

On toric generators in the unitary and special unitary bordism rings

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We construct a new family of toric manifolds generating the unitary bordism ring. Each manifold in the family is the complex projectivisation of the sum of a line bundle and a trivial bundle over a complex projective space. We also construct a family of special unitary quasitoric manifolds which contains polynomial generators of the special unitary bordism ring with 2 inverted in dimensions > 8 . Each manifold in the latter family is obtained from an iterated complex projectivisation of a sum of line bundles by amending the complex structure to make the first Chern class vanish.

[57R77](#); [14M25](#)

1 Introduction

Finding geometric representatives of bordism classes is a classical problem on the borders of geometry and topology. The theory of bordism and cobordism is one of the deepest and most influential parts of algebraic topology, which experienced a spectacular development in the 1960s. Although the original definition of bordism, going back to Pontryagin and Thom, was very geometric, it soon became clear that elaborate homotopy-theoretic, algebraic and number-theoretic techniques were required to obtain structural results on bordism groups and (co)bordism rings.

Most calculations of bordism rings of a point for the classical series of Lie groups were settled by coordinated efforts of many topologists by the end of the 1960s (with the notable exception of symplectic bordism, whose structure is still not described completely). These results were summarised in the monograph by Stong [16]. Nevertheless, it has remained a challenging task to describe particular geometric representatives for generators of bordism rings (which tend to be rings of polynomials when 2 is inverted) and other “special” bordism classes. The importance of this problem was much emphasised in the original works such as Conner and Floyd [7].

Over the rationals, the bordism rings are generated by projective spaces, but the integral generators are more subtle as they involve divisibility conditions on characteristic

numbers. One of the few general results on geometric representatives for bordism classes known from the early 1960s is that the complex bordism ring Ω^U , which is an integral polynomial ring, can be generated by the so-called *Milnor hypersurfaces* $H(n_1, n_2)$. These are hyperplane sections of the Segre embeddings of products $\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2}$ of complex projective spaces. Similar generators exist for unoriented and oriented bordism rings.

The early progress was impeded by the lack of examples of higher-dimensional (stably) complex manifolds for which the characteristic numbers can be calculated explicitly. With the appearance of *toric varieties* in the late 1970s and subsequent development of toric topology (see Buchstaber and Panov [2]), a host of concrete examples of complex manifolds with large symmetry groups has been produced for which characteristic numbers can be calculated effectively using combinatorial-geometric techniques.

In [5], Buchstaber and Ray constructed a set of generators for Ω^U consisting entirely of complex projective toric manifolds $B(n_1, n_2)$, which are projectivisations of sums of line bundles over bounded flag manifolds. Later it was shown in Buchstaber, Panov and Ray [3] that one can get a geometric representative in every complex bordism class if toric manifolds are relaxed to *quasitoric* ones, the latter still have a “large torus” action, but are only stably complex instead of being complex. Characteristic numbers of toric manifolds satisfy quite restrictive conditions (eg their Todd genus is always 1) which prevent the existence of a toric representative in every bordism class; quasitoric manifolds enjoy more flexibility. We note that representing *polynomial* generators of Ω^U by toric manifolds remains open; some progress has been made by Wilfong [17].

Here we consider a family of projective toric manifolds obtained by iterated projectivisation of sums of line bundles, starting from a complex projective space. Such iterated projectivisations are also known as *generalised Bott manifolds* (see Masuda and Suh [12], and Buchstaber and Panov [2, Section 7.8]). Our first result (Theorem 3.8) shows that the complex bordism ring Ω^U can be generated by the most simple nontrivial two-stage projectivisations: manifolds $L(n_1, n_2) = \mathbb{C}P(\xi)$, where ξ is the sum of a tautological line bundle and an n_2 -dimensional trivial bundle over $\mathbb{C}P^{n_1}$. This new toric generator set is somewhat simpler than either of the set of Milnor hypersurfaces $\{H(n_1, n_2)\}$ or Buchstaber and Ray’s toric set $\{B(n_1, n_2)\}$.

We proceed by providing explicit families of quasitoric SU -manifolds which contain polynomial generators of the SU -bordism ring $\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$ (Theorem 4.19). In fact, our quasitoric SU -manifolds are genuinely indecomposable and indivisible elements in Ω^{SU} (integrally, without inverting any prime), however Ω^{SU} is not a polynomial ring.

We recall that a stably complex (or unitary) manifold M is *special unitary* (an SU -manifold for short) if $c_1(M) = 0$. A renewed interest to this class of manifolds

has been stimulated by the development of geometry motivated by physics; the notion of a *Calabi–Yau manifold* plays a central role here. By a Calabi–Yau manifold one usually understands a Kähler SU–manifold; it has a Ricci flat metric by the theorem of Yau. We note however that our SU–manifolds are rarely Kähler.

As was observed by Lü and Wang in [11], quasitoric SU–manifolds can be constructed by taking iterated complex projectivisations (which are projective toric manifolds) and then amending the stably complex structure so that the first Chern class becomes zero. The underlying smooth manifold of the result is still toric, but the stably complex structure is not the standard one. Examples of this sort were known to Conner and Floyd and used in their constructions [7], however the existence of a torus action was not emphasised and their amended stably complex structures were actually not SU.

The characteristic numbers of SU–manifolds satisfy intricate divisibility conditions. Ochanine’s theorem [14] asserting that the signature of an $(8k+4)$ –dimensional SU–manifold is divisible by 16 is one of the most famous examples. We therefore find it quite miraculous that polynomial generators for the SU–bordism ring Ω^{SU} occur within the most basic families of examples that one can produce using toric methods: two-stage complex projectivisations, and three-stage projectivisations with the first stage being just $\mathbb{C}P^1$. The proof of [Theorem 4.19](#) involves calculating the characteristic numbers and checking various divisibility conditions. We use both classical and more recent results on binomial coefficients modulo a prime.

We note also that the existence of large torus actions indicates possible applications of our examples in the equivariant setting. Applicability of toric methods in equivariant bordism is currently being explored (see Buchstaber, Panov and Ray [4], Buchstaber and Panov [2, Chapter 9], and Lü [10]).

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2 Toric and quasitoric manifolds, cohomology and Chern classes

Here we collect the necessary information about toric varieties and quasitoric manifolds; the details can be found in [2].

A *toric variety* is a normal complex algebraic variety V containing an algebraic torus $(\mathbb{C}^\times)^n$ as a Zariski open subset in such a way that the natural action of $(\mathbb{C}^\times)^n$ on itself extends to an action on V . We only consider nonsingular complete (compact in the usual topology) toric varieties, also known as *toric manifolds*.

There is a bijective correspondence between the isomorphism classes of complex n -dimensional toric manifolds and complete regular fans in \mathbb{R}^n . A *fan* is a finite collection $\Sigma = \{\sigma_1, \dots, \sigma_s\}$ of strongly convex cones σ_i in \mathbb{R}^n such that every face of a cone in Σ belongs to Σ and the intersection of any two cones in Σ is a face of each. A fan Σ is *regular* if each of its cones σ_j is generated by part of a basis of the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ (we choose the standard lattice for simplicity). In particular, each 1-dimensional cone of Σ is generated by a primitive vector $\mathbf{a}_i \in \mathbb{Z}^n$. A fan Σ is *complete* if the union of its cones is the whole \mathbb{R}^n .

Projective toric varieties are particularly important. A projective toric manifold V is defined by a *lattice Delzant polytope* P . Given a simple n -dimensional polytope P with vertices in the lattice \mathbb{Z}^n , one defines the *normal fan* Σ_P as the fan whose n -dimensional cones σ_v correspond to the vertices v of P , and σ_v is generated by the primitive inside-pointing normals to the facets of P meeting at v . The polytope P is *Delzant* precisely when its normal fan Σ_P is regular. The fan Σ_P defines a projective toric manifold V_P . Different lattice Delzant polytopes with the same normal fan produce different projective embeddings of the same toric manifold.

Irreducible torus-invariant divisors on V are the toric subvarieties of complex codimension 1 corresponding to the 1-dimensional cones of Σ . When V is projective, they also correspond to the facets of P . We assume that there are m 1-dimensional cones (or facets), denote the corresponding primitive vectors by $\mathbf{a}_1, \dots, \mathbf{a}_m$, and denote the corresponding codimension-1 subvarieties by V_1, \dots, V_m .

