Monotonicity properties of interpolation spaces*

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Abstract

For any interpolation pair (A_0, A_1), Peetre's K-functional is defined by:

$$K(t, a;\, A_0, A_1) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\,\|a_1\|_{A_1}).$$

It is known that for several important interpolation pairs (A_0, A_1), all the interpolation spaces A of the pair can be characterised by the property of *K-monotonicity*, that is, if $a \in A$ and $K(t, b;\, A_0, A_1) \leqq K(t, a;\, A_0, A_1)$ for all positive t then $b \in A$ also.

We give a necessary condition for an interpolation pair to have its interpolation spaces characterized by K-monotonicity. We describe a weaker form of K-monotonicity which holds for all the interpolation spaces of any interpolation pair and show that in a certain sense it is the strongest form of monotonicity which holds in such generality. On the other hand there exist pairs whose interpolation spaces exhibit properties lying somewhere between K-monotonicity and weak K-monotonicity. Finally we give an alternative proof of a result of Gunnar Sparr, that all the interpolation spaces for (L^p_v, L^q_w) are K-monotone.

0. Introduction

In the study of interpolation spaces the point of departure is usually a pair of Banach spaces A_0 and A_1 which are both continuously embedded in some Hausdorff topological vector space $\mathscr{A}$. We refer to the couple (A_0, A_1) as an *interpolation pair*.

For such a pair the vector spaces $A_0 \cap A_1$ and $A_0 + A_1$ are well defined and, when normed by

$$\|a\|_{A_0 \cap A_1} = \max(\|a\|_{A_0}, \|a\|_{A_1})$$

and

$$\|a\|_{A_0+A_1} = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + \|a_1\|_{A_1}),$$

* Research supported by the C. N. R. S.

become Banach spaces continuously embedded in $\mathscr{A}$. A_0+A_1 can be equivalently renormed by Peetre's "K-functional"

$$K(t, a;\ A_0, A_1) = \inf_{a=a_0+a_1} (\|a_0\|_{A_0} + t\,\|a_1\|_{A_1})$$

for any positive number t. The abbreviated notation $K(t, a)$ is also used where there is no risk of ambiguity. For each fixed $a \in A_0+A_1$, $K(t, a)$ is a continuous non decreasing concave function of t (see [2] p. 167).

A vector space A is called *intermediate* if $A_0 \cap A_1 \subset A \subset A_0+A_1$, the inclusions being continuous embeddings if A is topologized. An intermediate space A is an *interpolation space* if all linear operators on A_0+A_1 which map A_0 continuously into itself and A_1 continuously into itself also map A into itself (continuously if A is topologized).

Several mathematicians have studied the problem of characterizing all interpolation spaces for a given pair (A_0, A_1). In [5] Calderón gave a solution for the pair (L^1, L^∞). Subsequently Lorentz and Shimogaki [11] treated the pair (L^p, L^∞) with $1<p<\infty$, and Sedaev and Semenov [15], [16], dealt with a pair of L^p spaces with different weights, and also with a pair of Hilbert spaces. Sparr [17, 18] generalised the weighted L^p case permitting two different values of p. Certain pairs of spaces of compact operators have been considered by Gapaillard [7]. In each of the above cases it was found that the interpolation spaces for the pair could be characterized as those spaces possessing a property which we shall call *K-monotonicity.*

Definition 1. The space A is *K-monotone* with respect to the pair (A_0, A_1) if whenever $a \in A$, $b \in A_0+A_1$ and $K(t, b;\ A_0, A_1) \leqq K(t, a;\ A_0, A_1)$ for all positive t, it follows that $b \in A$.

In view of the above series of results we also introduce the following terminology.

Definition 2. The interpolation pair (A_0, A_1) will be called a *Calderón pair* if every intermediate space is an interpolation space if and only if it is K-monotone.

In this paper we first study interpolation pairs which are not Calderón. In section 2 we describe a necessary condition for a pair to be Calderón and also show that it is not sufficient. In section 3 we show that for an arbitrary interpolation pair (A_0, A_1), every interpolation space A satisfies a weak form of K-monotonicity: if $a \in A$ and $b \in A_0+A_1$, then b is also in A if the inequality $K(t, b) \leqq w(t)K(t, a)$ holds for all positive t, where $w(t)$ is a positive measurable function satisfying $\int_0^\infty \min(\varepsilon, w(t))\,dt/t < \infty$ for some positive constant ε. This result seems very close to the best possible. It will be seen that the hypothesis on $w(t)$ cannot be weakened to $\int_0^\infty \min(\varepsilon, w(t)^p)\,dt/t < \infty$ for some $p>1$. At the same time we note that for some non-Calderón pairs the hypothesis on $w(t)$ can be weakened. We study a case where it suffices that $\lim_{t\to 0} w(t)=0$. Finally in section 4 we give an alternative and independently conceived proof of Gunnar Sparr's result that a pair of weighted L^p spaces is Calderón. We take this opportunity to thank Professor Jaak Peetre for having informed us that this result was already known, and also Dr. Sparr himself for having provided some details of his forthcoming paper [18] which will elaborate the work announced in [17]. Thanks are also due to Dr. Yoram Sagher for a stimulating discussion concerning this class of problems, and to the referee for some suggestions for improving the presentation of this paper. We remark in passing that the description of all interpolation spaces for the pair (L^p, L^q) is of interest in connection with norm convergence of Fourier series in rearrangement invariant Banach spaces. (See [6].)

