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# Nonself-adjoint operators with almost Hermitian spectrum: Cayley identity and some questions of spectral structure

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To the memory of Vladimir Veselov, who had contributed a lot to the spectral theory of nonself-adjoint operators

**Abstract.** Nonself-adjoint, non-dissipative perturbations of possibly unbounded self-adjoint operators with real purely singular spectrum are considered under an additional assumption that the characteristic function of the operator possesses a scalar multiple. Using a functional model of a nonself-adjoint operator (a generalization of a Sz.-Nagy–Foiaş model for dissipative operators) as a principle tool, spectral properties of such operators are investigated. A class of operators with almost Hermitian spectrum (the latter being a part of the real singular spectrum) is characterized in terms of existence of the so-called weak outer annihilator which generalizes the classical Cayley identity to the case of nonself-adjoint operators in Hilbert space. A similar result is proved in the self-adjoint case, characterizing the condition of absence of the absolutely continuous spectral subspace in terms of the existence of weak outer annihilation. An application to the rank-one nonself-adjoint Friedrichs model is given.

## 1. Introduction

In the present paper we consider([1]) nonself-adjoint, non-dissipative additive perturbations $L=A+iV$ of a self-adjoint operator $A$ acting in Hilbert space $H$. Operators of this class have been extensively studied, see, e.g., [18], [19], [31], [32] and [33] (see also [8], [9], [15], [21] and [22] for directly related results on the oper-

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([1]) The theory developed in this paper can be generalized to the case of operators with not necessarily additive imaginary part, that is, to the class of operators with non-empty resolvent set. However, this would lead to purely technical difficulties that would complicate the reading of the paper. Due to this, we have elected not to include this generalization into the present paper.

ators which are "close to unitary"). We are primarily interested in discussing the properties of the singular spectral subspace $N_i$ of the operator $L$ (see Section 2 for definitions), in particular, its subspace $N_i^0$, related to a part of the real spectrum.

Recall [20] that spectral subspaces $N_i^+, N_i^-, N_i^0 \subset N_i$ can be naturally singled out for an arbitrary nonself-adjoint operator $L$ of the class under consideration. Here $N_i^+$ ($N_i^-$) [19] corresponds to the point spectrum in the upper half-plane and a part of the real singular spectrum (resp. the point spectrum in the lower half-plane and a part of the real singular spectrum), and is directly analogous to the singular subspace of a dissipative (resp. adjoint to a dissipative) completely nonself-adjoint(2) operator. The subspace $N_i^0$ (introduced in [20], see also [31] and [33]) corresponds to a portion of the real singular spectrum and plays a special role in the spectral theory of a nonself-adjoint, non-dissipative operator. In a way most new features of non-dissipative operators (compared to dissipative and adjoint to dissipative ones) are related to the presence and properties of $N_i^0$. It is also worth mentioning that the subspace $N_i^0$ plays a special role in the investigation of the similarity of the operator $L$ to a dissipative operator, see [20]. We also mention that in the finite-dimensional case the condition $N_i^0 = H$ leads to the class of non-Hermitian matrices with real spectrum.

Under an additional assumption which effectively imposes a restriction of "weakness" on the interaction between the positive and negative parts of the perturbation (see [19]) it is possible to establish the identity $N_i^0 = \{0\}$ and to prove that the angles between the spectral subspaces $N_e$, $N_i^+$ and $N_i^-$ are positive. Moreover, it can be shown, that from the viewpoint of the similarity problem (to a self-adjoint or to a dissipative operator) the operator $L$ behaves essentially as an orthogonal sum of a dissipative and an anti-dissipative operator.

In the general case, however, the singular subspace $N_i^0$ is non-trivial, which can result in a zero angle between the subspaces $N_e$ and $N_i$ and even in non-triviality of their intersection [33]. On the other hand, the presence of $N_i^0$ as such does not yet lead to this. For example, in [31] and [32] it was shown that the similarity problem can be successfully resolved and the angles between spectral subspaces estimated from below even in the situation of a non-trivial $N_i^0$, see also [8] and [9].

In [12] yet another important property of the spectral subspace $N_i^0$ was established in the case when the operator $L$ is a trace-class nonself-adjoint perturbation of a bounded operator $A$. Namely, it was shown that both the operator $L$ and its adjoint $L^*$ are weakly annihilated (see Section 4 for rigorous definitions) by some scalar-valued bounded outer analytic function if and only if they both satisfy the condition $N_i^0 = H$. This result extends the well-known Cayley identity to the case of nonself-adjoint operators of the class under consideration. Further results on

(2) That is, the operator has no reducing self-adjoint parts, see also [28].

the properties of the spectral subspace $N_i^0$ were obtained in the matrix model case (i.e., rank-two nonself-adjoint perturbations of a bounded self-adjoint operator $A$) in [13].

It turns out that a similar consideration is also of considerable interest in the self-adjoint case, leading to an extension of the Cayley identity for arbitrary self-adjoint operators with purely singular spectrum. As a by-product of this analysis we suggest a new criterion for the absence of the absolutely continuous spectrum of a self-adjoint operator which is of independent interest.

Nonetheless, most of the questions on the spectral structure of operators such that $H=N_i^0$ remain open and the understanding of the structure of $N_i^0$ lags to a major extent behind the understanding of the other spectral components of a nonself-adjoint operator.

In the present paper we continue the discussion of the phenomenon of weak annihilation started in [12]. We consider a more general situation, when the operator $L$ is not only no longer necessarily bounded but also (which is the most technically demanding task) the *perturbation is no longer necessarily of trace-class.* The only significant restriction that we impose on the operator $L$ is that the four factors $\Theta_j(\lambda)$, $\Theta'_j(\lambda)$, $j=1,2$, of the characteristic function $\Theta(\lambda)$ of the operator $L$ possess scalar multiples (see Section 2 for definition). This is the most natural class of operators, generalizing trace-class perturbations.

The paper is organized as follows. In Section 2 we introduce the functional model of Sz.-Nagy–Foiaş in its symmetric form due to Pavlov [18], [23], [24] and [28] which we then use as a principle tool of the investigation. The singular spectral subspace $N_i^0$ is introduced in functional model terms (following [20] and [33]).

An equivalent (non-model) description of $N_i^0$ in terms of the original Hilbert space and operators acting in it is given in Section 3. This description is given in both "strong" (see [12]) and "weak" (first introduced in this paper) flavours, and their equivalence is proved.

In Section 4, we discuss the relationship between scalar multiples of the factors $\Theta_j(\lambda)$, $\Theta'_j(\lambda)$, $j=1,2$, of the characteristic function $\Theta(\lambda)$ under the assumption that the spectrum of the operator $L$ is almost Hermitian. Essentially, this section generalizes the results on the relationship between their determinants obtained in [33] in the case when $L$ is a trace-class perturbation of a bounded self-adjoint operator. These results are not only an essential ingredient in passing over from trace-class to non trace-class perturbations (in the case of trace-class perturbations, these analytical difficulties do not arise due to the existence of corresponding generalized determinants), but are also in our opinion of considerable independent interest.

Section 5 is devoted to the definition of weak outer annihilation and some immediate implications of this phenomenon, namely on the non-real part of the spectrum of the operator $L$. In this section, as well as in Sections 6 and 7, we attempt to follow the design of our paper [12], although the technique we use here is quite different.

In Sections 6 and 7 we prove that both the operator $L$ and its adjoint $L^*$ are weakly annihilated by some scalar-valued bounded outer analytic functions if and only if they both have almost Hermitian spectra, provided that the operator $L$ is such that all four operator-valued functions $\Theta_j(\lambda)$, $\Theta'_j(\lambda)$, $j=1,2$, possess scalar multiples. Thus the Cayley identity is generalized to the class of operators with almost Hermitian spectrum.

Section 8 is devoted to proving an analogous result in the self-adjoint case, namely, that an arbitrary self-adjoint operator $A$ possesses a weak outer annihilator if and only if its absolutely continuous spectral subspace is trivial. This result was absent until now in the spectral theory of self-adjoint operators. Thus it not only rather transparently demonstrates the spectral meaning of the almost Hermitian spectral component, but also has independent value. It is interesting to note that this self-adjoint result is obtained based on essentially nonself-adjoint methods and notions.

Finally, in Section 9 the results obtained are applied to the spectral analysis of the rank-one nonself-adjoint Friedrichs model operator, i.e., a rank-one nonself-adjoint perturbation of the multiplication operator, in $L_2(\mathbb{R}; d\sigma)$ over a measure $\sigma$ singular with respect to the Lebesgue measure. In essence, here the results originally presented in [12] without proofs and in a simplified form are generalized to the unbounded case, and complete proofs are supplied.

## 2. The functional model

In the present section we briefly recall the functional model of a nonself-adjoint operator constructed in [23] and [28] in the dissipative case and then extended in [16], [17], [18] and [26] to the case of a wide class of non-dissipative operators. As in [18], we consider a class of nonself-adjoint operators of the form $L=A+iV$, where $A$ is a self-adjoint operator in $H$ defined on the domain $D(A)$ and the perturbation $V$ admits the factorization $V=\alpha J\alpha/2$, where $\alpha$ is a non-negative self-adjoint operator in $H$, and $J$ is a unitary operator in an auxiliary Hilbert space $E$, defined as the closed range of the operator $\alpha$: $E\equiv\overline{R(\alpha)}$. This factorization corresponds to the polar decomposition of the operator $V$. It can also be easily generalized to the "node" case [30], where $J$ acts in an auxiliary Hilbert space $\mathfrak{H}$ and $V=\alpha^* J\alpha/2$, $\alpha$ being an operator acting from $H$ to $\mathfrak{H}$. In order for the expression $A+iV$ to be meaningful, we impose the condition that $V$ is $(A)$-bounded with relative bound less than 1, i.e., $D(A)\subset D(V)$ and for some $a$ and $b$ $(a<1)$ the condition $\|Vu\|\le a\|Au\|+b\|u\|$,

$u \in D(A)$, is satisfied, see [10]. Then the operator $L$ is well-defined on the domain $D(L) = D(A)$.

Alongside with the operator $L$ we are going to consider the maximal dissipative operator $L^{\|} = A + i\alpha^2/2$ and the one adjoint to it, $L^{-\|} \equiv L^{\|*} = A - i\alpha^2/2$. Since the functional model for the dissipative operator $L^{\|}$ will be used below, we require that $L^{\|}$ is completely nonself-adjoint, i.e., that it has no reducing self-adjoint parts. This requirement is not restrictive in our case due to [18, Proposition 1].

We also note that the functional model in the general case of operators with not necessarily additive imaginary part and with non-empty resolvent set has been developed in [26].

Now we are going to briefly describe the construction of the self-adjoint dilation of the completely nonself-adjoint dissipative operator $L^{\|}$, following [23] and [28], see also [18].

The characteristic function $S(\lambda)$ of the operator $L^{\|}$ is a contractive analytic operator-valued function acting in the Hilbert space $E$, defined for $\operatorname{Im} \lambda > 0$ by

$$S(\lambda) = I + i\alpha (L^{-\|} - \lambda)^{-1} \alpha. \tag{1}$$

In the case of an unbounded $\alpha$ the characteristic function is first defined by (1) on the manifold $E \cap D(\alpha)$ and then extended by continuity to the whole space $E$. The definition given above makes it possible to consider $S(\lambda)$ for $\operatorname{Im} \lambda < 0$ with $S(\bar{\lambda}) = S^*(\lambda)^{-1}$ provided that the inverse exists at the point $\lambda$. Finally, $S(\lambda)$ possesses boundary values on the real axis in the strong topology sense: $S(k) \equiv S(k + i0)$, $k \in \mathbb{R}$ (see [28]).

Consider the model space $\mathcal{H} = L_2\left(\begin{smallmatrix} I & S^* \\ S & I \end{smallmatrix}\right)$, which is defined in [23] (see also [22] for descriptions of general coordinate-free models) as Hilbert space of two-component vector-functions $(\tilde{g}, g)$ on the axis ($\tilde{g}(k), g(k) \in E, k \in \mathbb{R}$) with metric

$$\left\langle \begin{pmatrix} \tilde{g} \\ g \end{pmatrix}, \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \right\rangle = \int_{-\infty}^{\infty} \left\langle \begin{pmatrix} I & S^*(k) \\ S(k) & I \end{pmatrix} \begin{pmatrix} \tilde{g}(k) \\ g(k) \end{pmatrix}, \begin{pmatrix} \tilde{g}(k) \\ g(k) \end{pmatrix} \right\rangle_{E \oplus E} dk.$$

It is assumed here that the set of two-component functions has been factored by the set of elements with norm equal to zero. Although we consider $(\tilde{g}, g)$ as a symbol only, the formal expressions $g_- := (\tilde{g} + S^* g)$ and $g_+ := (S\tilde{g} + g)$ (the motivation for the choice of notation is self-evident from what follows) can be shown to represent some true $L_2(E)$-functions on the real line. In what follows we plan to deal mostly with these functions.

Define the following orthogonal subspaces in $\mathcal{H}$:

$$D_- \equiv \begin{pmatrix} 0 \\ H^2_-(E) \end{pmatrix}, \quad D_+ \equiv \begin{pmatrix} H^2_+(E) \\ 0 \end{pmatrix} \quad \text{and} \quad K \equiv \mathcal{H} \ominus (D_- \oplus D_+),$$

where $H^2_+(E)$ (resp. $H^2_-(E)$) denotes the Hardy class of analytic functions $f$ in the upper (resp. lower) half-plane taking values in the Hilbert space $E$ [28]. These subspaces are "incoming" and "outgoing" subspaces, respectively, in the sense of [14].

