Density of algebras generated by Toeplitz operators on Bergman spaces

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Abstract. In this paper it is shown that Toeplitz operators on Bergman space form a dense subset of the space of all bounded linear operators, in the strong operator topology, and that their norm closure contains all compact operators. Further, the C^*-algebra generated by them does not contain all bounded operators, since all Toeplitz operators belong to the essential commutant of certain shift. The result holds in Bergman spaces $A^2(\Omega)$ for a wide class of plane domains $\Omega\subset\mathbf{C}$, and in Fock spaces $A^2(\mathbf{C}^N)$, $N\geqq 1$.

1. Introduction

Let Ω be a domain in $\mathbf{C}^N$ equipped with the Lebesgue measure $d\lambda(z)$, or $\Omega=\mathbf{C}^N$ with the Gaussian measure $d\mu(z)=e^{-|z|^2/2}(2\pi)^{-N}d\lambda(z)$. We shall be concerned with operators on the subspace $A^2(\Omega)$ of $L^2(\Omega, d\lambda)$ consisting of functions analytic on Ω, and on the subspace $A^2(\mathbf{C}^N)$ of $L^2(\mathbf{C}^N, d\mu)$ consisting of entire functions on $\mathbf{C}^N$. The former is usually referred to as Bergman space, while the latter is known as the Segal—Bargmann space, or as the Fock space. For $\Omega\subsetneqq\mathbf{C}^N$, we shall assume throughout this paper that Ω has finite Lebesgue measure (otherwise the space $A^2(\Omega)$ becomes too small — it won't even contain nonzero constant functions).

For ϕ an essentially bounded function on Ω, we may define the multiplication operator

$$M_\phi\colon f\mapsto \phi f,$$

acting from $A^2(\Omega)$ into $L^2(\Omega, d\lambda)$ or $L^2(\mathbf{C}^N, d\mu)$. If P_+ stands for the orthogonal projection of L^2 onto the corresponding A^2, the Toeplitz operator T_ϕ and the Hankel operator H_ϕ with symbol ϕ are defined by

$$T_\phi f = P_+ M_\phi f \quad\text{and}\quad H_\phi f = (I-P_+)M_\phi f.$$

They are bounded linear operators from $A^2(\Omega)$ into $A^2(\Omega)$ and $L^2\ominus A^2(\Omega)$, respectively. These operators have been studied by numerous authors, e.g. by Axler, Conway and McDonald [2], McDonald and Sundberg [19] and Axler [1]. More recently, they became of interest due to their connection with pseudodifferential operators and with quantum mechanics, cf. Guillemin [16], Berezin [4], [5], [6], Berger and Coburn [7], [8], Coburn [11], Berger, Coburn and Zhu [9]. The problem of compactness of Toeplitz and Hankel operators has been solved by Stroethoff and Zheng [20], [22], [21].

In this paper, we shall be concerned with the question of "how many" operators on $A^2(\Omega)$ are Toeplitz. The classical Toeplitz operators on the Hardy space H^2 on the unit circle $\mathbf{T}$ are characterized by an intertwining relation (cf. [17], problem 194)

$$S^* TS = T,$$

S being the unilateral shift operator on H^2. It follows that Toeplitz operators on H^2 form a rather small w^*-closed subset of $\mathscr{B}(H^2)$ of infinite codimension. It can be shown that no such characterization (i.e. of the form "T is Toeplitz$\Leftrightarrow ATB=T$" for some operators A, B) is possible on $A^2(\mathbf{D})$ [13] and that Toeplitz operators on $A^2(\mathbf{D})$ are in fact dense in the space of all bounded linear operators in the strong operator topology [14]. For $A^2(\mathbf{C}^N)$, similar observation was made by Berger and Coburn [8], who also proved that the norm closure contains all compact linear operators, but not all bounded ones.

In this article, we first show that the above results remain valid for arbitrary $\Omega\subseteq\mathbf{C}^N$: the SOT closure of the set $\{T_\phi\colon \phi\in L^\infty(\Omega)\}$ contains all bounded linear operators, and its norm closure contains all compact ones (Section 2). An important ingredient in the proof is a simple interpolation property of Toeplitz operators (Theorem 2). The remaining sections deal with algebras generated by Toeplitz operators. It is shown that even the C^*-algebra generated by the set $\{T_\phi\colon \phi\in L^\infty(\Omega)\}$ does not contain all bounded linear operators. The reason is that Toeplitz operators essentially commute with an operator unitarily equivalent to a unilateral ($\Omega\subsetneqq\mathbf{C}$) or bilateral ($\Omega=\mathbf{C}^N$, $N\geqq 1$) shift. This result gives a negative answer to a conjecture of Berger and Coburn in [8]. The proof is first presented for $\Omega=\mathbf{D}$, the unit disc (Section 3), then for $\Omega\subsetneqq\mathbf{C}$ (Section 4), and finally for $\Omega=\mathbf{C}^N$ (Section 5); the case $\Omega\subsetneqq\mathbf{C}^N$, $N>1$, remains open.

In view of these results, it would be useful to know something more about the essential commutant of the unilateral and of the bilateral shift. Some work in this direction has been done by Barría and Halmos [3] and closely related topics appear also in Davidson [12] and Johnson and Parrot [18]. Otherwise, very little seems to be known; for instance, it is not even clear whether these two essential commutants are not in fact isomorphic as C^*-algebras.

As mentioned above, the results of sections 3—5 do not cover the case $\Omega\subsetneqq\mathbf{C}^N$,

$N>1$. The difference from the case $N=1$ is best illustrated by the behaviour of Hankel operators: if $N=1$, H_f is compact for arbitrary $f\in C(\bar{\Omega})$; for $N=2$ and $\Omega=\mathbf{D}\times\mathbf{D}$ (the bidisc in $\mathbf{C}^2$), even the operators H_{z_1} and H_{z_2} are not compact (see [11], p. 101).

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2. Density of Toeplitz operators

We begin with a simple interpolation property for Toeplitz operators. For $\Omega=\mathbf{D}$, this property was established in [14]. Denote by $\mathcal{D}(\Omega)$ the Schwarz space of all compactly supported infinitely differentiable functions on Ω.

