On the removal of singular sets for the tangential Cauchy—Riemann operator

G. Lupacciolu

1. Introduction

The present paper pursues the subject of removability of singular sets for the boundary values of holomorphic functions, which in the last years has been discussed in several papers. We refer to a recent article by E. L. Stout [10] for a comprehensive account and an ample list of references on this matter.

The notions of removability we shall be concerned with are as follows:

Definition 1.1. Let $D\subset\subset \mathbf{C}^n$, $n\geqq 2$, be an open domain and $E\subset bD$ a closed set, and assume that $bD\setminus E$ is a connected real hypersurface of class $\mathscr{C}^1$ in $\mathbf{C}^n\setminus E$. The set E is said to be *removable* (with respect to D) in case every continuous CR-function f on $bD\setminus E$ has a continuous extension $F\in\mathscr{C}^0(\bar{D}\setminus E)\cap\mathscr{O}(D)$.

Definition 1.2. Let $D\subset\subset \mathbf{C}^n$, $n\geqq 2$, be an open domain, with connected boundary bD of class $\mathscr{C}^1$, and $S\subset bD$ an arbitrary set. The set S is said to be *locally removable* at a point z in case there exists a compact neighbourhood E of z in S, co-connected in bD, which is removable according to Definition 1.1.

The motivation of this paper has been to get some conditions for the removability of sets of Hausdorff dimension $\leqq 2n-2$, and in particular to extend to general domains some results in this area which have already been obtained previously for C^2-bounded pseudoconvex or strongly pseudoconvex domains, namely that sets of Hausdorff $(2n-3)$-measure zero are removable (cf. [7]) and that $(2n-3)$-dimensional manifolds and $(2n-2)$-dimensional generic CR-manifolds are locally removable (cf. [4][1]).

[1] However the main result of [4] is another one, namely that, for $n=2$ and D C^2-bounded and strongly pseudoconvex, a totally real disc $\Delta\subset bD$ of class C^2 is removable. I have not been able to prove or disprove the extendability of this result to more general domains.

This is accomplished in Section 3 (Theorem 2 and its corollaries) and in Section 4 (Theorem 4).

Firstly we consider, in Section 2, a refinement of the notion of rational convexity and prove a quite general removability theorem in this setting (Theorem 1), which is indeed the main result of the paper and the base for the discussion of Section 3.

Moreover we show (Theorem 3) that every compact $(2n-3)$-dimensional generic CR-manifold of class C^2 is globally removable.

Acknowledgments. I wish to thank Professor E. Lee Stout for several useful conversations on the subject of removable singularities and Professor Telemachos Hatziafratis for informing me of his recent results on the Cauchy—Fantappié kernels [1], without which I should not have been able to carry out the proof of Theorem 1.

2. Rational $(n-1)$-convexity

As is well-known, given a compact set $K \subset \mathbf{C}^n$, the set of the points $z \in \mathbf{C}^n$ such that $p(z) \in p(K)$, for every holomorphic polynomial function $p\colon \mathbf{C}^n \to \mathbf{C}$, is called the rational convex hull of K; moreover K is said to be rationally convex in case it coincides with its rational convex hull, which amounts to there being a family P of polynomial functions such that

$$\mathbf{C}^n \setminus K = \bigcup_{p \in P} (\mathbf{C}^n \setminus p^{-1}(p(K))).$$

Here we consider the following weaker notion of convexity:

Definition 2.1. A compact set $K \subset \mathbf{C}^n$ is called *rationally* $(n-1)$-*convex* in case there is a family $\mathscr{H}$ of holomorphic polynomial mappings $h\colon \mathbf{C}^n \to \mathbf{C}^{n-1}$ such that

$$\mathbf{C}^n \setminus K = \bigcup_{h \in \mathscr{H}} (\mathbf{C}^n \setminus h^{-1}(h(K))).$$

Thus for $n=2$ rational $(n-1)$-convexity is the same as ordinary rational convexity; however for $n \geqq 3$ the former is indeed a weaker property than the latter. On the other hand for $n \geqq 3$ rational $(n-1)$-convexity seems to be a stronger property than convexity with respect to 1-dimensional varieties.[2]

Rational $(n-1)$-convexity is not sufficient for removability. As a matter of fact the set

$$E = \{z \in \mathbf{C}^n;\ |z_1|^2 + \ldots + |z_n|^2 \leqq 2,\ |z_1|^2 + \ldots + |z_{n-1}|^2 = 1\}$$

[2] A compact set $K \subset \mathbf{C}^n$ is said to be convex with respect to k-dimensional varieties $(1 \leqq k \leqq n-1)$ in case for every point $z \in \mathbf{C}^n \setminus K$ there is a k-dimensional closed analytic variety $V \subset \mathbf{C}^n \setminus K$ with $z \in V$ (cf. [10]).

is compact and rationally $(n-1)$-convex, but is not removable with respect to the domain

$$D = \{z \in \mathbf{C}^n;\ |z_1|^2 + \dots + |z_n|^2 < 2,\ |z_1|^2 + \dots + |z_{n-1}|^2 < 1\},$$

since the function $f(z_1, \dots, z_n) = z_n^{-1}$ is holomorphic on $bD \setminus E$, but has no holomorphic extension to D.