1. Preliminaries

(a) For any pair of Banach spaces A and B, $\mathscr{L}(A, B)$ will denote the class of all bounded linear operators mapping A into B, and $\mathscr{L}_\lambda(A, B)$ will denote the subclass of $\mathscr{L}(A, B)$ of operators with norm not exceeding λ. Let $\mathscr{L}(A)=\mathscr{L}(A, A)$ and $\mathscr{L}_\lambda(A)=\mathscr{L}_\lambda(A, A)$.

(b) Let $\mathbf{R}$ denote the real line and $\mathbf{R}_+$ the positive real line, each equipped with Lebesgue measure. Let $\mathbf{T}$ denote the circle group with Haar measure.

(c) The spaces L^p are defined in the usual way, and where it is necessary to indicate the underlying measure space (X, Σ, μ) we shall use any one of the following notations: $L^p(X)$, $L^p(\mu)$, $L^p(d\mu)$. For a positive weight function $w(x)$, the weighted L^p space L^p_w consists of all functions $f(x)$ such that $f(x)w(x)\in L^p$ with $\|f\|_{L^p_w}=\|fw\|_{L^p}$.

(d) Given an interpolation pair (A_0, A_1) there are two important special methods of constructing interpolation spaces.

(i) The real method (see for example [2] Chapter 3): For $0<\theta<1$ and $1\leq q<\infty$, the space $(A_0, A_1)_{\theta, q}$ is defined to consist of all elements $a\in A_0+A_1$ such that

$$\|a\|_{(A_0, A_1)_{\theta, q}} = \left(\int_0^\infty [t^{-\theta}K(t, a;\, A_0, A_1)]^q\, dt/t\right)^{1/q} < \infty .$$

$(A_0, A_1)_{\theta, \infty}$ is defined similarly by the norm $\sup_{t>0} t^{-\theta}K(t, a)$.

(ii) The complex method (see for example [4]): Let $\mathscr{F}(A_0, A_1)$ be the space of A_0+A_1-valued functions $f(z)$ continuous in the strip $0\leq \operatorname{Re} z\leq 1$ and analytic in its interior such that

$$\|f\|_{\mathscr{F}} = \max\Big\{ \sup_{-\infty<y<\infty} \|f(iy)\|_{A_0},\ \sup_{-\infty<y<\infty} \|f(1+iy)\|_{A_1}\Big\} < \infty .$$

Then the complex interpolation space $[A_0, A_1]_\theta$ is defined by $[A_0, A_1]_\theta=\{f(\theta)\,|\,f\in\mathscr{F}\}$, and as norm we usually take $\|a\|_\theta=\inf\{\|f(z)\|_{\mathscr{F}}\,|\,f(\theta)=a\}$.

(e) The notation $\Phi(t,f)\sim\Psi(t,f)$ shall mean that there exists a positive constant C independent of t and f such that $C^{-1}\Phi(t,f)\leqq\Psi(t,f)\leqq C\Phi(t,f)$.

(f) It is a simple matter to show that if $S\in\mathscr{L}_\alpha(A_0)\cap\mathscr{L}_\beta(A_1)$ and $a\in A_0+A_1$, then $K(t, Sa;\ A_0, A_1)\leqq\max(\alpha,\beta)K(t, a;\ A_0, A_1)$. Thus any K-monotone space is necessarily an interpolation space with respect to (A_0, A_1). The non trivial part of the proof that a given pair (A_0, A_1) is Calderón is to show that if f, g are in A_0+A_1 with $K(t, g)\leqq K(t,f)$ for all positive t, then there exists an operator $S\in\mathscr{L}(A_0)\cap\mathscr{L}(A_1)$ with $Sf=g$ and so every interpolation space is K-monotone.