Proposition 1. *Let Ω be a domain in $\mathbf{C}^N$ and $F(x, y)$ an analytic function on $\Omega\times\bar{\Omega}$ such that*

$$F(x, \bar{x}) = 0 \quad \forall x\in\Omega.$$

Then F vanishes identically on $\Omega\times\bar{\Omega}$.

Proof. For $\Omega=\mathbf{C}^N$, a proof may be found in [15] (Proposition 1.69); the idea, however, works for other domains as well. Without loss of generality, we may assume that Ω contains the origin. Put

$$G(u, v) = F(u+iv, u-iv)$$

whenever $u, v\in\mathbf{C}^N$ are such that $u+iv\in\Omega$, $u-iv\in\bar{\Omega}$. Then G is an analytic function on a domain $\tilde{\Omega}$ in $\mathbf{C}^{2N}$ containing the origin, and the condition $F(x, \bar{x})=0$ implies that $G(u, v)=0$ whenever $(u, v)\in\tilde{\Omega}$ and $u, v\in\mathbf{R}^N$. Expanding G into the Taylor series at the origin and looking at the coefficients, we find that this is only possible when $G\equiv 0$ identically. □

Theorem 2. *Let T be a bounded operator on $A^2(\Omega)$ and $F_i, G_i\in A^2(\Omega)$ $(i=1, 2, \ldots, k)$. Then there exists $\phi\in\mathcal{D}(\Omega)$ such that*

$$\langle T_\phi F_i, G_i\rangle = \langle TF_i, G_i\rangle, \quad i = 1, 2, \ldots, k.$$

Proof. We shall prove the theorem for the case $\Omega\neq\mathbf{C}^N$; the proof for the Fock space is perfectly similar. Let $f_1, f_2, \ldots, f_n$, resp. $g_1, g_2, \ldots, g_m$ be a basis of the subspace of $A^2(\Omega)$ spanned by $F_1, \ldots, F_k$, resp. $G_1, \ldots, G_k$. Clearly it's sufficient to find $\phi\in\mathcal{D}(\Omega)$ such that

$$\langle T_\phi f_i, g_j\rangle = \langle Tf_i, g_j\rangle \quad \text{for all} \quad i = 1, \ldots, n \quad \text{and} \quad j = 1, \ldots, m.$$

Consider the operator $R\colon \mathcal{D}(\Omega)\to\mathbf{C}^{n\times m}$, defined by the formula

$$(R\phi)_{ij} = \int_\Omega \phi(z) f_i(z)\overline{g_j(z)}\,d\lambda(z) = \langle T_\phi f_i, g_j\rangle.$$

Suppose some $u\in\mathbf{C}^{n\times m}$ is orthogonal to the range of R, i.e.

$$\sum_{i=1}^n \sum_{j=1}^m (R\phi)_{ij}\bar{u}_{ij} = 0 \quad \text{for all} \quad \phi\in\mathcal{D}(\Omega).$$

This means that

$$\int_\Omega \phi(z) \sum_{i=1}^n \sum_{j=1}^m \bar{u}_{ij} f_i(z)\overline{g_j(z)}\,d\lambda(z) = 0 \quad \text{for all} \quad \phi\in\mathcal{D}(\Omega),$$

which implies

$$\sum_{i=1}^n \sum_{j=1}^m \bar{u}_{ij} f_i(z)\overline{g_j(z)} = 0$$

$d\lambda$-almost everywhere in Ω; since the left-hand side is obviously continuously on Ω, this equality holds, in fact, on the whole of Ω. Thus, the function

$$F(x, y) = \sum_{i=1}^n \sum_{j=1}^m \bar{u}_{ij} f_i(x)\overline{g_j(\bar{y})},$$

which is analytic on $\Omega\times\bar{\Omega}$, equals zero whenever $x=\bar{y}$. By Proposition 1, this implies that F is identically zero on $\Omega\times\bar{\Omega}$. Because the functions f_i, $i=1, 2, \ldots, n$, are linearly independent, we have

$$\sum_{j=1}^m u_{ij} g_j(\bar{y}) = 0 \quad \text{for all} \quad y\in\Omega,\ i = 1, 2, \ldots, n;$$

but g_j, $j=1, 2, \ldots, m$, are also linearly independent, and so $u_{ij}=0$ for all i, j, i.e. $u=0$. This means that the range of R is all of $\mathbf{C}^{n\times m}$, which immediately yields the desired conclusion. □

Corollary 3. *The set $\mathcal{T}=\{T_\phi\colon \phi\in\mathcal{D}(\Omega)\}$ is dense in $\mathcal{B}(A^2(\Omega))$ in SOT (the strong operator topology).*

Proof. In view of the preceding theorem, it is certainly dense in WOT (the weak operator topology); and because $\mathcal{T}$ is a subspace, i.e. a convex set, its WOT- and SOT-closures coincide. □

A natural question arises at this point — namely, whether the Toeplitz operators are not actually *norm-dense* in $\mathcal{B}(A^2(\Omega))$. We shall see later that this is not the case — even the C^*-algebra generated by them is smaller than $\mathcal{B}(A^2(\Omega))$. It is true, however, that the norm closure of the Toeplitz operators contains all compact operators. The following two lemmas will be needed in the proof.

Lemma 4. *$A^2(\Omega)$ is a reproducing kernel space, i.e. for each $x\in\Omega$ there exists $g_x\in A^2(\Omega)$ such that*

$$\langle f, g_x\rangle = f(x) \quad \forall f\in A^2(\Omega).$$

The family $\{g_x\colon x\in\Omega\}$ *spans* $A^2(\Omega)$, *and* $\sup_{x\in K}\|g_x\|<+\infty$ *for every compact subset* K *of* Ω.