However we are going to show that there is a quite large class of rationally $(n-1)$-convex compact sets for which removability holds.

Let us give the following definition:

Definition 2.2. A rationally $(n-1)$-convex compact set $K \subset \mathbf{C}^n$ is said to be *strongly co-connected* in case it is possible to find a family $\mathscr{H}$ as in Definition 2.1 in such a way that all the open sets $\mathbf{C}^n \setminus h^{-1}(h(K))$, $h \in \mathscr{H}$ are connected.

The adverb "strongly" is motivated from the fact that any rationally $(n-1)$-convex compact set K is co-connected, since no affine algebraic variety of dimension $\geqq 1$ is compact.

The following theorem is the basic result of this paper.

Theorem 1. *Let D and E be as in Definition* 1.1 *and let E be rationally* $(n-1)$-*convex and strongly co-connected. Then E is removable.*

Proof. We shall use a method based on integral representations involving the Martinelli—Bochner kernel and certain suitable $\bar{\partial}$-primitives of this kernel constructed by Hatziafratis [1].

Basically this method is the same as in some earlier papers (cf. [1], [5]); there are however some significant technical differences, in particular the real-analyticity of the coefficients of the differential forms involved plays here an essential rôle.

We will denote by $\zeta_1, \dots, \zeta_n$ the standard coordinates in $\mathbf{C}^n$ and by $\omega(\cdot, z)$ the Martinelli—Bochner kernel relative to a point $z \in \mathbf{C}^n$. We assume that the positive orientation of $\mathbf{C}^n$ is the one given by the form $(i/2)^n\, d\zeta_1 \wedge d\bar{\zeta}_1 \dots d\zeta_n \wedge d\bar{\zeta}_n$ and that $\omega(\cdot, z)$ is normalized so that

$$\int_{b\mathbf{B}_n(z)} \omega(\cdot, z) = 1.$$

We recall from [1] that, given a holomorphic mapping

$$h = (h_1, \dots, h_{n-1})\colon \mathbf{C}^n \to \mathbf{C}^{n-1},$$

it is possible to construct a real-analytic $(n, n-2)$-form $\mu_h(\cdot, z)$ on $\mathbf{C}^n \setminus h^{-1}(h(z))$ which is a $\bar{\partial}$-primitive of $\omega(\cdot, z)$, that is

$$\bar{\partial}_\zeta \mu_h(\cdot, z) = \omega(\cdot, z) \quad \text{on} \quad \mathbf{C}^n \setminus h^{-1}(h(z)).$$

In fact $\mu_h(\cdot, z)$ is explicitly written in [1] in terms of a chosen matrix (h_{jl}) of holomorphic functions on $\mathbf{C}^n \times \mathbf{C}^n$ such that, for every $(\zeta, z) \in \mathbf{C}^n \times \mathbf{C}^n$,

$$h_j(\zeta) - h_j(z) = \sum_{l=1}^n h_{jl}(\zeta, z)(\zeta_l - z_l), \quad j = 1, \dots, n-1.$$

Giving here the explicit expression of $\mu_h(\cdot, z)$ is needless; we only point out that $\mu_h(\cdot, z)$ depends in a real-analytic fashion also on the point-parameter z.

That being stated, let us fix a family $\mathscr{H}$ of holomorphic polynomial mappings $\mathbf{C}^n \to \mathbf{C}^{n-1}$ as in Definition 2.2, and for every $h \in \mathscr{H}$, let $\mu_h(\cdot, z)$ be a $\bar{\partial}$-primitive of $\omega(\cdot, z)$ of the type mentioned above.

Moreover let us fix a sequence $\{\Gamma_s\}_{s \in \mathbf{N}}$ of real hypersurfaces with boundary, connected and of class $\mathscr{C}^1$, such that

$$\Gamma_s \subset \Gamma_{s+1} \setminus b\Gamma_{s+1}, \quad \forall s \in \mathbf{N}; \quad \bigcup_{s \in \mathbf{N}} \Gamma_s = bD \setminus E.$$

For every $s \in \mathbf{N}$, let $E_s = \overline{bD \setminus \Gamma_s}$, so that $\{E_s\}_{s \in \mathbf{N}}$ is a decreasing sequence of closed neighbourhoods of E in bD with

$$\Gamma_s \cap E_s = b\Gamma_s, \quad \forall s \in \mathbf{N}; \quad \bigcap_{s \in \mathbf{N}} E_s = E.$$

We are to prove that, given a continuous CR-function f on $bD \setminus E$, there exists a function $F \in \mathscr{C}^0(\bar{D} \setminus E) \cap \mathscr{O}(D)$ with $F|_{bD \setminus E} = f$.

