Weyl multipliers, Bochner—Riesz means and special Hermite expansions

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Dedicated to Prof. K. G. Ramanathan on his 70th birthday

1. Introduction

Let $\hat{f}$ denote the Fourier transform of f on $\mathbf{R}^n$. Let B_R^δ be the Bochner—Riesz means for the Laplacian on $\mathbf{R}^n$ defined by

$$(1.1) \qquad (B_R^\delta f)\hat{}\,(\xi) = \left(1-\frac{|\xi|^2}{R^2}\right)_+^\delta \hat{f}(\xi).$$

These operators are related to the summability of Fourier integrals and multiple Fourier series. It is well-known that these operators are not uniformly bounded on $L^p(\mathbf{R}^n)$ unless p lies in the interval $\frac{2n}{n+1+2\delta}<p<\frac{2n}{n-1-2\delta}$. When $\delta>\frac{n-1}{2}$, these operators are uniformly bounded on all $L^p(\mathbf{R}^n)$, for $1\leqq p\leqq\infty$. When $0<\delta\leqq\frac{n-1}{2}$, the conjecture is that B_R^δ are uniformly bounded if p lies in the above interval. When $n=2$ the conjecture is proved and when $n>2$ it is only proved for $\delta>\frac{n-1}{2(n+1)}$ (see [2]). Moreover, in a celebrated work [3] C. Fefferman has shown that the conjecture is false when $\delta=0$.

In this paper we like to treat a similar problem for the Weyl transform. The Weyl transform W takes functions on $\mathbf{C}^n$ into operators bounded on $L^2(\mathbf{R}^n)$. W enjoys many properties of the Fourier transform and is closely related to expansions in terms of Laguerre, Hermite and special Hermite functions. So it will be interesting to study multipliers for the Weyl transform analogous to Fourier multipliers. In [8] Mauceri has studied general multipliers for the Weyl transform. In [15] the author considered multipliers of the form $\varphi(H)$ where H is the Hermite operator $(-\Delta+|x|^2)$.

Here we propose to study multipliers of very special form. Consider the Riesz means S_R^δ of order δ of the Hermite expansions. These are defined by

$$S_R^\delta = \sum \left(1-\frac{2k+n}{R}\right)_+^\delta P_k \tag{1.2}$$

where P_k are the projections of $L^2(\mathbf{R}^n)$ onto the k^{th} eigenspace of H. S_R^δ is a bounded operator on $L^2(\mathbf{R}^n)$. We define the multiplier operator T_R^δ by setting

$$W(T_R^\delta f) = S_R^\delta W(f). \tag{1.3}$$

Analogous to the Fourier transform case one has the following conjecture. T_R^δ are uniformly bounded on $L^p(\mathbf{C}^n)$ iff $\frac{4n}{2n+1+2\delta}<p<\frac{4n}{2n-1-2\delta}$. In [17] we verified this conjecture when f is radial function. In fact, when f is radial, $W(f)$ reduces to the Laguerre transform and $T_R^\delta f$ is nothing but the Riesz means for the expansions interms of the Laguerre functions $L_N^{n-1}(r^2)e^{-(1/2)r^2}$.

In this paper we prove that the conjecture is true when $\delta>1/2$. That is, if $\delta>\frac{1}{2}$ and $\frac{4n}{2n+1+2\delta}<p<\frac{4n}{2n-1-2\delta}$, then

$$\|T_R^\delta f\|_p \leqq C\,\|f\|_p \tag{1.4}$$

with a constant C independent of R. This result can be interpreted as a summability result for the special Hermite expansions. Using transference results we can deduce from (1.4) a summability result for ordinary Hermite expansions on $\mathbf{R}^n$ and also a result for the Bochner—Riesz means for the Laplacian on $\mathbf{R}^{2n}$.

To prove the uniform bounds (1.4) we adapt a method of Fefferman—Stein [4] to reduce matters to proving L^p-L^2 bounds for certain projection operators. This technique was well developed in Sogge [10] where the convergence of the Riesz means for the eigenfunction expansions associated to second order elliptic differential operators on a compact manifold were studied. The same ideas were used by the author in [16] to prove the convergence of the Riesz means for the Hermite series on $\mathbf{R}^{2n}$ for radial functions.

The paper is organised as follows. In the next section we collect the background material and state the main results. In Section 3 we get an estimate for the Laguerre functions and in Section 4 we prove the kernel estimate and the L^p-L^2 bounds for the projections Q_k. Finally in Section 5 we sketch the proof of the main result.