The subspace $K$ can be described as

$$K=\{(\tilde g,g)\in\mathcal{H}:g_-\equiv\tilde g+S^*g\in H^2_-(E) \text{ and } g_+\equiv S\tilde g+g\in H^2_+(E)\}.$$

Let $P_K$ be the orthogonal projection of the space $\mathcal{H}$ onto $K$, then

$$P_K\begin{pmatrix}\tilde g\\ g\end{pmatrix}=\begin{pmatrix}\tilde g-P_+(\tilde g+S^*g)\\ g-P_-(S\tilde g+g)\end{pmatrix},$$

where $P_\pm$ are the orthogonal Riesz projections of the space $L_2(E)$ onto $H^2_\pm(E)$.

The following theorem holds true, see [28] and [23].

**Theorem 2.1.** *The operator* $(L^{\|}-\lambda_0)^{-1}$ *is unitarily equivalent to the operator* $P_K(k-\lambda_0)^{-1}|_K$ *in the space* $K$ *for all* $\lambda_0$ *with* $\operatorname{Im}\lambda_0<0$.

This means, that the operator of multiplication by $k$ in $\mathcal{H}$ serves as a minimal ($\operatorname{clos}_{\operatorname{Im}\lambda\neq0}(k-\lambda)^{-1}K=\mathcal{H}$) self-adjoint dilation of the operator $L^{\|}$ [28].

Provided that the non-real spectrum of the operator $L$ is countable, the characteristic function of the operator $L$ is defined for $\operatorname{Im}\lambda\neq0$ by the expression

$$\Theta(\lambda)\equiv I+iJ\alpha(L^*-\lambda)^{-1}\alpha$$

and under the additional assumption that $V$ is a relatively compact perturbation([3]) can be shown to be a meromorphic $J$-contractive ($\Theta^*(\lambda)J\Theta(\lambda)\le J$, $\operatorname{Im}\lambda>0$) operator-valued function [4]. The characteristic function $\Theta(\lambda)$ admits, see [1] and [16], a factorization (also called the Ginzburg–Potapov factorization of a $J$-contractive function [2]) in the form of a ratio of two bounded analytic operator-valued functions (in the corresponding half-planes $\operatorname{Im}\lambda<0$ and $\operatorname{Im}\lambda>0$) triangular with respect to decomposition of the space $E$ into the orthogonal sum $E=\mathcal{X}_+E\oplus\mathcal{X}_-E$, $\mathcal{X}_\pm:=(I\pm J)/2$:

$$\Theta(\lambda)=\Theta_1'^*(\bar\lambda)(\Theta_2'^*)^{-1}(\bar\lambda),\ \operatorname{Im}\lambda>0;\quad \Theta(\lambda)=\Theta_2^*(\bar\lambda)(\Theta_1^*)^{-1}(\bar\lambda),\ \operatorname{Im}\lambda<0, \tag{2}$$

where the factors $\Theta_{1,2}$ and $\Theta'_{1,2}$ are introduced as follows [17]:

$$\begin{aligned}\Theta_1(\lambda)&=\mathcal{X}_-+S(\lambda)\mathcal{X}_+, &\Theta_2(\lambda)&=\mathcal{X}_++S(\lambda)\mathcal{X}_-;\\ \Theta_1'(\lambda)&=\mathcal{X}_-+S^*(\bar\lambda)\mathcal{X}_+, &\Theta_2'(\lambda)&=\mathcal{X}_++S^*(\bar\lambda)\mathcal{X}_-,\end{aligned} \tag{3}$$

and $S(\lambda)$ is the characteristic function of the dissipative operator $L^{\|}$.

([3]) This assumption guarantees that the non-real spectrum of $L$ is discrete.

We will assume throughout the present paper that all four operator-valued functions appearing in the factorization (2) possess scalar multiples [28], [18]. Recall that the scalar multiple of an analytic operator-valued function $Q(\lambda)\colon E\to E$ bounded in the half-plane $\mathbb{C}_+$ (resp. $\mathbb{C}_-$) is a scalar-valued analytic function $d(\lambda)\not\equiv 0$ bounded in the upper (resp. lower) half-plane such that

$$\Omega(\lambda)Q(\lambda)=Q(\lambda)\Omega(\lambda)=d(\lambda)I, \quad \lambda\in\mathbb{C}_+ \text{ (resp. } \lambda\in\mathbb{C}_-),$$

where $\Omega(\lambda)$ is some analytic operator-valued function bounded in the upper (resp. lower) half-plane.

We remark that if $(L^{\|}-\lambda_0)^{-1}-(L^{-\|}-\lambda_0)^{-1}\in\mathfrak{S}_1$ for some $\lambda_0$, $\operatorname{Im}\lambda_0\neq 0$, then our restriction is satisfied, i.e., all the operator functions $\Theta_j,\Theta_j'$, $j=1,2$, possess scalar multiples in their respective half-planes [18].

Following [17], we define the linear sets $\widehat{N}_\pm$ in $\mathcal{H}$ as follows:

$$\widehat{N}_\pm\equiv\left\{\begin{pmatrix}\tilde g\\ g\end{pmatrix}:\begin{pmatrix}\tilde g\\ g\end{pmatrix}\in\mathcal{H} \text{ and } P_\pm(\Theta_1'^*\tilde g+\Theta_2^* g)\equiv P_\pm(\mathcal{X}_+g_++\mathcal{X}_-g_-)=0\right\} \tag{4}$$

and introduce the subspaces $N_\pm=\operatorname{clos}P_K\widehat{N}_\pm$. Then, as is shown in [18], one gets for $\operatorname{Im}\lambda<0$ (resp. $\operatorname{Im}\lambda>0$) and $(\tilde g,g)\in\widehat{N}_-$ (resp. $(\tilde g,g)\in\widehat{N}_+$),

$$(L-\lambda)^{-1}P_K\begin{pmatrix}\tilde g\\ g\end{pmatrix}=P_K\frac{1}{k-\lambda}\begin{pmatrix}\tilde g\\ g\end{pmatrix}. \tag{5}$$

Conversely, the property (5) for $\operatorname{Im}\lambda<0$ (resp. $\operatorname{Im}\lambda>0$) guarantees that the vector $(\tilde g,g)$ belongs to the set $\widehat{N}_-$ (resp. $\widehat{N}_+$).

Absolutely continuous and singular subspaces of the nonself-adjoint operator $L$ were defined in [16]: let $N\equiv\widehat{N}_+\cap\widehat{N}_-$, $\widetilde{N}_\pm\equiv P_K\widehat{N}_\pm$ and $\widetilde{N}_e\equiv\widetilde{N}_+\cap\widetilde{N}_-$. Then([4])

$$N_e\equiv\operatorname{clos}(\widetilde{N}_+\cap\widetilde{N}_-)=\operatorname{clos}P_KN\equiv\operatorname{clos}\widetilde{N}_e \quad\text{and}\quad N_i\equiv K\ominus N_e(L^*), \tag{6}$$

where $N_e(L^*)$ denotes the absolutely continuous subspace of the operator $L^*$, which can be easily described in a similar way in terms of the same model space $\mathcal{H}$.

One can also ascertain that the linear sets $\widetilde{N}_\pm$ can be characterized in terms, independent of the functional model, in the following way:

$$\widetilde{N}_\pm=\{u\in H:\mathcal{X}_\pm\alpha(L-\lambda)^{-1}u\in H^2_\pm(E)\}. \tag{7}$$

Here $\mathcal{X}_\pm\alpha(L-\lambda)^{-1}u$([5]) is treated as an analytic vector function of $\lambda\in\mathbb{C}_\pm$ taking values in the auxiliary Hilbert space $E$. It can be verified that the projections $\mathcal{X}_\pm$

([4]) The linear set $\widetilde{N}_e$ is called the set of "smooth" vectors of the operator $L$ (see [18]).

([5]) That is, analytic continuations of the vector $\mathcal{X}_\pm\alpha(L-\lambda)^{-1}u$ from the domain of analyticity of the resolvent to the half-plane $\mathbb{C}_\pm$.

can be dropped altogether in the definition (7), see [18]. The existence of this description gives ground to calling the vectors belonging to the named linear sets “smooth”.

The definition (6) in the case of maximal dissipative operators leads to the same subspace as the classical definition by L. A. Sahnovich [27] (the latter definition introduces the absolutely continuous subspace as the maximal invariant subspace reducing the operator $L$ to an operator with purely outer characteristic function) and was later developed by V. A. Ryzhov (in the case of more general non-dissipative operators) [26] and A. S. Tikhonov [29] (the so-called weak definition of the absolutely continuous subspace). Recently it turned out that the weak definition coincides with the strong one (6) (see [25]).

The subspaces $N_\pm(L^*)$ for the operator $L^*$ adjoint to $L$ are defined in a similar way using the same model representation.

The singular spectral subspace $N_i$ can be characterized in the following way [33]:

$$N_i = \left\{ \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \in K : \Theta_1^{-1}(k+i0) g_+(k) = (\Theta_2')^{-1}(k-i0) g_-(k) \text{ for a.a. } k \in \mathbb{R} \right\}. \tag{8}$$

Here the inverses $\Theta_1^{-1}(k+i0)$ and $(\Theta_2')^{-1}(k-i0)$ exist almost everywhere on the real line due to the existence of scalar multiples and the uniqueness theorem for bounded analytic functions [7].

The non-model description of it, see [15] and [33], is that $N_i$ consists of all vectors $u \in H$ that ensure the zero jump of the resolvent in the weak sense,

$$\begin{aligned} N_i = \{u \in H : \langle [(L-k-i\varepsilon)^{-1} - (L-k+i\varepsilon)^{-1}]u, v\rangle \to 0 \\ \text{for a.a. } k \in \mathbb{R} \text{ as } \varepsilon \to 0 \text{ and for all } v \in H\}. \end{aligned} \tag{9}$$

Essentially for almost all real $k$ the fact that the jump of the resolvent vanishes weakly (9) is equivalent to the condition in (8) for the boundary values of analytic vector-valued functions in the model representation [15], [33].

Following [19] we also introduce the subspaces $N_i^\pm \subset N_i$ as

$$N_i^+ = \operatorname{clos} \widetilde{N}_i^+, \quad \widetilde{N}_i^+ = P_K \begin{pmatrix} H_-^2(E) \ominus \Theta_1' H_-^2(E) \\ 0 \end{pmatrix},$$

$$N_i^- = \operatorname{clos} \widetilde{N}_i^-, \quad \widetilde{N}_i^- = P_K \begin{pmatrix} 0 \\ H_+^2(E) \ominus \Theta_2 H_+^2(E) \end{pmatrix}.$$

It can be shown that $\widetilde{N}_i^+ \subset \widetilde{N}_-$, whereas $\widetilde{N}_i^- \subset \widetilde{N}_+$.

The subspace $N_i^+$ (resp. $N_i^-$) corresponds to the point spectrum in the upper half-plane and a part of the real singular spectrum (resp. the point spectrum in the lower half-plane and a part of the real singular spectrum) [19].

Finally, singular spectral subspaces $N_i^0 \subset N_i$, $N_i^0(L^*) \subset N_i(L^*)$ were introduced in [20], see also [33], as

$$N_i^0 \equiv N_i^0(L) := K \ominus (N_+(L^*) \vee N_-(L^*)) \quad \text{and} \quad N_i^0(L^*) := K \ominus (N_+ \vee N_-). \tag{10}$$

One should again emphasize that the analytic properties of the spectral subspace $N_i^0$ differ drastically from those of the singular subspaces of completely nonself-adjoint dissipative operators.

We call an operator, for which the identities $H = N_i^0 = N_i^0(L^*)$ hold, an operator with *almost Hermitian spectrum.* The spectrum of the operator $L|_{N_i^0}$ is real [33], but there are much deeper reasons for the name used than this.

First, in the matrix case (i.e., when $\dim(H) < \infty$) the class of operators with almost Hermitian spectrum coincides with the class of completely nonself-adjoint matrices with real spectrum.

A number of results (in the case of trace-class perturbations of bounded self-adjoint operators, some remaining valid in the general case also), linking the properties of $N_i^0$ to those of the singular spectral subspace of a self-adjoint (rather than dissipative!) operator $B$, have been obtained in [33]. In particular, it can be shown, that the subspace $N_i^0$ consists of, at least, all eigenvectors and root vectors of the operator $L$, corresponding to real values of the spectral parameter $\lambda$.

Further, if $N_i^0 = H$ then the determinant of the characteristic function $\Theta(\lambda)$ is trivial:

$$\det \Theta(\lambda) \equiv 1, \quad \operatorname{Im} \lambda \neq 0.$$

In terms of similarity, one can also prove that if $L$ is similar to a self-adjoint operator $B$ (i.e., there exists a bounded, boundedly invertible operator $X$ such that $L = XBX^{-1}$), then $N_i^0 = XH_s$, where $H_s$ is the singular spectral subspace of the operator $B$ [33].

Yet another way to reveal the similarity between $N_i^0$ and the singular spectral subspace of a self-adjoint operator $B$ is in terms of the weak annihilator, see Section 8 below. We also plan to give a much more detailed coverage of this topic in a forthcoming publication.