Proof. The proof is standard but short, so it is given here for completeness. For $\Omega=\mathbf{C}^N$, we have $g_x(z)=e^{\bar{x}z/2}$ and $\|g_x\|=e^{|x|^2/4}$ (see [7]). For $\Omega\subsetneqq\mathbf{C}^N$ and $x\in\Omega$, denote R the largest radius for which the polydisc $B(x,R)=\{z\in\mathbf{C}^N\colon |x_k-z_k|<R,\ k=1,2,\dots,N\}$ lies entirely in Ω. Let χ be the characteristic function of this polydisc. Owing to the mean value property of analytic functions,

$$|B(x,R)|\cdot f(x)=\int_\Omega \chi(z)f(z)\,d\lambda(z)=\langle f,\chi\rangle_{L^2(\Omega)}=\langle f,P_+\chi\rangle_{A^2(\Omega)}\quad \forall f\in A^2(\Omega),$$

and so $g_x=\dfrac{1}{|B(x,R)|}P_+\chi$ will do. Further, $\|P_+\chi\|\leqq\|\chi\|=|B(x,R)|^{1/2}=\pi^{N/2}R^N$. Consequently,

$$\sup_{z\in K}\|g_x\|\leqq\frac{1}{\pi^{N/2}\cdot\operatorname{dist}(K,\mathbf{C}^N\setminus\Omega)^N}<+\infty$$

for every compact subset K of Ω.

Finally, if $\Omega\subsetneqq\mathbf{C}^N$ and $f\perp g_x\ \forall x\in\Omega$, then $f(x)=\langle f,g_x\rangle=0$ on Ω, i.e. $f=0$. It follows that the linear span of the functions g_x, $x\in\Omega$, is dense in $A^2(\Omega)$. □

Denote by $L^\infty_C(\Omega)$ the subspace of $L^\infty(\Omega)$ consisting of functions supported on compact subsets of Ω, and let $V(\Omega)$ be the closure of $L^\infty_C(\Omega)$ in $L^\infty(\Omega)$. Thus, loosely speaking, $V(\Omega)$ consists of bounded functions on Ω which vanish on the boundary $\partial\Omega$ and — if Ω is unbounded — at infinity.

Lemma 5. *For* $\phi\in V(\Omega)$, T_ϕ *is a compact operator.*

Proof. It suffices to prove the assertion for $\phi\in L^\infty_C(\Omega)$. So suppose that ϕ vanishes on $\Omega\setminus K$, where $K\subset\Omega$ is compact, and let $\{f_n\}\subset A^2(\Omega)$ be a sequence converging weakly to zero. Such a sequence must be bounded — say, $\|f_n\|\leqq c\ \forall n$. Owing to Lemma 4,

$$|f_n(z)|\leqq\|f_n\|\cdot\|g_z\|\leqq c\sup_{z\in K}\|g_z\|\equiv C<+\infty\quad\forall z\in K,$$

i.e. the family of functions $\{f_n\}$ is uniformly bounded on K. On the other hand, $f_n\xrightarrow{w}0$ implies $f_n(z)=\langle f_n,g_z\rangle\to0\ \forall z\in\Omega$, and we may apply the Lebesgue dominated convergence theorem to conclude that

$$\|T_\phi f_n\|^2\leqq\|\phi f_n\|^2=\int_K|\phi(z)f_n(z)|^2\,d\lambda(z)\to0$$

as $n\to\infty$. It follows that T_ϕ maps weakly convergent sequences into norm convergent ones, and so must be compact. □

Theorem 6. *The closure of the set* $\mathcal{T}=\{T_\phi\colon \phi\in\mathcal{D}(\Omega)\}$ *in the operator norm topology coincides with* $\operatorname{Comp}(A^2(\Omega))$, *the space of all compact linear operators on* $A^2(\Omega)$.

Proof. According to the last lemma, we have $\operatorname{clos}_{\|\cdot\|}\mathcal{T}\subset\operatorname{Comp}$; it remains to prove the reverse inclusion.

Recall that the dual of $\operatorname{Comp}(H)$, where H is a separable Hilbert space, may be identified with $\operatorname{Trace}(H)$, the space of all trace class operators on H equipped with the trace norm $\|\cdot\|_{\mathrm{Tr}}$; the pairing is given by $(K, T)\mapsto\operatorname{Tr}(KT)=\operatorname{Tr}(TK)$, Tr being the trace.

Suppose that $\Omega\subsetneqq\mathbf{C}^N$ and that $\operatorname{clos}_{\|\cdot\|}\mathcal{T}$ is a proper subset of $\operatorname{Comp}(A^2(\Omega))$. By the Hahn—Banach theorem, there exists $T\in\operatorname{Trace}(A^2(\Omega))$, $T\neq 0$, such that

$$\operatorname{Tr}(TT_\phi)=0\quad\forall\phi\in\mathcal{D}(\Omega).$$

Let A, B be two Hilbert—Schmidt operators such that $T=AB^*$, $\{e_n\}_{n=0}^\infty$ an orthonormal basis for $A^2(\Omega)$, $f_n=Ae_n$, $g_n=Be_n$. Then $\operatorname{Tr}(TT_\phi)=\operatorname{Tr}(B^*T_\phi A)=\sum_{n=0}^\infty\langle B^*T_\phi Ae_n, e_n\rangle$, and so the last condition may be rewritten as

$$\sum_{n=0}^\infty\langle T_\phi f_n, g_n\rangle=0\quad\forall\phi\in\mathcal{D}(\Omega),$$

or

$$\sum_{n=0}^\infty\int_\Omega\phi(z)f_n(z)\overline{g_n(z)}\,d\lambda(z)=0\quad\forall\phi\in\mathcal{D}(\Omega)$$

(if $\Omega=\mathbf{C}^N$, replace $d\lambda$ by $d\mu$). Because ϕ has compact support, $\sum\|f_n\|^2<+\infty$ and $\sum\|g_n\|^2<+\infty$, we may interchange the integration and summation signs, which yields

$$\int_\Omega\phi(z)F(z,\bar z)\,d\lambda(z)\ (\text{or } d\mu(z))=0\quad\forall\phi\in\mathcal{D}(\Omega),$$

where

$$F(x,y)=\sum_{n=0}^\infty f_n(z)\overline{g_n(\bar y)}=\operatorname{Tr}(TG_{x,y}),$$

$$G_{x,y}=\langle\cdot, g_x\rangle g_{\bar y}.$$

It follows that $F(z,\bar z)=0$ for almost all $z\in\Omega$; in other words, the function $F(x,y)$, analytic on $\Omega\times\bar\Omega$, vanishes when $x=\bar y$. Appealing to Proposition 1, we conclude that $F=0$ everywhere on $\Omega\times\bar\Omega$, i.e. $\operatorname{Tr}(TG_{x,\bar y})=0\ \forall x,y\in\Omega$. In view of Lemma 4, $\operatorname{Tr}(TK)=0$ for all rank one operators K; by linearity and continuity, $\operatorname{Tr}(TK)=0$ for all compact K, whence $T=0$ — a contradiction. The proof is complete. □

Remark. For $\Omega=\mathbf{D}$, a different proof may be found in [14]. For $\Omega=\mathbf{C}^N$, Theorem 6 was proved independently by Berger and Coburn ([8], Theorem 9). Their proof makes use of a sophisticated machinery developed by Berezin, but very likely could be reduced to the one given above. On page 38 of [8], Berger and Coburn consider the set of all Toeplitz operators with bounded symbols

$$\{T_\phi\colon \phi\in L^\infty(\mathbf{C}^N)\} \tag{1}$$

and prove that it is *not* dense in $\mathscr{B}(A^2(\mathbf{C}^N))$ in the norm topology. They conjectured, however, that "the C^*-algebra generated by the set (1) ... could contain all bounded operators". The subsequent sections show that this conjecture is not true.