(g) For any measurable function f on (X,Σ,μ) we let $f^*(t)$ denote the non-increasing rearrangement of $|f|$ on $\mathbf{R}_+$. Then

$$K(t,f;\ L^1, L^\infty) = \int_0^t f^*(s)\,ds \quad \text{(Peetre [12])}$$

and

$$K(t,f;\ L^p, L^\infty) \sim \Big(\int_0^{t^p} f^*(s)^p\,ds\Big)^{1/p} \quad \text{(Krée [10])}.$$

For $0<p<q<\infty$, Holmstedt [8] has shown that:

$$K(t,f;\ L^p, L^q) \sim \Big(\int_0^{t^\alpha} f^*(s)^p\,ds\Big)^{1/p}+t\Big(\int_{t^\alpha}^\infty f^*(s)^q\,ds\Big)^{1/q}$$

where $1/\alpha=1/p-1/q$.

2. Interpolation pairs which are not Calderón

Define $A_0+\infty\cdot A_1$ to be the space of all elements $a\in A_0+A_1$ for which $\|a\|_{A_0+\infty\cdot A_1}=\lim_{t\to\infty}K(t, a;\ A_0, A_1)$ is finite.

Let $A_1+\infty\cdot A_0$ be defined analogously, so that

$$\|a\|_{A_1+\infty\cdot A_0}=\lim_{t\to\infty}K(t, a;\ A_1, A_0)=\lim_{t\to 0}\frac{1}{t}K(t, a;\ A_0, A_1).$$

It is not very difficult to see that $A_0+\infty\cdot A_1$ is a Banach space which contains A_0, and that for each $a\in A_0$ $\|a\|_{A_0+\infty\cdot A_1}\leqq\|a\|_{A_0}$. In fact $A_0+\infty\cdot A_1$ can be thought of as a sort of closure of A_0 with respect to A_1, as the following lemma shows.

Lemma 1. *An element a of A_0+A_1 is in $A_0+\infty\cdot A_1$ if and only if there exists a sequence $(a_n)_{n=1}^\infty$ in A_0 with $\sup_n\|a_n\|_{A_0}<\infty$ and $\lim_{n\to\infty}\|a-a_n\|_{A_1}=0$. For each such a, $\|a\|_{A_0+\infty\cdot A_1}=\inf\{\sup_n\|a_n\|_{A_0}\}$ where the infimum is taken over all sequences (a_n) in A_0 for which $\lim_{n\to\infty}\|a-a_n\|_{A_1}=0$.*

Proof. We leave the details to the reader.

Lemma 2. *For all $a\in A_0+A_1$ and all positive t, $K(t, a;\ A_0+\infty\cdot A_1, A_1+\infty\cdot A_0)=K(t, a;\ A_0, A_1)$.*

Proof. Fix a and t, and let $b \in A_0 + \infty \cdot A_1$ and $c \in A_1 + \infty \cdot A_0$ be such that $a = b + c$ and

$$\|b\|_{A_0+\infty\cdot A_1} + t\|c\|_{A_1+\infty\cdot A_0} \leqq K(t, a;\ A_0 + \infty \cdot A_1, A_1 + \infty \cdot A_0) + \varepsilon$$

for some arbitrarily small positive number ε. Let $(b_n)_{n=1}^{\infty}$ and $(c_n)_{n=1}^{\infty}$ be sequences which approximate b and c in A_1 and A_0 norms respectively such that

$$\sup_n \|b_n\|_{A_0} \leqq \|b\|_{A_0+\infty\cdot A_1} + \varepsilon \quad \text{and} \quad \sup_n \|c_n\|_{A_1} \leqq \|c\|_{A_1+\infty\cdot A_0} + \varepsilon.$$

Then

$$K(t, a;\ A_0, A_1) \leqq \|b_n + c - c_n\|_{A_0} + t\|c_n + b - b_n\|_{A_1} \leqq$$
$$\leqq \|b\|_{A_0+\infty\cdot A_1} + t\|c\|_{A_1+\infty\cdot A_0} + (1+t)\varepsilon + O(n).$$

It follows that $K(t, a;\ A_0, A_1) \leqq K(t, a;\ A_0 + \infty \cdot A_1, A_1 + \infty \cdot A_0)$. The reverse inequality is an immediate consequence of the inequalities $\|a\|_{A_0+\infty\cdot A_1} \leqq \|a\|_{A_0}$, $\|a\|_{A_1+\infty\cdot A_0} \leqq \leqq \|a\|_{A_1}$.