## 3. Non-model description of the spectral subspace $N_i^0$

In the present section, we will present two equivalent descriptions of the spectral subspace $N_i^0$ in non-model terms. The first of these (the "strong" one, see also [12]) is analogous to (7) for the smooth vectors of the operator $L$, whereas the second (the "weak" one) is an analogue of the "weak" description of the absolutely continuous subspace [25] and [29].

**Theorem 3.1.** (Strong description of $N_i^0$) *Let $L$ be a nonself-adjoint operator such that all four operator-valued functions appearing in the factorization* (2) *possess scalar multiples, see* [18] *and* [28]. *Then the following statements are equivalent*:

(i) *The vector $u\in H$ belongs to the singular spectral subspace $N_i^0$*;

(ii) *The vector $u\in H$ belongs to the singular spectral subspace $N_i$ and the vector $\alpha(L-\lambda)^{-1}u$ belongs to the vector Smirnov classes*([6]) *$N^2_\pm(E)$, see* [22], *i.e., it can be represented as $h_\pm(\lambda)/\delta_\pm(\lambda)$, where $h_\pm\in H^2_\pm(E)$ and $\delta_\pm(\lambda)$ are scalar-valued bounded outer analytic functions in the half-planes $\mathbb{C}_\pm$. Here the functions $\delta_\pm$ can be chosen independently of the vector $u$.*

*Proof.* Recall, that in [33] it was proved, that in the model representation the following characterization of the subspace $N_i^0$ holds (cf. (8)):

$$N_i^0=\left\{\begin{pmatrix}\tilde g\\ g\end{pmatrix}\in K:\ \begin{aligned}&g_+(k+i0)=(\Theta_1)_i(k+i0)v_+(k+i0),\\ &g_-(k-i0)=(\Theta_2')_i(k-i0)v_-(k-i0),\\ &\text{where } v_\pm\in H^2_\pm(E),\ \text{and for a.a. } k\in\mathbb{R} \text{ the equality}\\ &(\widetilde\Theta_1)_e^{-1}(k+i0)v_+(k+i0)=(\widetilde\Theta_2')_e^{-1}(k-i0)v_-(k-i0)\ \text{holds}\end{aligned}\right\}. \tag{11}$$

Here the canonical factorization, see [28], of the operator-valued functions $\Theta_1$ and $\Theta_2'$ into a product of their inner and outer factors is used, $\Theta_1=(\Theta_1)_i(\widetilde\Theta_1)_e$ and $\Theta_2'=(\Theta_2')_i(\widetilde\Theta_2')_e$. Assuming the conditions of Theorem 3.1 all the operator-valued functions appearing in this canonical factorization also possess scalar multiples [28]. Therefore, by the uniqueness theorem for scalar-valued bounded analytic functions [7], the inverses in (11) exist for almost all real $k$.

Let now $u\in N_i^0$. Using the identities (see [18])

$$\begin{aligned}\sqrt{2\pi}g_+(\lambda)&=-\Theta_1(\lambda)\alpha(L-\lambda)^{-1}u,\quad \operatorname{Im}\lambda>0,\\ \sqrt{2\pi}g_-(\lambda)&=-\Theta_2'(\lambda)\alpha(L-\lambda)^{-1}u,\quad \operatorname{Im}\lambda<0,\end{aligned}\tag{12}$$

and the representation (11), we immediately obtain, that for all $\lambda\in\mathbb{C}_+$,

$$\alpha(L-\lambda)^{-1}u=-\sqrt{2\pi}(\Theta_1)_e^{-1}(\lambda)v_+(\lambda)=-\sqrt{2\pi}\delta_+^{-1}(\lambda)\Omega(\lambda)v_+(\lambda),$$

where $\Omega(\lambda)(\Theta_1)_e(\lambda)=(\Theta_1)_e(\lambda)\Omega(\lambda)=\delta_+(\lambda)I$, i.e., $\delta_+$ is a scalar multiple of the operator-valued function $(\Theta_1)_e$ and therefore can be chosen to be an outer analytic function in the upper half-plane [28]. In turn, $\Omega(\lambda)$ is a bounded operator-valued

([6]) That is, analytic continuations of the vector $\alpha(L-\lambda)^{-1}u$ from the domain of analyticity of the resolvent to the half-planes $\mathbb{C}_\pm$ exist and belong to the corresponding Smirnov classes there.

function in $\mathbb{C}_+$. An application of a similar argument to the vector $g_-$ completes the proof of the implication (i)$\Rightarrow$(ii).

Conversely, if (ii) holds, then for all $\lambda\in\mathbb{C}_+$,

$$g_+(\lambda) = -\frac{1}{\sqrt{2\pi}}\Theta_1(\lambda)\frac{h_+(\lambda)}{\delta_+(\lambda)}$$

with some scalar-valued analytic function $\delta_+$, outer in the upper half-plane, and $E$-valued function $h_+\in H^2_+(E)$.

Put

$$v_+(\lambda) = -\frac{1}{\sqrt{2\pi}}(\widetilde{\Theta}_1)_e(\lambda)\frac{h_+(\lambda)}{\delta_+(\lambda)}.$$

It then suffices to prove (since the condition $u\in N_i$ guarantees that the last equality in (11) holds, see (8)) that $v_+\in H^2_+(E)$. Since $\delta_+(\lambda)$ is an outer function in the upper half-plane, this follows immediately from the vector version of Smirnov's theorem [7] and the fact that $g_+\in H_2^+$, whereas the operator-valued function $(\Theta_1)_i(t+i0)$ is unitary for a.a. $t\in\mathbb{R}$. An analogous consideration of the vector $g_-(\lambda)$ completes the proof. $\square$

**Theorem 3.2.** (Weak description of $N_i^0$) *Let $L$ be a nonself-adjoint operator such that all four operator-valued functions appearing in the factorization* (2) *possess scalar multiples, see* [18] *and* [28]. *Then the following statements are equivalent*:

(i) *The vector $u\in H$ belongs to the singular spectral subspace $N_i^0$*;

(ii) *The vector $u\in H$ belongs to the singular spectral subspace $N_i$ and the function $\langle (L-\lambda)^{-1}u, v\rangle$ belongs to the Smirnov classes $N^1_\pm$ for all $v\in H$, i.e., it can be represented as $h_\pm(\lambda)/\delta_\pm(\lambda)$, where $h_\pm\in H^1_\pm$ and $\delta_\pm(\lambda)$ are bounded scalar-valued outer analytic functions in the half-planes $\mathbb{C}_\pm$. Here the functions $\delta_\pm$ are independent of $v\in H$ and can be chosen independently of the vector $u$.*

*Proof.* Let $u\in N_i^0$ and $\lambda\in\mathbb{C}_+$. Then in the model representation the following formula describes the action of the resolvent $(L-\lambda)^{-1}$ on the vector $u$ [18]:

$$(L-\lambda)^{-1}\begin{pmatrix}\tilde g\\ g\end{pmatrix} = P_K\frac{1}{k-\lambda}\begin{pmatrix}\tilde g\\ g\end{pmatrix} - P_K\frac{1}{k-\lambda}\begin{pmatrix}\mathcal{X}_+\Theta_1^{-1}(\lambda)g_+(\lambda)\\ 0\end{pmatrix}, \tag{13}$$

where $(\tilde g, g)\in K$ is the model representation of the vector $u$. Then

$$\langle (L-\lambda)^{-1}u, v\rangle = \left\langle \frac{1}{k-\lambda}\begin{pmatrix}\tilde g\\ g\end{pmatrix}, \begin{pmatrix}\tilde f\\ f\end{pmatrix}\right\rangle - \left\langle \frac{1}{k-\lambda}\begin{pmatrix}\mathcal{X}_+\Theta_1^{-1}(\lambda)g_+(\lambda)\\ 0\end{pmatrix}, \begin{pmatrix}\tilde f\\ f\end{pmatrix}\right\rangle,$$

where $(\tilde f, f)$ is the model image of the vector $v$. The first term in the right-hand side is clearly the Cauchy transform of an $L_1$-function, whereas the second one can be rewritten by residue calculus as

$$\left\langle \frac{1}{k-\lambda}\begin{pmatrix}\mathcal{X}_+\Theta_1^{-1}(\lambda)g_+(\lambda)\\ 0\end{pmatrix}, \begin{pmatrix}\tilde f\\ f\end{pmatrix}\right\rangle = -2\pi i\langle \mathcal{X}_+\Theta_1^{-1}(\lambda)g_+(\lambda), f_-(\bar\lambda)\rangle_E.$$

By Theorem 3.1 the vector $u$ is such that $\alpha(L-\lambda)^{-1}u\in N_+^2(E)$, and therefore by (12) $\Theta_1^{-1}(\lambda)g_+(\lambda)=h_+(\lambda)/\delta_+(\lambda)$ (the last formula is first established outside of the point spectrum of the operator $L$ in $\mathbb{C}_+$ and then extended by analyticity to the whole of $\mathbb{C}_+$) for some $h_+\in H_+^2(E)$ and some outer function $\delta_+$ bounded in the upper half-plane. It follows that if one puts $\nu(\lambda):=1/(\lambda+i)$, then

$$\begin{aligned}\langle (L-\lambda)^{-1}u, v\rangle &= k_1(\lambda) - 2\pi i\frac{1}{\delta_+(\lambda)}\langle \mathcal{X}_+h_+(\lambda), f_-(\bar\lambda)\rangle_E\\ &\equiv \frac{1}{\delta_+(\lambda)\nu(\lambda)}[k_1(\lambda)\delta_+(\lambda)\nu(\lambda) - 2\pi i\langle \mathcal{X}_+h_+(\lambda), f_-(\bar\lambda)\rangle_E\nu(\lambda)]\in N_+^1,\end{aligned}$$

since $k_1(\lambda):=\left\langle (k-\lambda)^{-1}\binom{\tilde g}{g}, \binom{\tilde f}{f}\right\rangle$ and $\overline{f_-(\bar\lambda)}\in H_+^2(E)$.

Conversely, if (ii) holds, let first $\alpha$ be a bounded operator in $H$. Then for all $v\in H$ one has $\langle \alpha(L-\lambda)^{-1}u, v\rangle\in N_+^1(E)$. It follows, see [22], that the vector-valued function $\delta_+(\lambda)\alpha(L-\lambda)^{-1}u$ belongs to the vector Hardy space $H_+^1(E)$ and consequently $\alpha(L-\lambda)^{-1}u=h_+(\lambda)/\delta_+(\lambda)$ for some vector-valued function $h_+\in H_+^1(E)$ and scalar-valued outer function $\delta_+$ bounded in $\mathbb{C}_+$, independent of $v\in H$.

Put, as in the proof of Theorem 3.1,

$$v_+(\lambda) = -\frac{1}{\sqrt{2\pi}}(\widetilde\Theta_1)_e(\lambda)\frac{h_+(\lambda)}{\delta_+(\lambda)}.$$

Since $u\in N_i$ and $g_+(k)=(\Theta_1)_i(k)v_+(k)$, as in the proof of the previous theorem it suffices to show that $v_+\in H_+^2(E)$. Clearly by its definition $v_+\in N_+^1(E)$; on the other hand its boundary values belong to the space $L_2(E)$ along with the function $g_+(k)$. Therefore, by the vector version of Smirnov's theorem [22] $v_+\in H_+^2(E)$ indeed.

In the case of an unbounded $\alpha$ a simple regularization $u_\tau:=i\tau(L+i\tau)^{-1}u$, $\tau\gg 1$ reduces the situation to the one already considered, where we have taken into account that $u_\tau\to u$ as $\tau\to\infty$.

An analogous argument applied to the case of $\mathbb{C}_-$ completes the proof. $\square$

*Remark* 3.3. The results presented in the present section make it possible to reveal the major differences between the parts of $N_i^\pm$ corresponding to the real singular spectrum of the operator $L$, on the one hand, and $N_i^0$, on the other. That is, the vectors from $N_i^\pm$ corresponding to real values of the spectral parameter are "smooth" in one of the half-planes with strong singularity in the other, whereas the vectors from the subspace $N_i^0$ cannot be "smooth" in any of the half-planes (since by [33] the subspace $N_i^0$ contains no smooth vectors), but their singularities on the real line have to be relatively weak (as weak as zeros of an outer function on the real line can be).

## 4. Relationship between the scalar multiples in the case of almost Hermitian spectrum

In the case when the operator $L$ is a trace-class perturbation of a bounded self-adjoint operator $A$, the determinant of the characteristic function of $L$ is identically equal to 1 provided that the spectrum of the operator $L$ is almost Hermitian, i.e., $N_i^0=H$ [33]. For the operator-valued functions (3) appearing in the factorization (2) this identity immediately yields that $\det\Theta_1(\lambda)=\det\Theta_2(\lambda)$, $\operatorname{Im}\lambda>0$, and $\det\Theta_1'(\lambda)=\det\Theta_2'(\lambda)$, $\operatorname{Im}\lambda<0$. Here $\det\Theta_1(\lambda)=\det\mathcal{X}_+S(\lambda)\mathcal{X}_+$ (and $\det\Theta_2(\lambda)=\det\mathcal{X}_-S(\lambda)\mathcal{X}_-$), where the last operator is treated as an operator in the auxiliary Hilbert space $\mathcal{X}_+E$ (resp. $\mathcal{X}_-E$) and similar formulae hold for the other two determinants (in the lower half-plane) [18]. Therefore,

$$\det\mathcal{X}_+S(\lambda)\mathcal{X}_+=\det\mathcal{X}_-S(\lambda)\mathcal{X}_-,\quad \operatorname{Im}\lambda>0,$$

provided that $N_i^0=H$.