Theorem 2.1 *Let V be a toric manifold of complex dimension n , with the corresponding complete regular fan Σ . The cohomology ring $H^*(V; \mathbb{Z})$ is generated by the degree-two classes v_i dual to the invariant submanifolds V_i , and is given by*

$$H^*(V; \mathbb{Z}) \cong \mathbb{Z}[v_1, \dots, v_m]/\mathcal{I}, \quad \deg v_i = 2,$$

where \mathcal{I} is the ideal generated by elements of the following two types:

- (a) $v_{i_1} \cdots v_{i_k}$ such that $\mathbf{a}_{i_1}, \dots, \mathbf{a}_{i_k}$ do not span a cone of Σ ;
- (b) $\sum_{i=1}^m \langle \mathbf{a}_i, \mathbf{x} \rangle v_i$, for any vector $\mathbf{x} \in \mathbb{Z}^n$.

It is convenient to consider the integer $n \times m$ -matrix

$$(2-1) \quad \Lambda = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

whose columns are the vectors \mathbf{a}_i written in the standard basis of \mathbb{Z}^n . Then the ideal (b) of Theorem 2.1 is generated by the n linear forms $a_{j_1}v_1 + \dots + a_{j_m}v_m$ corresponding to the rows of Λ .

Theorem 2.2 *There is the following isomorphism of complex vector bundles:*

$$\mathcal{T}V \oplus \underline{\mathbb{C}}^{m-n} \cong \rho_1 \oplus \dots \oplus \rho_m,$$

where $\mathcal{T}V$ is the tangent bundle, $\underline{\mathbb{C}}^{m-n}$ is the trivial $(m-n)$ -plane bundle, and ρ_i is the line bundle corresponding to V_i , with $c_1(\rho_i) = v_i$. In particular, the total Chern class of V is given by

$$c(V) = (1 + v_1) \cdots (1 + v_m).$$

Example 2.3 A basic example of a toric manifold is the complex projective space $\mathbb{C}P^n$. The cones of the corresponding fan are generated by proper subsets of the set of $m = n + 1$ vectors $\mathbf{e}_1, \dots, \mathbf{e}_n, -\mathbf{e}_1 - \dots - \mathbf{e}_n$, where $\mathbf{e}_i \in \mathbb{Z}^n$ is the i^{th} standard basis vector. It is the normal fan of the lattice simplex Δ^n with the vertices at $\mathbf{0}$ and $\mathbf{e}_1, \dots, \mathbf{e}_n$. The matrix (2-1) is given by

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & 1 & -1 \end{pmatrix}.$$

Theorem 2.1 gives the cohomology of $\mathbb{C}P^n$ as

$$H^*(\mathbb{C}P^n) \cong \mathbb{Z}[v_1, \dots, v_{n+1}] / (v_1 \cdots v_{n+1}, v_1 - v_{n+1}, \dots, v_n - v_{n+1}) \cong \mathbb{Z}[v] / (v^{n+1}),$$

where v is any of the v_i . Theorem 2.2 gives the standard decomposition

$$\mathcal{T}\mathbb{C}P^n \oplus \underline{\mathbb{C}} \cong \bar{\eta} \oplus \dots \oplus \bar{\eta} \quad (n + 1 \text{ summands}),$$

where $\eta = \mathcal{O}(-1)$ is the tautological (Hopf) line bundle over $\mathbb{C}P^n$, and $\bar{\eta} = \mathcal{O}(1)$ is its conjugate, or the line bundle corresponding to a hyperplane $\mathbb{C}P^{n-1} \subset \mathbb{C}P^n$.

Example 2.4 An example which will be important for our constructions is the complex projectivisation of a sum of line bundles over projective space.

Given two positive integers n_1, n_2 and a sequence of integers (i_1, \dots, i_{n_2}) , consider the projectivisation $V = \mathbb{C}P(\eta^{\otimes i_1} \oplus \dots \oplus \eta^{\otimes i_{n_2}} \oplus \underline{\mathbb{C}})$, where $\eta^{\otimes i}$ denotes the i^{th} tensor power of η over $\mathbb{C}P^{n_1}$ when $i \geq 0$ and the i^{th} tensor power of $\bar{\eta}$ otherwise. The manifold V is the total space of a bundle over $\mathbb{C}P^{n_1}$ with fibre $\mathbb{C}P^{n_2}$. It is also a projective toric manifold with the corresponding matrix (2-1) given by

The relation above is just $c_n(\bar{\gamma} \otimes p^*\xi) = 0$.

In the case considered above, $\xi = \eta^{\otimes i_1} \oplus \dots \oplus \eta^{\otimes i_{n_2}} \oplus \mathbb{C}$ over $X = \mathbb{C}P^{n_1}$. We then have $H^*(X) = \mathbb{Z}[u]/(u^{n_1+1})$ where $u = c_1(\bar{\eta})$, so that equation (2-4) becomes $v(v - i_1 u) \dots (v - i_{n_2} u) = 0$ and the ring $H^*(\mathbb{C}P(\xi))$ given by Theorem 2.5 is precisely (2-2). Further, the total Chern class of $p^*TX \oplus (\bar{\gamma} \otimes p^*\xi)$ is given by (2-3).

The quotient of the projective toric manifold V_P by the action of the compact torus $T^n \subset (\mathbb{C}^\times)^n$ is the polytope P .

A quasitoric manifold over a combinatorial simple n -dimensional polytope P is a manifold M of dimension $2n$ with a locally standard action of T^n such that the quotient M/T^n is homeomorphic, as a manifold with corners, to P . (An action of T^n on M^{2n} is *locally standard* if every point $x \in M^{2n}$ is contained in a T^n -invariant neighbourhood equivariantly homeomorphic to an open subset in \mathbb{C}^n with the standard coordinatewise action of T^n twisted by an automorphism of the torus; the orbit space of a locally standard action is a manifold with corners.) We therefore have a projection $\pi: M \rightarrow P$ whose fibres are orbits of the T^n -action.

Not every simple polytope can be the quotient of a quasitoric manifold. Nevertheless, quasitoric manifolds constitute a much larger family than projective toric manifolds, and enjoy more flexibility for topological applications.

If F_1, \dots, F_m are facets of P , then each $M_i = \pi^{-1}(F_i)$ is a quasitoric submanifold of M of codimension 2, called a *characteristic submanifold*. The characteristic submanifolds $M_i \subset M$ are analogues of the invariant divisors V_i on a toric manifold V . Each M_i is fixed pointwise by a closed 1-dimensional subgroup (a subcircle) $T_i \subset T^n$ and therefore corresponds to a primitive vector $\lambda_i \in \mathbb{Z}^n$ defined up to a sign. Choosing a direction of λ_i is equivalent to choosing an orientation for the normal bundle $\nu(M_i \subset M)$ or, equivalently, choosing an orientation for M_i , provided that M itself is oriented. An *omniorientation* of a quasitoric manifold M consists of a choice of orientation for M and each characteristic submanifold M_i , $1 \leq i \leq m$.

The vectors λ_i are analogues of the generators \mathbf{a}_i of the 1-dimensional cones of the fan corresponding to a toric manifold V (or analogues of the normal vectors to the facets of P when V is projective). However, the λ_i need not be the normal vectors to the facets of P in general.

There is an analogue of Theorem 2.1 for quasitoric manifolds:

Theorem 2.6 *Let M be an omnioriented quasitoric manifold of dimension $2n$ over a polytope P . The cohomology ring $H^*(M; \mathbb{Z})$ is generated by the degree-two*

classes v_i dual to the oriented characteristic submanifolds M_i , and is given by

$$H^*(M; \mathbb{Z}) \cong \mathbb{Z}[v_1, \dots, v_m]/\mathcal{I}, \quad \deg v_i = 2,$$

where \mathcal{I} is the ideal generated by elements of the following two types:

- (a) $v_{i_1} \cdots v_{i_k}$ such that $F_{i_1} \cap \cdots \cap F_{i_k} = \emptyset$ in P ;
- (b) $\sum_{i=1}^m \langle \lambda_i, x \rangle v_i$, for any vector $x \in \mathbb{Z}^n$.