Lemma 3. *If (A_0, A_1) is a Calderón pair, then $A_0 = A_0 + \infty \cdot A_1$ and $A_1 = A_1 + \infty \cdot A_0$.*

Proof. Let $a \in A_0 + \infty \cdot A_1$ and let $(a_n)_{n=1}^{\infty}$ be a bounded sequence in A_0 with $\lim_{n\to\infty} \|a - a_n\|_{A_1} = 0$. For all positive t $K(t, a) \leqq \|a\|_{A_0+\infty\cdot A_1}$, and also for any fixed n

$$K(t, a) \leqq K(t, a - a_n) + K(t, a_n)$$
$$\leqq t\|a - a_n\|_{A_1} + K(t, a_n).$$

So $K(t, a) \leqq K(t, a_n) + \min(t\|a - a_n\|_{A_1}, \|a\|_{A_0+\infty\cdot A_1})$. $K(t, a)$ is a positive non decreasing concave function and so for a sufficiently large positive number λ, $K(t, a) \leqq \leqq \lambda K(t, a_n)$. But, by hypothesis, A_0 as an interpolation space must be K-monotone and $\lambda a_n \in A_0$. Thus $a \in A_0$ and $A_0 = A_0 + \infty \cdot A_1$. Similarly $A_1 = A_1 + \infty \cdot A_0$.

Remark. It can be seen that if $A_0 \neq A_0 + \infty \cdot A_1$ then interpolation spaces other than the "end point" spaces A_0 and A_1 may also fail to be K-monotone.

Examples. Let $C(\mathbf{R})$ be the space of continuous bounded functions on $\mathbf{R}$ with supremum norm, and let $W^{1,1}(\mathbf{R})$ be the Sobolev space of $L^1(\mathbf{R})$ functions f whose first derivatives f' (in the distribution sense) are also in $L^1(\mathbf{R})$. (The norm is $\|f\|_{W^{1,1}} = = \|f\|_{L^1} + \|f'\|_{L^1}$.) Then $C(\mathbf{R}) + \infty \cdot L^1 = L^\infty$ and, for example, $C(\mathbf{R})$ and $C(\mathbf{R}) \cap L^1$ are not K-monotone. $(L^1, W^{1,1})$ also fails to be a Calderón pair. In fact $W^{1,1} + \infty L^1 = = BV \cap L^1$, the space of integrable functions of bounded variation on $\mathbf{R}$ with norm $\|f\|_{BV\cap L^1} = \operatorname{var}(f) + \|f\|_{L^1}$.

The last example will show that Lemma 3 does not have a converse. Let $\mathbf{T}$ denote the circle group with Haar measure and $W^{1,p}(\mathbf{T})$ the Sobolev space of functions f in $L^p(\mathbf{T})$ whose (distributional) first derivatives f' are also in $L^p(\mathbf{T})$. As norm take $\|f\|_{W^{1,p}}=\|f\|_{L^p}+\|f'\|_{L^p}$. For $1\leqq p<\infty$ we have an estimate of Peetre,

$$K(t, f;\, L^p(T), W^{1,p}(T)) \sim \Psi(t, f)$$

where

$$\Psi(t,f) = \sup_{0\leqq |h|\leqq t} \|f(x+h)-f(x)\|_{L^p}+t\|f\|_{L^p} \quad \text{for} \quad t<1$$

$$= \|f\|_{L^p} \quad \text{for} \quad t\geqq 1.$$

See [12], [13], also [2] p. 258. Though these proofs are given for $(L^p(\mathbf{R}^n),\ W^{1,p}(\mathbf{R}^n))$ rather than for spaces taken on the circle or n-torus, the result for $\mathbf{T}$ or $\mathbf{T}^n$ can be readily deduced. (For example construct an operator $S\in\mathscr{L}(L^p(\mathbf{T}^n),\ L^p(\mathbf{R}^n))\cap \cap\mathscr{L}(W^{1,p}(\mathbf{T}^n),\ W^{1,p}(\mathbf{R}^n))$ where Sf is the periodic extension of f multiplied by a suitable C^∞ function of compact support, so that $Sf|_{\mathbf{T}^n}=f$ and $\|Sf\|_{L^p(\mathbf{R}^n)}\sim\|f\|_{L^p(\mathbf{T}^n)}$, $\|Sf\|_{W^{1,p}(\mathbf{R}^n)}\sim\|f\|_{W^{1,p}(\mathbf{T}^n)}$ and $\Psi(t,\, Sf)\sim\Psi(t,f)$.)

Let $H^p_\alpha(\mathbf{T})$ be the space of tempered distributions f on $\mathbf{T}$ whose Fourier coefficients $\hat{f}(n)$ are given by $\hat{f}(n)=(1+|n|^2)^{-\alpha/2}\hat{\varphi}(n)$, where φ is a function in $L^p(\mathbf{T})$. Let $\|f\|_{H^p_\alpha}=\|\varphi\|_{L^p}$. There are analogous definitions for H^p_α on $\mathbf{T}^n$ and $\mathbf{R}^n$. For our purposes it suffices to consider the parameter α in the range $(0, 1)$ and in this case $H^p_\alpha=[L^p,\ W^{1,p}]_\alpha$, as was shown by Calderón ([3], [4]). However H^p_α is not K-monotone, at least if $2<p<\infty$ and $1/p<\alpha<1$. This may be seen with the help of some special functions used by Taibleson [19] to show non-inclusions between H^p_α and certain generalised Lipschitz or Besov spaces $(L^p,\ W^{1,p})_{\alpha,q}$.