In our case, the operator-valued functions (3) might have no determinants. The present section is devoted to the generalization of the result mentioned above to the case when these four operator-valued functions possess just scalar multiples in their respective half-planes.

We begin with the following lemma.

**Lemma 4.1.** *Suppose that all four operator-valued functions appearing in the factorization* (2) *possess scalar multiples in their respective half-planes. Further let the characteristic function $S(\lambda)$ possess a scalar multiple itself*(7)*. Let $S(\lambda)$ be inner in the upper half-plane, i.e., its boundary values are unitary almost everywhere on the real line, $S^*(k+i0)S(k+i0)=S(k+i0)S^*(k+i0)=I$ for a.a. $k\in\mathbb{R}$. Then the scalar multiple of $S(\lambda)$ can be chosen to be an inner analytic function in $\mathbb{C}_+$.*

*Proof.* Since $S(\lambda)$ possesses a scalar multiple $\delta(\lambda)$ in $\mathbb{C}$,

$$S(\lambda)\Omega(\lambda)=\Omega(\lambda)S(\lambda)=\delta(\lambda)I \tag{14}$$

(7) In general, the existence of a scalar multiple for $S(\lambda)$ does not follow from the existence of scalar multiples for the operator-valued functions in (2).

for some analytic operator-valued function $\Omega(\lambda)$ bounded in $\mathbb{C}_+$. Multiplying this identity by $S^*(\lambda)$ and taking into account that $S(\lambda)$ is inner in $\mathbb{C}_+$, we obtain $\Omega(k+i0)/\delta(k+i0)=S^*(k+i0)$ for a.a. real $k$. It follows, that the boundary values of the functions $\Omega(k+i0)/\delta(k+i0)$ and, consequently, $\Omega(k+i0)/\delta_e(k+i0)$, where $\delta_e$ is the outer part, see [7], of the function $\delta$, are uniformly bounded for a.a. real $k$. Then by the operator version of Smirnov's theorem [22] we obtain that $\Omega(\lambda)/\delta_e(\lambda)$ is a bounded analytic operator-valued function in $\mathbb{C}_+$. Dividing the identity (14) by $\delta_e(\lambda)$ concludes the proof. $\square$

By [33], in our setting the characteristic function $\Theta(k)$ is a.e. $J$-unitary on the real line, i.e., $\Theta^*(k+i0)J\Theta(k+i0)=\Theta(k+i0)J\Theta^*(k+i0)=J$ for a.a. real $k$, provided that the operator $L$ has a trivial absolutely continuous subspace, $N_e=\{0\}$. From the identity (see [33])

$$\Theta_2'(\bar{\lambda})(J-\Theta^*(\lambda)J\Theta(\lambda))\Theta_2'^*(\bar{\lambda})=I-S^*(\lambda)S(\lambda), \quad \lambda\in\mathbb{C}_+,$$

it follows then that $S^*(k+i0)S(k+i0)=I$ for a.a. real $k$. A similar identity involving $J-\Theta(\lambda)J\Theta^*(\lambda)$ and $I-S(\lambda)S^*(\lambda)$ yields the identity $S(k+i0)S^*(k+i0)=I$. Consequently, if $N_i^0=H$ (which readily implies that the absolutely continuous subspace of the operator $L$ is trivial), the characteristic function $S(\lambda)$ is an inner operator-valued function in $\mathbb{C}_+$. Therefore, we arrive at the following corollary.

**Corollary 4.2.** *Suppose that all four operator-valued functions appearing in the factorization* (2) *possess scalar multiples in their respective half-planes. Further let the characteristic function $S(\lambda)$ possess a scalar multiple itself. Let the spectrum of the operator $L$ be almost Hermitian, i.e., $N_i^0=H$. Then the scalar multiple of the characteristic function $S(\lambda)$ can be chosen to be an inner analytic function in $\mathbb{C}_+$.*

The following theorem generalizes the result of [33] on the equality of determinants in the case of almost Hermitian spectrum (see the very beginning of this section) to the case considered in the present paper. We recall that if an operator-valued function possesses a well-defined bounded determinant, it can be chosen as a scalar multiple as well. In the case treated in the present section, the equalities for determinants transfer into the corresponding equalities for the properly chosen scalar multiples. Although we use this result as a lemma in the next three sections, it is also of clear independent interest.

**Theorem 4.3.** *Suppose that all four operator-valued functions appearing in the factorization* (2) *possess scalar multiples in their respective half-planes. Let the spectrum of the operator $L$ be almost Hermitian, i.e., $N_i^0=H$. Then the scalar mul-*

*tiples*(8) $\gamma_1(\lambda)$, $\gamma_2(\lambda)$, $\gamma_1'(\lambda)$, $\gamma_2'(\lambda)$ *of the operator-valued functions* $\Theta_1(\lambda)$, $\Theta_2(\lambda)$, $\Theta_1'(\lambda)$, $\Theta_2'(\lambda)$, *respectively, can be chosen in such a way that*

(i) $\gamma_1(\lambda)$ *and* $\gamma_2(\lambda)$ *are* (*scalar-valued*) *bounded outer functions in* $\mathbb{C}_+$; $\gamma_1'(\lambda)$ *and* $\gamma_2'(\lambda)$ *are* (*scalar-valued*) *bounded outer functions in* $\mathbb{C}_-$;

(ii) *The following relations between them hold*:

$$\gamma_1(\lambda)\equiv\gamma_2(\lambda),\quad \gamma_1(\lambda)\equiv\overline{\gamma_1'(\bar\lambda)}\quad \textit{and}\quad \gamma_2(\lambda)\equiv\overline{\gamma_2'(\bar\lambda)}; \tag{15}$$

(iii) *The operator-valued function* $\gamma_1(\lambda)(\Theta_2')^{-1}(\lambda)$ *is analytic and bounded in the upper half-plane, and the operator-valued function* $\gamma_2'(\lambda)\Theta_1^{-1}(\lambda)$ *is analytic and bounded in the lower half-plane.*

*Proof.* First observe that the condition $N_i^0=H$ guarantees, see [33], that the characteristic function $\Theta(\lambda)$ is $J$-unitary a.e. on the real line and, consequently, the characteristic function $S(\lambda)$ is inner in the upper half-plane (see above).

Next, by [33] again, the condition $N_i^0=H$ further implies that all the operator-valued functions (3) are outer in their respective half-planes. Therefore, see [18] and [28], their scalar multiples can be chosen to be outer bounded functions as well.

We are now going to prove that the choice $\gamma_1(\lambda)=\overline{\gamma_1'(\bar\lambda)}$ is possible. By [18], the scalar multiple of the operator-valued function $\Theta_1(\lambda)$ can be chosen equal to the scalar multiple of the contractive operator-valued function $\mathcal{X}_+S(\lambda)\mathcal{X}_+$, treated as an operator in the Hilbert space $\mathcal{X}_+E$. Indeed, if $\gamma_1(\lambda)$ is a scalar multiple of the latter operator-valued function,

$$\Omega_1^+(\lambda)(\mathcal{X}_+S(\lambda)\mathcal{X}_+)=(\mathcal{X}_+S(\lambda)\mathcal{X}_+)\Omega_1^+(\lambda)=\gamma_1(\lambda)I,$$

then

$$\begin{aligned}(\gamma_1(\lambda)\mathcal{X}_-+(I-\mathcal{X}_-S(\lambda)\mathcal{X}_+)\Omega_1^+(\lambda)\mathcal{X}_+)\Theta_1(\lambda)\\ =\Theta_1(\lambda)(\gamma_1(\lambda)\mathcal{X}_-+(I-\mathcal{X}_-S(\lambda)\mathcal{X}_+)\Omega_1^+(\lambda)\mathcal{X}_+)=\gamma_1(\lambda)I.\end{aligned}$$

Conversely, it is obvious (since $\mathcal{X}_+\Omega_1(\lambda)\mathcal{X}_-\equiv0$, where $\Omega_1(\lambda)$ is the operator-valued factor in the definition of the scalar multiple of $\Theta_1(\lambda)$) that the scalar multiple $\gamma_1(\lambda)$ of $\Theta_1(\lambda)$ is a scalar multiple of the operator-valued function $\mathcal{X}_+S(\lambda)\mathcal{X}_+$ as well, hence for a.a. real $k$,

$$\Omega_1^+(k+i0)\mathcal{X}_+S(k+i0)\mathcal{X}_+=\mathcal{X}_+S(k+i0)\mathcal{X}_+\Omega_1^+(k+i0)=\gamma_1(k+i0)I_{\mathcal{X}_+E}$$

for an analytic operator-valued function $\Omega_1^+(\lambda):=\mathcal{X}_+\Omega_1(\lambda)\mathcal{X}_+$ bounded in $\mathbb{C}_+$. Passing over to the adjoint equality, putting $\Omega_1^{+\prime}:=\Omega_1^{+*}(\bar\lambda)$ and $\gamma_1':=\overline{\gamma_1(\bar\lambda)}$, we arrive at

(8) The notation $\gamma_1'(\lambda)$ and $\gamma_2'(\lambda)$ should not be mistaken for the derivatives of the functions $\gamma_1(\lambda)$ and $\gamma_2(\lambda)$.

the conclusion that for a.a. real $k$,

$$\Omega_1^{+\prime}(k-i0)\mathcal{X}_+S^*(k+i0)\mathcal{X}_+=\mathcal{X}_+S^*(k+i0)\mathcal{X}_+\Omega_1^{+\prime}(k-i0)=\gamma_1'(k-i0)I_{\mathcal{X}_+E}.$$

Since by construction $\Omega_1^{+\prime}$ is an analytic bounded function in $\mathbb{C}_-$, whereas $\gamma_1'$ is an outer bounded function in $\mathbb{C}_-$ as long as the function $\gamma_1$ is outer in $\mathbb{C}_+$, we have proved that the function $\overline{\gamma_1(\overline{\lambda})}$ can be chosen as a scalar multiple of the operator-valued function $\mathcal{X}_+S^*(\overline{\lambda})\mathcal{X}_+$ in the lower half-plane. Thus by an argument similar to the one presented above this function is also a scalar multiple for the operator-valued function $\Theta_1'(\lambda)$ in $\mathbb{C}_-$.

Since quite analogously in the case of $H=N_i^0$ the scalar multiple $\gamma_2'$ can be chosen as $\gamma_2'=\overline{\gamma_2(\overline{\lambda})}$, it remains to verify (iii) and to prove that the function $\overline{\gamma_1(\overline{\lambda})}$ can be chosen as a scalar multiple of the operator-valued function $\Theta_2'(\lambda)$ in the lower half-plane.

Note that $\Theta_1(k+i0)\Omega_1(k+i0)=\Omega_1(k+i0)\Theta_1(k+i0)=\gamma_1(k+i0)I$ and $S(\lambda)$ is inner in $\mathbb{C}_+$. Therefore $S(k+i0)=\Theta_1(k+i0)(\Theta_2')^{-1}(k-i0)$ for a.a. real $k$ (see [33], the expression $(\Theta_2')^{-1}(k-i0)$ is meaningful for a.a. real $k$ due to the existence of a scalar multiple), and we immediately obtain that

$$\Omega_1(k+i0)S(k+i0)=\gamma_1(k+i0)(\Theta_2')^{-1}(k-i0) \tag{16}$$

a.e. on the real line. On the other hand, since $N_i^0=H$ and

$$(\Theta_2')^{-1}(\lambda)=I+i\alpha(L-\lambda)^{-1}\alpha\mathcal{X}_-,$$

it is easy to see that for a.a. real $k$, $(\Theta_2')^{-1}(k+i0)=(\Theta_2')^{-1}(k-i0)$ (this follows from the a.e.-zero jump of the resolvent over the real line). From (16) and the uniqueness theorem for analytic functions possessing boundary values almost everywhere, see [7], it follows now that $\gamma_1(\lambda)(\Theta_2')^{-1}(\lambda)$ is a bounded analytic operator-valued function in the upper half-plane, equal to $\Omega_1(\lambda)S(\lambda)$. The corresponding statement for $\gamma_2'(\lambda)\Theta_1^{-1}(\lambda)$ can be verified along the same lines.

On the other hand, the operator-valued function $\Theta_2'$ possesses an outer bounded scalar multiple $\gamma'(\lambda)$ in the lower half-plane and therefore

$$\Theta_2'(\lambda)\Omega'(\lambda)=\Omega'(\lambda)\Theta_2'(\lambda)=\gamma'(\lambda)I, \quad \operatorname{Im}\lambda<0,$$

with some bounded analytic operator-valued function $\Omega'(\lambda)$ in the lower half-plane.