By analogy with (2-1), we consider the integer $n \times m$ -matrix

$$(2-5) \quad \Lambda = \begin{pmatrix} \lambda_{11} & \cdots & \lambda_{1m} \\ \vdots & \ddots & \vdots \\ \lambda_{n1} & \cdots & \lambda_{nm} \end{pmatrix}$$

whose columns are the vectors λ_i written in the standard basis of \mathbb{Z}^n . Changing a basis in the lattice results in multiplying Λ from the left by a matrix from $GL(n, \mathbb{Z})$. The ideal (b) of Theorem 2.6 is generated by the n linear forms $\lambda_{j1}v_1 + \cdots + \lambda_{jm}v_m$ corresponding to the rows of Λ . Also, Λ has the property that $\det(\lambda_{i_1}, \dots, \lambda_{i_n}) = \pm 1$ whenever the facets F_{i_1}, \dots, F_{i_n} intersect at a vertex of P .

There is also an analogue of Theorem 2.2:

Theorem 2.7 For a quasitoric manifold M of dimension $2n$, there is an isomorphism of real vector bundles:

$$(2-6) \quad \mathcal{T}M \oplus \underline{\mathbb{R}}^{2(m-n)} \cong \rho_1 \oplus \cdots \oplus \rho_m,$$

where ρ_i is the real 2-plane bundle corresponding to the orientable characteristic submanifold $M_i \subset M$, so that $\rho_i|_{M_i} = \nu(M_i \subset M)$.

3 Unitary bordism

Here we provide a new set of toric generators for the unitary bordism ring. The general information about unitary (or complex) bordism can be found in [16].

Elements of the unitary bordism ring Ω^U are the complex bordism classes of stably complex manifolds. A *stably complex manifold* is a pair $(M, c_{\mathcal{T}})$ consisting of a smooth manifold M and a *stably complex structure* $c_{\mathcal{T}}$, where the latter is determined by a choice of an isomorphism

$$(3-1) \quad c_{\mathcal{T}}: \mathcal{T}M \oplus \underline{\mathbb{R}}^N \xrightarrow{\cong} \xi$$

between the stable tangent bundle of M and a complex vector bundle ξ . We omit $c_{\mathcal{T}}$ in the notation when it is clear from the context. We denote by $[M] \in \Omega^U$ the bordism class of a stably complex manifold M . The sum in Ω^U is the disjoint union, and the product is induced by the Cartesian product of manifolds. The ring Ω^U is graded by the dimension of manifolds.

A complex manifold M (in particular, a toric manifold) has a canonical stably complex structure arising from the complex structure on $\mathcal{T}M$. An omniorientation of a quasitoric manifold M gives it a stably complex structure by means of the isomorphism of [Theorem 2.7](#), because a choice of orientation for each real 2-plane bundle ρ_i is equivalent to endowing it with a complex structure.

Example 3.1 The canonical stably complex structure on $\mathbb{C}P^n$ (as a complex manifold) is given by the isomorphism

$$\mathcal{T}\mathbb{C}P^n \oplus \underline{\mathbb{R}}^2 \cong \bar{\eta} \oplus \cdots \oplus \bar{\eta} \quad (n + 1 \text{ summands}).$$

On the other hand, $\mathbb{C}P^n$, viewed as a quasitoric manifold over Δ^n , has $n + 1$ characteristic submanifolds, and therefore 2^{n+2} different omniorientations. Each of these omniorientations gives rise to a stably complex structure, obtained by replacing some of the line bundles $\bar{\eta}$ above with η , or by reversing the global orientation. Some of these stably complex structures are equivalent, of course.

We have $H^*(BU(n)) \cong \mathbb{Z}[c_1, \dots, c_n]$, $\deg c_i = 2i$, where the c_i are the universal Chern characteristic classes. For any sequence $\omega = (i_1, \dots, i_n)$ of nonnegative integers, there is the monomial $c_\omega = c_1^{i_1} \cdots c_n^{i_n}$ of degree $2\|\omega\| = 2\sum_{k=1}^n k i_k$ and the corresponding characteristic class $c_\omega(\xi)$ of a complex n -plane bundle ξ . The corresponding tangential Chern *characteristic number* of a stably complex manifold M is defined by $c_\omega[M] = c_\omega(\mathcal{T}M)\langle M \rangle$. Here $\langle M \rangle$ is the fundamental homology class of M , and $\mathcal{T}M$ is regarded as a complex bundle via the isomorphism (3-1). The number $c_\omega[M]$ is assumed to be zero when $2\|\omega\| \neq \dim M$.

Theorem 3.2 *Two stably complex manifold M and N represent the same bordism classes in Ω^U if and only if their sets of Chern characteristic numbers coincide.*

Another important characteristic class is s_n . It is defined as the polynomial in c_1, \dots, c_n obtained by expressing the symmetric polynomial $x_1^n + \cdots + x_n^n$ via the elementary symmetric functions $\sigma_i(x_1, \dots, x_n)$ and then replacing each σ_i by c_i . Define the corresponding characteristic number as $s_n[M] = s_n(\mathcal{T}M)\langle M \rangle$.

The ring Ω^U was described by Milnor and Novikov (see [13], and Stong [16]):

Theorem 3.3 *The ring Ω^U is a polynomial ring on generators in every even degree:*

$$\Omega^U \cong \mathbb{Z}[a_i, i > 0], \quad \deg a_i = 2i.$$

Then the bordism class of a stably complex manifold M^{2i} may be taken to be the $2i$ -dimensional generator a_i if and only if

$$s_i[M^{2i}] = \begin{cases} \pm 1 & \text{if } i + 1 \neq p^s \text{ for any prime } p, \\ \pm p & \text{if } i + 1 = p^s \text{ for some prime } p \text{ and integer } s > 0. \end{cases}$$

There is no universal description of connected manifolds representing the polynomial generators $a_n \in \Omega^U$. However, there are known explicit families of manifolds whose bordism classes generate the whole ring Ω^U .

The classical family of generators for the ring Ω^U consists of the *Milnor hypersurfaces* $H(n_1, n_2)$. Each $H(n_1, n_2)$ is a hyperplane section of the Segre embedding $\mathbb{C}P^{n_1} \times \mathbb{C}P^{n_2} \rightarrow \mathbb{C}P^{(n_1+1)(n_2+1)-1}$ and may be given explicitly by the equation

$$z_0 w_0 + \dots + z_{n_1} w_{n_1} = 0$$

in the homogeneous coordinates $[z_0 : \dots : z_{n_1}] \in \mathbb{C}P^{n_1}$ and $[w_0 : \dots : w_{n_2}] \in \mathbb{C}P^{n_2}$, assuming that $n_1 \leq n_2$. Also, $H(n_1, n_2)$ can be identified with the projectivisation $\mathbb{C}P(\zeta)$ of a certain n_2 -plane bundle over $\mathbb{C}P^{n_1}$. The bundle ζ is not a sum of line bundles when $n_1 > 1$, so $H(n_1, n_2)$ is *not* a toric manifold in this case (see [2, Section 9.1]).

Buchstaber and Ray [5] introduced a family $B(n_1, n_2)$ of *toric* generators of Ω^U . Each $B(n_1, n_2)$ is the projectivisation of a sum of n_2 line bundles over the *bounded flag manifold* BF_{n_1} . Then $B(n_1, n_2)$ is a toric manifold, because BF_{n_1} is toric and the projectivisation of a sum of line bundles over a toric manifold is toric.

We have $H(0, n_2) = B(0, n_2) = \mathbb{C}P^{n_2-1}$, so

$$s_{n_2-1}[H(0, n_2)] = s_{n_2-1}[B(0, n_2)] = n_2.$$

Furthermore,

$$(3-2) \quad s_{n_1+n_2-1}[H(n_1, n_2)] = s_{n_1+n_2-1}[B(n_1, n_2)] = -\binom{n_1+n_2}{n_1} \quad \text{for } n_1 > 1;$$

see [2, Section 9.1] for the details.

We shall need the following two facts from number theory:

Theorem 3.4 (Lucas) *Let p be a prime, and let*

$$\begin{aligned} n &= n_0 + n_1 p + \cdots + n_{k-1} p^{k-1} + n_k p^k, \\ m &= m_0 + m_1 p + \cdots + m_{k-1} p^{k-1} + m_k p^k \end{aligned}$$

be the base p expansions of positive integers m and n . Then

$$\binom{n}{m} \equiv \binom{n_0}{m_0} \binom{n_1}{m_1} \cdots \binom{n_k}{m_k} \pmod{p}.$$

Here the standard convention $\binom{m}{n} = 0$ if $m < n$ is used.

For the proof, see eg [15, Lemma 2.6].