For every $h \in \mathscr{H}$ and $s \in \mathbf{N}$ we can define a complex-valued function F_h^s on $(\mathbf{C}^n \setminus h^{-1}(h(E_s))) \setminus bD$ by setting

$$F_h^s(z) = \int_{\Gamma_s} f\omega(\cdot, z) - \int_{b\Gamma_s} f\mu_h(\cdot, z).$$

F_h^s is real-analytic, since so are the coefficients of both $\omega(\cdot, z)$ and $\mu_h(\cdot, z)$ with respect to z; moreover for $z \in (\mathbf{C}^n \setminus h^{-1}(h(E_s))) \setminus bD$ one has, by Stokes' theorem,

$$F_h^{s+1}(z) = F_h^s(z)\,[3].$$

It follows that there is a real-analytic function F_h defined on $(\mathbf{C}^n \setminus h^{-1}(h(E))) \setminus bD$ which is the coherent union of the F_h^s's for $s \in \mathbf{N}$.

We shall prove the following facts:

(i) $F_h \equiv 0$ on $(\mathbf{C}^n \setminus h^{-1}(h(E))) \setminus \bar{D}$;

(ii) F_h is holomorphic;

(iii) For every $h_1, h_2 \in \mathscr{H}$, $F_{h_1} \equiv F_{h_2}$ on

$$[(\mathbf{C}^n \setminus h_1^{-1}(h_1(E))) \cap (\mathbf{C}^n \setminus h_2^{-1}(h_2(E)))] \setminus bD.$$

[3] In the present case, where f is assumed to be merely continuous, the possibility of applying Stokes' theorem is not obvious. For this point we refer to [5, Proposition 1.9].

Proof of (i). Let z be a point in $\mathbf{C}^n \setminus h^{-1}(h(\bar{D}))$. Then $z \in (\mathbf{C}^n \setminus h^{-1}(h(E_1))) \setminus \bar{D}$ and hence

$$F_h(z) = F_h^1(z) = \int_{\Gamma_1} f\omega(\cdot, z) - \int_{b\Gamma_1} f\mu_h(\cdot, z).$$

Moreover, since the singular set $h^{-1}(h(z))$ of $\mu_h(\cdot, z)$ does not meet Γ_1, Stokes' theorem applies (cf. the footnote 3) and gives:

$$\int_{b\Gamma_1} f\mu_h(\cdot, z) = \int_{\Gamma_1} f\bar{\partial}_\zeta \mu_h(\cdot, z) = \int_{\Gamma_1} f\omega(\cdot, z),$$

and it follows that $F_h(z)=0$. Therefore, as F_h is real-analytic, it vanishes identically on the connected components of $\mathbf{C}^n \setminus h^{-1}(h(E)) \setminus \bar{D}$ which meet $\mathbf{C}^n \setminus h^{-1}(h(\bar{D}))$. On the other hand, since $\mathbf{C}^n \setminus E$, $bD \setminus E$, $\mathbf{C}^n \setminus h^{-1}(h(E))$ are all connected, one can readily see that so too is $(\mathbf{C}^n \setminus h^{-1}(h(E))) \setminus \bar{D}$, and (i) follows.

Proof of (ii). We have to show that, for every point $z \in (\mathbf{C}^n \setminus h^{-1}(h(E))) \setminus bD$, the Wirtinger derivatives

$$\frac{\partial F_h}{\partial \bar{z}_j}(z) = 0, \quad j = 1, \ldots, n.$$

Let us recall from [5] that the $(n, n-1)$-form $\dfrac{\partial \omega}{\partial \bar{z}_j}(\cdot, z)$ obtained by performing the derivative $\dfrac{\partial}{\partial \bar{z}_j}$ of each coefficient of $\omega(\cdot, z)$ is $\bar{\partial}_\zeta$-exact in $\mathbf{C}^n \setminus \{z\}$, i.e. there is a $(n, n-2)$-form $K_j(\cdot, z)$ such that

$$\frac{\partial \omega}{\partial \bar{z}_j}(\cdot, z) = \bar{\partial}_\zeta K_j(\cdot, z) \quad \text{on} \quad \mathbf{C}^n \setminus \{z\}.$$

Moreover $K_j(\cdot, z)$ can be chosen so as to be real-analytic with respect to z.