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2. Preliminaries and the main results

The Weyl transform $W(f)$ of a function f on $\mathbf{C}^n$ is a bounded operator on $L^2(\mathbf{R}^n)$ and is defined by

$$W(f)\varphi(\xi) = \int_{\mathbf{C}^n} f(z)\, e^{ix((1/2)y+\xi)}\, \varphi(\xi+y)\, dz \tag{2.1}$$

where $z=x+iy$. The Weyl transform enjoys many properties of the Fourier transform. For example we have an analogue of the Fourier inversion formula

$$f(z) = (2\pi)^{-n}\operatorname{tr}\big(W(z)^* W(f)\big)$$

where $W(z)$ is the operator valued function

$$W(z)\varphi(\xi) = e^{ix((1/2)y+\xi)}\, \varphi(\xi+y)$$

and a Plancherel formula

$$\|f\|_2^2 = (2\pi)^{-n}\|W(f)\|_{\mathrm{HS}}^2. \tag{2.2}$$

Given a bounded operator M on $L^2(\mathbf{R}^n)$ we can define an operator T_M on $C_0^\infty(\mathbf{C}^n)$ by

$$W(T_M f) = MW(f). \tag{2.3}$$

We say that M is an L^p multiplier for the Weyl transform if T_M extends to a bounded operator on $L^p(\mathbf{C}^n)$.

Let Φ_α be the normalised Hermite functions on $\mathbf{R}^n$ which are eigenfunctions of the Hermite operator H. Let P_k be the projection defined by

$$P_k f = \sum_{|\alpha|=k} (f, \Phi_\alpha)\, \Phi_\alpha.$$

We define the Riesz means S_R^δ by

$$S_R^\delta = \sum \left(1-\frac{2k+n}{R}\right)_+^\delta P_k. \tag{2.4}$$

Observe that S_R^δ is a bounded operator on $L^2(\mathbf{R}^n)$. Let $T_R^\delta = T_M$ where $M=S_R^\delta$. We prove the following theorem.

Theorem 2.1. *Let* $1\leqq p \leqq \dfrac{2n}{n+1}$ *and* $\delta > \delta(p) = 2n\left(\dfrac{1}{p}-\dfrac{1}{2}\right)-\dfrac{1}{2}$. *Then we have* $\|T_R^\delta f\|_p \leqq C\|f\|_p$ *for all* $f\in L^p(\mathbf{C}^n)$ *where* C *is independent of* R.

Corollary. *Assume that* $\delta>1/2$. *Then the uniform estimates* (1.4) *hold iff* $\dfrac{4n}{2n+1+2\delta} < p < \dfrac{4n}{2n-1-2\delta}$.

The necessity of the condition has been proved in [17]. The corollary immediately follows from the theorem. To prove the theorem we need to get an estimate for the kernel of T_R^δ and also to prove bounds for certain projection operators which we are going to define presently.

Let $\varphi_k^{n-1}(z)=(2\pi)^{-n}L_k^{n-1}(\frac{1}{2}|z|^2)\,e^{-(1/4)|z|^2}$ where L_k^{n-1} is the Laguerre polynomial of type $(n-1)$. Then in [9] Peetre has shown that the projection P_k is just the Weyl transform of φ_k^{n-1}. Now recall the definition of the twisted convolution $f\times g$ of two functions

$$f\times g(z) = \int_{\mathbf{C}^n} f(z-w)g(w)\,e^{(i/2)\,\mathrm{Im}\, z.\bar{w}}\,dw.$$

Then it is well-known that $\varphi_k^{n-1}\times\varphi_k^{n-1}=\varphi_k^{n-1}$. So, if we set $Q_k f=\varphi_k^{n-1}\times f$ then Q_k is a projection operator. Moreover since $W(f\times g)=W(f)W(g)$ we have $W(Q_k f)=P_k W(f)$ from which it follows that

$$W(T_R^\delta f) = W\left(\sum\left(1-\frac{2k+n}{R}\right)_+^\delta \varphi_k^{n-1}\right)W(f)$$

or

(2.5) $$T_R^\delta f = s_R^\delta \times f$$

where

(2.6) $$s_R^\delta(z) = \sum\left(1-\frac{2k+n}{R}\right)_+^\delta \varphi_k^{n-1}(z).$$

In Section 4 we will prove the following estimate for the kernel $s_R^\delta(z)$.

(2.7) $$|s_R^\delta(z)| \leqq CR^n(1+R^{1/2}|z|)^{-\delta-n-(1/3)}.$$

For the projection operators we prove the following bounds.