3. Toeplitz algebra on D

Let us start with $A^2(\mathbf{D})$, where the proof is most transparent. Modifying the definition a little, we replace the Lebesgue measure $d\lambda(z)$ on $\mathbf{D}$ by its multiple

$$dv(z) = \frac{1}{\pi}\,d\lambda(z)$$

chosen so that $\mathbf{D}$ has measure 1. If $f=\sum_{n=0}^\infty f_n z^n$ is a function analytic on $\mathbf{D}$, its norm in $L^2(\mathbf{D}, dv)$ is easily seen to be given by

$$\|f\|^2 = \sum_{n=0}^\infty \frac{|f_n|^2}{n+1},$$

and so $f\in A^2(\mathbf{D})$ iff this quantity is finite. For $f, g\in A^2(\mathbf{D})$, we have

$$\langle f, g\rangle_{A^2(\mathbf{D})} = \sum_{n=0}^\infty \frac{f_n\overline{g_n}}{n+1}.$$

Thus, the set

$$e_n(z) = \sqrt{n+1}\,z^n,\quad n\in\mathbf{N}, \tag{2}$$

constitutes an orthonormal basis for $A^2(\mathbf{D})$.

Define

$$\mathscr{A}(T_z) := \{T\in\mathscr{B}(A^2(\mathbf{D}))\colon\ T-T_z^*TT_z\in\mathrm{Comp}\}.$$

There is an alternative definition of $\mathscr{A}(T_z)$:

Proposition 7. $\mathscr{A}(T_z)=\{T\in\mathscr{B}(A^2(\mathbf{D}))\colon\ [T, T_z]\in\mathrm{Comp}\}$.

Proof. The operators

$$I-T_z^*T_z = \mathrm{diag}\left(1-\frac{n+1}{n+2}\right) \quad\text{and}\quad I-T_zT_z^* = \mathrm{diag}\left(1-\frac{n}{n+1}\right)$$

are compact (diagonality is understood with respect to the basis (2)). Consequently,

$$T-T_z^*TT_z\in\text{Comp}\Rightarrow T_z(T-T_z^*TT_z)=(T_zT-TT_z)+(I-T_zT_z^*)TT_z\in\text{Comp}$$
$$\Rightarrow T_zT-TT_z\in\text{Comp}.$$

Similarly,

$$T_zT-TT_z\in\text{Comp}\Rightarrow T_z^*(T_zT-TT_z)=(T-TT_z^*T_z)-(I-T_z^*T_z)T\in\text{Comp}$$
$$\Rightarrow T-T_z^*TT_z\in\text{Comp},$$

and the assertion follows. □

Theorem 8. (i) *$\mathscr{A}(T_z)$ is a C^*-algebra.*
(ii) $\forall\phi\in L^\infty(\mathbf{D})$: $T_\phi\in\mathscr{A}(T_z)$.

Proof. (i) It's clear that $\mathscr{A}(T_z)$ is a linear and selfadjoint set, which is moreover closed in the norm topology; so the only thing that remains to be checked is that it is closed under multiplication. But

$$[AB,T_z]=A(BT_z-T_zB)+(AT_z-T_zA)B=A[B,T_z]+[A,T_z]B,$$

which is compact if $[A,T_z]$ and $[B,T_z]$ are.
(ii) If $\phi\in L^\infty(\mathbf{D})$, then

$$T_\phi-T_z^*T_\phi T_z=T_\phi-T_{\bar z}T_\phi T_z=T_{\phi-\bar z\phi z}=T_{(1-|z|^2)\phi(z)}.$$

But $(1-|z|^2)\phi(z)\in V(\mathbf{D})$ and so the last operator is compact by Lemma 5. □

Corollary 9. *The C^*-algebra generated by $\{T_\phi\colon\phi\in L^\infty(\mathbf{D})\}$ is strictly smaller than $\mathscr{B}(A^2(\mathbf{D}))$.*

Proof. In view of the preceding theorem, it suffices to find an operator not in $\mathscr{A}(T_z)$; one of them is

$$J=\operatorname{diag}(-1)^n,$$

since

$$J-T_z^*JT_z=\operatorname{diag}\left((-1)^n-\frac{n+1}{n+2}(-1)^{n+1}\right)$$

certainly is not compact. □

Theorem 8 carries over trivially to the classical Hardy space H^2. Indeed, when T is a Toeplitz operator on H^2, then (see [17])

$$T=S^*TS,$$

where S is the usual (forward) shift operator on H^2. Thus, if we define

$$\mathscr{A}(S):=\{T\in\mathscr{B}(H^2)\colon T-S^*TS\in\text{Comp}(H^2)\},$$

then the following assertions are immediate.

Proposition 10. (i) $\mathscr{A}(S)=\{T\in\mathscr{B}(H^2)\colon [T, S]\in\mathrm{Comp}\,(H^2)\}$.