The function $k=k_{\alpha,1/2}$ which has the Fourier series $\sum_1^\infty 2^{-n\alpha}n^{-1/2}\cos 2^n x$ does not belong to H^p_α ([19] p. 473 paragraph (d)). Using the estimate on p. 472 of [19] we see that $\|k(x+h)-k(x)\|_{L^p(\mathbf{T})}\leqq M_1|h|^\alpha\log^{-1/2}(1/|h|)$ for some constant M_1. However the function $f=f_{\alpha+1/p',\,1/p+\varepsilon}$ with Fourier series $\sum_2^\infty n^{-\alpha-1/p'}\log^{-1/p-\varepsilon}n\cos nx$ is in H^p_α for each $\varepsilon>0$ and furthermore $\|f(x+h)-f(x)\|_{L^p(\mathbf{T})}\geqq M_2|h|^\alpha\log^{-1/p-\varepsilon}(1/|h|)$ for some constant M_2 ([19] pp. 473—474, paragraph (h)). We choose $\varepsilon=1/2-1/p$ and clearly $K(t,k)\leqq K(t,\lambda f)$ for all t and some constant λ.

(It is easy to deduce that $H^p_\alpha(\mathbf{R}^n)$ and $H^p_\alpha(\mathbf{T}^n)$ are also not K-monotone for the above ranges of values of p and α using Lemmas 23, 24 and 25 of [19]. Incidentally, by using interpolation methods with an operator of the form S as above, one can give an immediate proof of Lemma 25.)

Obviously $L^p+\infty\cdot W^{1,p}=L^p$, and from the weak compactness of the unit ball of L^p for $1<p\leqq\infty$ we may readily deduce that $W^{1,p}+\infty\cdot L^p=W^{1,p}$. Let us summarize the results of this section.

Theorem. *Every Calderón pair* (A_0, A_1) *has the "mutual closure" property* $A_0=A_0+\infty\cdot A_1$, $A_1=A_1+\infty\cdot A_0$, *but this property is not a sufficient condition for an interpolation pair to be Calderón.*

3. Weak K-monotonicity

Having observed that there are at least two different "mechanisms" which may prevent an interpolation space from being K-monotone, we now turn to the study of a monotonicity property weaker than K-monotonicity which holds in all interpolation spaces.

Lemma 1. *Let $w(t)$ be a positive measurable function such that $\int_0^\infty w(t)\,dt/t<\infty$, and let (A_0, A_1) be an interpolation pair. Let $f, g\in A_0+A_1$ such that $K(t, g)=w(t)K(t,f)$ for all positive t. Then there exists an operator $S\in\mathscr{L}_\lambda(A_0)\cap\mathscr{L}_\lambda(A_1)$ such that $Sf=g$. λ may be taken to be any number greater than $\min_{\alpha>1}(2\alpha/\log\alpha)\int_0^\infty w(t)\,dt/t$.*

Remark. In connection with this lemma we should draw attention to the work of Jöran Bergh [1]. It has been pointed out to us that Bergh's Theorem 2.1, when combined with the so-called "fundamental lemma" (see for example [2] page 172, Lemma 3.2.10) which relates Peetre's K and J functionals, can readily give a proof of a discretized version of Lemma 1, in which $K(2^n, g)\leqq w_n K(2^n,f)$ for $n=0, \pm 1, \pm 2, \dots$ and $\sum_{-\infty}^{\infty} w_n<\infty$ imply the existence of $S\in\mathscr{L}(A_0)\cap\mathscr{L}(A_1)$ with $Sf=g$. In fact, by arguments similar to those in the proof of Theorem 1 below, the discrete and continuous versions of Lemma 1 imply each other to within a constant; the passage from either version to the other worsens the estimates for the norms of S. The proof which follows will implicitly contain proofs of both the "fundamental lemma" and Bergh's result, together with a "rescaling" which gives better norm estimates for S.

Proof. Let $r>1$ be such that $\min_{\alpha>1} 2\alpha/\log\alpha=2r/\log r$. Choose a number $\varepsilon>0$. For each $n=0, \pm 1, \pm 2, \dots$, let $g=a_n+b_n$, where $a_n\in A_0$, $b_n\in A_1$, and $\|a_n\|_{A_0}+$ $+r^n\|b_n\|_{A_1}\leqq(1+\varepsilon)K(r^n, g)$. We shall need two estimates:

$$\|a_n-a_{n-1}\|_{A_0}\leqq(1+\varepsilon)\frac{(1+r)}{\log r}\Big(\int_{r^{n-1}}^{r^n} w(t)\,dt/t\Big)K(r^n,f). \tag{1}$$

$$\|a_n-a_{n-1}\|_{A_1}\leqq(1+\varepsilon)\frac{2r^{-n+1}}{\log r}\Big(\int_{r^{n-1}}^{r^n} w(t)\,dt/t\Big)K(r^n,f). \tag{2}$$