(2.8) $$\|Q_k f\|_2 \leqq Ck^{n(1/p-1/2)-1/2}\|f\|_p$$

for $f\in L^p(\mathbf{C}^n)$, $1\leqq p\leqq\dfrac{2n}{n+1}$. Once we have (2.7) and (2.8) we can proceed as in [10] or [16] to the proof of Theorem 2.1.

We will now bring out the connection between the special Hermite functions. These functions $\Phi_{\alpha,\beta}$ are defined by

$$\Phi_{\alpha,\beta}(x, y) = \pi^{-n/2}\int_{\mathbf{R}^n} e^{ix.\xi}\,\Phi_\beta\left(\xi+\frac{y}{2}\right)\Phi_\alpha\left(\xi-\frac{y}{2}\right)d\xi.$$

These functions appear as the entry functions of the Schrodinger representation π_1 of the Heisenberg group (see Strichartz [12]). They form an orthonormal system for $L^2(\mathbf{R}^{2n})$. Let L be the differential operator

$$L = -\Delta_x-\Delta_y+\frac{1}{4}(|x|^2+|y|^2)-iN$$

where $N=\sum_{j=1}^{n}\left(x_j\frac{\partial}{\partial y_j}-y_j\frac{\partial}{\partial x_j}\right)$. It has been proved in [12] that $\Phi_{\alpha,\beta}$ are the eigenfunctions of the operator L. In fact

$$L(\Phi_{\alpha,\beta}) = (2|\alpha|+n)\Phi_{\alpha,\beta}.$$

Given a function $f(z)=f(x, y)$ in $L^2(\mathbf{C}^n)$ we have the special Hermite expansion

$$f(x, y) = \sum_{\alpha,\beta}(f, \Phi_{\alpha,\beta})\Phi_{\alpha,\beta}.$$

The series converges in the L^2 norm if $f\in L^2$. But for other L^p functions it need not converge. Let us set

$$\tilde{Q}_k f = \sum_{|\alpha|=k}\sum_{\beta}(f, \Phi_{\alpha,\beta})\Phi_{\alpha,\beta}$$

and

$$\tilde{S}_R^\delta(z) = \sum\left(1-\frac{2k+n}{R}\right)_+^\delta \tilde{Q}_k f$$

be the Riesz means of order δ of the special Hermite expansion. We claim that $\tilde{Q}_k f=f\times\varphi_k^{n-1}$. This can be seen as follows.

Consider the differential operators

$$Z_j = \frac{\partial}{\partial z_j}+\frac{1}{4}\bar{z}_j,\ \bar{Z}_j = \frac{\partial}{\partial \bar{z}_j}-\frac{1}{4}z_j$$

on $\mathbf{C}^n$. Then it is easily seen that

$$L = -\frac{1}{2}\sum_1^n(Z_j\bar{Z}_j+\bar{Z}_jZ_j).$$

An easy calculation shows that

$$W(Lf) = W(f)H.$$

From this one can prove that for any reasonable function φ,

$$W(\varphi(L)f) = W(f)\varphi(H).$$

Choosing φ in such a way that $\varphi(H)=P_k$ and $\varphi(L)=\tilde{Q}_k$ we obtain

$$\tilde{Q}_k f = f\times\varphi_k^{n-1}.$$

This proves the claim. Therefore, we can interpret Theorem 2.1 as a summability result for the special Hermite expansions.

Theorem 2.2. *Let* $\delta>\frac{1}{2}$ *and* $\frac{4n}{2n+1+2\delta}<p<\frac{4n}{2n-1-2\delta}$. *Then one has* $\|\tilde{S}_R^\delta f\|_p\le C\|f\|_p$. *Moreover, the Riesz means* $\tilde{S}_R^\delta f$ *converge to* f *in the norm.*

This theorem states that the operators $\left(1-\frac{L}{R}\right)_+^\delta$ are uniformly bounded on $L^p(\mathbf{R}^{2n})$ when $\delta>\frac{1}{2}$ and p is in the prescribed range. Using a transference result of Kenig—Stanton—Tomas [6] we can prove the following result concerning the Bochner—Riesz means B_R^δ for the Laplacian on $\mathbf{R}^{2n}$.

Theorem 2.3. *Let* $\delta>\frac{1}{2}$ *and* $\frac{4n}{2n+1+2\delta}<p<\frac{4n}{2n-1-2\delta}$. *Then one has* $\|B_R^\delta f\|_p\leqq C\|f\|_p$ *for all* $f\in L^p(\mathbf{R}^{2n})$.