Proposition 3.5 *For any integer $n > 0$, we have*

$$\gcd\left\{\binom{n}{i}, 0 < i < n\right\} = \begin{cases} 1 & \text{if } n \neq p^s \text{ for any prime } p, \\ p & \text{if } n = p^s \text{ for some prime } p \text{ and integer } s > 0. \end{cases}$$

Proof Assume $n = p^s$. Then each $\binom{n}{i}$ with $0 < i < n$ is divisible by p . On the other hand, $\binom{p^s}{p^{s-1}}$ is not divisible by p^2 , eg by Kummer’s theorem.

Now assume $n \neq p^s$. Write the base p expansion

$$n = n_0 + n_1 p + \cdots + n_{k-1} p^{k-1} + n_k p^k,$$

where we may assume $n_k > 0$. Take

$$i = n_0 + n_1 p + \cdots + n_{k-1} p^{k-1} + (n_k - 1) p^k.$$

Then $i \neq 0$ as otherwise $n = p^k$. By Theorem 3.4, $\binom{n}{i} \equiv n_k \not\equiv 0 \pmod{p}$. □

The fact that each of the families $\{[H(n_1, n_2)]\}$ and $\{[B(n_1, n_2)]\}$ generates the unitary bordism ring Ω^U follows from (3-2), Proposition 3.5 and Theorem 3.3.

We proceed to describe another family of toric generators for Ω^U .

Construction 3.6 Given two positive integers n_1 and n_2 , we define the manifold $L(n_1, n_2)$ as the projectivisation $\mathbb{C}P(\eta \oplus \mathbb{C}^{n_2})$, where η is the tautological line bundle over $\mathbb{C}P^{n_1}$. This $L(n_1, n_2)$ is a particular case of manifolds from Example 2.4, so it is a projective toric manifold with the corresponding matrix (2-1) given by

Proof Assuming $[L(n_1, n_2)] = 0$ when $n_1 < 0$, we calculate using Lemma 3.7:

$$\begin{aligned} s_{n_1+n_2} [L(n_1, n_2) - 2L(n_1 - 1, n_2 + 1) + L(n_1 - 2, n_2 + 2)] \\ = (-1)^{n_1-1} \binom{n_1+n_2}{n_1-1} + (-1)^{n_1} \binom{n_1+n_2}{n_1} - 2(-1)^{n_1-1} \binom{n_1+n_2}{n_1-1} \\ = (-1)^{n_1} \binom{n_1+n_2+1}{n_1}. \end{aligned}$$

The result follows from Proposition 3.5 and Theorem 3.3. □

Theorem 3.8 implies that any unitary bordism class can be represented by a disjoint union of products of projective toric manifolds. Products of toric manifolds are toric, but disjoint unions are not, as toric manifolds are connected. In bordism theory, a disjoint union may be replaced by a connected sum, representing the same bordism class. However, connected sum is not an algebraic operation, and a connected sum of two algebraic varieties is rarely algebraic. This can be remedied by appealing to quasitoric manifolds, as explained next. Recall that an omnioriented quasitoric manifold has an intrinsic stably complex structure, arising from the isomorphism of Theorem 2.7. One can form the equivariant connected sum of quasitoric manifolds, as explained in Davis and Januszkiewicz [8], but the resulting invariant stably complex structure does not represent the cobordism sum of the two original manifolds. A more intricate connected sum construction is needed, as outlined below. The details can be found in [3] or [2, Section 9.1].

Construction 3.9 The construction applies to two omnioriented $2n$ -dimensional quasitoric manifolds M and M' over n -polytopes P and P' respectively. The connected sum will be taken at the fixed points of M and M' corresponding to vertices $v \in P$ and $v' \in P'$. We need to assume that v is the intersection of the first n facets of P , ie $v = F_1 \cap \dots \cap F_n$, and the corresponding characteristic matrix (2-5) of M is in the refined form, ie

$$\Lambda = (I \mid \Lambda_\star) = \begin{pmatrix} 1 & 0 & 0 & \lambda_{1,n+1} & \cdots & \lambda_{1,m} \\ 0 & \ddots & 0 & \vdots & \ddots & \vdots \\ 0 & 0 & 1 & \lambda_{n,n+1} & \cdots & \lambda_{n,m} \end{pmatrix},$$

where I is the unit matrix and Λ_\star is an $n \times (m-n)$ -matrix. The same assumptions are made for M', P', v' and Λ' .

The next step depends on the signs of the fixed points, $\sigma(v)$ and $\sigma(v')$. The sign of v is determined by the omniorientation data; it is $+1$ when the orientation of $\mathcal{T}_v M$ induced from the global orientation of M coincides with the orientation arising from $\rho_1 \oplus \dots \oplus \rho_n|_v$, and is -1 otherwise.

If $\sigma(v) = -\sigma(v')$, then we take the connected sum $M \# M'$ at v and v' . It is a quasitoric manifold over $P \# P'$ with the characteristic matrix $(\Lambda_\star \mid I \mid \Lambda'_\star)$.

If $\sigma(v) = \sigma(v')$, then we need an additional connected summand. Consider the quasitoric manifold $S = S^2 \times \cdots \times S^2$ over the n -cube I^n , where each S^2 is the quasitoric manifold over the segment I with the characteristic matrix $(1 \mid 1)$. It represents zero in Ω^U , and may be thought of as $\mathbb{C}P^1$ with the stably complex structure given by the isomorphism $\mathcal{T}C P^1 \oplus \mathbb{R}^2 \cong \bar{\eta} \oplus \eta$. The characteristic matrix of S is therefore $(I \mid I)$. Now consider the connected sum $M \# S \# M'$. It is a quasitoric manifold over $P \# I^n \# P'$ with the characteristic matrix $(\Lambda_\star \mid I \mid I \mid \Lambda'_\star)$.

In either case, the resulting omnioriented quasitoric manifold $M \# M'$ or $M \# S \# M'$ with the canonical stably complex structure represents the sum of bordism classes $[M] + [M'] \in \Omega_{2n}^U$.

The conclusion, which can be derived from the above construction and any of the toric generating sets $\{B(n_1, n_2)\}$ or $\{L(n_1, n_2)\}$ for Ω^U , is as follows:

Theorem 3.10 [3] *In dimensions > 2 , every unitary bordism class contains a quasitoric manifold, necessarily connected, whose stably complex structure is induced by an omniorientation, and is therefore compatible with the torus action.*

4 Special unitary bordism

Basics

A *special unitary structure* (an *SU-structure* for short) on a manifold M is a stably complex structure $c_{\mathcal{T}}$ with a choice of an *SU-structure* on the complex bundle ξ ; see (3-1). A stably complex manifold $(M, c_{\mathcal{T}})$ admits an *SU-structure* if and only if $c_1(\xi) = 0$. Bordism classes of *SU-manifolds* form the *special unitary bordism ring* Ω^{SU} .

The ring structure of Ω^{SU} is more subtle than that of Ω^U . Novikov [13] described $\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$ (it is a polynomial ring). The 2-torsion was described by Conner and Floyd [7]. For the description of the ring structure in Ω^{SU} (which is not a polynomial ring, even modulo torsion), see [16]. We shall need the following facts:

Theorem 4.1 (a) *The kernel of the forgetful map $\Omega^{SU} \rightarrow \Omega^U$ consists of torsion elements.*

(b) *Every torsion element in Ω^{SU} has order 2.*

(c) *$\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}]$ is a polynomial algebra on generators in every even degree > 2 :*

$$\Omega^{SU} \otimes \mathbb{Z}[\frac{1}{2}] \cong \mathbb{Z}[\frac{1}{2}][y_i : i > 1], \quad \deg y_i = 2i.$$

For the further analysis of the ring Ω^{SU} we need to consider an auxiliary ring \mathcal{W} , apparently named after C T C Wall. We describe it following [7] and [16].