Choose $\gamma_2'(\lambda):=\overline{\gamma_1(\overline{\lambda})}$. We will show that this function can be chosen as a scalar multiple of the operator-valued function $\Theta_2'$ in the lower half-plane. Let $\Omega_2'(\lambda):=\overline{\gamma_1(\overline{\lambda})}(\Theta_2')^{-1}(\lambda)$. If we prove that this is a bounded operator-valued function in the lower half-plane, then $\gamma_2'$ is a scalar multiple of $\Theta_2'(\lambda)$ in $\mathbb{C}_-$. Since for a.a. real $k$

$\Omega_2'(k-i0)=\overline{\gamma_1(k+i0)}(\Theta_2')^{-1}(k-i0)$, the latter expression is uniformly bounded for a.a. real $k$ (see (16)), but at the same time

$$\Omega_2'(\lambda)=\overline{\gamma_1(\overline{\lambda})}\frac{\Omega'(\lambda)}{\gamma'(\lambda)}$$

with an outer function $\gamma'$ in the lower half-plane. Then by the operator version of Smirnov's theorem, see [22], the operator-valued function $\Omega_2'$ is itself bounded in the lower half-plane, which completes the proof. □

*Remark* 4.4. From the proof given it clearly follows that in choosing scalar multiples whose existence is claimed by the theorem one can start with an arbitrary outer bounded scalar multiple of any of the four operator-valued functions (3). The other three can then be obtained based on (15). It of course does not imply the existence of the other three scalar multiples, which has to be pre-assumed.

## 5. Weak annihilation: the definition

In the present section we give the definition of weak outer annihilation (see [12], where the case of trace-class perturbations of bounded self-adjoint operators was considered) and discuss its immediate implications. As established in the paper cited above, this phenomenon plays an important role in the spectral theory of nonself-adjoint operators with almost Hermitian spectrum. In fact, the class of these operators can be independently described in terms of existence of a weak outer annihilator. In Sections 6 and 7 below we will generalize these results to the case when the operator $L$ is not necessarily bounded.

Throughout the present section (unless explicitly stated otherwise) we will assume that the spectrum of the operator $L$ is real.

In the case of trace-class perturbations of bounded self-adjoint operators the following definition was given in [12].

*Definition* 5.1. Let $\gamma(\lambda)$ be an outer, see [7], bounded scalar-valued analytic function in the upper half-plane $\mathbb{C}_+$. We call this function a *weak annihilator* of a nonself-adjoint operator (with real spectrum) $L$, if

$$\operatorname*{w-lim}_{\varepsilon\downarrow 0}\gamma(L+i\varepsilon)=0. \tag{17}$$

For a bounded operator $L$ the function $\gamma(L+i\varepsilon)$ can be defined using standard Riesz–Dunford functional calculus [5]:

$$\gamma(L+i\varepsilon)u=-\frac{1}{2\pi i}\oint_\Gamma \gamma(\lambda)(L+i\varepsilon-\lambda)^{-1}u\,d\lambda, \tag{18}$$

where the contour of integration $\Gamma\subset\mathbb{C}_+$ encircles the spectrum of the operator $L+i\varepsilon$ in the upper half-plane.

Based on this definition, the scalar-valued function $\langle\gamma(L+i\varepsilon)u,v\rangle$, $u,v\in H$, can be rewritten, using the functional model of a nonself-adjoint operator (see Section 2 above) in the following way([9])

$$\begin{aligned}\left\langle\gamma(L+i\varepsilon)\begin{pmatrix}\tilde g\\ g\end{pmatrix},\begin{pmatrix}\tilde f\\ f\end{pmatrix}\right\rangle &= \left\langle\gamma(k+i\varepsilon)\begin{pmatrix}\tilde g\\ g\end{pmatrix},\begin{pmatrix}\tilde f\\ f\end{pmatrix}\right\rangle\\ &+\int_{-\infty}^{\infty}\gamma\Bigl(t+i\frac{\varepsilon}{2}\Bigr)\Bigl\langle\mathcal{X}_-(\Theta_2')^{-1}\Bigl(t-i\frac{\varepsilon}{2}\Bigr)g_-\Bigl(t-i\frac{\varepsilon}{2}\Bigr),f_+\Bigl(t+i\frac{\varepsilon}{2}\Bigr)\Bigr\rangle_E\,dt\end{aligned}\tag{19}$$

for all $(\tilde g,g)$ and $(\tilde f,f)$ in $K$, the model representation of the Hilbert space $H$. Although it is tempting to use (19) as the definition of the operator $\gamma(L+i\varepsilon)$ for an unbounded operator $L$, it can be easily seen that the latter expression is only well-defined on a linear set dense in $H$, and even that only under some additional assumptions on the asymptotic behaviour of the operator-valued function $(\Theta_2')^{-1}(\lambda)$ in the lower half-plane along lines parallel to the real line. These assumptions cannot be stated in terms of its scalar multiple due to the fact that the latter is not uniquely defined.

Due to these reasons, we introduce a regularization for the (in general, unbounded) operator $L$. Consider the function $w_\varepsilon(z):=(z+i\varepsilon)/(1-i\varepsilon z)$ for $\varepsilon>0$. This analytic function in the upper half-plane maps $\mathbb{C}_+$ conformally onto the interior of a large circle in $\mathbb{C}_+$. Moreover, as $\varepsilon$ tends to zero, this circle enlarges to the whole $\mathbb{C}_+$ and for every real $z$ we have $w_\varepsilon(z)\to z$ as $\varepsilon\to 0$.

We introduce a bounded operator $L_\varepsilon$ defined by the formula

$$L_\varepsilon:=w_\varepsilon(L)\equiv(L+i\varepsilon)(1-i\varepsilon L)^{-1}\equiv\frac{i}{\varepsilon}+\frac{1-\varepsilon^2}{\varepsilon^2}\Bigl(L+\frac{i}{\varepsilon}\Bigr)^{-1}\tag{20}$$

(the boundedness of the operator $L_\varepsilon$ follows from the latter representation). The spectrum of the operator $L_\varepsilon$ is clearly a subset of the image of the real line under the transformation $w_\varepsilon$ and lies entirely in the upper half-plane. Moreover, in the case when then the operator $L$ is bounded it is easy to see that $\|L_\varepsilon-L\|\to 0$ as $\varepsilon\to 0$. A similar regularization of the operator $L$ has been used in the proof of the Hille–Yosida theorem in [14] in the definition of the group of exponentials.

Since the operator $L_\varepsilon$ is a bounded operator with spectrum in $\mathbb{C}_+$, for any bounded analytic function $\gamma(\lambda)$ in the upper half-plane we can define the oper-

([9]) We skip the details of this calculation since they are essentially the same as in [12]; see also Section 7 below.

ator $\gamma(L_\varepsilon)$ based on the Riesz–Dunford calculus:

$$\gamma(L_\varepsilon)u = -\frac{1}{2\pi i}\oint_\Gamma \gamma(\lambda)(L_\varepsilon-\lambda)^{-1}u\,d\lambda, \tag{21}$$

where the integration contour $\Gamma$ is an arbitrary piecewise smooth contour lying entirely in $\mathbb{C}_+$ and encircling the spectrum of $L_\varepsilon$. For example, such a contour can be chosen as the image of the real line under the transformation $w_{\varepsilon/2}$.

We now give the following definition.

*Definition* 5.2. Let $\gamma(\lambda)$ be an outer, see [7], bounded scalar-valued analytic function in the upper half-plane $\mathbb{C}_+$. We call this function a *weak annihilator* of a nonself-adjoint operator $L$ (with real spectrum), if

$$\operatorname*{w-lim}_{\varepsilon\downarrow 0}\gamma(w_\varepsilon(L)) = 0. \tag{22}$$

Note that, if there exists a weak annihilator for the operator $L$ in the upper half-plane, there also exists a weak annihilator for the operator $L^*$ due to the following almost obvious lemma which we will find useful in the next section.

**Lemma 5.3.** *Suppose that a bounded outer scalar-valued function $\gamma(\lambda)$ in the upper half-plane weakly annihilates the nonself-adjoint operator $L$ (with real spectrum) in the sense of Definition* 5.2. *Then the bounded outer scalar-valued function $\gamma_*(\lambda) := \overline{\gamma(\overline{\lambda})}$ in the lower half-plane weakly annihilates the adjoint operator $L^*$ in the lower half-plane, i.e.,*

$$\operatorname*{w-lim}_{\varepsilon\downarrow 0}\gamma(w'_\varepsilon(L)) = 0,$$

*where $w'_\varepsilon(z) := \overline{w_\varepsilon(\overline{z})}$.*

The proof can be easily obtained from (21) and (22) by passing over to the adjoint operator in the inner product in the integral.

Finally, the following lemma allows us to "forget" about the non-real part of the spectrum of the operator $L$ in the case when both operators $L$ and $L^*$ possess weak annihilators. In order to still be able to use the Riesz–Dunford functional calculus (21) in the case when the operator $L$ may possess countable non-real spectrum (in particular, in the lower half-plane), we proceed as follows. For any $\varepsilon>0$ choose the integration contour $\Gamma := w_{\varepsilon/2}(\mathbb{R})$ in (21). If its intersection with the point spectrum of the operator $L$ is non-trivial (it might contain at most a finite number of points) pick an $\varepsilon'$ such that $0<\varepsilon'\ll\varepsilon$ and a new contour $\Gamma' := w_{\varepsilon/2+\varepsilon'}(\mathbb{R})$ so that the intersection of $\Gamma'$ with the point spectrum is empty. Recall that under our assumptions it is possible that the point spectrum of the operator $L$ accumulates at every point of the real line (from below in particular, see [18]). Nevertheless, we have only a finite number of eigenvalues outside the contour $\Gamma'$. According to the Riesz–Dunford calculus for piecewise-analytic functions we can now assume that the function $\gamma(\lambda)$ is identically equal to zero in some neighbourhood of this finite set of points.

**Lemma 5.4.** (See [12] for the bounded case) *Let $L=A+iV$ be a nonself-adjoint operator such that all four operator-valued functions* (3) *possess scalar multiples.*

(i) *If there exists an outer bounded weak annihilator*([10]) $\gamma(\lambda)$ *for the operator $L$, the discrete spectrum of $L$ in the upper half-plane is empty, $\sigma(L)\cap\mathbb{C}_+=\varnothing$.*

(ii) *If there exists an outer bounded weak annihilator $\gamma_*(\lambda)$ for the operator $L^*$, the discrete spectrum of the operator $L$ in the lower half-plane is empty, $\sigma(L)\cap\mathbb{C}_-=\varnothing$.*

*Proof.* We will check the implication (i); the proof of (ii) is carried out by passing from the operator to its adjoint.

Due to the conditions imposed on the operator $L$ it follows, see [10], that the spectrum of the operator $L$ is discrete in the open upper and lower half-planes of the complex plane. Let $\lambda_0$, $\operatorname{Im}\lambda_0>0$, be an eigenvalue of the operator $L$ and $u_0$ be the corresponding eigenvector. Then clearly $\gamma(L_\varepsilon)u_0=\gamma(w_\varepsilon(\lambda_0))u_0$. The weak limit of this expression, as $\varepsilon\to 0$, exists and is equal to $\gamma(\lambda_0)u_0$. Therefore, since $\gamma(\lambda)$ is an outer function in the upper half-plane, $u_0\equiv 0$. □

## 6. The implications of weak annihilations on the spectral structure

In the present section we will show how the existence of a weak annihilator for the operator $L$ affects its spectral structure. We will reveal the links between the annihilation phenomenon and the class of operators with almost Hermitian spectrum by proving, that if both the operator $L$ and its adjoint $L^*$ are weakly annihilated in the upper half-plane by some bounded outer scalar-valued functions, then both $L$ and $L^*$ have purely almost Hermitian spectrum. We start with the following statement treating the absolutely continuous subspace of $L$ and $L^*$.

**Theorem 6.1.** *Let $L=A+iV$ be a nonself-adjoint operator such that all four operator-valued functions* (3) *possess scalar multiples. Let the spectrum of the operator $L$ be real. Then existence of either an outer bounded weak annihilator $\gamma(\lambda)$ for the operator $L$ or a corresponding annihilator for the operator $L^*$ implies, that the operators $L$ and $L^*$ have trivial absolutely continuous spectral subspaces, $N_e(L)=N_e(L^*)=\{0\}$.*

([10]) The precise way to calculate $\gamma(L_\varepsilon)$ will be clarified in the proof.

*Proof.* Assume that $\gamma(\lambda)$ is a weak annihilator for the operator $L$. The case when there exists a weak annihilator for the adjoint operator is reduced to a similar situation in $\mathbb{C}_-$ by conjugation. Suppose that the absolutely continuous subspace of the operator $L$ is non-trivial. Then there exists a non-zero smooth vector $(\tilde{g}, g)\in N$, for which the formula (5) holds for all non-real $\lambda$. Then, taking into account the assumption of absence of non-real spectrum, one gets

$$\begin{aligned}\gamma(L_\varepsilon)P_K\begin{pmatrix}\tilde{g}\\ g\end{pmatrix} &= -\frac{1}{2\pi i}\oint_\Gamma \gamma(\lambda)(L_\varepsilon-\lambda)^{-1}P_K\begin{pmatrix}\tilde{g}\\ g\end{pmatrix}d\lambda\\ &= -\frac{1}{2\pi i}\oint_\Gamma \gamma(\lambda)P_K\frac{1}{w_\varepsilon(k)-\lambda}\begin{pmatrix}\tilde{g}\\ g\end{pmatrix}d\lambda\\ &= P_K\left[-\frac{1}{2\pi i}\oint_\Gamma \gamma(\lambda)\frac{1}{w_\varepsilon(k)-\lambda}\,d\lambda\right]\begin{pmatrix}\tilde{g}\\ g\end{pmatrix},\end{aligned}\tag{23}$$

where the contour of integration $\Gamma$ encircles the spectrum of the operator $L_\varepsilon$ and lies inside the upper half-plane $\mathbb{C}_+$.