This theorem is not new. As we have mentioned in the introduction the above theorem has been proved in the bigger range $\delta>\frac{2n-1}{2(2n+1)}$. Finally consider the Riesz means $S_R^\delta f$ for the Hermite expansions on $\mathbf{R}^n$,

$$S_R^\delta f=\sum\left(1-\frac{2k+n}{R}\right)_+^\delta P_k f$$

where $f\in L^p(\mathbf{R}^n)$. In [14] we proved that when $\delta>\frac{n-1}{2}$ one has the uniform estimates

$$\|S_R^\delta f\|_p\leqq C\|f\|_p$$

for all p, $1\leqq p\leqq\infty$. Here we want to deduce a result for S_R^δ when $0<\delta<\frac{n-1}{2}$. The following transference theorem has been proved in [8].

Theorem (Mauceri). *If M is an L^p multiplier for the Weyl transform then M is a bounded operator on $L^p(\mathbf{R}^n)$.*

In view of this theorem we immediately deduce the following result from Theorem 2.1.

Theorem 2.4.
Let $\delta>\frac{1}{2}$ *and* $\frac{4n}{2n+1+2\delta}<p<\frac{4n}{2n-1-2\delta}$. *Then one has* $\|S_R^\delta f\|_p\leqq C\|f\|_p$ *for all* $f\in L^p(\mathbf{R}^n)$.

3. An estimate for the Laguerre functions

The Laguerre polynomials $L_k^\alpha(x)$ satisfy the following equation (see page 107 of [13]).

$$L_k^\alpha(x)=\frac{(-1)^k\pi^{1/2}\Gamma(k+\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right)(2k)!}\int_{-1}^{1}(1-t^2)^{\alpha-(1/2)}H_{2k}(xt)\,dt \tag{3.1}$$

where H_{2k} is the $2k^{\text{th}}$ Hermite polynomial. The above representation is valid for $\alpha>-\frac{1}{2}$. The right-hand side of (3.1) can be viewed as an analytic function of α, analytic for $\operatorname{Re}\alpha>-\frac{1}{2}$. Let us consider the functions

$$\psi_k^\alpha(x)=\frac{\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma(k+\alpha+1)}L_k^\alpha(x)e^{-(1/2)x},\quad x\geqq 0. \tag{3.2}$$

In the next section we require the following estimate for the above functions.

Proposition 3.1.

$$|\psi_k^\alpha(x)|\leqq C(1+|\tau|)^{1/2} \tag{3.3}$$

for all x *uniformly in* $0\leqq\sigma\leqq n$ *where* $\alpha=\sigma+i\tau$.

Proof. Let $\varphi_{2k}(t)$ be the normalised Hermite function defined by

$$\varphi_{2k}(t)=\left(2^{2k}(2k)!\sqrt{\pi}\right)^{-1/2}H_{2k}(t)e^{-(1/2)t^2}. \tag{3.4}$$

We will prove the following estimates:

$$\int_{-1}^{1}(1-t^2)^{-1/2}|\varphi_{2k}(xt)|\,dt\leqq C(2k+1)^{-1/4} \tag{3.5}$$

$$\left|\frac{2^k\Gamma(k+1)\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right)(\Gamma(2k+1))^{1/2}}\right|\leqq C(2k+1)^{1/4}(1+|\tau|)^{1/2}. \tag{3.6}$$

It is clear that the proposition follows from these two estimates.

Recall the Stirling's formula which states that

$$\Gamma(z)=z^{z-(1/2)}e^{-z}e^{J(z)} \tag{3.7}$$

where $J(z)$ tends to zero as $z\to\infty$. From (3.7) it is clear that

$$\frac{2^k\Gamma(k+1)}{(\Gamma(2k+1))^{1/2}}\leqq C(2k+1)^{1/4}. \tag{3.8}$$

To estimate $\Gamma(\alpha+1)/\Gamma\left(\alpha+\frac{1}{2}\right)$, we use Legendre's duplication formula, namely,

$$\sqrt{\pi}\,\Gamma(2\alpha)=2^{2\alpha-1}\Gamma(\alpha)\Gamma\left(\alpha+\tfrac{1}{2}\right). \tag{3.9}$$

In view of this

$$\frac{\sqrt{\pi}\,\Gamma(\alpha+1)}{\Gamma\left(\alpha+\frac{1}{2}\right)}=\frac{2^{2\alpha-1}\Gamma(\alpha)\Gamma(\alpha+1)}{\Gamma(2\alpha)}=\frac{2^{2\alpha-1}\alpha(\Gamma(\alpha))^2}{\Gamma(2\alpha)}.$$

Again by Stirling's formula this gives

$$|\Gamma(\alpha+1)|\leqq C(1+|\tau|)^{1/2}\left|\Gamma\left(\alpha+\tfrac{1}{2}\right)\right|, \tag{3.10}$$

uniformly in $0 \leqq \sigma \leqq n$. This proves (3.6). The estimate (3.5) will be proved in the next lemma.