Let $\partial: \Omega_{2n}^{\text{U}} \rightarrow \Omega_{2n-2}^{\text{U}}$ be the homomorphism sending a bordism class $[M^{2n}]$ to the bordism class $[V^{2n-2}]$ of a submanifold $V^{2n-2} \subset M$ dual to $c_1(M)$. There is a line bundle γ over M corresponding to $c_1(M)$, and the restriction of γ to V is the normal bundle $\nu(V \subset M)$. The stably complex structure on V is defined via the isomorphism $\mathcal{T}M|_V \cong \mathcal{T}V \oplus \nu(V \subset M)$. Then V is an SU -manifold, so $\partial^2 = 0$. The homomorphism ∂ is not a derivation of Ω^{U} though; it satisfies the identity

$$\partial(a \cdot b) = a \cdot \partial b + \partial a \cdot b - [\mathbb{C}P^1] \cdot \partial a \cdot \partial b.$$

Let \mathcal{W}_{2n} be the subgroup of Ω_{2n}^{U} consisting of bordism classes $[M^{2n}]$ such that every Chern number of M^{2n} of which c_1^2 is a factor vanishes. The forgetful homomorphism decomposes as $\Omega_{2n}^{\text{SU}} \rightarrow \mathcal{W}_{2n} \rightarrow \Omega_{2n}^{\text{U}}$, and the restriction of the boundary homomorphism $\partial: \mathcal{W}_{2n} \rightarrow \mathcal{W}_{2n-2}$ is defined.

The direct sum $\mathcal{W} = \bigoplus_{i \geq 0} \mathcal{W}_{2i}$ is *not* a subring of Ω^{U} : one has $[\mathbb{C}P^1] \in \mathcal{W}_2$, but $c_1^2[\mathbb{C}P^1 \times \mathbb{C}P^1] = 8 \neq 0$, so $[\mathbb{C}P^1] \times [\mathbb{C}P^1] \notin \mathcal{W}_4$. However, \mathcal{W} becomes a commutative ring with unit with respect to the *twisted product*

$$(4-1) \quad a * b = a \cdot b + 2[V^4] \cdot \partial a \cdot \partial b,$$

where \cdot denotes the product in Ω^{U} and where V^4 is a stably complex manifold with $c_1^2[V^4] = -1$. One may take $V^4 = \mathbb{C}P^1 \times \mathbb{C}P^1 - \mathbb{C}P^2$ with the standard complex structure, or $V^4 = \overline{\mathbb{C}P^2}$ with the stably complex structure defined by the isomorphism $\mathcal{T}\mathbb{C}P^2 \oplus \mathbb{R}^2 \cong \bar{\eta} \oplus \bar{\eta} \oplus \eta$.

We shall use the notation

$$m_i = \begin{cases} 1 & \text{if } i + 1 \neq p^s \text{ for any prime } p, \\ p & \text{if } i + 1 = p^s \text{ for some prime } p \text{ and integer } s > 0, \end{cases}$$

so that $[M^{2i}] \in \Omega_{2i}^{\text{U}}$ represents a polynomial generator whenever $s_i[M^{2i}] = \pm m_i$.

Theorem 4.2 \mathcal{W} is a polynomial ring on generators in every even degree except 4:

$$\mathcal{W} \cong \mathbb{Z}[x_1, x_i : i > 2], \quad x_1 = [\mathbb{C}P^1], \quad \deg x_i = 2i,$$

with $s_i[x_i] = m_i m_{i-1}$, and the boundary operator $\partial: \mathcal{W} \rightarrow \mathcal{W}$, $\partial^2 = 0$, given by

$$\partial x_1 = 2, \quad \partial x_{2i} = x_{2i-1},$$

satisfies the identity

$$\partial(a * b) = a * \partial b + \partial a * b - x_1 * \partial a * \partial b.$$

The forgetful map $\alpha: \Omega^{\text{SU}} \rightarrow \mathcal{W}$ is a ring homomorphism; this follows from (4-1) because $\partial\alpha(x) = 0$ for any $x \in \Omega^{\text{SU}}$.

The fundamental result relating Ω^{SU} and \mathcal{W} is as follows:

Theorem 4.3 *There is an exact sequence of groups*

$$0 \longrightarrow \Omega_{2n-1}^{\text{SU}} \xrightarrow{\theta} \Omega_{2n}^{\text{SU}} \xrightarrow{\alpha} \mathcal{W}_{2n} \xrightarrow{\beta} \Omega_{2n-2}^{\text{SU}} \xrightarrow{\theta} \Omega_{2n-1}^{\text{SU}} \longrightarrow 0,$$

where θ is the multiplication by the generator $\theta \in \Omega_1^{\text{SU}} \cong \mathbb{Z}_2$, α is the forgetful homomorphism, and $\alpha\beta = -\partial$.

Analysing the exact sequence above, one obtains the following information about the free and torsion parts of Ω^{SU} :

Theorem 4.4 (a) *Torsion(Ω_n^{SU}) = 0 unless $n = 8k + 1$ or $8k + 2$, in which case Torsion(Ω_n^{SU}) is a \mathbb{Z}_2 -vector space of rank equal to the number of partitions of k .*

(b) *$\Omega_{2i}^{\text{SU}} / \text{Torsion}$ is isomorphic to $\text{Ker}(\partial: \mathcal{W} \rightarrow \mathcal{W})$ if $2i \not\equiv 4 \pmod{8}$ and is isomorphic to $\text{Im}(\partial: \mathcal{W} \rightarrow \mathcal{W})$ if $2i \equiv 4 \pmod{8}$.*

(c) *There exist SU-bordism classes $w_{4k} \in \Omega_{8k}^{\text{SU}}$, $k \geq 1$, such that $\text{Im } \alpha / \text{Im } \partial \cong \mathbb{Z}_2[w_{4k}]$. Every torsion element of Ω^{SU} is uniquely expressible in the form $P \cdot \theta$ or $P \cdot \theta^2$ where P is a polynomial in w_{4k} with coefficients 0 or 1.*

Note that we have

$$(4-2) \quad \mathcal{W} \otimes \mathbb{Z}\left[\frac{1}{2}\right] \cong \mathbb{Z}\left[\frac{1}{2}\right][x_1, x_{2k-1}, 2x_{2k} - x_1x_{2k-1} : k > 1],$$

where $x_1^2 = x_1 * x_1$ is a ∂ -cycle, and each $x_{2k-1}, 2x_{2k} - x_1x_{2k-1}$ is a ∂ -cycle.

Theorem 4.5 *There exist elements $y_i \in \Omega_{2i}^{\text{SU}}$, $i > 1$, such that $s_i(y_i) = m_i m_{i-1}$ if i is odd, $s_2(y_2) = -48$, and $s_i(y_i) = 2m_i m_{i-1}$ if i is even and $i > 2$. These elements are mapped as follows under the forgetful homomorphism $\alpha: \Omega^{\text{SU}} \rightarrow \mathcal{W}$:*

$$y_2 \mapsto 2x_1^2, \quad y_{2k-1} \mapsto x_{2k-1}, \quad y_{2k} \mapsto 2x_{2k} - x_1x_{2k-1}, \quad k > 1,$$

where the x_i are polynomial generators of \mathcal{W} . In particular, $\Omega^{\text{SU}} \otimes \mathbb{Z}\left[\frac{1}{2}\right]$ embeds into (4-2) as the polynomial subring generated by x_1^2, x_{2k-1} and $2x_{2k} - x_1x_{2k-1}$.

Quasitoric SU–manifolds

Omnioriented quasitoric manifolds whose stably complex structures are SU can be detected using the following simple criterion:

Proposition 4.6 [4] *An omnioriented quasitoric manifold M has $c_1(M) = 0$ if and only if there exists a linear function $\varphi: \mathbb{Z}^n \rightarrow \mathbb{Z}$ such that $\varphi(\lambda_i) = 1$ for $i = 1, \dots, m$. Here the λ_i are the columns of matrix (2-5).*

In particular, if some n vectors of $\lambda_1, \dots, \lambda_m$ form the standard basis e_1, \dots, e_n , then M is SU if and only if the column sums of Λ are all equal to 1.

Proof By Theorem 2.7, $c_1(M) = v_1 + \dots + v_m$. By Theorem 2.6, $v_1 + \dots + v_m$ is zero in $H^2(M)$ if and only if

$$v_1 + \dots + v_m = \sum_i \varphi(\lambda_i)v_i$$

for some linear function $\varphi: \mathbb{Z}^n \rightarrow \mathbb{Z}$, whence the result follows. □

Corollary 4.7 *A toric manifold V cannot be SU.*

Proof If $\varphi(\lambda_i) = 1$ for all i , then the vectors λ_i lie in the positive halfspace of φ , so they cannot span a complete fan. □

A more subtle result also rules out low-dimensional quasitoric manifolds:

Theorem 4.8 [4, Theorem 6.13] *A quasitoric SU–manifold M^{2n} represents 0 in Ω_{2n}^U whenever $n < 5$.*

The reason for this is that the Krichever genus $\varphi_K: \Omega^U \rightarrow R_K$ vanishes on quasitoric SU–manifolds, but φ_K is an isomorphism in dimensions < 10 .