For (1),

$$\|a_n-a_{n-1}\|_{A_0}\leqq\|a_n\|_{A_0}+\|a_{n-1}\|_{A_0}\leqq(1+\varepsilon)\big(K(r^n, g)+K(r^{n-1}, g)\big)$$
$$\leqq(1+\varepsilon)(1+r)K(r^{n-1}, g),$$

since $K(t, g)/t$ is non-increasing,

$$\leqq(1+\varepsilon)(1+r)\frac{\int_{r^{n-1}}^{r^n} K(t, g)\,dt/t}{\int_{r^{n-1}}^{r^n} dt/t},$$

since $K(t, g)$ is non-decreasing,

$$\leqq (1+\varepsilon)\frac{(1+r)}{\log r}\left(\int_{r^{n-1}}^{r^n} w(t)\,dt/t\right)K(r^n, f).$$

For (2),

$$\begin{aligned}\|a_n - a_{n-1}\|_{A_1} = \|b_{n-1}-b_n\|_{A_1} &\leqq (1+\varepsilon)\big(r^{-n+1}K(r^{n-1}, g)+r^{-n}K(r^n, g)\big)\\ &\leqq (1+\varepsilon)\big(2r^{-n+1}K(r^{n-1}, g)\big)\\ &\leqq (1+\varepsilon)\frac{2r^{-n+1}}{\log r}\left(\int_{r^{n-1}}^{r^n} w(t)\,dt/t\right)K(r^n, f).\end{aligned}$$

For each n $\|h\| = K(r^n, h)$ is a norm on A_0+A_1 and thus there exists a continuous linear functional l_n on A_0+A_1 such that $l_n(f)=K(r^n, f)$ and $|l_n(h)|\leqq K(r^n, h)$ for all $h\in A_0+A_1$. The operator S will be given by

$$Sh = \sum_{n=-\infty}^{\infty}\frac{l_n(h)}{K(r^n, f)}(a_n-a_{n-1}) \quad \text{for all} \quad h\in A_0+A_1.$$

If $h\in A_0$, Sh is given by an absolutely convergent A_0-valued series, since

$$\sum_{n=-\infty}^{\infty}\frac{|l_n(h)|}{K(r^n, f)}\|a_n-a_{n-1}\|_{A_0} \leqq (1+\varepsilon)\frac{(1+r)}{\log r}\int_0^\infty w(t)\,dt/t\,\|h\|_{A_0} \text{ from (1).}$$

Similarly if $h\in A_1$

$$\sum_{n=-\infty}^{\infty}\frac{|l_n(h)|}{K(r^n, f)}\|a_n-a_{n-1}\|_{A_1} \leqq (1+\varepsilon)\frac{2r}{\log r}\int_0^\infty w(t)\,dt/t \text{ by (2)}$$

and so $Sh\in A_1$, and indeed $S\in \mathscr{L}_\lambda(A_0)\cap\mathscr{L}_\lambda(A_1)$ for every λ greater than $(1+\varepsilon)2r/\log r\int_0^\infty w(t)\,dt/t$. Since $f\in A_0+A_1$, $Sf=\sum_{-\infty}^{\infty}(a_n-a_{n-1})$ is a series converging absolutely in A_0+A_1 norm.

$$\begin{aligned}Sf &= \sum_{-\infty}^{0}(a_n-a_{n-1})+\sum_{1}^{\infty}(b_{n-1}-b_n)\\ &= a_0-\lim_{n\to-\infty}a_{n-1}+b_0-\lim_{n\to\infty}b_n.\end{aligned}$$

As in the proof of (1),

$$K(r^n, g)\leqq \frac{1}{\log r}\left(\int_{r^n}^{r^{n+1}} w(t)\,dt/t\right)K(r^{n+1}, f).$$

Thus as $n\to-\infty$ $K(r^n, g)\to 0$ and as $n\to+\infty$ $K(r^n, g)/r^n\to 0$. From

$$\|a_n\|_{A_0}+r^n\|b_n\|_{A_1}\leqq (1+\varepsilon)K(r^n, g)$$

we have

$$\lim_{n\to-\infty}\|a_n\|_{A_0+A_1}\leqq \lim_{n\to-\infty}\|a_n\|_{A_0}=0, \quad \lim_{n\to\infty}\|b_n\|_{A_0+A_1}\leqq\lim_{n\to\infty}\|b_n\|_{A_1}=0.$$

So $Sf=a_0+b_0=g$.

