A uniqueness theorem of Beurling for Fourier transform pairs

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There are many theorems known which state that a function and its Fourier transform cannot simultaneously be very small at infinity, such as various forms of the uncertainty principle and the basic results on quasianalytic functions. One such theorem is stated on page 372 in volume II of the collected works of Arne Beurling [1]. Although it is not in every respect the most precise result of its kind, it has a simplicity and generality which make it very attractive. The editors state that no proof has been preserved. However, in my files I have notes which I made when Arne Beurling explained this result to me during a private conversation some time during the years 1964—1968 when we were colleagues at the Institute for Advanced Study. I shall reproduce these notes here in English translation with only minor details added where my notes are too sketchy.

Theorem. *Let* $f\in L^1(\mathbf{R})$ *and assume that*

$$\iint_{\mathbf{R}^2} |f(x)\hat{f}(y)|e^{|xy|}\,dx\,dy < \infty$$

where the Fourier transform $\hat{f}$ *is defined by*

$$\hat{f}(y) = \int_{-\infty}^{\infty} e^{-ixy} f(x)\,dx.$$

Then it follows that $f=0$.

Corollary. *If* φ *and* ψ *are conjugate convex functions then*

$$\int_{-\infty}^{\infty} |f(x)|e^{\varphi(x)}\,dx < \infty, \quad \int_{-\infty}^{\infty} |\hat{f}(y)|e^{\psi(y)}\,dy < \infty \Rightarrow f = 0.$$

Proof of the Theorem. Set

$$\widehat{M}(y) = \int_{-\infty}^{\infty} |f(x)|e^{|xy|}\,dx, \quad M(x) = \int_{-\infty}^{\infty} |\hat{f}(y)|e^{|xy|}\,dy.$$

$\widehat{M}(y)$ and $M(x)$ are increasing functions of $|y|$ and $|x|$ respectively, and

(1) $\int_{-\infty}^{\infty} |\hat{f}(y)|\widehat{M}(y)\,dy = \int_{-\infty}^{\infty} |f(x)|M(x)\,dx = \iint |f(x)\hat{f}(y)|e^{|xy|}\,dx\,dy < \infty.$

1) Assume at first that f has compact support. Then $\widehat{M}$ must grow exponentially if $f \neq 0$. Hence f is analytic in a band containing the real axis and f must be 0 after all.

2) If $\hat{f}$ has compact support, we can argue in the same way.

3) If neither f nor $\hat{f}$ has compact support, then M and $\widehat{M}$ must grow faster than any exponential function, so Fourier's inversion formula shows that f and $\hat{f}$ are entire functions,

$$|f(z)| \leqq M(\operatorname{Im} z), \quad |\hat{f}(z)| \leqq \widehat{M}(\operatorname{Im} z); \quad z \in \mathbf{C}. \tag{2}$$

Let

$$F(z) = \int_0^z f(z)f(iz)\,dz. \tag{3}$$

Since

$$\int_{-\infty}^{\infty} |f(x)|\,|f(\pm ix)|\,dx \leqq \int_{-\infty}^{\infty} |f(x)|\,M(x)\,dx < \infty,$$

the entire function F is bounded on both the real and the imaginary axes. If now

$$\varlimsup_{|x|\to\infty} M(x)e^{-cx^2/2} = 0$$

for every $c>0$, it would follow from the Phragmén—Lindelöf theorem that f is a bounded function, hence equal to 0. If f is not 0 we can therefore find $c>0$ and similarly $\hat{c}>0$ such that

$$M(x) > e^{cx^2/2}, \quad \widehat{M}(y) > e^{\hat{c}y^2/2}, \quad \text{if} \quad x,\ y \quad \text{are large.}$$

Since

$$f(z) = (2\pi)^{-1}\int_{-\infty}^{\infty} \hat{f}(y)\,\widehat{M}(y)e^{izy}/\widehat{M}(y)\,dy$$

we conclude that

$$|f(z)| \leqq K \sup_y e^{|y||\operatorname{Im} z| - \hat{c}y^2/2} = Ke^{|\operatorname{Im} z|^2/2\hat{c}}.$$

This proves that f is exactly of order 2 or identically 0.