(ii) *$\mathscr{A}(S)$ is a C^*-subalgebra of $\mathscr{B}(H^2)$.*

(iii) *$T_\phi\in\mathscr{A}(S)$ for every Toeplitz operator T_ϕ on H^2.*

(iv) *The C^*-algebra generated by the Toeplitz operators in $\mathscr{B}(H^2)$ is strictly smaller than $\mathscr{B}(H^2)$.*

The proofs are similar to those for Proposition 7—Corollary 9, and actually a lot simpler. In the Corollary, the same operator J works (this time, of course, diagonality is understood with respect to the standard orthonormal basis $\{z^n\}_{n=0}^\infty$ of H^2). The algebras $\mathscr{A}(T_z)$ and $\mathscr{A}(S)$ are, in fact, isomorphic; moreover, the isomorphism $\mathscr{A}(T_z)\to\mathscr{A}(S)$ may be chosen to be spatial, i.e. of the form

$$T\mapsto W^*TW,$$

where W is a fixed unitary operator from H^2 onto $A^2(\mathbf{D})$. To see this, let W be the operator mapping the standard basis $\{z^n\}_{n\in\mathbf{N}}$ of H^2 onto the basis $\{\sqrt{n+1}\,z^n\}_{n\in\mathbf{N}}$ of $A^2(\mathbf{D})$,

$$W\colon \sum_{n=0}^\infty f_n z^n \to \sum_{n=0}^\infty f_n \sqrt{n+1}z^n.$$

Then

$$T\in\mathscr{A}(T_z) \Leftrightarrow [T, T_z]\in\mathrm{Comp} \Leftrightarrow W^*TT_zW-W^*T_zTW\in\mathrm{Comp}$$
$$\Leftrightarrow (W^*TW)(W^*T_zW)-(W^*T_zW)(W^*TW)\in\mathrm{Comp}$$
$$\Leftrightarrow (W^*TW)S-S(W^*TW)\in\mathrm{Comp} \Leftrightarrow W^*TW\in\mathscr{A}(S); \tag{3}$$

here T_z is the Toeplitz operator on $A^2(\mathbf{D})$, not on H^2, and the last-but-one equivalence is due to the fact that

$$W^*T_zW-S = S\cdot\mathrm{diag}\left(\sqrt{\frac{n+1}{n+2}}-1\right)$$

is a compact operator (diagonality is understood with respect to the standard basis of H^2).

In general, we may define

$$\mathscr{A}(M) := \{T\in\mathscr{B}(H)\colon [M, T]\in\mathrm{Comp}\,(H)\}$$

for arbitrary operator M on a Hilbert space H. The following theorem generalizes the considerations of the previous paragraph.

Theorem 11. (i) *$\mathscr{A}(M)=\mathscr{A}(M+K)$ for arbitrary compact operator K.*

(ii) *Suppose that M is essentially normal, $\sigma_e(M)=\mathbf{T}$, the unit circle, and* $\mathrm{ind}\,M=-1$. *Then there exists a unitary operator $W\colon H^2\to H$ such that the transformation*

$$T\mapsto W^*TW$$

is a C^*-algebra isomorphism of $\mathcal{A}(M)$ onto $\mathcal{A}(S)$. In particular, $\mathcal{A}(M)$ is a proper C^*-subalgebra of $\mathcal{B}(H)$.

Proof. (i) is immediate (actually, it has already been used in the end of the last-but-one paragraph).

(ii) According to the Brown—Douglas—Fillmore theory [10], an operator M satisfying these conditions is unitarily equivalent to S modulo the compacts, i.e. there exists a unitary operator W: $H^2 \to H$ and a compact operator $K \in \mathrm{Comp}(H)$ such that

$$WSW^* = M + K.$$

Owing to part (i), $\mathcal{A}(M) = \mathcal{A}(M+K)$, and repeating the argumentation from (3) — with $M+K$ in place of T_z — leads to the desired conclusion. □

4. Toeplitz algebras on $\Omega \subsetneqq \mathbf{C}$

Now we are in a position to prove the analogue of Theorem 8 for a general domain $\Omega \subsetneqq \mathbf{C}$. In case Ω is simply connected, a short proof may be given using the Riemann mapping theorem. We present it first and then, in case Ω is bounded (but not necessarily simply connected), we exhibit another proof based on the results of Axler, Conway and McDonald [2]. So suppose $\Omega \subsetneqq \mathbf{C}$ is simply connected and let Φ: $\Omega \to \mathbf{D}$ be the Riemann mapping function.

Lemma 12. $\sigma_e(T_\Phi) = \mathbf{T}$ *and* $\operatorname{ind} T_\Phi = -1$.

Proof. For arbitrary $x \in \mathbf{D}$, the operator $T_{\Phi - x}$ is injective and its range clearly consists exactly of functions vanishing at $\Phi^{-1}(x)$ (since $\Phi(z) - x$, loosely speaking, behaves like $z - \Phi^{-1}(x)$ in a sufficiently small neighbourhood of $\Phi^{-1}(x)$). Hence, $\operatorname{ind} T_{\Phi - x} = -1$ for $x \in \mathbf{D}$. On the other hand, $\|T_\Phi\| \leqq \|\Phi\|_\infty = 1$ and so $\operatorname{ind} T_{\Phi - x} = 0$ if $|x| > 1$. Since

$$x \mapsto \operatorname{ind} T_{\Phi - x}$$

is a continuous function on $\mathbf{C} \setminus \sigma_e(T_\Phi)$, necessarily $\sigma_e(T_\Phi) = \mathbf{T}$. □

Lemma 13. *Assume T is a Fredholm operator and $I - T^*T \in \mathrm{Comp}$. Then also $I - TT^* \in \mathrm{Comp}$ and T is essentially normal.*

Proof. (Cf. [10], proof of Theorem 3.1.) By assumption, $I - T^*T \in \mathrm{Comp}$, and on multiplying by the inverse of $I + (T^*T)^{1/2}$, we find that $I - (T^*T)^{1/2} \in \mathrm{Comp}$. If $T = W(T^*T)^{1/2}$ is the polar decomposition, it follows that T is a compact perturbation of the partial isometry W. Now $I - WW^*$ is the projection onto $(\operatorname{Ran} W)^\perp = (\operatorname{Ran} T)^\perp$, which is a subspace of finite dimension (since T is Fredholm); hence,

$I-WW^*$ is a finite rank operator. Because T is a compact perturbation of W, $I-TT^*$ must be a compact operator. □

Theorem 14. *Assume* $\Omega\subset\mathbf{C}$ *is a simply connected domain (of finite Lebesgue measure) and let* $\Phi\colon \Omega\to\mathbf{D}$ *be the Riemann mapping function. Then* $T_f\in\mathcal{A}(T_\Phi)$ $\forall f\in L^\infty(\Omega)$ *and there exists a unitary operator* $W\colon H^2\to A^2(\Omega)$ *such that the transformation* $T\mapsto W^*TW$ *establishes a* C^**-isomorphism of* $\mathcal{A}(T_\Phi)$ *onto* $\mathcal{A}(S)$. *In particular, the* C^**-algebra generated by the Toeplitz operators* T_f, $f\in L^\infty(\Omega)$, *is a proper subalgebra of* $\mathcal{B}(A^2(\Omega))$.