Theorem 1. *Let $w(t)$ be a positive measurable function such that for some positive number ε, $\int_0^\infty \min(\varepsilon, w(t))\,dt/t<\infty$. Let A be an interpolation space for (A_0, A_1). Then if $f\in A$ and $g\in A_0+A_1$ such that $K(t,g)\leqq w(t)K(t,f)$ for all $t>0$ it follows that $g\in A$.*

Proof. We change to a notation in which A_0 and A_1 appear more symmetrically. Let $K_*(x,a)=e^{-x/2}K(e^x,a)$ and $w_*(x)=w(e^x)$, so that $K_*(x,g)\leqq w_*(x)K_*(x,f)$ for all $x\in(-\infty,\infty)$ and $\int_{-\infty}^\infty \min(\varepsilon, w_*(x))\,dx<\infty$. For any $a\in A_0+A_1$ and any real x and y we see that $K_*(x+y,a)\leqq e^{|y|/2}K_*(x,a)$. Let $H(x)=K_*(x,g)/K_*(x,f)$. We deduce immediately that $H(x+y)\geqq e^{-|y|}H(x)$ for all real x and y. Further, since $H(x)\leqq w_*(x)$, the set $\{x\,|\,H(x)>\eta\}$ must have finite Lebesgue measure for any positive η. It follows that $\lim_{|x|\to\infty}H(x)=0$, and that $\int_{-\infty}^\infty H(x)\,dx<\infty$. Let $w_1(t)=$ $=H(\log t)$. $\int_0^\infty w_1(t)\,dt/t<\infty$ and $K(t,g)\leqq w_1(t)K(t,f)$. Using Lemma 1 we conclude that $g\in A$.

Remarks. In some particular cases the condition $K(t,g)\leqq w(t)K(t,f)$ for $f\in A$ forces g to be in a class much smaller than A. For example if $f\in(A_0,A_1)_{\theta,\infty}$ g must be in $(A_0,A_1)_{\theta,1}$, and if $f\in A_0$ or A_1 then g must be zero. This seems to suggest that the above theorem is rather crude and that for example, it should be possible to weaken the conditions imposed on $w(t)$ and still have $g\in A$. Bearing in mind that for some interpolation pairs we only need $w(t)$ to be bounded, we ask if it is possible to weaken the requirement $\int_0^\infty \min(\varepsilon, w(t))\,dt/t<\infty$ to something corresponding to a slower convergence of $w(t)$ to zero as $t\to 0$ and $t\to\infty$, for example $\int_1^\infty \min(\varepsilon, w(t)^p)\,dt/t<\infty$ for some $p>1$. We shall construct an example which shows that such a sharpening of the theorem is in fact impossible. On the other hand there are specific cases of non-Calderón pairs where a weaker condition on $w(t)$ certainly does suffice. As a second example we shall consider the pair $(L^1(\mathbf{T}), C(\mathbf{T}))$ for which it suffices that $\lim_{t\to 0}w(t)=0$.

Example 1. Let $\{B_n\}_{n=1}^\infty$ be a sequence of Banach spaces. For $1\leqq p\leqq\infty$ define the space $l^p\{B_n\}$ to consist of all vector valued sequences $\{a_n\}_{n=1}^\infty$ satisfying $a_n\in B_n$ for each n, and $\|\{a_n\}\|_{l^p\{B_n\}}=\left(\sum_{n=1}^\infty \|a_n\|_{B_n}^p\right)^{1/p}<\infty$. The usual modification is made for $p=\infty$.

Lemma 2. *Let (B_n, C_n) $n=1,2,\ldots$ be a sequence of interpolation pairs. Then $(l^1\{B_n\}, l^1\{C_n\})$ is an interpolation pair and*

(i) $K(t, \{a_n\}; l^1\{B_n\}, l^1\{C_n\}) = \sum_1^\infty K(t, a_n; B_n, C_n)$

(ii) $l^1\{(B_n, C_n)_{\theta,q}\}\subset(l^1\{B_n\},\ l^1\{C_n\})_{\theta,q}$ *for* $0<\theta<1$ *and* $1\leqq q\leqq\infty$.

The inclusion is an equality for $q=1$.

(iii) $[l^1\{B_n\}, l^1\{C_n\}]_\theta = l^1\{[B_n, C_n]_\theta\}$ for $0<\theta<1$.