Lemma 3.1.

$$\int_{-1}^{1} (1-t^2)^{-1/2} |\varphi_n(xt)|\, dt \leqq C(2n+1)^{-1/4}.$$

Proof. Let us set $N=2n+1$. By making a change of variable it is enough to show that

$$\int_0^x (x^2-t^2)^{-1/2} |\varphi_n(t)|\, dt \leqq CN^{-1/4}. \tag{3.11}$$

To do this we need the asymptotic properties of the Hermite function φ_n for various regions (which can be seen from Askey—Wainger [1]). We need to establish

$$I = \int_a^x (x^2-t^2)^{-1/2} |\varphi_n(t)|\, dt \leqq CN^{-1/4}$$

where $a<x\leqq b$ for various values of a and b. We have to treat several cases.

Case i). $a=0$, $b=\frac{1}{2} N^{1/2}$.
When $0<t\leqq \frac{1}{2} N^{1/2}$ we have the estimate

$$|\varphi_n(t)| \leqq CN^{-1/8}(N^{1/2}-t)^{-1/4}. \tag{3.12}$$

Therefore,

$$I \leqq CN^{-1/4} \int_0^x (x^2-t^2)^{-1/2}\, dt \leqq CN^{-1/4}.$$

Case ii). $a=\frac{1}{2} N^{1/2}$, $b=N^{1/2}-N^{1/6}$.
In this range also we have the same estimate (3.12) and so

$$I \leqq CN^{-1/8} \int_a^x (x^2-t^2)^{-1/2}(N^{1/2}-t)^{-1/4}\, dt.$$

As $\frac{1}{2} N^{1/2} \leqq x < N^{1/2}$ and $N^{1/2}-t \geqq x-t$ we have

$$I \leqq CN^{-3/8} \int_0^x (x-t)^{-3/4}\, dt \leqq CN^{-3/8} x^{1/4} \leqq CN^{-1/4}.$$

Case iii). $a=N^{1/2}-N^{-1/6}$, $b=N^{1/2}+N^{-1/6}$.
In this range we have the estimate

$$|\varphi_n(t)| \leqq CN^{-1/12} \tag{3.13}$$

and hence we have

$$I \leqq CN^{-1/12} N^{-1/4} (x-N^{1/2}+N^{-1/6})^{1/2}.$$

As $x-N^{1/2}+N^{-1/6} \leqq 2N^{-1/6}$ we immediately get $I \leqq CN^{-1/4}$.

Case iv). $a=N^{1/2}+N^{-1/6}$, $b=(2N)^{1/2}$.

In this range there is a constant $\varepsilon>0$ such that

$$|\varphi_n(t)| \leqq CN^{-1/8}(t-N^{1/2})^{-1/4}\exp\{-\varepsilon N^{1/4}(t-N^{1/2})^{3/2}\}. \tag{3.14}$$

The integral I is bounded by

$$I \leqq CN^{-1/8}\int_a^x (x^2-t^2)^{-1/2}(t-N^{1/2})^{-1/4}\exp\{-\varepsilon N^{1/4}(t-N^{1/2})^{3/2}\}\,dt.$$

Applying Hölder's inequality with $4/3<p<2$ the above integral is bounded by AB where

$$A = \Big(\int_a^x (x^2-t^2)^{-p/2}\,dt\Big)^{1/p}$$

and

$$B = \Big(\int_a^x (t-N^{1/2})^{-q/4}\exp\{-\varepsilon qN^{1/4}(t-N^{1/2})^{3/2}\}\,dt\Big)^{1/q}$$

where $p+q=pq$. A simple calculation shows that $A\leqq N^{-1/2+1/2p}$. By a change of variable B gives the estimate

$$B \leqq C\Big(\int_0^\infty t^{-q/4}\exp\{-\gamma N^{1/4}t^{3/2}\}\,dt\Big)^{1/q} \leqq CN^{1/24-1/6q}.$$

Finally, we have the estimate

$$I \leqq CN^{-1/4}N^{-1/2+1/2p}N^{1/24-1/6q}$$

which after simplification becomes, as $4/3<p<2$,

$$I \leqq CN^{-1/4}N^{-1/2+2/3p} \leqq CN^{-1/4}.$$

Case v). $a=(2N)^{1/2}$, $b=\infty$.

In this range we have

$$|\varphi_n(t)| \leqq Ce^{-\delta t^2}$$

and hence

$$I \leqq Ce^{-2\delta N}\int_0^x (x^2-t^2)^{-1/2}\,dt \leqq Ce^{-2\delta N}.$$

This completes the proof of the lemma.