Proof. Let $f\in L^\infty(\Omega)$. We have

$$T_f-T_\Phi^*T_fT_\Phi = T_{f-\bar\Phi f\Phi} = T_{(1-|\Phi|^2)f}.$$

For arbitrary $\delta>0$, the set

$$\{y\in\Omega\colon |\Phi(y)|\leqq 1-\delta\}$$

is a compact subset of Ω. Consequently, $(1-|\Phi|^2)f\in V(\Omega)$, and Lemma 5 implies that

$$T_f-T_\Phi^*T_fT_\Phi\in\text{Comp}\quad \forall f\in L^\infty(\Omega). \tag{4}$$

Taking $f=1$, we see that $I-T_\Phi^*T_\Phi$ must be compact, and an application of Lemma 12 and Lemma 13 shows that $I-T_\Phi T_\Phi^*$ is compact as well. Multiplying (4) by T_Φ from the left yields (cf. the proof of Proposition 7)

$$T_\Phi T_f-T_fT_\Phi\in\text{Comp},$$

i.e. $T_f\in\mathcal{A}(T_\Phi)$. It remains only to make use of Lemma 12, Lemma 13 and Theorem 11. □

Now assume that $\Omega\subsetneqq\mathbf{C}$ is bounded, but not necessarily simply connected. Let us recall briefly the pertinent results of [2]. A point $x\in\partial\Omega$ is called removable if there exists a neighbourhood V of x such that every function $f\in A^2(\Omega)$ can be analytically continued to V. (For instance, every isolated point of $\partial\Omega$ is removable, by a variant of Riemann's removable singularity theorem.) The collection of all removable boundary points is called $\partial_r\Omega$, the *removable boundary* of Ω; $\partial_e\Omega:=\partial\Omega\setminus\partial_r\Omega$ is the *essential boundary*.

The following assertions are proved in [2].

(A) ([2], Proposition 3) $\partial_e\Omega\supset\partial\bar\Omega$, *the boundary of the closure of* Ω.

(B) ([2], Proposition 8) *If* $f\in C(\bar\Omega)$, *then* $H_f\in\text{Comp}\,(A^2(\Omega))$.

(C) ([2], Corollary 10) *If* $f\in C(\bar\Omega)$, *then* $\sigma_e(T_f)=f(\partial_e\Omega)$.

Now we are ready to prove the main theorem of this section.

Theorem 15. *Assume that Ω is a bounded domain in* **C**. *Then there exists a unitary operator W: $H^2 \to A^2(\Omega)$ such that the transformation*

$$T \mapsto W^* T W, \quad \mathscr{B}(A^2(\Omega)) \to \mathscr{B}(H^2),$$

sends every Toeplitz operator T_f, $f \in L^\infty(\Omega)$, on $A^2(\Omega)$ to an element of $\mathscr{A}(S)$. In particular, the C^*-algebra generated by the Toeplitz operators on $A^2(\Omega)$ is a proper subalgebra of $\mathscr{B}(A^2(\Omega))$.

Proof. Without loss of generality we may assume $\mathbf{D} \subset \Omega$. Let $\Phi \in L^\infty(\Omega)$ be the function $z/|z|$ adjusted in a small neighbourhood of 0 so as to be continuous on $\bar{\Omega}$; for instance, take

$$\Phi(re^{it}) = \begin{cases} e^{it} & \text{if } r \geqq 1, \\ re^{it} & \text{if } r \leqq 1. \end{cases}$$

Because $\Phi \in C(\bar{\Omega})$, the Hankel operators H_Φ and $H_{\bar{\Phi}}$ are compact in view of (B), and, consequently, so is the operator

$$[T_f, T_\Phi] = H_{\bar{\Phi}}^* H_f - H_{\bar{f}}^* H_\Phi$$

for arbitrary $f \in L^\infty(\Omega)$. Thus, $T_f \in \mathscr{A}(T_\Phi)$. In particular, $T_{\bar{\Phi}} \in \mathscr{A}(T_\Phi)$, i.e. T_Φ is essentially normal. In view of (A) and (C), $\sigma_e(T_\Phi) = \Phi(\partial_e \Omega) = \Phi(\partial \bar{\Omega}) = \mathbf{T}$. If we prove that $\operatorname{ind} T_\Phi = -1$, we can apply Theorem 11 and the desired conclusions will follow.

For $0 \leqq \theta \leqq 1$, define

$$\Phi_\theta(re^{it}) = \begin{cases} r^\theta e^{it} & \text{for } r \geqq 1, \\ re^{it} & \text{for } r \leqq 1. \end{cases}$$

Then $\Phi_\theta \in L^\infty(\Omega)$, and so the operators T_{Φ_θ} are defined. Moreover, for arbitrary $\theta_1, \theta_2 \in \langle 0, 1 \rangle$,

$$\|T_{\Phi_{\theta_1}} - T_{\Phi_{\theta_2}}\| \leqq \|\Phi_{\theta_1} - \Phi_{\theta_2}\|_\infty = \sup_{1 \leqq r \leqq \operatorname{diam} \Omega} |r^{\theta_1} - r^{\theta_2}| \leqq c \cdot |\theta_1 - \theta_2|,$$

where $\operatorname{diam} \Omega$ is the diameter of Ω and

$$c = \sup_{\substack{1 \leqq r \leqq \operatorname{diam} \Omega \\ 0 \leqq \theta \leqq 1}} |r^\theta \ln r|$$

is independent of r and θ. This shows that the mapping $\theta \mapsto T_{\Phi_\theta}$ is continuous. Besides, $\sigma_e(T_{\Phi_\theta}) = \Phi_\theta(\partial_e \Omega) \not\ni 0$, i.e. all T_{Φ_θ} are Fredholm operators and so their index is defined. Since "ind" is a continuous integer-valued function, it must be constant along the path $\theta \mapsto T_{\Phi_\theta}$, whence $\operatorname{ind} T_{\Phi_{\theta_0}} = \operatorname{ind} T_{\Phi_{\theta_1}}$, or

$$\operatorname{ind} T_\Phi = \operatorname{ind} T_z.$$

But $\ker T_z = \{0\}$, while $\operatorname{Ran} T_z$ consists of all functions from $A^2(\Omega)$ that vanish at 0. Consequently, $\operatorname{ind} T_z = -1$, and the proof is complete. □

5. Toeplitz algebra on $\mathbf{C}^N$

We have seen that for $\Omega \subsetneqq \mathbf{C}$, all Toeplitz operators belong to the essential commutator of certain unilateral shift. This turns out to hold for $\Omega = \mathbf{C}$ as well, but not for $\Omega = \mathbf{C}^N$, $N > 1$.

