product to [1]. Note that the 1–skeleton $X^{(1)}$ of X defines a π–equivariant stratification $\widetilde{X}^{(1)} \subset \widetilde{X}$ of the universal cover. This stratification allows us to treat f as a stratified locally trivial bundle with the fiber $(\widetilde{X}, \widetilde{X}^{(1)})$. We note that all conditions of Lemma 3.2 are satisfied for $r = 1$. Therefore,

$$\mathrm{cat}_{\mathrm{LS}}(\widetilde{X} \times_\pi E\pi) \le \dim B\pi + \left\lceil \frac{\mathrm{hd}(\widetilde{X} \times_\pi E\pi) - 1}{2} \right\rceil.$$

Since g is a fibration with homotopy trivial fiber, the space $\widetilde{X} \times_\pi E\pi$ is homotopy equivalent to X. Thus, $\mathrm{cat}_{\mathrm{LS}}(\widetilde{X} \times_\pi E\pi) = \mathrm{cat}_{\mathrm{LS}} X$ and $\mathrm{hd}(\widetilde{X} \times_\pi E\pi) = \mathrm{hd}(X)$. In view of the results of Eilenberg and Ganea [7] (see also Brown [2]) we may assume that $\dim B\pi = \mathrm{cd}(\pi)$ if $\mathrm{cd}(\pi) > 2$. The case when $\mathrm{cd}(\pi) \le 2$ is treated in [5]. □

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References

[1] **G E Bredon**, *Introduction to compact transformation groups*, 46, Academic Press, New York (1972) MR0413144

[2] **K S Brown**, *Cohomology of groups*, Graduate Texts in Math. 87, Springer, New York (1982) MR672956

[3] **O Cornea**, **G Lupton**, **J Oprea**, **D Tanré**, *Lusternik–Schnirelmann category*, Math. Surveys and Monogr. 103, Amer. Math. Soc. (2003) MR1990857

[4] **A Dold**, *Partitions of unity in the theory of fibrations*, Ann. of Math. (2) 78 (1963) 223–255 MR0155330

[5] **A N Dranishnikov**, *On the Lusternik–Schnirelmann category of spaces with 2–dimensional fundamental group*, Proc. Amer. Math. Soc. 137 (2009) 1489–1497 MR2465675

[6] **A N Dranishnikov**, **M G Katz**, **Y B Rudyak**, *Small values of the Lusternik–Schnirelmann category for manifolds*, Geom. Topol. 12 (2008) 1711–1727 MR2421138

[7] **S Eilenberg**, **T Ganea**, *On the Lusternik–Schnirelmann category of abstract groups*, Ann. of Math. (2) 65 (1957) 517–518 MR0085510

[8] **D P Grossman**, *An estimation of the category of Lusternik–Schnirelmann*, C. R. (Doklady) Acad. Sci. URSS (N.S.) 54 (1946) 109–112 MR0018820

[9] **W Hurewicz**, *On the concept of fiber space*, Proc. Nat. Acad. Sci. USA 41 (1955) 956–961 MR0073987

[10] **P A Ostrand**, *Dimension of metric spaces and Hilbert's problem* 13, Bull. Amer. Math. Soc. 71 (1965) 619–622 MR0177391

[11] **E H Spanier**, *Algebraic topology*, McGraw-Hill, New York (1966) MR0210112

[12] **J Stallings**, *Groups of dimension 1 are locally free*, Bull. Amer. Math. Soc. 74 (1968) 361–364 MR0223439

[13] **J Strom**, *Lusternik–Schnirelmann category of spaces with free fundamental group*, Algebr. Geom. Topol. 7 (2007) 1805–1808 MR2366179

[14] **A S Švarc**, *The genus of a fibered space*, Trudy Moskov. Mat. Obšč. 10 (1961) 217–272 MR0154284

[15] **R G Swan**, *Groups of cohomological dimension one*, J. Algebra 12 (1969) 585–610 MR0240177

[16] **G W Whitehead**, *The homology suspension*, from: "Colloque de topologie algébrique, Louvain, 1956", Georges Thone, Liège (1957) 89–95 MR0094794

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