4. Estimate for the kernel and bounds for the projections Q_k

As we have already mentioned the two ingredients to prove the multiplier theorem are the following: the kernel estimate

$$|s_R^\delta(z)| \leqq CR^n(1+R^{1/2}|z|)^{-\delta-1} \tag{4.1}$$

and the L^p-L^2 bounds for the projections

$$\|Q_k f\|_2 \leqq C k^{n(1/p-1/2)-1/2}\|f\|_p. \tag{4.2}$$

Our aim in this section is to prove (4.1) and (4.2). Once we have these two ingredients we can proceed as in [16].

First we will prove the kernel estimate. Let σ_N^δ denote the Cesàro means

$$\sigma_N^\delta = \frac{1}{A_N^\delta}\sum_{k=0}^N A_{N-k}^\delta P_k. \tag{4.3}$$

The kernel $s_R^\delta(z)$ can be expressed in terms of the kernel $\sigma_N^\delta(z)$ corresponding to the multiplier σ_N^δ. In fact the following formula (see Gergen [5]) connects the two kernels.

$$S_R^\delta(z) = R^{-\delta}\sum_{k=0}^R v(R-k)A_k^\delta\sigma_k^\delta(z)$$

where v satisfies the estimate

$$|v(t)| \leqq C(1+t^2)^{-1}.$$

In view of the above formula it suffices to prove the following estimate.

Proposition 4.1.

$$|\sigma_k^\delta(z)| \leqq C k^n (1+k^{1/2}|z|)^{-\delta-n-(1/3)}. \tag{4.4}$$

Proof. The kernel $\sigma_k^\delta(z)$ is given by

$$\sigma_k^\delta(z) = \frac{1}{A_k^\delta}\sum_{j=0}^k A_{k-j}^\delta \varphi_j^{n-1}(z).$$

In view of the formula

$$\sum_{j=0}^k A_{k-j}^\delta L_j^\alpha(z) = L^{\alpha+\delta+1}(z)$$

we get that

$$\sigma_k^\delta(z) = \frac{1}{A_k^\delta} L_k^{n+\delta}\left(\frac{1}{2}|z|^2\right)e^{-1/4|z|^2}.$$

If we let $\mathscr{L}_k^\alpha(r)=k^{-\alpha/2}L_k^\alpha(r)e^{-r/2}r^{\alpha/2}$ then the following estimates are well-known [7].

$$|\mathscr{L}_k^\alpha(r)| \leqq C\begin{cases}(r\nu)^{\alpha/2} & \text{if } 0\leqq r\leqq \dfrac{1}{\nu}\\[2mm] (r\nu)^{-1/4} & \text{if } \dfrac{1}{\nu}\leqq r\leqq \dfrac{\nu}{2}\\[2mm] \{\nu(\nu^{1/3}+|\nu-r|)\}^{-1/4} & \text{if } \dfrac{\nu}{2}\leqq r\leqq \dfrac{3\nu}{2}\\[2mm] \exp\{-\gamma r\} & \text{if } r\geqq \dfrac{3\nu}{2}\end{cases}$$

where $\nu=4k+2\alpha+2$. Using these estimates it is now an easy matter to establish the estimate (4.4).

Now we will prove the bounds (4.2) for the projection operators.

Proposition 4.2. *Let* $1\leqq p\leqq\frac{2n}{n+1}$ *and* $f\in L^p(\mathbf{C}^n)$. *Then*

$$\|Q_k f\|_2\leqq Ck^{n(1/p-1/2)-1/2}\|f\|_p$$

where C is independent of k.

Proof. In view of Riesz—Thorin convexity theorem the estimate for Q_k is a consequence of the following two estimates.

$$\|Q_k f\|_2\leqq Ck^{(n-1)/2}\|f\|_1, \tag{4.5}$$

$$\|Q_k f\|_2\leqq C\|f\|_{2n/(n+1)}. \tag{4.6}$$

Since $Q_k f=\varphi_k^{n-1}\times f$ and $\|\varphi_k^{n-1}\|_2\leqq Ck^{(n-1)/2}$ (4.5) is a consequence of the Young's inequality. Since Q_k are projections

$$\|Q_k f\|_2^2\leqq\|Q_k f\|_{p'}\|f\|_p$$

and hence (4.6) will be proved once we show that

$$\|Q_k f\|_{2n/(n-1)}\leqq C\|f\|_{2n/(n+1)}. \tag{4.7}$$

To prove this inequality we are going to use Stein's interpolation theorem for analytic families of operators.