Then, if we take $s \in \mathbf{N}$ large enough so that $z \in (\mathbf{C}^n \setminus h^{-1}(h(E_s))) \setminus bD$, we get

$$\frac{\partial F_h}{\partial \bar{z}_j}(z) = \frac{\partial F_h^s}{\partial \bar{z}_j}(z) = \int_{b\Gamma_s} f\left(K_j(\cdot, z) - \frac{\partial \mu_h}{\partial \bar{z}_j}(\cdot, z)\right),$$

which shows $\dfrac{\partial F_h}{\partial \bar{z}_j}$ to extend as a real-analytic function to the all of $\mathbf{C}^n \setminus h^{-1}(h(E))$. It follows, since (i) holds and $\mathbf{C}^n \setminus h^{-1}(h(E))$ is connected, that $\dfrac{\partial F_h}{\partial \bar{z}_j} \equiv 0$.

Proof of (iii). For every $s \in \mathbf{N}$ and $z \in [(\mathbf{C}^n \setminus h_1^{-1}(h_1(E_s))) \cap (\mathbf{C}^n \setminus h_2^{-1}(h_2(E_s)))] \setminus bD$ we have

$$F_{h_1}^s(z) - F_{h_2}^s(z) = \int_{b\Gamma_s} f(\mu_{h_2}(\cdot, z) - \mu_{h_1}(\cdot, z)),$$

which implies that $F_{h_1} - F_{h_2}$ extends as a real-analytic function to all of $(\mathbf{C}^n \setminus h_1^{-1}(h_1(E))) \cap (\mathbf{C}^n \setminus h_2^{-1}(h_2(E)))$. On account of (i), $F_{h_1} - F_{h_2}$ is zero on

$[(\mathbf{C}^n \setminus h_1^{-1}(h_1(E))) \cap (\mathbf{C}^n \setminus h_2^{-1}(h_2(E)))] \setminus \bar{D}$. On the other hand no connected components of $(\mathbf{C}^n \setminus h_1^{-1}(h_1(E))) \cap (\mathbf{C}^n \setminus h_2^{-1}(h_2(E)))$ can be contained in D, for, if $z \in D \setminus (h_1^{-1}(h_1(E)) \cup h_2^{-1}(h_2(E)))$, then $h_1^{-1}(h_1(z)) \cup h_2^{-1}(h_2(z))$ is an affine algebraic variety of dimension $\geqq 1$, no component of which can be contained in D. It follows that (iii) holds.

Finally, let F denote the coherent union of the F_h's for $h \in \mathcal{H}$. Then F is a holomorphic function on $\mathbf{C}^n \setminus bD$ and is zero on $\mathbf{C}^n \setminus \bar{D}$. Arguing as in [5, Proposition 2.7] one can show that, for each $z^0 \in bD \setminus E$, $\lim F(z) = f(z^0)$ as z approaches z^0 in D; which concludes the proof of Theorem 1.

Q.e.d.

Remarks. (1) Theorem 1 still holds, essentially with the same proof, if the assumption that E is rationally $(n-1)$-convex and strongly co-connected is replaced by the following weaker assumption: There exists a rationally $(n-1)$-convex and strongly co-connected compact set K such that $K \cap \bar{D} = E$. (2) If K is a compact set of $\mathbf{C}^n$ and Ω is a connected Stein open set containing K, one can say that K is *meromorphically $(n-1)$-convex and strongly co-connected in Ω* just by a formal adaptation of Definitions 2.1 and 2.2: namely the family $\mathcal{H}$ has to be made by holomorphic mappings $h\colon \Omega \to \mathbf{C}^{n-1}$ such that

$$\Omega \setminus K = \bigcup_{h \in \mathcal{H}} (\Omega \setminus h^{-1}(h(K))),$$

and each $\Omega \setminus h^{-1}(h(K))$ is connected. It is still possible to prove Theorem 1 under such a weaker assumption on E, provided $\bar{D} \subset \Omega$.

(2) For $n=2$, if K is meromorphically $(n-1)$-convex and strongly co-connected in Ω, then K is $\mathcal{O}(\Omega)$-convex. As a matter of fact, if $z \in \Omega \setminus K$, there is a $h \in \mathcal{O}(\Omega)$ such that $h(z) \notin h(K)$ and $\Omega \setminus h^{-1}(h(K))$ is connected; hence $h(\Omega) \setminus h(K)$ is connected too, which implies, by the Runge theorem, that $h(K)$ is $\mathcal{O}(h(\Omega))$-convex. Then there is a $g \in \mathcal{O}(h(\Omega))$ with $|g \circ h(z)| > \max_{h(K)} |g| = \max_K |g \circ h|$, so that $z \in \Omega \setminus \hat{K}_\Omega$.

It follows that for $n=2$ Theorem 1 is contained in the result of [5]. On the contrary for $n \geqq 3$ it seems to be a new result.