Examples of quasitoric SU–manifolds representing nonzero bordism classes in Ω_{2n}^U for all $n \geq 5$, except $n = 6$, were constructed in [11]. We modify this construction to present two particular families of quasitoric SU–manifolds representing nonzero bordism classes in Ω_{2n}^U for all $n \geq 5$, including $n = 6$.

Construction 4.9 Assume that $n_1 = 2k_1$ is positive even and $n_2 = 2k_2 + 1$ is positive odd, and consider the manifold $L(n_1, n_2)$ from Construction 3.6. We change the stably complex structure (3-5) to the following:

$$\begin{aligned} & \mathcal{T}L(n_1, n_2) \oplus \mathbb{R}^4 \\ & \cong \underbrace{p^*\bar{\eta} \oplus p^*\eta \oplus \dots \oplus p^*\bar{\eta} \oplus p^*\eta \oplus p^*\bar{\eta}}_{2k_1} \oplus (\bar{\gamma} \otimes p^*\eta) \oplus \underbrace{\bar{\gamma} \oplus \gamma \oplus \dots \oplus \bar{\gamma} \oplus \gamma \oplus \gamma}_{2k_2} \end{aligned}$$

Also make the corresponding definitions for m_j, M_j, r_j, R_j . Let e_j be the number of indices $i \geq j$ for which $n_i < m_i$. Then

$$\frac{1}{p^{e_0}} \binom{n}{m} \equiv (\pm 1)^{e_{q-1}} \frac{N_0!_p}{M_0!_p R_0!_p} \cdot \frac{N_1!_p}{M_1!_p R_1!_p} \cdots \frac{N_k!_p}{M_k!_p R_k!_p} \pmod{p^q},$$

where ± 1 is -1 except if $p = 2$ and $q \geq 3$, and $n!_p$ denotes the product of those positive integers $\leq n$ that are not divisible by p .

Lemma 4.12 For $n_1 = 2k_1 > 0$ and $n_2 = 2k_2 + 1 > 0$, we have

$$s_{n_1+n_2}[\tilde{L}(n_1, n_2)] = -\binom{n_1+n_2}{1} + \binom{n_1+n_2}{2} - \cdots - \binom{n_1+n_2}{n_1-1} + \binom{n_1+n_2}{n_1}.$$

Proof Using (4-3) and (3-4) we calculate

$$\begin{aligned} s_{n_1+n_2}(\tilde{L}(n_1, n_2)) &= (v-u)^{n_1+n_2} + (k_2+1)(-1)^{n_1+n_2}v^{n_1+n_2} + k_2v^{n_1+n_2} \\ &= (v-u)^{n_1+n_2} - v^{n_1+n_2} \\ &= \left(-\binom{n_1+n_2}{1} + \binom{n_1+n_2}{2} - \cdots - \binom{n_1+n_2}{n_1-1} + \binom{n_1+n_2}{n_1}\right)u^{n_1}v^{n_2}, \end{aligned}$$

and the result follows by evaluating at $\langle \tilde{L}(n_1, n_2) \rangle$. □

Note that $s_3(\tilde{L}(2, 1)) = 0$ in accordance with Theorem 4.8. On the other hand, $s_{2+n_2}(\tilde{L}(2, n_2)) \neq 0$ for $n_2 > 1$, providing an example of a non-bounding quasitoric SU-manifold in each dimension $4k + 2$ with $k > 1$.

Lemma 4.13 For $k > 1$, there is a linear combination y_{2k+1} of SU-bordism classes $[\tilde{L}(n_1, n_2)]$ with $n_1 + n_2 = 2k + 1$ such that $s_{2k+1}(y_{2k+1}) = m_{2k+1}m_{2k}$.

Proof By the previous lemma,

$$s_{n_1+n_2}[\tilde{L}(n_1, n_2) - \tilde{L}(n_1-2, n_2+2)] = \binom{n_1+n_2}{n_1} - \binom{n_1+n_2}{n_1-1}.$$

The result follows from the next lemma. □

Lemma 4.14 For any integer $k > 1$, we have

$$\gcd\left\{\binom{2k+1}{2i} - \binom{2k+1}{2i-1}, 0 < i \leq k\right\} = m_{2k+1}m_{2k}.$$

Remark This result has been obtained independently in a recent work of Buchstaber and Ustinov on the coefficient rings of universal formal group laws [6, Section 9].

Proof of Lemma 4.14 We need to establish the following two facts:

- (a) The largest power of 2 which divides each number $\binom{2k+1}{2i} - \binom{2k+1}{2i-1}$ with $0 < i \leq k$ is 2 if $2k + 2 = 2^s$ and is 1 otherwise.
- (b) The largest power of odd prime p which divides each number $\binom{2k+1}{2i} - \binom{2k+1}{2i-1}$ with $0 < i \leq k$ is p if $2k + 1 = p^s$ and is 1 otherwise.

We prove (a) first.

Case 1 ($2k + 2 = 2^s$) Then $s > 2$, as $k > 1$. For $0 < i \leq k$, we have

$$\binom{2k+1}{2i} - \binom{2k+1}{2i-1} \equiv \binom{2^s-1}{2i} + \binom{2^s-1}{2i-1} = \binom{2^s}{2i} \equiv 0 \pmod{2}$$

by Proposition 3.5. On the other hand,

$$\binom{2^s-1}{2} - \binom{2^s-1}{1} = (2^s - 1)(2^{s-1} - 1 - 1) = 2(2^s - 1)(2^{s-2} - 1) \not\equiv 0 \pmod{4}.$$

Case 2 ($2k + 2 \neq 2^s$) Write the base 2 expansion

$$2k + 2 = n_1 2 + \dots + n_{l-1} 2^{l-1} + 2^l$$

with $n_i = 1$ or 0 . Set $2i = n_1 2 + \dots + n_{l-1} 2^{l-1}$. Then we have $2i \neq 0$, as otherwise $2k + 2 = 2^l$. Then $\binom{2k+1}{2i} - \binom{2k+1}{2i-1} \equiv \binom{2k+2}{2i} \equiv 1 \pmod{2}$ by Theorem 3.4.

Now we prove (b).

Case 1 ($2k + 1 = p^s$) Then $\binom{2k+1}{2i} - \binom{2k+1}{2i-1} \equiv 0 \pmod{p}$ for $0 < i \leq k$ by Proposition 3.5. On the other hand, setting $2i = p^{s-1} + 1$, we get

$$\begin{aligned} \binom{2k+1}{2i} - \binom{2k+1}{2i-1} &= \frac{2k-4i+2}{2i} \binom{2k+1}{2i-1} \\ &= \frac{p^s - 2p^{s-1} - 1}{p^{s-1} + 1} \binom{p^s}{p^{s-1}} \not\equiv 0 \pmod{p^2}. \end{aligned}$$

This follows from the fact that $p^s - 2p^{s-1} - 1 > 0$ as $k > 1$, and $\binom{p^s}{p^{s-1}}$ is not divisible by p^2 by Kummer's theorem.

Case 2 ($2k + 1 \neq p^s$) Write the base p expansion

$$2k + 1 = n_0 + n_1 p + \dots + n_{l-1} p^{l-1} + n_l p^l$$

with $0 \leq n_i \leq p - 1$ (for $i = 0, \dots, l$) and $n_l > 0$.

Assume that $n_0 > 1$. Then we set

$$2i = n_0 + n_1 p + \dots + n_{l-1} p^{l-1} + (n_l - 1) p^l.$$

We have $\binom{2k+1}{2i} \equiv n_l \pmod p$ by [Theorem 3.4](#). Also,

$$2i - 1 = (n_0 - 1) + n_1 p + \dots + n_{l-1} p^{l-1} + (n_l - 1) p^l > 0$$

and $\binom{2k+1}{2i-1} \equiv n_l n_0 \pmod p$. Therefore, $\binom{2k+1}{2i} - \binom{2k+1}{2i-1} \equiv n_l(1 - n_0) \not\equiv 0 \pmod p$.