Consider the family of operators

$$G_k^\alpha f=\psi_k^{n\alpha}\times f. \tag{4.8}$$

Then clearly G_k^α is an analytic family of operators, analytic in $\operatorname{Re}\alpha>-\frac{1}{2n}$ and

$$Q_k f=\frac{\Gamma(k+n)\Gamma(n)}{\Gamma(k+1)}G_k^{(n-1)/n}f. \tag{4.9}$$

If f and g are simple functions then using the estimate (3.3) we have

$$\left|\int G_k^\alpha f(z)g(z)\,dz\right|\leqq C(1+|\tau|)^{1/2}\|f\|_1\|g\|_1$$

where $\alpha=\sigma+i\tau$ uniformly in σ, $0\leqq\sigma\leqq1$. This shows that the family G_k^α is admissible in the sense of Stein [11]. Since

$$\frac{\Gamma(k+n)\Gamma(n)}{\Gamma(k+1)}\leqq Ck^{n-1}$$

the estimate (4.7) will follow from the following lemma by analytic interpolation.

Lemma 4.1.

$$\|G_k^{i\tau}f\|_\infty \leqq C(1+|\tau|)^{1/2}\|f\|_1 \tag{4.10}$$

$$\|G_k^{1+i\tau}f\|_2 \leqq C(1+|\tau|)^n k^{-n}\|f\|_2. \tag{4.11}$$

Proof. The estimate (4.10) follows from (3.3). To prove (4.11) we use the Plancherel theorem for the Weyl transform. If T and S are bounded operators on $L^2(\mathbf{R}^n)$ then

$$\|TS\|_{HS} \leqq \|T\|\,\|S\|_{HS}$$

where $\|T\|$ is the operator norm of T and $\|S\|_{HS}$ is the Hilbert—Schmidt norm of S. By applying Plancherel theorem for the Weyl transform

$$\begin{aligned}\|G_k^{1+i\tau}f\|_2^2 &= (2\pi)^{-n}\|W(\psi_k^{n+ni\tau})W(f)\|_{HS}^2 \\ &\leqq \|T\|^2\,\|f\|_2^2\end{aligned} \tag{4.12}$$

where $T=W(\psi_k^{n+ni\tau})$. So it is enough to show that

$$\|T\| \leqq C(1+|\tau|)^n k^{-n}. \tag{4.13}$$

In order to do this we make use of the following formula.

$$L_k^{\alpha+\beta+1}(r) = \sum_{j=0}^k A_{k-j}^\beta L_j^\alpha(r) \tag{4.14}$$

where

$$A_k^\alpha = \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1)}.$$

This formula is stated in [7] for real values of α and β but it remains true for complex values also. So we have

$$\psi_k^{n+ni\tau}(z) = \frac{\Gamma(k+1)\Gamma(n+1+ni\tau)}{\Gamma(k+n+1+ni\tau)}\sum_{j=0}^k A_{k-j}^{ni\tau}\varphi_j^{n-1}(z). \tag{4.15}$$

Since $W(\varphi_j^{n-1})=P_j$ this gives

$$T = \frac{\Gamma(k+1)\Gamma(n+1+ni\tau)}{\Gamma(k+n+1+ni\tau)}\sum_{j=0}^k A_{k-j}^{ni\tau}P_j.$$

Now

$$(n+1+ni\tau)\dots(n+k+ni\tau)\Gamma(n+1+ni\tau) = \Gamma(k+n+ni\tau+1)$$

and hence

$$\frac{\Gamma(k+1)\Gamma(n+1+ni\tau)}{\Gamma(k+n+1+ni\tau)} = \frac{\Gamma(k+1)(1+ni\tau)\dots(n+ni\tau)}{(1+ni\tau)\dots(k+ni\tau)(k+1+ni\tau)\dots(n+k+ni\tau)}.$$

Similarly we have

$$A_j^{ni\tau} = \frac{\Gamma(j+1+ni\tau)}{\Gamma(j+1)\Gamma(ni\tau+1)} = \frac{(j+ni\tau)\dots(1+ni\tau)}{\Gamma(j+1)}.$$

Together we have

$$\frac{\Gamma(k+1)\Gamma(n+1+ni\tau)}{\Gamma(k+n+1+ni\tau)} A_j^{ni\tau} = \frac{(j+1)\dots(k-1)k}{(j+1+ni\tau)\dots(k+ni\tau)} \frac{(1+ni\tau)\dots(n+ni\tau)}{(k+1+ni\tau)\dots(k+n+ni\tau)}.$$

Thus we have the estimate

$$\left|\frac{\Gamma(k+1)\Gamma(n+1+ni\tau)}{\Gamma(k+n+1+ni\tau)} A_j^{ni\tau}\right| \leqq C(1+|\tau|)^n k^{-n}.$$

Therefore, in view of (4.15) we have (4.13). This completes the proof of Lemma 4.1.