3. Removable sets of $(2n-2)$-measure zero

Throughout this section we shall use the notations that Λ^α denotes the α-dimensional Hausdorff measure in $\mathbf{C}^n$ and that $\mathrm{Epi}(\mathbf{C}^n, \mathbf{C}^{n-1})$ denotes the set of all surjective $\mathbf{C}$-linear mappings $\varphi\colon \mathbf{C}^n \to \mathbf{C}^{n-1}$. $\mathrm{Epi}(\mathbf{C}^n, \mathbf{C}^{n-1})$ is in a natural way a complex manifold, biholomorphically equivalent to a Zariski open subset of $\mathbf{C}^{(n-1)n}$, and such that the mapping

$$k\colon \operatorname{Epi}(\mathbf{C}^n, \mathbf{C}^{n-1}) \to \mathbf{P}^{n-1}, \quad k(\varphi) = \text{kernel of } \varphi$$

is holomorphic and open.

The following theorem is the starting point for the discussion of this section.

Theorem 2. *Let D and E be as in Definition* 1.1 *and let E satisfy the following two conditions:*

(i) $\Lambda^{2n-2}(E)=0$;

(ii) *There is an open set $U\subset \operatorname{Epi}(\mathbf{C}^n, \mathbf{C}^{n-1})$ such that $\mathbf{C}^{n-1}\setminus\varphi(E)$ is connected for almost every $\varphi\in U$.*

Then E is removable.

Proof. We shall prove that, given any point $z\in\mathbf{C}^n\setminus E$, there is a $\varphi\in\operatorname{Epi}(\mathbf{C}^n, \mathbf{C}^{n-1})$ such that $z\in\mathbf{C}^n\setminus\varphi^{-1}(\varphi(E))$ and $\mathbf{C}^n\setminus\varphi^{-1}(\varphi(E))$ is connected. Plainly this will imply that E is rationally $(n-1)$-convex and strongly co-connected, so that the thesis will follow at once from Theorem 1.

Let A_z be the subset of $\mathbf{P}^{n-1}$ consisting of all the complex lines l of $\mathbf{C}^n$ through the origin such that the parallel of l through z does not intersect E. By our assumption that $\Lambda^{2n-2}(E)=0$, a result proved in Shiffman [9] yields that $\mathbf{P}^{n-1}\setminus A_z$ has zero measure; moreover the compactness of E implies that A_z is open. Therefore A_z is a dense open subset of $\mathbf{P}^{n-1}$.

It follows that also the subset $k^{-1}(A_z)$ of $\operatorname{Epi}(\mathbf{C}^n, \mathbf{C}^{n-1})$ is open and dense. Clearly $k^{-1}(A_z)$ consists of all the $\varphi\in\operatorname{Epi}(\mathbf{C}^n, \mathbf{C}^{n-1})$ such that $z\in\mathbf{C}^n\setminus\varphi^{-1}(\varphi(E))$.

Since $k^{-1}(A_z)$ is open and dense, $U_z=U\cap k^{-1}(A_z)$ is a non-empty open set, and the condition (ii) implies that $\mathbf{C}^{n-1}\setminus\varphi(E)$ is connected for almost every $\varphi\in U_z$.

Finally, since $\mathbf{C}^n\setminus\varphi^{-1}(\varphi(E))$ is biholomorphically equivalent to $\mathbf{C}\times(\mathbf{C}^{n-1}\setminus\varphi(E))$, we conclude that, for $\varphi\in U_z$, $\mathbf{C}^n\setminus\varphi^{-1}(\varphi(E))$ is connected.

Q.e.d.

As a first consequence of Theorem 2 we have:

Corollary 1. *Let D and E be as in Definition* 1.1 *and let E satisfy the condition*

$$\Lambda^{2n-3}(E) = 0.$$

Then E is removable.

Proof. All we have to show is that the condition (ii) of Theorem 2 is satisfied, since plainly so is the condition (i).

As a matter of fact, for every $\varphi\in\operatorname{Epi}(\mathbf{C}^n, \mathbf{C}^{n-1})$, we have

$$\Lambda^{2n-3}(\varphi(E)) = 0,$$

which implies that $\varphi(E)$ has covering dimension $\leqq 2n-4$; hence $\varphi(E)$ cannot disconnect $\mathbf{C}^{n-1}$ (cf. [2]).

Q.e.d.

The above result has been previously obtained in [7] under the additional assumptions that D is pseudoconvex and that bD is of class $\mathscr{C}^2$.

Corollary 1 implies in particular the removability of any compact real manifold of class $\mathscr{C}^1$ and dimension $\leqq 2n-4$, for such a manifold has Hausdorff dimension $<2n-3$. Hence we may state:

Corollary 2. *Let D and E be as in Definition* 1.1 *and let* $E=M$ *be a compact real manifold of class* $\mathscr{C}^1$ *and dimension* $\leqq 2n-4$ *(possibly with boundary and not necessarily connected).*

Then M is removable.