Assume that $n_0 = 1$. Then we set $2i = 2k$. We have

$$\binom{2k+1}{2k} = 2k + 1 \equiv 1 \pmod p \quad \text{and} \quad \binom{2k+1}{2k-1} = k(2k + 1) \equiv 0 \pmod p,$$

so that

$$\binom{2k+1}{2k} - \binom{2k+1}{2k-1} \not\equiv 0 \pmod p.$$

Finally, assume that $n_0 = 0$. Then we set

$$\begin{aligned} 2i &= n_0 + n_1 p + \dots + n_{l-1} p^{l-1} + (n_l - 1) p^l \\ &= n_q p^q + \dots + n_{l-1} p^{l-1} + (n_l - 1) p^l, \end{aligned}$$

where $q > 0$ and $n_q > 0$. We have $2i > 0$, as otherwise $2k + 1 = p^l$. Then $\binom{2k+1}{2i} \equiv n_l \pmod p$. Also,

$$2i - 1 = (p-1) + (p-1)p + \dots + (p-1)p^{q-1} + (n_q - 1)p^q + \dots + n_{l-1} p^{l-1} + (n_l - 1) p^l,$$

and $\binom{2k+1}{2i-1} \equiv 0 \pmod p$ by [Theorem 3.4](#). Therefore, $\binom{2k+1}{2i} - \binom{2k+1}{2i-1} \not\equiv 0 \pmod p$. \square

Now we turn our attention to the manifolds $\tilde{N}(n_1, n_2)$ from [Construction 4.10](#).

Lemma 4.15 *For $n_1 = 2k_1 > 0$ and $n_2 = 2k_2 + 1 > 0$, set $n = n_1 + n_2 + 1$, so that $\dim \tilde{N}(n_1, n_2) = 2n = 4(k_1 + k_2 + 1)$. Then*

$$s_n[\tilde{N}(n_1, n_2)] = 2\left(-\binom{n}{1} + \binom{n}{2} - \dots - \binom{n}{n_1-1} + \binom{n}{n_1} - n_1\right).$$

Proof Using [\(4-5\)](#) and [\(4-4\)](#) we calculate

$$\begin{aligned} (4-6) \quad s_n(\tilde{N}(n_1, n_2)) &= 2(w - u)^n + (v + w)^n + (2k_2 - 1)w^n \\ &= 2w^n - 2nuw^{n-1} + w^n + \binom{n}{1}vw^{n-1} + \dots \\ &\quad + \binom{n}{2k_1}v^{2k_1}w^{2k_2+2} + (2k_2 - 1)w^n \\ &= -2nuw^{n-1} + (n - n_1)w^n + \binom{n}{1}vw^{n-1} + \dots + \binom{n}{n_1}v^{n_1}w^{n-n_1}. \end{aligned}$$

Now we have to express each monomial above via $uv^{n_1}w^{n_2}$ using the identities in (4-4), namely

$$(4-7) \quad u^2 = 0, \quad v^{n_1+1} = 0, \quad w^{n_2+1} = 2uw^{n_2} - vw^{n_2} + 2uvw^{n_2-1}.$$

We have

$$(4-8) \quad \begin{aligned} uw^{n-1} &= uw^{n_1-1}w^{n_2+1} = uw^{n_1-1}(2uw^{n_2} - vw^{n_2} + 2uvw^{n_2-1}) \\ &= -uvw^{n-2} = \dots = (-1)^j uv^j w^{n-j-1} = \dots = uv^{n_1}w^{n_2}. \end{aligned}$$

Also, we show that

$$(4-9) \quad v^j w^{n-j} = (-1)^j 2uv^{n_1}w^{n_2}, \quad 0 \leq j \leq n_1,$$

by verifying the identity successively for $j = n_1, n_1 - 1, \dots, 0$. Indeed,

$$v^{n_1}w^{n-n_1} = v^{n_1}w^{n_2+1} = 2uv^{n_1}w^{n_2}$$

by (4-7). Now, we have

$$\begin{aligned} v^{j-1}w^{n-j+1} &= v^{j-1}w^{n_1+1-j}w^{n_2+1} = v^{j-1}w^{n_1+1-j}(2uw^{n_2} - vw^{n_2} + 2uvw^{n_2-1}) \\ &= 2uv^{j-1}w^{n-j} - v^jw^{n-j} + 2uv^jw^{n-1-j} = -v^jw^{n-j}, \end{aligned}$$

where the last identity holds because of (4-8). The identity (4-9) is therefore verified completely. Plugging (4-8) and (4-9) into (4-6) we obtain

$$\begin{aligned} s_n(\tilde{N}(n_1, n_2)) &= \left(-2n + 2(n - n_1) - 2\binom{n}{1} + 2\binom{n}{2} - \dots - 2\binom{n}{n_1-1} + 2\binom{n}{n_1}\right)uv^{n_1}w^{n_2}. \end{aligned}$$

The result follows by evaluating at $\langle \tilde{N}(n_1, n_2) \rangle$. □

Note that $s_4(\tilde{N}(2, 1)) = 0$ in accordance with Theorem 4.8. On the other hand, $s_n(\tilde{N}(2, n_2)) = n^2 - 3n - 4 > 0$ for $n > 4$, providing an example of a non-bounding quasitoric SU-manifold in each dimension $4k$ with $k > 2$. This includes a 12-dimensional quasitoric SU-manifold $\tilde{N}(2, 3)$, which was missing in [11].

Lemma 4.16 *For $k > 2$, there is a linear combination y_{2k} of SU-bordism classes $[\tilde{N}(n_1, n_2)]$ with $n_1 + n_2 + 1 = 2k$ such that $s_{2k}(y_{2k}) = 2m_{2k}m_{2k-1}$.*

Proof The result follows from Lemma 4.15 and Lemmata 4.17 and 4.18 below. □

Lemma 4.17 *For $k > 2$, the largest power of 2 which divides each number*

$$a_i = -\binom{2k}{1} + \binom{2k}{2} - \dots - \binom{2k}{2i-1} + \binom{2k}{2i} - 2i, \quad 0 < i < k,$$

is 2 if $2k = 2^s$ and is 1 otherwise.

Proof First assume that $2k = 2^s$. Then $a_i \equiv 0 \pmod 2$ by Proposition 3.5. On the other hand, we have

$$a_1 = -2^s + 2^{s-1}(2^s - 1) - 2 \not\equiv 0 \pmod 4,$$

because $s > 2$.

Now assume that $2k \neq 2^s$. We have $a_i - a_{i-1} \equiv \binom{2k}{2i} \pmod 2$, so it is enough to find i such that $\binom{2k}{2i} \not\equiv 0 \pmod 2$. This was done in the proof of Lemma 4.14. \square

Lemma 4.18 For $k > 2$, the largest power of odd prime p which divides each

$$a_i = -\binom{2k}{1} + \binom{2k}{2} + \dots - \binom{2k}{2i-1} + \binom{2k}{2i} - 2i, \quad 0 < i < k,$$

is p if $2k + 1 = p^s$ and is 1 otherwise.

Proof Using the identity

$$2 + \sum_{j=1}^{2k-1} (-1)^j \binom{2k}{j} = 0$$

we obtain $a_{k-1} = 0$ and

$$(4-10) \quad a_i + a_{k-i-1} = \binom{2k}{2i+1} - 2k, \quad 0 < i < k - 1.$$

Case 1 ($2k + 1 = p^s$) We have

$$\binom{2k}{2i+1} = \binom{p^s-1}{2i+1} = \binom{p^s-1}{2i-1} \frac{(p^s-2i)(p^s-2i-1)}{2i(2i+1)} \equiv -1 \pmod p$$

by induction starting from $i = 0$. Therefore,

$$a_i = a_{i-1} - \binom{2k}{2i-1} + \binom{2k}{2i} - 2 = a_{i-1} + \frac{p^s-4i}{2i} \binom{p^s-1}{2i-1} - 2 \equiv 0 \pmod p$$

by induction starting from $a_0 = 0$. In view of (4-10), it suffices to find i , where $0 < i < k - 1$, such that

$$\binom{2k}{2i+1} - 2k \not\equiv 0 \pmod{p^2}.$$

If $s = 1$, then $p > 5$ as $k > 2$. We set $2i + 1 = 3$, so that

$$\binom{2k}{2i+1} - 2k = \binom{p-1}{3} - (p-1) = \frac{p(p-1)(p-5)}{6} \not\equiv 0 \pmod{p^2}.$$