Let us first consider $\Omega = \mathbf{C}$. Recall that the Fock space $A^2(\mathbf{C})$ has an orthonormal basis $\{e_n\}_{n=0}^{\infty}$,

$$e_n(z) := (n!\,2^n)^{-1/2} z^n. \tag{5}$$

Denote Z the forward shift operator with respect to this basis, and let

$$\Phi(z) = \frac{z}{|z|} = e^{i \arg z}.$$

Theorem 16. (i) *The operator T_Φ is a compact perturbation of Z. Consequently, $\mathscr{A}(T_\Phi) = \mathscr{A}(Z)$.*

(ii) $T_f \in \mathscr{A}(T_\Phi)$, *i.e.* $T_f T_\Phi - T_\Phi T_f \in \mathrm{Comp}$, *for every* $f \in L^\infty(\mathbf{C})$.

(iii) *There exists a unitary operator $W\colon H^2 \to A^2(\mathbf{C})$ such that the transformation $T \mapsto W^* T W$ is a C^*-isomorphism of $\mathscr{A}(Z)$ onto $\mathscr{A}(S)$.*

In particular, the C^*-algebra generated by all T_f, $f \in L^\infty(\mathbf{C})$, is a proper subset of $\mathscr{B}(A^2(\mathbf{C}))$.

Proof. (i) A direct computation reveals that

$$\langle T_\Phi z^n, z^m \rangle = \begin{cases} 0 & m \neq n+1, \\ 2^{n+(3/2)} \Gamma\left(n+\frac{3}{2}\right) & m = n+1, \end{cases}$$

where Γ is Euler's gamma-function. Thus $T_\Phi e_n = c_n e_{n+1}$, where

$$c_n = \frac{\Gamma\left(n+\frac{3}{2}\right)}{\Gamma(n+1)^{1/2} \cdot \Gamma(n+2)^{1/2}}.$$

It follows that

$$Z - T_\Phi = Z \cdot \operatorname{diag}(1 - c_n)$$

(diagonality is understood with respect to the basis (5)). An application of Stirling's formula shows that $c_n \to 1$, and so $Z - T_\Phi$ is a compact operator.

(ii) Recall the formulas

$$T_{fg} - T_f T_g = H_{\bar f}^* H_g, \quad T_f T_g - T_g T_f = H_{\bar g}^* H_f - H_{\bar f}^* H_g \tag{6}$$

which hold for arbitrary $f, g \in L^\infty(\mathbf{C})$. Owing to the second one,

$$T_f T_\Phi - T_\Phi T_f = H_{\bar\Phi}^* H_f - H_{\bar f}^* H_\Phi$$

will be compact for arbitrary $f\in L^\infty(\mathbf{C})$ if H_Φ, $H_{\bar\Phi}\in$Comp. The latter is equivalent to $H_\Phi^*H_\Phi$, $H_{\bar\Phi}^*H_{\bar\Phi}\in$Comp, respectively, and the first formula in (6) shows that this in turn is equivalent to

$$I-T_\Phi^*T_\Phi \quad \text{and} \quad I-T_\Phi T_\Phi^*\in\text{Comp},$$

respectively. Owing to part (i), the last two operators are compact perturbations of $I-Z^*Z=0$ and $I-ZZ^*=\langle .,e_0\rangle e_0$, respectively, and the result follows.

(iii) Define $W\colon H^2\to A^2(\mathbf{C})$ by mapping the standard basis of H^2 onto the basis $\{e_n\}_{n\in\mathbf{N}}$ of $A^2(\mathbf{C})$,

$$W\colon z^n\in H^2\mapsto\frac{z^n}{\sqrt{n!2^n}}\in A^2(\mathbf{C}).$$

This operator is unitary and the transformation $T\mapsto W^*TW$ maps Z to S; hence, as before, it induces a C^*-isomorphism of $\mathscr{A}(Z)=\mathscr{A}(T_\Phi)$ onto $\mathscr{A}(S)$. The proof is complete. □

A variant of this result may also be obtained for $A^2(\mathbf{C}^N)$, $N\geqq 2$; however, things come off a little differently this time — the corresponding C^*-algebra is no longer spatially isomorphic to $\mathscr{A}(S)$. All the same, it is still a proper subset of $\mathscr{B}(A^2(\mathbf{C}^N))$.

We shall need some results of Berger and Coburn [7]. Define

$$ESV:=\{\Phi\in L^\infty(\mathbf{C}^N)\colon \lim_{|z|\to+\infty}\ \operatorname*{ess-sup}_{|z-w|\leqq 1}|\Phi(z)-\Phi(w)|=0\}$$

and

$$BC\,ESV:=\{\Phi\in ESV\colon \Phi \text{ is continuous on } \mathbf{C}^N\}.$$

Here, as usual,

$$|x|=\left(\sum\nolimits_{n=1}^N|x_n|^2\right)^{1/2} \quad\text{for}\quad x=(x_1,x_2,\dots,x_N)\in\mathbf{C}^N.$$

Further, let $\mathscr{S}:=\{x\in\mathbf{C}^N\colon |x|=1\}$ be the unit sphere in $\mathbf{C}^N$.