On the contrary it is not true in general that a compact real manifold of dimension $2n-3$ is removable. In fact, if $M=bD\cap\{g=0\}$, where $g\in\mathscr{C}^0(\bar{D})\cap\mathscr{O}(D)$ and $D\cap\{g=0\}\neq\emptyset$, then M is not removable, since the CR-function $f=g^{-1}|_{bD\setminus M}$ does not extend holomorphically to D.

Nevertheless there are manifolds of dimension $2n-3$ which are removable. In particular for $n=2$ every arc is removable (cf. [10]) and for general n Jöricke [4] proved that, if the domain D is $\mathscr{C}^2$-bounded and strongly pseudoconvex, every $(2n-3)$-dimensional manifold of class $\mathscr{C}^2$ is locally removable (according to Definition 1.2).

As another consequence of Theorem 2, we can readily prove the following result on the global removability of $(2n-3)$-dimensional manifolds:

Corollary 3. *Let D and E be as in Definition* 1.1 *and let* $E=M$ *be a compact real manifold with boundary of dimension* $2n-3$ *and of class* $\mathscr{C}^1$ *embedded in* $\mathbf{C}^n$ *(not necessarily connected).*

Assume that the following two conditions are satisfied:

(a) $H^{2n-3}(M, \mathbf{R})=0$;

(b) *There is a* $\varphi_0\in\operatorname{Epi}(\mathbf{C}^n, \mathbf{C}^{n-1})$ *such that* $\varphi_0|_M\colon M\to\mathbf{C}^{n-1}$ *is an embedding.*

Then M is removable.

Proof. Since $\varphi_0|_M\colon M\to\mathbf{C}^{n-1}$ is an embedding of class C^1 and M is compact, it is possible to choose a small real number $\delta>0$ such that the δ-neighbourhood of $\varphi_0|_M$ in the fine C^1 topology of the space of C^1 mappings of M into $\mathbf{C}^{n-1}$ is made up by embeddings (cf. [8, pp. 29—33]). This means that, if $g\colon M\to\mathbf{C}^{n-1}$ is any mapping of class C^1 such that

$$\|g(z)-\varphi_0(z)\|<\delta,\quad \|dg(\vec{v})-d\varphi_0(\vec{v})\|<\delta\|\vec{v}\|,$$

for every $z\in M$ and $\vec{v}\in T_z(M)$ with $\vec{v}\neq\vec{0}$, then g is an embedding.

It follows easily that there is an open neighbourhood U of φ_0 in $\operatorname{Epi}(\mathbf{C}^n, \mathbf{C}^{n-1})$ such that, for every $\varphi \in U$, $\varphi|_M\colon M \to \mathbf{C}^{n-1}$ is an embedding.

Then, by the condition (a) we have, for every $\varphi \in U$,

$$H^{2n-3}(\varphi(M), \mathbf{R}) = 0,$$

and consequently we infer, via the cohomology sequence with compact supports

$$\ldots \to H^{2n-3}(\varphi(M), \mathbf{R}) \to H_c^{2n-2}(\mathbf{C}^{n-1} \setminus \varphi(M), \mathbf{R}) \to H_c^{2n-2}(\mathbf{C}^{n-1}, \mathbf{R}) \to \ldots$$

that $\mathbf{C}^{n-1} \setminus \varphi(M)$ is connected.

It follows that M verifies the two conditions of Theorem 2, and hence is removable.

Q.e.d.

Let us point out that Corollary 3 implies that the local removability of real $(2n-3)$-dimensional manifolds holds in greater generality than it has been proved in [4]. Indeed we have

Corollary 4. *Let D and S be as in Definition 1.2 and let $S=M$ be a real manifold of dimension $2n-3$ and of class $\mathscr{C}^1$ embedded in $\mathbf{C}^n$ (not necessarily closed in $\mathbf{C}^n$).*

Then M is locally removable at every point.

Proof. Let $z \in M$ and let T be the real $(2n-3)$-plane of $\mathbf{C}^n$ through the origin parallel to $T_z(M)$. Let l be a complex line of $\mathbf{C}^n$ through the origin such that $l \cap T = \emptyset$. Then, if $\varphi_0 \in k^{-1}(l)$, $\varphi_0|_M$ is an immersion near z and therefore we can find a compact $(2n-3)$-dimensional manifold $M' \subset M$ containing z and satisfying the conditions of Corollary 3. Since M' is then removable, we conclude that M is locally removable at z.

Q.e.d.