5. Proof of the multiplier theorem

We will give only a sketch of the proof referring to [16] for details. We take a partition of unity $\sum_{j=-\infty}^{\infty} \varphi(2^j t)=1$, $t>0$ where $\varphi\in C_0^\infty(\frac{1}{2}, 2)$. For each j we set

(5.1) $$\varphi_{R,j}^\delta(t) = \varphi\left(2^j\left(1-\frac{t}{R}\right)\right)\left(1-\frac{t}{R}\right)^\delta$$

and define for $j=1, 2, \dots$

(5.2) $$T_{R,j}^\delta f = \sum \varphi_{R,j}^\delta(2k+n) Q_k f,$$

(5.3) $$T_{R,0}^\delta f = \sum \varphi_0\left(1-\frac{2k+n}{R}\right)\left(1-\frac{2k+n}{R}\right)^\delta Q_k f$$

where $\varphi_0(t)=1-\sum_{j=1}^\infty \varphi(2^j t)$. Then we have

$$T_R^\delta f = T_{R,0}^\delta f+\sum_{j=1}^{[\log\sqrt{R}]} T_{R,j}^\delta f+V_R^\delta f.$$

We will show that there exists an $\varepsilon>0$ such that

(5.4) $$\|T_{R,j}^\delta f\|_p \leqq C2^{-\varepsilon j}\|f\|_p,$$

(5.5) $$\|V_R^\delta f\|_p \leqq C2^{-\varepsilon j}\|f\|_p.$$

Then by summing we will get the theorem.

Proceeding as in [16] and using the kernel estimate (4.1) we can obtain the following. Given $\gamma>0$ there is an $\varepsilon>0$ such that

(5.6) $$\int_{|z|>2^{j(1+\gamma)}/\sqrt{R}} |s_R^\delta(z)|\, dz \leqq C2^{-\varepsilon j}.$$

Next using the L^p-L^2 bounds (4.2) we can show that

(5.7) $$\|T_{R,j}^\delta f\|_2 \leqq C(2^{-j}\sqrt{R})^{1/2}2^{-\delta j}(\sqrt{R})^{\delta(p)}\|f\|_p.$$

From this it follows that if B is a ball of radius $2^{j(1+\gamma)}/\sqrt{R}$ then

$$\|T^{\delta}_{R,j}f\|_{L^p(B)} \leqq C2^{-j(\delta+1/2)}\,2^{j(1+\gamma)(\delta(p)+1/2)}\,\|f\|_p.$$

Since $\delta>\delta(p)$ we can choose a $\gamma>0$ such that $\delta+\frac{1}{2}>(1+\gamma)\left(\delta(p)+\frac{1}{2}\right)$ and with this choice it is clear that there is an $\varepsilon>0$ such that

$$\|T^{\delta}_{R,j}f\|_{L^p(B)} \leqq C2^{-\varepsilon j}\|f\|_p. \tag{5.8}$$

Now split the kernel into two parts $s^{\delta}_{R}(z)=K_1(z)+K_2(z)$ where

$$K_1(z)\begin{cases} = s^{\delta}_R(z) & \text{if } \ |z| \leqq 2^{j(1+\gamma)}/\sqrt{R}, \\ = 0 & \text{otherwise.}\end{cases}$$

Then (5.6) immediately gives that

$$\|K_2 f\|_p \leqq C2^{-\varepsilon j}\,\|f\|_p.$$

To get a similar estimate for $K_1 f$ we proceed as follows. Let $h\in C^n$ and $B_r(h)$ be the ball of radius $r2^{j(1+\gamma)}/\sqrt{R}$ centred at h. Write $f=f_1+f_2+f_3$ where

$$f_1 = f\chi_{B_{3/4}(h)},\quad f_2 = f\chi_{B_{5/4}(h)/B_{3/4}(h)}.$$

Then it is clear that

$$K_1 f = K_1 f_1 + K_1 f_2.$$

Using (5.6) and (5.8) we can show that

$$\int_{B_{1/4}(h)} |K_1 f(z)|^p\,dz \leqq C2^{-\varepsilon jp}\int_{B_{5/4}(h)} |f(z)|^p\,dz.$$

Integrating with respect to h we get

$$\|K_1 f\|_p \leqq C2^{-\varepsilon j}\,\|f\|_p.$$

This completes the proof.

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