Proposition 17. *Let* $G\colon\mathscr{S}\to\mathbf{C}$ *be a continuous function on* $\mathscr{S}$. *Define*

$$\Phi(rx)=\begin{cases}G(x) & \text{if}\ \ r\geqq 1,\\ rG(x) & \text{if}\ \ r\leqq 1,\end{cases}\quad x\in\mathscr{S},\ 0\leqq r<+\infty. \tag{7}$$

Then

(i) $\Phi\in BCESV$, *and*

(ii) *the Hankel operators* H_Φ, $H_{\bar\Phi}$ *are compact.*

Assume further that

$$G(\mathscr{S})=\mathbf{T}. \tag{8}$$

Then also

(iii) $\sigma_e(T_\Phi)=\mathbf{T}$ *and*

(iv) $\operatorname{ind} T_\Phi=0$ *if* $N\geqq 2$, *and* $\operatorname{ind} T_\Phi$ *is minus the winding number of the function* $G\colon \mathbf{T}\to\mathbf{T}$ *(with respect to the origin) when* $N=1$.

Proof. (i) Φ is continuous and bounded since G is, and $\Phi\in ESV$ in view of [7], Theorem 3(i).

(ii) Theorem 11 of [7] says that H_Φ and $H_{\bar\Phi}$ are compact for arbitrary $\Phi\in ESV$.

(iii) & (iv) Immediate consequences of [7], Theorem 19. □

Remark. It is possible to prove part (iv) in another way, using the idea from the end of the proof of Theorem 15. Suppose that $\theta\mapsto G_\theta$, $\theta\in\langle 0,1\rangle$, $G_\theta\in C(\mathscr{S}\to\mathbf{T})$, is a homotopy between G_0 and G_1; construct functions Φ_θ according to (7) and consider the Toeplitz operators T_{Φ_θ}. It can be shown that T_{Φ_θ} are Fredholm operators $\forall\theta\in\langle 0,1\rangle$, and, consequently, $\operatorname{ind} T_{\Phi_0}=\operatorname{ind} T_{\Phi_1}$. If $N\geqq 2$, $\mathscr{S}$ is simply connected, and so the homotopy group $\pi(\mathscr{S},\mathbf{T})=\pi_{2N-1}(\mathbf{T})$ is trivial; hence, there is a homotopy connecting $G_0=G$ to $G_1=1$. It follows that

$$\operatorname{ind} T_\Phi = \operatorname{ind} T_1 = \operatorname{ind} I = 0.$$

If $N=1$, $\pi(\mathbf{T},\mathbf{T})=\pi_1(\mathbf{T})$ is isomorphic to $\mathbf{Z}$; an isomorphism is given by $G\mapsto \operatorname{wind} G$. It follows that there is a homotopy connecting $G_0=G$ to G_1, $G_1(e^{it})=e^{kit}$, $k=\operatorname{wind} G$, and

$$\operatorname{ind} T_\Phi = \operatorname{ind} T_{\Phi_1} = -k.$$

Thus the occurrence of two cases — $N=1$ versus $N\geqq 2$ — in the part (iv) is of topological nature, being related to (non)vanishing of the homotopy groups $\pi_n(\mathbf{T})$.

Theorem 18. *Assume that the functions* $G\colon\mathscr{S}\to\mathbf{C}$ *and* $\Phi\colon \mathbf{C}^N\to\mathbf{C}$ *satisfy the conditions (7), (8), and that either* $N\geqq 2$ *or* $N=1$ *and* $\operatorname{wind} G=0$. *Then*

(a) $T_f\in\mathscr{A}(T_\Phi)$ *for all* $f\in L^\infty(\mathbf{C}^N)$.

(b) *There exists a unitary operator* $W\colon L^2(\mathbf{T})\to A^2(\mathbf{C}^N)$ *such that the transformation* $T\mapsto W^*TW$ *is a* C^**-isomorphism of* $\mathscr{A}(T_\Phi)$ *onto* $\mathscr{A}(U)$, *where* U *is the bilateral (forward) shift operator (multiplication by* z*) on the Lebesgue space* $L^2(\mathbf{T})$. *In particular,* $\mathscr{A}(T_\Phi)$ *is a* C^**-algebra.*

(c) *The operator* $J\colon L^2(\mathbf{T})\to L^2(\mathbf{T})$, $Jf(z):=f(-z)$, *does not belong to* $\mathscr{A}(U)$. *Consequently,* $\mathscr{A}(T_\Phi)$ *is a proper* C^**-subalgebra of* $\mathscr{B}(A^2(\mathbf{C}^N))$.

Proof. (a) For arbitrary $f\in L^\infty(\mathbf{C}^N)$,

$$T_fT_\Phi - T_\Phi T_f = H_{\bar\Phi}^* H_f - H_{\bar f}^* H_\Phi,$$

and the operators H_Φ, $H_{\bar\Phi}$ are compact by Proposition 17, (ii).

(b) Taking $f=\bar{\Phi}$ in part (a) shows that T_Φ is essentially normal. According to Proposition 17, (iii) and (iv), $\sigma_e(T_\Phi)=\mathbf{T}=\sigma_e(U)$ and $\operatorname{ind} T_\Phi=0=\operatorname{ind} U$. Hence, by the Brown—Douglas—Fillmore theory [10], there exists a unitary operator $W\colon L^2(\mathbf{T})\to A^2(\mathbf{C}^N)$ such that

$$W^* T_\Phi W = U+K,$$

where $K\in\mathrm{Comp}$. The result follows in the same way as in the proof of Theorem 11, with S replaced by U.

(c) With respect to the standard orthonormal basis $\{e_n\}_{n\in\mathbf{Z}}$, $e_n(z)=z^n$, $z\in\mathbf{T}$, of $L^2(\mathbf{T})$, the operators J and U are given by

$$Ue_n = e_{n+1}, \quad Je_n = (-1)^n e_n \quad (n\in\mathbf{Z}).$$

It follows that $UJ-JU=2UJ$; but the operator UJ is unitary, and so certainly not compact. □

To be precise, we ought to check that there exist functions G and Φ satisfying the conditions (7) and (8). As an example, take $G(z)=e^{4i\operatorname{Re} z_1}$.

The argument above also applies in the case $N=1$, wind $G\neq 0$; one has only to replace $L^2(\mathbf{T})$ by H^2 and U by S^k or $S^{*(-k)}$ when $k=-\operatorname{ind} T_\Phi=\operatorname{wind} G$. In particular, if $G\colon \mathbf{T}\to\mathbf{T}$ is the identity, we get another proof of Theorem 16.

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