4. Removability of generic CR-manifolds

Let M be a real manifold of class $\mathscr{C}^1$ and of dimension m embedded in $\mathbf{C}^n$, with $m \geqq n$. Let us recall that, given a point $z \in M$, one defines the CR-dimension of M at z as follows:

$$CR-\dim_z(M) = \dim_{\mathbf{C}}(T_z(M) \cap JT_z(M)).$$

Thus $m-n \leqq CR\text{-}\dim_z(M) \leqq [m/2]$. If $CR\text{-}\dim_z(M)$ attains at every point $z \in M$ the same value, then M is called a CR-manifold, and if this value is the minimum possible, i.e. $m-n$, M is called a generic CR-manifold (or, when $m=n$, a totally real manifold).

Here we shall be concerned with generic CR-manifolds of dimensions $m=2n-3$ (hence $n \geqq 3$) and $m=2n-2$, in which cases the CR-dimensions are $n-3$ and $n-2$ respectively.

The results of this section relate to the removability of such kind of manifolds and are independent of the discussion of the previous two sections.

We shall prove two theorems, the first of which is:

Theorem 3. *Let* $n \geqq 3$ *and let D and E be as in Definition* 1.1. *Assume that* $E=M$ *is a compact non-bounded* $(2n-3)$*-dimensional generic CR-manifold of class* $\mathscr{C}^2$ *embedded in* $\mathbf{C}^n$ *(not necessarily connected).*

Then M is removable.

Proof. We shall apply the following result on removability, which is proved in [6] in a more general setting: Let D and E be as in Definition 1.1 and assume that E has a neighbourhood basis of $(n-3)$-complete open sets[4]. Then E is removable.

Now we will show that, if $M_1, \ldots, M_k$ are the connected components of M, each M_j does in fact admit a neighbourhood basis of $(n-3)$-complete open sets. Let $\varrho_1=0$, $\varrho_2=0$, $\varrho_3=0$ be equations of M_j, with ϱ_1, ϱ_2, ϱ_3 being real-valued functions of class $\mathscr{C}^2$ in an open neighbourhood U of M_j, such that $\partial\varrho_1 \wedge \partial\varrho_2 \wedge \partial\varrho_3 \neq 0$ at every point of M_j. Let $\varphi=\varrho_1^2+\varrho_2^2+\varrho_3^2$. Then one can verify (cf. [3]) that φ is strongly $(n-3)$-plurisubharmonic in an open neighbourhood $U' \subset U$ of M_j, from which it follows that M_j has the asserted property (cf. [11]).

Hence M has a neighbourhood basis of $(n-3)$-complete open sets and the desired conclusion follows at once from the result recalled at the beginning.

Q.e.d.

Next let us consider a generic CR-manifold M of dimension $2n-2$.

The theorem below generalizes to an arbitrary domain D a result previously obtained by Jöricke [4] for the case that D is $\mathscr{C}^2$-bounded and strongly pseudoconvex.

Theorem 4. *Let D and S be as in Definition* 1.2 *and let* $S=M$ *be a non-bounded* $(2n-2)$*-dimensional generic CR-manifold of class* $\mathscr{C}^2$ *embedded in* $\mathbf{C}^n$ *(not necessarily closed in* $\mathbf{C}^n$*).*

Then M is locally removable at each point.

Proof. We shall resort to the following result on removability, which is proved in [6] in a more general setting: Let D and E be as in Definition 1.1 and assume that E admits a neighbourhood basis $\mathscr{U}$ of relatively compact $(n-2)$-complete open sets such that, for each $U \in \mathscr{U}$ the homomorphism $H_c^2(U, \mathcal{O}) \to H_c^2(\mathbf{C}^n, \mathcal{O})$ induced by inclusion is injective.[5] Then E is removable.

[4] We recall that an open set $\Omega \subset \mathbf{C}^n$ is called *q-complete* $(0 \leqq q \leqq n-1)$ if there is an exhaustion function of class $\mathscr{C}^2 u$: $\Omega \to \mathbf{R}$ such that the Levi form of u has at each point at least $n-q$ positive eigenvalues. Such a function is called *strongly q-plurisubharmonic.*

[5] For $n \geqq 3$, since $H_c^2(\mathbf{C}^n, \mathcal{O})=0$, the last condition is simply that $H_c^2(U, \mathcal{O})=0$.

Let z be an arbitrarily fixed point of M and let $\varrho_1=0$, $\varrho_2=0$ be local equations of M near z, with ϱ_1 and ϱ_2 being real-valued functions of class $\mathscr{C}^2$ in an open neighbourhood U of z, such that $\partial\varrho_1\wedge\partial\varrho_2\neq 0$ at every point of $U\cap M$. Let $\varphi=\varrho_1^2+\varrho_2^2$. Then, after shrinking U, φ is strongly $(n-2)$-plurisubharmonic on U (cf. [3]).

Now let r, R be small positive real numbers with $r<R$ and R such that $\overline{\mathbf{B}}_n(z,R)\subset U$ and $M\cap\overline{\mathbf{B}}_n(z,R)$ is compact. Let

$$E = M\cap\overline{\mathbf{B}}_n(z,r).$$

We may assume that $bD\setminus E$ is connected, provided r is small enough. We shall prove that E is removable.