In order to prove the second equality in (23) we have used the formula (5) and the following representation for the resolvent $(L_\varepsilon-\lambda)^{-1}$, which one can easily check,

$$\begin{aligned}(L_\varepsilon-\lambda)^{-1} &= -i\varepsilon\left(L+\frac{i}{\varepsilon}\right)((1+i\lambda\varepsilon)L-(\lambda-i\varepsilon))^{-1}\\ &= -\frac{i\varepsilon}{1+i\lambda\varepsilon}\left[1+i\frac{1-\varepsilon^2}{\varepsilon(1+i\lambda\varepsilon)}\left(L-\frac{\lambda-i\varepsilon}{1+i\lambda\varepsilon}\right)^{-1}\right].\end{aligned}\tag{24}$$

Therefore, by Cauchy's formula we obtain the formula, for any vector $(\tilde{g}, g)\in N$,

$$\gamma(L_\varepsilon)P_K\begin{pmatrix}\tilde{g}\\ g\end{pmatrix} = P_K\gamma(w_\varepsilon(k))\begin{pmatrix}\tilde{g}\\ g\end{pmatrix}.\tag{25}$$

The right-hand side here has a limit in $\mathcal{H}$ by the Lebesgue dominated convergence theorem. Since $\gamma(\lambda)$ weakly annihilates the operator $L$ it clearly follows that on all vectors $(\tilde{g}, g)$ from the linear set $N$ this limit is equal to zero and therefore the following identity holds:

$$P_K\gamma(k)\begin{pmatrix}\tilde{g}\\ g\end{pmatrix} = 0,\tag{26}$$

where $\gamma(k):=\lim_{\varepsilon\downarrow 0}\gamma(w_\varepsilon(k))$ exist for a.a. $k\in\mathbb{R}$ due to Fatou's theorem [7].

Recall [18], that the linear set $N$ consists of vectors $(\tilde{g}, g)\in\mathcal{H}$ such that $\Theta_1'^*(k-i0)\tilde{g}+\Theta_2^*(k+i0)g=0$. This identity is meaningful due to (4). Therefore, for any bounded scalar-valued Borel function $\beta(k)$ the vector $\beta(k)(\tilde{g}, g)$ also belongs to the linear set $N$. Consider a sequence of measurable sets $\delta_n\subset\mathbb{R}$, defined for all natural $n$ as $\delta_n := \{k \in \mathbb{R} : |\gamma(k)| \geq 1/n\}$, and a sequence of corresponding characteristic functions $\mathcal{X}_n$. Clearly, for every $n$ the function $\mathcal{X}_n(k)/\gamma(k)$ is a bounded function on the real line. Therefore,

$$P_K \gamma(k) \left[\frac{\mathcal{X}_n(k)}{\gamma(k)}\right] \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} \equiv P_K \mathcal{X}_n(k) \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} = 0,$$

and finally, passing to the limit as $n \to \infty$, $P_K(\tilde{g}, g) = 0$, which implies that $\widetilde{N}_e = \{0\}$.

For the adjoint operator $L^*$, consider an outer bounded analytic function $\gamma_*(\lambda)$ in the lower half-plane, which weakly annihilates the operator $L^*$ due to Lemma 5.3. Therefore, a computation (in $\mathbb{C}_-$ instead of $\mathbb{C}_+$) similar to the one presented above may be used to show, that the absolutely continuous subspace of the operator $L^*$ is also trivial. This completes the proof. $\square$

The following theorem establishes implications of existence of weak annihilators on the smooth vectors of the operators $L$ and $L^*$.

**Theorem 6.2.** *Let $L = A + iV$ be a nonself-adjoint operator such that all four operator-valued functions* (3) *possess scalar multiples. Let the spectrum of the operator $L$ be real.*

(i) *If a bounded outer scalar-valued function $\gamma(\lambda)$ in the upper half-plane weakly annihilates the operator $L$, then the linear sets $\widetilde{N}_-(L)$ and $\widetilde{N}_+(L^*)$ are trivial.*

(ii) *If a bounded outer scalar-valued function $\gamma'(\lambda)$ in the upper half-plane weakly annihilates the operator $L^*$, then the linear sets $\widetilde{N}_+(L)$ and $\widetilde{N}_-(L^*)$ are trivial.*

*Proof.* We shall prove that $\widetilde{N}_-(L) = \widetilde{N}_+(L^*) = \{0\}$, the assertion (ii) can be established analogously.

As in the proof of the previous theorem, one can easily establish the formula

$$\begin{aligned} \gamma(L_\varepsilon) P_K \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} &= -\frac{1}{2\pi i} \oint_\Gamma \gamma(\lambda)(L_\varepsilon - \lambda)^{-1} P_K \begin{pmatrix} \tilde{g} \\ g \end{pmatrix} d\lambda \\ &= P_K \left[-\frac{1}{2\pi i} \oint_\Gamma \gamma(\lambda) \frac{1}{w_\varepsilon(k) - \lambda}\, d\lambda\right] \begin{pmatrix} \tilde{g} \\ g \end{pmatrix}, \end{aligned}$$

where $(\tilde{g}, g) \in \widetilde{N}_-$ and the contour of integration $\Gamma$ is chosen accordingly in $\mathbb{C}_+$.

Calculating the integral in the latter expression, we arrive at the formula (25). Since $\gamma(\lambda)$ annihilates the operator $L$ (see p. 111), we conclude, that for all vectors $(\tilde{g}, g) \in \widetilde{N}_-$ the identity (26) holds.

Recall [18], that the linear set $\widetilde{N}_-$ consists of vectors $(\tilde{g}, g) \in \mathcal{H}$ such that

$$P_-(\Theta_1'^*(k - i0)\tilde{g} + \Theta_2^*(k + i0)g) = 0$$

(see (4) for the rigourous meaning of this formal expression). Therefore, for any $H^\infty(\mathbb{C}_+)$-function $\beta(k)$ the vector $\beta(k)(\tilde{g},g)$ also belongs to the linear set $N$.

Analogously to the proof of Theorem 6.1, consider a sequence of bounded analytic functions $\mathcal{X}'_n(k)$ in the upper half-plane such that for every natural $n$ the function $\mathcal{X}'_n(k)/\gamma(k)\in H^\infty(\mathbb{C}_+)$ and $\mathcal{X}'_n(k)\to 1$ for a.e. $k$ on the real line.

Such a sequence exists (cf. [33]) due to the following argument. Since $\gamma(\lambda)$ is an outer function in the upper half-plane, it admits the representation, see [7],

$$\gamma(\lambda)=e^{i\delta}\exp\biggl(\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{1+t\lambda}{t-\lambda}\frac{\log|\gamma(t)|}{1+t^2}\,dt\biggr),$$

where $\delta$ is a real constant. Let the sequence $\varphi_n(\lambda)$ be defined by the formula

$$\varphi_n(\lambda)=e^{-i\delta}\exp\biggl(-\frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{1+t\lambda}{t-\lambda}\frac{\log(|\gamma(t)|+1/n)}{1+t^2}\,dt\biggr).$$

Clearly, $\varphi_n(t)\in H^\infty_+$ for all integers $n$ and furthermore $|\varphi_n(t)\gamma(t)|\le 1$ for a.a. real $t$. By Kolmogorov's theorem [7], there is a subsequence $\varphi_{n_l}(\lambda)$ such that $\varphi_{n_l}(t)\gamma(t)\to 1$, as $l\to\infty$, for a.a. real $t$. Finally, we put $\mathcal{X}'_l(\lambda):=\varphi_{n_l}(\lambda)\gamma(\lambda)$.

Therefore,

$$P_K\gamma(k)\left[\frac{\mathcal{X}'_n(k)}{\gamma(k)}\right]\begin{pmatrix}\tilde{g}\\ g\end{pmatrix}\equiv P_K\mathcal{X}'_n(k)\begin{pmatrix}\tilde{g}\\ g\end{pmatrix}=0,$$

which implies that $\widetilde{N}_-=\{0\}$.

The corresponding result for $\widetilde{N}_+(L^*)$ follows from consideration of the bounded outer function in the lower half-plane $\gamma_*(\lambda):=\overline{\gamma(\overline{\lambda})}$, analogously to that in the proof of Theorem 6.1. $\square$

Note that Theorem 6.1 is a particular case of Theorem 6.2 due to the fact that $\widetilde{N}_e\subset\widetilde{N}_-$ and $\widetilde{N}_e(L^*)\subset\widetilde{N}_+(L^*)$.

Taking into account the definition of the spectral subspace $N_i^0$ (see Section 2) and Lemma 5.4, we can now formulate a corollary of the results proven in the present section in the following form.

**Theorem 6.3.** *Let $L=A+iV$ be a nonself-adjoint operator such that all four operator-valued functions* (3) *possess scalar multiples. Suppose that two outer bounded functions $\gamma(\lambda)$ and $\gamma_*(\lambda)$ in the upper half-plane weakly annihilate the operator $L$ and the operator $L^*$, respectively*([11])*. Then the operators $L$ and $L^*$ both have almost Hermitian spectra, i.e., $H=N_i^0(L)=N_i^0(L^*)$.*

([11]) For the rigourous definition of weak annihilation in the case of non-triviality of the non-real spectrum see Section 5, p. 109.

## 7. Existence of a weak annihilator in the case of almost Hermitian spectrum. Generalized Cayley identity

In the present section we establish the fact that a weak annihilator exists for both operators $L$ and $L^*$, provided that the spectrum of the operator $L$ is almost Hermitian. We begin with the following result.

**Theorem 7.1.** *Let $L=A+iV$ be a nonself-adjoint operator such that all four operator-valued functions* (3) *possess scalar multiples. Suppose that the operator $L$ has almost Hermitian spectrum, i.e., $H=N_i^0$. Then there exists a bounded outer function $\gamma(\lambda)$ in the upper half-plane, weakly annihilating the operator $L$. Moreover, this function can be chosen to be equal to a scalar multiple of the operator-valued function $\Theta_1(\lambda)$*(12)*. There also exists a bounded outer function $\gamma_*(\lambda)$ in the upper half-plane, weakly annihilating the adjoint operator $L^*$, and this function can be chosen to be equal to an outer bounded scalar multiple of the operator-valued function $\Theta_2(\lambda)$. Due to Theorem* 4.3 *this function can be chosen equal to the function $\gamma(\lambda)$ above.*

*Proof.* Observe first, that in conditions of the theorem the function $\gamma(\lambda)$ can be selected as a bounded outer function in the upper half-plane. Indeed, since $N_i^0=H$, all operator-valued functions appearing in the factorization (2) are outer and bounded in the corresponding half-planes [33] and therefore their scalar multiples can also be chosen outer and bounded [18] and [28].

Let $\gamma(\lambda)$ be an outer bounded scalar multiple of the operator-valued function $\Theta_1(\lambda)$. As in Lemma 6.2, we can obtain by Riesz–Dunford calculus that

$$\left\langle \gamma(L_\varepsilon)P_K\begin{pmatrix}\tilde g\\ g\end{pmatrix}, \begin{pmatrix}\tilde f\\ f\end{pmatrix}\right\rangle = -\frac{1}{2\pi i}\oint_\Gamma \gamma(\lambda)\left\langle (L_\varepsilon-\lambda)^{-1}\begin{pmatrix}\tilde g\\ g\end{pmatrix}, \begin{pmatrix}\tilde f\\ f\end{pmatrix}\right\rangle d\lambda, \tag{27}$$

where the contour of integration $\Gamma$ in $\mathbb{C}_+$ has been chosen suitably.

We use the formulae obtained in [18] for the action of the resolvent of the operator $L$ in the model representation on the whole space $K$,

$$(L-\lambda)^{-1}\begin{pmatrix}\tilde g\\ g\end{pmatrix} = P_K\frac{1}{k-\lambda}\begin{pmatrix}\tilde g\\ g\end{pmatrix} - P_K\frac{1}{k-\lambda}\begin{pmatrix}0\\ \mathcal{X}_-(\Theta_2')^{-1}(\lambda)g_-(\lambda)\end{pmatrix},$$

where $(\tilde g, g)\in K$ and $\lambda\in\mathbb{C}_-$, and

$$(L-\lambda)^{-1}\begin{pmatrix}\tilde g\\ g\end{pmatrix} = P_K\frac{1}{k-\lambda}\begin{pmatrix}\tilde g\\ g\end{pmatrix} - P_K\frac{1}{k-\lambda}\begin{pmatrix}\mathcal{X}_+\Theta_1^{-1}(\lambda)g_+(\lambda)\\ 0\end{pmatrix},$$

(12) This scalar multiple can in turn be chosen equal to a scalar multiple of the operator-valued function $\mathcal{X}_+S(\lambda)\mathcal{X}_+$, treated as an operator acting in Hilbert space $\mathcal{X}_+H$, see Section 4.

where $(\tilde{g}, g)\in K$, $\lambda\in\mathbb{C}_+$, and $g_+(\lambda)\equiv(S\tilde{g}+g)(\lambda)$ and $g_-(\lambda)\equiv(\tilde{g}+S^*g)(\lambda)$ are values at the point $\lambda$ of analytic continuations of the functions $g_\pm\in H_2^\pm(E)$ into the upper and lower half-planes, respectively. By Cauchy's formula we then obtain using (24),

(28)

$$\left\langle \gamma(L_\varepsilon)\begin{pmatrix}\tilde{g}\\ g\end{pmatrix}, \begin{pmatrix}\tilde{f}\\ f\end{pmatrix}\right\rangle = \left\langle \gamma(w_\varepsilon(k))\begin{pmatrix}\tilde{g}\\ g\end{pmatrix}, \begin{pmatrix}\tilde{f}\\ f\end{pmatrix}\right\rangle - \frac{1}{2\pi i}$$

$$\times \oint_\Gamma \gamma(\lambda)\left\langle \frac{1-\varepsilon^2}{(1+i\lambda\varepsilon)^2}\frac{1}{k-\dfrac{\lambda-i\varepsilon}{1+i\lambda\varepsilon}}\begin{pmatrix}0\\ \mathcal{X}_-(\Theta_2')^{-1}\left(\dfrac{\lambda-i\varepsilon}{1+i\lambda\varepsilon}\right)g_-\left(\dfrac{\lambda-i\varepsilon}{1+i\lambda\varepsilon}\right)\end{pmatrix}, \begin{pmatrix}\tilde{f}\\ f\end{pmatrix}\right\rangle d\lambda.$$

Note that in the above formula we have actually used the model representation of the action of resolvent in the *lower* half-plane. This is due to the fact that the exterior of the contour $w_\varepsilon(\mathbb{R})$ (of which the operators' spectrum is a subset), where the contour of integration belongs to, corresponds to the lower half-plane. Here $(\Theta_2')^{-1}(\lambda)=\Omega_2'(\lambda)/\gamma_2'(\lambda)$ with some analytic operator-valued function $\Omega_2'(\lambda)$, bounded in the lower half-plane (see [28] and [18]), where $\gamma_2'(\lambda)$ is as before an outer scalar multiple of the function $\Theta_2'(\lambda)$. On the other hand, by Theorem 4.3 the scalar multiple $\gamma_2'(\lambda)$ can be chosen so that $\gamma_2'(\lambda)=\overline{\gamma(\overline{\lambda})}$ since the spectrum of the operator $L$ is almost Hermitian.