Now assume that $s > 1$. We set $2i + 1 = p^{s-1}$ and use [Theorem 4.11](#) to calculate $\binom{2k}{2i+1} \pmod{p^2}$. In the notation of [Theorem 4.11](#), we have $q = 2$,

$$n = p^s - 1 = n_0 + n_1 p + \dots + n_{s-2} p^{s-2} + n_{s-1} p^{s-1}$$

$$= (p - 1) + (p - 1)p + \dots + (p - 1)p^{s-2} + (p - 1)p^{s-1},$$

$$N_0 = \dots = N_{s-2} = p^2 - 1, \quad N_{s-1} = p - 1,$$

$$m = p^{s-1} = m_0 + m_1 p + \dots + m_{s-2} p^{s-2} + m_{s-1} p^{s-1},$$

$$M_0 = \dots = M_{s-3} = 0, \quad M_{s-2} = p, \quad M_{s-1} = 1,$$

$$r = p^s - p^{s-1} - 1 = r_0 + r_1 p + \dots + r_{s-2} p^{s-2} + r_{s-1} p^{s-1}$$

$$= (p - 1) + (p - 1)p + \dots + (p - 1)p^{s-2} + (p - 2)p^{s-1},$$

$$R_0 = \dots = R_{s-3} = p^2 - 1, \quad R_{s-2} = p^2 - p - 1, \quad R_{s-1} = p - 2,$$

and $e_0 = e_1 = 0$. Therefore, [Theorem 4.11](#) gives

$$\binom{p^s - 1}{p^{s-1}} \equiv \frac{(p^2 - 1)!_p}{p!_p (p^2 - p - 1)!_p} \cdot \frac{(p - 1)!_p}{1!_p (p - 2)!_p} = \frac{(p^2 - 1) \dots (p^2 - p + 1)}{(p - 1)!} \cdot (p - 1)$$

$$\equiv p - 1 \pmod{p^2},$$

and we obtain

$$\binom{2k}{2i+1} - 2k = \binom{p^s - 1}{p^{s-1}} - (p^s - 1) \equiv p \pmod{p^2}.$$

Case 2 ($2k + 1 \neq p^s$) In view of [\(4-10\)](#), it suffices to find i , where $0 < i < k - 1$, such that $\binom{2k}{2i+1} - 2k \not\equiv 0 \pmod{p}$. Write the base p expansion

$$2k = n_0 + n_1 p + \dots + n_{l-1} p^{l-1} + n_l p^l$$

with $0 \leq n_i \leq p - 1$ (for $i = 0, \dots, l$) and $n_l > 0$. We have $2k \equiv n_0 \pmod{p}$.

Assume that $n_0 = 0$. Then we set

$$2i + 1 = n_0 + n_1 p + \dots + n_{l-1} p^{l-1} + (n_l - 1) p^l.$$

We have $\binom{2k}{2i+1} - 2k \equiv n_l \not\equiv 0 \pmod{p}$.

Assume that $0 < n_0 < p - 2$. If $2k < p$, then $n_0 = 2k > 5$. We set $2i + 1 = 3$, so that

$$(4-11) \quad \binom{2k}{2i+1} - 2k \equiv \binom{n_0}{3} - n_0 = \frac{n_0(n_0 - 4)(n_0 + 1)}{6} \not\equiv 0 \pmod{p}.$$

If $2k > p$, then we set

$$2i + 1 = \begin{cases} n_0 + 1 & \text{if } n_0 \text{ is even,} \\ n_0 + 2 & \text{if } n_0 \text{ is odd.} \end{cases}$$

We have $2i + 1 < 2k$ and $\binom{2k}{2i+1} - 2k \equiv -n_0 \not\equiv 0 \pmod p$.

Assume that $n_0 = p - 2$. If $p = 3$, then $n_0 = 1$. We set $2i + 1 = 5 < 2k$, so that

$$\binom{2k}{2i+1} - 2k \equiv \binom{n_0}{2} \binom{n_1}{1} - 1 = -1 \not\equiv 0 \pmod p.$$

If $p > 3$, then we set $2i + 1 = 3$, so that $\binom{2k}{2i+1} - 2k \not\equiv 0 \pmod p$ by (4-11).

Assume that $n_0 = p - 1$ and $n_l < p - 1$. Then we set

$$2i + 1 = n_0 + n_1 p + \dots + n_{l-1} p^{l-1} + (n_l - 1) p^l.$$

We have $\binom{2k}{2i+1} - 2k \equiv n_l - n_0 \not\equiv 0 \pmod p$.

Finally, assume that $n_0 = p - 1$ and $n_l = p - 1$. As $2k \neq p^s - 1$, there exists j , where $0 < j < l$, such that $n_j < p - 1$. Then we set

$$2i + 1 = n_0 + n_1 p + \dots + n_{j-1} p^{j-1} + (n_j + 1) p^j.$$

We have $2i + 1 < 2k$ and $\binom{2k}{2i+1} - 2k \equiv -n_0 \not\equiv 0 \pmod p$. □

We now can prove our main result:

Theorem 4.19 *There exist quasitoric SU–manifolds M^{2i} , $i \geq 5$, with $s_i(M^{2i}) = m_i m_{i-1}$ if i is odd and $s_i(M^{2i}) = 2m_i m_{i-1}$ if i is even. These quasitoric manifolds represent polynomial generators of $\Omega^{\text{SU}} \otimes \mathbb{Z}[\frac{1}{2}]$.*

Proof It follows from Lemmata 4.13 and 4.16 that there exist linear combinations of SU–bordism classes represented by quasitoric SU–manifolds with the required properties. We observe that application of Construction 3.9 to two quasitoric SU–manifolds M and M' produces a quasitoric SU–manifold representing their bordism sum. Also, the SU–bordism class $-[M]$ can be represented by the omnioriented quasi-toric SU–manifold obtained by reversing the global orientation of M . Therefore, we can replace the linear combinations obtained using Lemmata 4.13 and 4.16 by appropriate connected sums, which are quasitoric SU–manifolds. □

Concluding remarks

By analogy with [Theorem 3.10](#), we may ask the following:

Question 4.20 Which SU–bordism classes of dimension > 8 can be represented by quasitoric SU–manifolds?

[Theorem 4.19](#) provides quasitoric representatives for the elements $y_i \in \Omega_{2i}^{\text{SU}}$ described in [Theorem 4.5](#) for $i \geq 5$. The elements y_2, y_3, y_4 cannot be represented by quasitoric manifolds because of [Theorem 4.8](#). Any polynomial in these elements cannot be represented by a quasitoric manifold for the same reason: the Krichever genus $\varphi_K: \Omega^{\text{U}} \rightarrow R_K$ vanishes on quasitoric SU–manifolds, but φ_K is nonzero on any polynomial in y_2, y_3 , and y_4 . We thank Michael Wiemeler for this observation.

The element $x_1^2 \in \mathcal{W}_4$ (see [Theorem 4.2](#)) is represented by $9\mathbb{C}P^1 \times \mathbb{C}P^1 - 8\mathbb{C}P^2$, which is also the bordism class of a toric manifold over a 12–gon, with characteristic numbers $c_1^2 = 0$ and $c_2 = 12$ (so $s_2 = -24$). The element $y_2 = 2x_1^2 \in \Omega_4^{\text{SU}}$ is represented by a *K3 surface*, but not by a toric manifold.

The 6–sphere S^6 has a T^2 –invariant almost complex structure as the homogeneous space $G_2/\text{SU}(3)$ of the exceptional Lie group G_2 (see [\[1\]](#)), and therefore represents an SU–bordism class in Ω_6^{SU} . Its characteristic numbers are $c_1^3 = c_1c_2 = 0$ and $c_3 = 2$. Therefore, $s_3[S^6] = 6 = m_3m_2$, so S^6 represents $y_3 \in \Omega_6^{\text{SU}}$.

It would be interesting to find good geometric representatives for $y_4 \in \Omega_8^{\text{SU}}$, and also for the elements $w_{4k} \in \Omega_{8k}^{\text{SU}}$ that control the 2–torsion in [Theorem 4.4\(c\)](#). The image of w_{4k} under the forgetful homomorphism $\alpha: \Omega_{8k}^{\text{SU}} \rightarrow \mathcal{W}_{8k}$ is x_1^{4k} , so it is decomposable in Ω^{U} and has $s_{4k}[w_{4k}] = 0$. The conditions on the characteristic numbers specifying w_{4k} are given in [\[7, \(19.3\)\]](#).

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