Let p be a $\mathscr{C}^\infty$ strongly plurisubharmonic exhaustion function on $\mathbf{B}_n(z,R)$ such that

$$\overline{\mathbf{B}}_n(z,r) = \{p\leqq 0\}$$

and let $\{\varphi_\nu\}_{\nu\in\mathbf{N}}$ be a sequence of $\mathscr{C}^\infty$ strongly $(n-2)$-plurisubharmonic functions on a neighbourhood of $\overline{\mathbf{B}}_n(z,R)$ such that

$$\max_{\overline{\mathbf{B}}_n(z,R)}|\varphi-\varphi_\nu| < \frac{1}{(\nu+1)^4},\quad \forall\nu\in\mathbf{N}.$$

The possibility of considering such a sequence arises from the fact that, since φ is strongly $(n-2)$-plurisubharmonic in a neighbourhood of $\overline{\mathbf{B}}_n(z,R)$, any $\mathscr{C}^\infty$ function which approximates uniformly φ together with the first and second order derivatives on $\overline{\mathbf{B}}_n(z,R)$ is also strongly $(n-2)$-plurisubharmonic on a neighbourhood of $\overline{\mathbf{B}}_n(z,R)$, provided the approximation is close enough.

Then, for each $\nu\in\mathbf{N}$, let us set

$$u_\nu = \varphi_\nu+\frac{1}{\nu}\left(p-\frac{1}{\nu}\right).$$

Since φ_ν is bounded from below by $-\dfrac{1}{(\nu+1)^4}$, u_ν is a $\mathscr{C}^\infty$ strongly $(n-2)$-plurisubharmonic exhaustion function on $\mathbf{B}_n(z,R)$; moreover one can verify that

$$\{u_\nu<0\}\supset\supset\{u_{\nu+1}<0\},\quad \forall\nu\in\mathbf{N}$$

and

$$E = \bigcap\nolimits_{\nu\in\mathbf{N}}\{u_\nu<0\}.$$

Now, if we set $U_\nu=\{u_\nu<0\}$ for each $\nu\in\mathbf{N}$, the family $\mathscr{U}=\{U_\nu\}_{\nu\in\mathbf{N}}$ is a neighbourhood basis of E of $(n-2)$-complete open sets such that the homomorphisms $H_c^2(U_\nu,\mathscr{O})\to H_c^2(\mathbf{B}_n(z,R),\mathscr{O})$, $\nu\in\mathbf{N}$ induced by inclusion are injective (cf. [6, Proof of Theorem 2.3]). Since also the homomorphism $H^2_c(\mathbf{B}_n(z, R), \mathcal{O}) \to H^2_c(\mathbf{C}^n, \mathcal{O})$ induced by inclusion is injective, we conclude, by the result on removability recalled at the beginning of the proof, that E is removable.

Q.e.d.

References

1. HATZIAFRATIS, T., On Certain Integrals Associated to *CR*-functions, to appear in *Trans. Amer. Math. Soc.*
2. HUREWICZ, W. and WALLMAN, H., *Dimension Theory,* Princeton University Press, Princeton, N.J., 1948.
3. IORDAN, A., Zero-sets of strictly q-pseudoconvex functions and maximum modulus sets for *CR*-functions, *Rev. Roumaine Math. Pures Appl.*, **31** (1986), 303—307.
4. JÖRICKE, B., Removable singularities of *CR*-functions, *Arkiv för Mat.*, **26** (1988), 117—143.
5. LUPACCIOLU, G., A theorem on holomorphic extension of *CR*-functions, *Pac. J. Math.*, **124** (1986), 177—191.
6. LUPACCIOLU, G., Some global results on extension of *CR*-objects in complex manifolds, to appear in *Trans. Amer. Math. Soc.*
7. LUPACCIOLU, G. and STOUT, E. L., Removable Singularities for $\bar{\partial}_b$, to appear in the *Proceedings of the special year in Several Complex Variables at the Mittag—Leffler Institute during the year 1987/88.*
8. MUNKRES, J. R., Elementary Differential Topology, *Annals of Math. Studies,* 54, Princeton Univ. Press, Princeton, N.J., 1958.
9. SHIFFMAN, B., On the removal of singularities of analytic functions, *Mich. Math. J.*, **15** (1968), 111—120.
10. STOUT, E. L. Removable singularities for the boundary values of holomorphic functions, to appear.
11. VIGNA SURIA, G., q-pseudoconvex and q-complete domains, *Comp. Math.*, **53** (1984), 105—111.

Received May 2, 1989

G. Lupacciolu
Dipartimento di Matematica
Università di Roma
"La Sapienza"
00185 Roma