Using Cauchy's formula once again, it is easy to see that the expression (28) can be rewritten in the following form:

(29)

$$\left\langle \gamma(L_\varepsilon)\begin{pmatrix}\tilde{g}\\ g\end{pmatrix}, \begin{pmatrix}\tilde{f}\\ f\end{pmatrix}\right\rangle = \left\langle \gamma(w_\varepsilon(k))\begin{pmatrix}\tilde{g}\\ g\end{pmatrix}, \begin{pmatrix}\tilde{f}\\ f\end{pmatrix}\right\rangle$$

$$+\oint_\Gamma \frac{\gamma(\lambda)}{\overline{\overline{\gamma\left(\dfrac{\lambda-i\varepsilon}{1+i\lambda\varepsilon}\right)}}}\frac{1-\varepsilon^2}{(1+i\lambda\varepsilon)^2}\left\langle \mathcal{X}_-\Omega_2'\left(\frac{\lambda-i\varepsilon}{1+i\lambda\varepsilon}\right)g_-\left(\frac{\lambda-i\varepsilon}{1+i\lambda\varepsilon}\right), f_+\left(\overline{\frac{\lambda-i\varepsilon}{1+i\lambda\varepsilon}}\right)\right\rangle_E d\lambda.$$

Choose the integration contour $\Gamma$ so that

$$\frac{\lambda-i\varepsilon}{1+i\lambda\varepsilon}=t-\frac{i\varepsilon}{2}\quad\text{or, equivalently,}\quad \lambda=\frac{t+i\varepsilon/2}{(1-\varepsilon^2/2)-i\varepsilon t},$$

and pass over to the variable $t\in\mathbb{R}$ utilizing analytic properties of the functions $g_\pm\in H_2^\pm(E)$ and $f_\pm\in H_2^\pm(E)$. Then we see that the expression (29) has a limit as $\varepsilon$ tends to 0 and by Lebesgue's dominated convergence theorem and Cauchy–Schwarz' inequality

$$\lim_{\varepsilon\downarrow 0}\left\langle \gamma(L_\varepsilon)\begin{pmatrix}\tilde g\\ g\end{pmatrix},\begin{pmatrix}\tilde f\\ f\end{pmatrix}\right\rangle \tag{30}$$
$$=\left\langle \gamma\begin{pmatrix}\tilde g\\ g\end{pmatrix},\begin{pmatrix}\tilde f\\ f\end{pmatrix}\right\rangle+\int_{-\infty}^{\infty}\frac{\gamma(t+i0)}{\overline{\gamma(t+i0)}}\langle \mathcal{X}_-\Omega_2'(t-i0)g_-(t-i0),f_+(t+i0)\rangle_E\,dt.$$

In order to prove that this limit is actually equal to zero, we recall (see Section 2) that for all $(\tilde g,g)\in N_i$ and for all $(\tilde f,f)\in K$

$$\left\langle [(L-k-i\varepsilon)^{-1}-(L-k+i\varepsilon)^{-1}]\begin{pmatrix}\tilde g\\ g\end{pmatrix},\begin{pmatrix}\tilde f\\ f\end{pmatrix}\right\rangle\to 0,\quad \text{as } \varepsilon\to 0, \tag{31}$$

for a.a. real $k$. Again taking into account formulae describing the action of the resolvent of the operator $L$ in the model representation in upper and lower half-planes, consider the following expression for arbitrary vectors $(\tilde g,g),(\tilde f,f)\in K\equiv N_i^0\subset N_i$:

$$\frac{1}{2\pi i}\gamma(t+i\varepsilon)\left\langle [(L-t-i\varepsilon)^{-1}-(L-t+i\varepsilon)^{-1}]\begin{pmatrix}\tilde g\\ g\end{pmatrix},\begin{pmatrix}\tilde f\\ f\end{pmatrix}\right\rangle$$
$$\begin{aligned}&=\frac{\gamma(t+i\varepsilon)}{2\pi i}\int_{-\infty}^{\infty}\frac{2i\varepsilon}{(k-t)^2+\varepsilon^2}\left\langle\begin{pmatrix}\tilde g\\ g\end{pmatrix},\begin{pmatrix}\tilde f\\ f\end{pmatrix}\right\rangle dk\\ &\quad+\langle \mathcal{X}_+\Omega_1(t+i\varepsilon)g_+(t+i\varepsilon),f_-(t-i\varepsilon)\rangle_E\\ &\quad+\frac{\gamma(t+i\varepsilon)}{\overline{\gamma(t+i\varepsilon)}}\langle \mathcal{X}_-\Omega_2'(t-i\varepsilon)g_-(t-i\varepsilon),f_+(t+i\varepsilon)\rangle_E\end{aligned}\tag{32}$$

(cf. (29)). Here the operator-valued function $\Omega_1(\lambda)$ is the factor in the definition of the scalar multiple $\gamma_1(\lambda)\equiv\gamma(\lambda)$ of $\Theta_1(\lambda)$, i.e., $\Theta_1(\lambda)\Omega_1(\lambda)=\Omega_1(\lambda)\Theta_1(\lambda)=\gamma_1(\lambda)I$ in $\mathbb{C}_+$. The expression on the right-hand side of (32) has a limit for a.a. $t\in\mathbb{R}$, equal to the integrand in (30) but for the term $\langle \mathcal{X}_+\Omega_1(t+i0)g_+(t+i0),f_-(t-i0)\rangle_E$, the integral of which over the real line is equal to zero due to the orthogonality of $H^2_+(E)$ and $H^2_-(E)$ in $L_2(E)$. On the other hand, from (31) it follows, that this limit is identically equal to zero for a.a. $t$. Thus we have completed the proof of the fact that the function $\gamma(\lambda)$ weakly annihilates the operator $L$.

Recall, that the condition that the operator $L$ has almost Hermitian spectrum is equivalent to the fact that the spectrum of the adjoint operator $L^*$ is also almost Hermitian. Indeed, [33, Lemma 3.3] implies that the subspace $N_i^0$ does not contain smooth vectors, i.e., $N_i^0\cap\widetilde N_-=N_i^0\cap\widetilde N_+=\{0\}$. Then $N_i^0(L^*)=K\ominus(N_+\vee N_-)=K$ by the definition of $N_i^0$. Thus, the assertion concerning the operator $L^*$ can be verified along similar lines. This observation completes the proof. □

*Remark* 7.2. We remark that due to the specific choice of the function $\gamma(\lambda)$ provided by Theorem 7.1 we can use Definition 5.1 of the weak outer annihilation instead of Definition 5.2 in the statement of Theorem 7.1. The proof in this case is obtained along the same lines as in [12].

A combination of Theorems 6.3 and 7.1 and Lemma 5.4 implies in particular the following statement.

**Theorem 7.3.** *Let $L=A+iV$ be a nonself-adjoint operator such that all four operator-valued functions* (3) *possess scalar multiples. Then both operators $L$ and $L^*$ are weakly annihilated by some scalar-valued bounded outer analytic functions if and only if one or* (*equivalently*) *both of them has almost Hermitian spectra.*

## 8. The self-adjoint case

The present section is devoted to the proof of one statement concerning the self-adjoint situation. It effectively shows, that in terms of weak outer annihilation the singular spectral subspace $N_i^0$ behaves in exactly the same way as the singular spectral subspace of a self-adjoint operator. Moreover, due to this result it would seem reasonable to include the singular component of the self-adjoint part of the operator $L$ (in the general case, when $L$ is not necessarily completely nonself-adjoint) into the singular subspace $N_i^0$. It is also worth mentioning that not only the proof of this theorem exploits essentially nonself-adjoint (in particular, functional model related) techniques, but even certain crucial objects of the nonself-adjoint spectral theory appear already in its statement.

**Theorem 8.1.** *Let $A$ be a* (*possibly, unbounded*) *self-adjoint operator in the Hilbert space $H$. Then the following two statements are equivalent*:

(i) *the spectrum of $A$ is purely singular*;

(ii) *there exists an outer bounded scalar-valued function $\gamma_A(\lambda)$ in the upper half-plane, weakly annihilating the operator $A$, i.e.,*

$$\operatorname*{w-lim}_{\varepsilon\downarrow 0} \gamma_A(A+i\varepsilon)=0.$$

*Moreover, the function $\gamma_A$ can be chosen as*([13])

$$\gamma_A(\lambda)=\det\bigl(I+i\sqrt{V}(A-iV-\lambda)^{-1}\sqrt{V}\bigr)$$

*for any trace-class* (*or relatively trace-class*) *non-negative operator $V$ in $H$ such that $\bigvee_{\operatorname{Im}\lambda\neq 0}(A-\lambda)^{-1}VH=H$.*

([13]) It is easy to see that $\gamma_A(\lambda)$ in fact coincides with the perturbation determinant $D_{A/A-iV}(\lambda)$ of the pair $(A, A-iV)$, see [6].

*Proof.* Choose $V$ to be a trace-class non-negative self-adjoint operator in the Hilbert space $H$ such that

$$\bigvee_{\operatorname{Im}\lambda\neq 0}(A-\lambda)^{-1}VH=H. \tag{33}$$

Clearly, such a choice is always possible.

Consider the functional model developed in [18] for the operators $L$ admitting a representation $L_\varkappa=A+\alpha\varkappa\alpha/2$, where $\alpha\geq 0$ is a non-negative operator in the Hilbert space $H$ and $\varkappa$ is a bounded operator in the subspace $E$, $E$ being the closure of the range of $\alpha$. Choose $\alpha$ to be a Hilbert–Schmidt class operator defined by the formula $\alpha=\sqrt{2V}\in\mathfrak{S}_2$. Then the operator $L_\varkappa$ is well defined on the domain $D(L_\varkappa)=D(A)$. Moreover, $L_\varkappa\equiv A$ when $\varkappa=0$, i.e., $L_0\equiv A$. Consider the dissipative operator $L^{\|}\equiv A+iV$ (this operator coincides with $L_{iI}$). Clearly, it is a maximal dissipative operator in $H$; moreover, it is easy to see that the condition (33) guarantees that it is also completely nonself-adjoint.

Construct the functional model based on the operator $L^{\|}$ (see Section 2 above). In the corresponding dilation space $\mathcal{H}$ the following formulae describe the action of the resolvent $(A-\lambda)^{-1}$ on all vectors $(\tilde g,g)\in K$, as above $K$ being the model image of $H$ (see [18]),

$$(A-\lambda)^{-1}\begin{pmatrix}\tilde g\\ g\end{pmatrix}=P_K\frac{1}{(k-\lambda)}\begin{pmatrix}\tilde g\\ g-\frac12\big(I+(S^*(\overline{\lambda})-I)\frac12\big)^{-1}g_-(\lambda)\end{pmatrix},\quad \operatorname{Im}\lambda<0, \tag{34}$$

$$(A-\lambda)^{-1}\begin{pmatrix}\tilde g\\ g\end{pmatrix}=P_K\frac{1}{(k-\lambda)}\begin{pmatrix}\tilde g-\frac12\big(I+(S(\lambda)-I)\frac12\big)^{-1}g_+(\lambda)\\ g\end{pmatrix},\quad \operatorname{Im}\lambda>0. \tag{35}$$

Here $S(\lambda)$ is the characteristic function of the completely nonself-adjoint maximal dissipative operator $L^{\|}$, all the other notation has already been introduced above.

We introduce the following notation for the operator-valued functions, appearing in this representation: $\Theta_A(\lambda):=I+(S(\lambda)-I)\frac12$ and $\Theta'_A(\lambda):=I+(S^*(\overline{\lambda})-I)\frac12$. The functions $\Theta_A$ and $\Theta'_A$ are bounded analytic operator-valued functions in the half-planes $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively.