If α is real, it follows from (1) and (2) that

$$\int_0^{\infty} |f(r)f(re^{i\alpha})|\,dr \leqq \int_0^{\infty} |f(r)|\,M(r)\,dr = K' < \infty.$$

Since f is of order $\leqq 2$, the Phragmén—Lindelöf theorem yields

$$\int_0^{\infty} |f(re^{i\alpha/2})|^2\,dr \leqq K', \quad \text{if} \quad 0 < \alpha < \pi/2. \tag{4}$$

In fact, if $\varphi \in C_0(\mathbf{R}_+)$, $|\varphi| \leqq 1$, then the absolute value of the analytic function

$$z \mapsto z \int f(tz)\overline{f(e^{i\alpha} t\bar{z})}\varphi(t)\,dt$$

is bounded by K' on the boundaries of the angle $0 < \arg z < \alpha$, and since the function is of order 2 the same bound holds in the interior of the angle. When $z = e^{i\alpha/2}$ the inequality (4) is obtained. Letting $\alpha \to \pi/2$ we conclude that

$$\int_0^\infty |f(re^{i\pi/4})|^2\,dr \leqq K'.$$

In the same way we obtain $\int_0^\infty |f(\pm re^{\pm i\pi/4})|^2 dr < \infty$. Hence $|F(\pm re^{\pm i\pi/4})|$ is bounded, and the Phragmén—Lindelöf theorem now gives that F is a constant. Hence $f(z)f(iz) = 0$, which implies that $f = 0$. The proof is complete.

Arne Beurling remarked that using results of Lindelöf one can find examples which show that the corollary is rather precise. Here my notes end, and they do not reveal what references he had in mind, so I shall add some comments of my own.

The simplest example showing the precision of the corollary is perhaps a Gaussian

$$f(x) = e^{-x^2/2a}, \quad \hat{f}(\xi) = \sqrt{2\pi a}\, e^{-a\xi^2/2}.$$

More generally, Gelfand and Shilov [2], [3] introduced for $1 < p < \infty$ the space Z_p^p of entire functions ψ such that for some constants A, B, C

$$|\psi(z)| \leqq Ce^{A|\operatorname{Im} z|^p - B|\operatorname{Re} z|^p}, \quad z \in \mathbf{C}.$$

They proved that the Fourier transform of Z_p^p is equal to $Z_{p'}^{p'}$ where $1/p + 1/p' = 1$. If $\psi \in Z_p^p$ it follows that

$$|\psi(x)| \leqq Ce^{-B|x|^p}, \quad x \in \mathbf{R}; \quad |\hat{\psi}(\xi)| \leqq C'e^{-B'|\xi|^{p'}}, \quad \xi \in \mathbf{R},$$

so Beurling's theorem gives $(Bp)^{1/p}(B'p')^{1/p'} \geqq 1$ if $\psi \not\equiv 0$. When $2 \leqq p \leqq 4$ one can easily find such a function ψ by noting that if φ is an even function in $\mathbf{C}$ with

$$\varphi(z) = -\operatorname{Re} z^p, \quad \text{when} \quad \operatorname{Re} z \geqq 0,$$

then φ is subharmonic. In fact, from refined indicator theorems which are available also for analytic functions of several variables it follows that there exist analytic functions $\psi \not\equiv 0$ with $|\psi(z)| \leqq e^{\varphi(z)}$, $z \in \mathbf{C}$, hence $\psi \in Z_p^p$. Since $z \mapsto \psi(z^2)$ is in Z_{2p}^{2p} if $\psi \in Z_p^p$, it follows that Z_p^p is non-trivial when $2 \leqq p < \infty$, and taking Fourier transforms we obtain the same statement when $1 < p \leqq 2$.

References

1. Beurling, A., *The collected works of Arne Beurling*, 1—2, Birkhäuser, Boston, 1989.
2. Gel'fand, I. M. and Shilov, G. E., Fourier transforms of rapidly increasing functions and questions of uniqueness of the solution of Cauchy's problem, *Uspekhi Mat. Nauk* **8**: 6 (1953), 3—54.
3. Gel'fand, I. M. and Shilov, G. E., *Generalized functions*, 2, Moscow, 1958 (Russian). English translation, Academic Press, 1968.

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