Observe that the characteristic function $S(\lambda)$ is an inner operator-valued function in the upper half-plane. Indeed, it suffices to show that the boundary values of $S(\lambda)$ on the real axis are a.e.-unitary there. By the Hilbert identity,

$$S^*(\lambda)S(\lambda)-I=\Theta'_A(\overline{\lambda})\alpha[(A-\lambda)^{-1}-(A-\overline{\lambda})^{-1}]\alpha\Theta'^*_A(\overline{\lambda}).$$

On the other hand, since the spectrum of the operator $A$ is purely singular, one has the following property for all $u,v\in H$ as $\varepsilon\downarrow 0$:

$$\langle[(A-k-i\varepsilon)^{-1}-(A-k+i\varepsilon)^{-1}]u,v\rangle\to 0\quad\text{for a.a. } k\in\mathbb{R}. \tag{36}$$

Combining these two facts with the existence of boundary values of $S(\lambda)$ in the strong topology, it is easy to see that $S^*(k+i0)S(k+i0)=I$ almost everywhere on the real line. The second identity $S(k+i0)S^*(k+i0)=I$ can be verified along similar lines.

It follows that, since $\Theta_A(\lambda)=(I+S(\lambda))/2$ and $\Theta'_A(\lambda)=(I+S^*(\overline{\lambda}))/2$, they are also outer (see [28]) in the half-planes $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively.

By the definition of $S(\lambda)$, both operator-valued functions also have well-defined outer (see [28] and [33]) determinants $\gamma_A$ and $\gamma'_A$, bounded (in fact, contractive) in their respective half-planes, which can therefore be also chosen as their scalar multiples [18]. It is also clear that $\gamma'_A(\lambda)=\overline{\gamma_A(\overline{\lambda})}$.

We will prove that $\gamma_A(\lambda)$ weakly annihilates the self-adjoint operator $A$, provided that its spectrum is purely singular. The function $\gamma'_A(\lambda)$, as can be easily seen, will then annihilate the operator $A$ in the lower half-plane. We remark, that $\gamma_A(\lambda)$ is a clearly non-zero function since $\lim_{\tau\to\infty}\gamma_A(i\tau)=1$.

The contractive (due to von Neumann's inequality [28] or, alternatively, due to the spectral theorem) operator $\gamma_A(A+i\varepsilon)$ is defined by the Riesz–Dunford integral,

$$\langle\gamma_A(A+i\varepsilon)u,v\rangle=\frac{1}{2\pi i}\left(\int_{-\infty+3i\varepsilon/2}^{\infty+3i\varepsilon/2}-\int_{-\infty+i\varepsilon/2}^{\infty+i\varepsilon/2}\right)\gamma_A(\lambda)\langle(A+i\varepsilon-\lambda)^{-1}u,v\rangle\,d\lambda.$$

Using the model representation (34) we then immediately obtain

(37)

$$\begin{aligned}\left\langle\gamma_A(A+i\varepsilon)\begin{pmatrix}\tilde g\\ g\end{pmatrix},\begin{pmatrix}\tilde f\\ f\end{pmatrix}\right\rangle&=\left\langle\gamma_A(k+i\varepsilon)\begin{pmatrix}\tilde g\\ g\end{pmatrix},\begin{pmatrix}\tilde f\\ f\end{pmatrix}\right\rangle\\ &+\frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma_A\Big(t+i\frac{\varepsilon}{2}\Big)\left\langle\frac{1}{k-\big(t-i\frac12\varepsilon\big)}\begin{pmatrix}0\\ \frac12(\Theta'_A)^{-1}\big(t-i\frac12\varepsilon\big)g_-\big(t-i\frac12\varepsilon\big)\end{pmatrix},\begin{pmatrix}\tilde f\\ f\end{pmatrix}\right\rangle dt\\ &-\frac{1}{2\pi i}\int_{-\infty}^{\infty}\gamma_A\Big(t+i\frac{3\varepsilon}{2}\Big)\left\langle\frac{1}{k-\big(t+i\frac12\varepsilon\big)}\begin{pmatrix}\frac12\Theta_A^{-1}\big(t+i\frac12\varepsilon\big)g_+\big(t+i\frac12\varepsilon\big)\\ 0\end{pmatrix},\begin{pmatrix}\tilde f\\ f\end{pmatrix}\right\rangle dt.\end{aligned}$$

Proceeding exactly as in the previous section and using the property (36) once again, we can quite easily ascertain that $\gamma_A$ is indeed weakly annihilating the operator $A$ in the upper half-plane.

Conversely, let the self-adjoint operator $A$ possess a weak outer bounded annihilator $\gamma_A(\lambda)$. Let the vector $u\neq0$ belong to the absolutely continuous spectral subspace $H_{\mathrm{ac}}$. Then, by the spectral theorem and by Lebesgue's dominated convergence theorem it is easy to see that

$$\int_\delta\gamma_A(k)\,d\mu_{u,v}(k)=0$$

(by taking $E_A(\delta)v$ instead of $v$) for an arbitrary Borel set $\delta\subset\mathbb{R}$ and the finite absolutely continuous complex measure $d\mu_{u,v}(k):=\langle dE_A(k)u,v\rangle$, see [3], where $E_A$

is the operator-valued spectral measure of the operator $A$ and $v$ is an arbitrary element of $H$. Since the boundary values of $\gamma_A$ are non-zero almost everywhere on the real line, this implies that the measure $d\mu_{u,v}\equiv 0$ for all $v\in H$, which completes the proof. □

*Remark* 8.2. Note that the existence of a non-zero *analytic* bounded annihilator of the operator $A$ is clearly sufficient for the pure singularity of its spectrum. Nevertheless, Theorem 8.1 asserts that this function can be chosen to be outer in $\mathbb{C}_+$ as well.

*Remark* 8.3. Suppose that the operator $A$ is a self-adjoint operator with simple spectrum. Then the trace-class operator $V$ of Theorem 8.1 due to (33) can clearly be chosen as a rank-one operator in Hilbert space $H$ [3]. In this situation, the statement of Theorem 8.1 can be modified in the part concerning the choice of the annihilator in the following way: the annihilator can be chosen as

$$\gamma_A(\lambda) := \frac{1}{1-i(D(\lambda)-1)},$$

where $D(\lambda):=1+\langle (A-\lambda)^{-1}\varphi,\varphi\rangle$ is the perturbation determinant of the pair $(A, A+\langle\,\cdot\,,\varphi\rangle\varphi)$ and $\varphi$ is the generating vector for the operator $A$.

The proof is a straightforward application of the explicit formula for the resolvent of a rank-one perturbation of a self-adjoint operator, based on the Hilbert identity.

## 9. Application

Consider an operator of rank-one nonself-adjoint Friedrichs model (see also [12] and [13] for the bounded operator version of this example), i.e., a rank-one nonself-adjoint perturbation of the operator of multiplication $A$ in $L_2(\mathbb{R}; d\sigma)$, where $d\sigma$ is assumed to be a positive Borel measure on the real line $\mathbb{R}$, singular with respect to the Lebesgue measure:

$$\begin{aligned} (Au)(x) &= xu(x), \\ (Lu)(x) &= (Au)(x)+\langle u,\varphi\rangle\psi(x), \end{aligned} \tag{38}$$

$u,\varphi,\psi\in L_2(\mathbb{R}; d\sigma)$, $\langle\cdot,\cdot\rangle$ denoting the inner product in $L_2(\mathbb{R}; d\sigma)$. Suppose also that the operator $L$ is completely nonself-adjoint, i.e., that $|\varphi|+|\psi|\neq 0$ a.e. on $\mathbb{R}$ with respect to $d\sigma$ (this condition is not restrictive since if it is not true one could easily reduce the measure $\sigma$ to its part corresponding to the completely nonself-adjoint operator. The self-adjoint part in the Langer decomposition will be then reduced to the corresponding part of the operator $A$).

Let $\varphi(x)\overline{\psi(x)}=0$ $d\sigma$-a.e. on the real line (in this case the perturbation determinant is identically equal to 1 [11]). Then the operator $\alpha$ is a diagonal operator in $E\equiv\bigvee\{\varphi,\psi\}$:

$$\alpha=\left(\frac{\|\psi\|}{\|\varphi\|}\right)^{1/2}\frac{1}{\|\varphi\|}\langle\,\cdot\,,\varphi\rangle\varphi+\left(\frac{\|\varphi\|}{\|\psi\|}\right)^{1/2}\frac{1}{\|\psi\|}\langle\,\cdot\,,\psi\rangle\psi,$$
$$J=\frac{1}{i\|\varphi\|\|\psi\|}(\langle\,\cdot\,,\varphi\rangle\psi-\langle\,\cdot\,,\psi\rangle\varphi).$$

Further, the resolvent $(L-\lambda)^{-1}$ satisfies the identity

$$((L-\lambda)^{-1}u)(x)=\frac{u(x)}{x-\lambda}-\left\langle\frac{u(t)}{t-\lambda},\varphi(t)\right\rangle\frac{\psi(x)}{x-\lambda}. \tag{39}$$

We will now prove that the operator $L$ is an operator with almost Hermitian spectrum.

**Proposition 9.1.** (See [13] for the bounded case) *Let $L$ be a completely nonself-adjoint operator of rank-one nonself-adjoint Friedrichs model in $L_2(\mathbb{R};d\sigma)$, where $d\sigma$ is assumed to be a positive Borel measure on the real line, singular with respect to the Lebesgue measure. Let the two functions $\varphi$ and $\psi$ determining the perturbation have disjoint supports: $\varphi(x)\overline{\psi(x)}=0$ $d\sigma$-a.e. on the real line. Then the operator $L$ is an operator with almost Hermitian spectrum, i.e., $L_2(\mathbb{R};d\sigma)=N_i^0(L)=N_i^0(L^*)$.*

*Proof.* It suffices to check that the operator $L$ has no smooth vectors, i.e., that $\widetilde{N}_-=\widetilde{N}_+=\{0\}$ [33]. Assume that $\widetilde{N}_-$ is non-trivial (the proof in the case of $\widetilde{N}_+$ is conducted along similar lines). Then there exists a vector $u\in L_2(\mathbb{R};d\sigma)$ such that $\alpha(L-\lambda)^{-1}u\in H^2_+(E)$ [18].

A direct computation yields

$$\begin{aligned}\alpha(L-\lambda)^{-1}u=&\left(\frac{\|\psi\|}{\|\varphi\|}\right)^{1/2}\frac{1}{\|\varphi\|}\left\langle\frac{u(x)}{x-\lambda},\varphi(x)\right\rangle\varphi\\&+\left(\frac{\|\varphi\|}{\|\psi\|}\right)^{1/2}\frac{1}{\|\psi\|}\left[\left\langle\frac{u(x)}{x-\lambda},\psi(x)\right\rangle-\left\langle\frac{u(x)}{x-\lambda},\varphi(x)\right\rangle\left\langle\frac{\psi(x)}{x-\lambda},\psi(x)\right\rangle\right]\psi.\end{aligned} \tag{40}$$

Since $\langle\varphi,\psi\rangle=0$, it follows that for this vector to belong to the space $H^2_+(E)$ it is necessary that the scalar-valued analytic function $v(\lambda):=\langle u(x)/(x-\lambda),\varphi(x)\rangle$ belongs to the space $H^2_+$. This in turn implies, see [7], that

$$\int_{\mathbb{R}}\frac{\overline{\varphi}(x)u(x)}{x-\lambda}\,d\sigma(x)=\int_{\mathbb{R}}\frac{f(x)}{x-\lambda}\,dx$$

for some function $f\in L_2(\mathbb{R})$. The left-hand side of the latter identity represents the Cauchy transform of a singular measure, whereas the right-hand side represents the Cauchy transform of an absolutely continuous one, from where on the basis of the F. and M. Riesz theorem, see [7], we conclude that $\overline{\varphi}u=0$ a.e. with respect to $d\sigma$. Applying an analogous argument to the second term in the right-hand side of (40) (since the second term in the square brackets vanishes identically) we conclude that $\overline{\psi}u=0$ a.e. with respect to $d\sigma$, which completes the proof due to the assumption of complete nonself-adjointness of the operator $L$. □

Let

$$a(\lambda)=\frac{2}{2-i\varkappa r_\varphi(\lambda)-i\varkappa^{-1}r_\psi(\lambda)-r_\varphi(\lambda)r_\psi(\lambda)}, \tag{41}$$

where $r_\varphi(\lambda):=\langle\varphi(x)/(x-\lambda),\varphi(x)\rangle$ and $r_\psi(\lambda):=\langle\psi(x)/(x-\lambda),\psi(x)\rangle$. The following result is an application of Theorem 7.1, see [11] for the details of the computation of the characteristic function $S(\lambda)$.

**Theorem 9.2.** *Let $L$ be a completely nonself-adjoint operator of rank-one nonself-adjoint Friedrichs model in $L_2(\mathbb{R};d\sigma)$, where $d\sigma$ is assumed to be a positive Borel measure on the real line, singular with respect to the Lebesgue measure. Let the two functions $\varphi$ and $\psi$ determining the perturbation have disjoint supports: $\varphi(x)\overline{\psi(x)}=0$ $d\sigma$-a.e. on the real line. Then the operator $L$ is weakly annihilated by the outer bounded analytic function $a(\lambda)$ in the upper half-plane defined by* (41).

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