Proof. It is easy to see that $l^1\{B_n\}$ and $l^1\{C_n\}$ are each Banach spaces continuously embedded in $l^1\{B_n+C_n\}$. The proofs of (i) and (ii) are left to the reader. For (iii) it is convenient to use a different construction for the complex interpolation space $[A_0, A_1]_\theta$. Let $\mathscr{I}_{1,1}(A_0, A_1)$ be the space of A_0+A_1-valued analytic functions $f(z)$ defined in the strip $0<\operatorname{Re} z<1$ such that as $x\to j$ $(j=0, 1)$ $f(x+iy)$ converges in the weak topology of tempered A_0+A_1-valued distributions on $\mathbf{R}$ to a strongly measurable A_j-valued function $f(j+iy)$ for which $\int_{-\infty}^{\infty} \|f(j+iy)\|_{A_j} dy<\infty$. Then $[A_0, A_1]_\theta$ consists of all elements $a\in A_0+A_1$ such that $a=f(\theta)$ for some $f(z)\in\mathscr{I}_{1,1}(A_0, A_1)$ and may be normed by

$$\|a\|_{[A_0, A_1]_\theta} = \inf_{a=f(\theta)} \int_{-\infty}^{\infty} \{\|f(iy)\|_{A_0} + \|f(1+iy\|_{A_1}\} dy.$$

Using ideas implicit in section 9.4 of [4] which are further explained in [14] (Lemma 1.1) it can be seen that this construction gives the same space to within equivalence of norm as that obtained from the original definition.

Let $\{a_n\}\in l^1\{[B_n, C_n]_\theta\}$. There exist analytic functions $f_n(z)\in\mathscr{I}_{1,1}(B_n, C_n)$ such that $f_n(\theta)=a_n$ and

$$\|a_n\|_{[B_n, C_n]_\theta} \geqq (1-\varepsilon)\int_{-\infty}^{\infty} \{\|f_n(iy)\|_{B_n} + \|f_n(1+iy)\|_{C_n})\} dy.$$

Let $\{f_{n,m}(z)\}$ and $\{a_{n,m}\}$ be truncated sequences, that is $f_{n,m}(z)=f_n(z)$, $a_{n,m}=a_n$ for $n\leqq m$ and $f_{n,m}(z)=0$, $a_{n,m}=0$ for $n>m$. Noting that $l^1\{B_n\}+l^1\{C_n\} = l^1\{B_n+C_n\}$ has dual space $l^\infty\{B_n'\cap C_n'\}$, we see that for each m, $\{f_{n,m}(z)\}\in\mathscr{I}_{1,1}(l^1\{B_n\}, l^1\{C_n\})$ and so $\{a_{n,m}\}\in[l^1\{B_n\}, l^1\{C_n\}]_\theta$ with norm

$$\|\{a_{n,m}\}\|_{[l^1\{B_n\}, l^1\{C_n\}]_\theta} \leqq \sum_{n=1}^{m} \int_{-\infty}^{\infty} \{\|f_n(iy)\|_{B_n} + \|f_n(1+iy)\|_{C_n}\} dy.$$

By similar estimates $\{a_{n,m}\}$ is a Cauchy sequence with respect to m in $[l^1\{B_n\}, l^1\{C_n\}]_\theta$. Thus its limit $\{a_n\}$ in $l^1\{B_n\}+l^1\{C_n\}$ must also be in $[l^1\{B_n\}, l^1\{C_n\}]_\theta$. This shows that $l^1\{[B_n, C_n]_\theta\}\subset[l^1\{B_n\}, l^1\{C_n\}]_\theta$. We leave the proof of the reverse inclusion to the reader.

We can now construct the required interpolation pair with the help of Lemma 2. Let $\{r_n\}_{n=1}^{\infty}$ be a sequence including all the rational numbers in $(1, \infty)$. Let us take $B_n=L^{r_n}(\mathbf{R}_+)$ and $C_n=L^\infty(\mathbf{R}_+)$ for $n=1, 2, \ldots$ and let $A_0=l^1\{L^{r_n}(\mathbf{R}_+)\}$, $A_1=$ $=l^1\{L^\infty(\mathbf{R}_+)\}$. Then $[A_0, A_1]_\theta=l^1\{[L^{r_n}, L^\infty]_\theta\}=l^1\{L^{r_n/(1-\theta)}\}$ ([4], 13.5, 13.6). We next observe that

$$(3) \qquad (A_0, A_1)_{\theta,q} \not\subset [A_0, A_1]_\theta \quad \text{for all} \quad q > 1/(1-\theta).$$

The space $(l^1\{L^{r_n}\}, l^1\{L^\infty\})_{\theta,q}$ includes sequences $\{a_n\}$ such that $a_n=0$ for all $n\neq m$ and $a_m\in(L^{r_m}, L^\infty)_{\theta,q}=L(r_m/(1-\theta), q)\not\subset L^{r_m/(1-\theta)}$ if m is such that $q>r_m/(1-\theta)$. (See [2], p. 187 and [9] p. 225.)

Now let us suppose that there exists a number $p>1$ such that the conclusion of Theorem 1 holds when $w(t)$ satisfies the weakened integrability condition $\int_0^\infty \min(\varepsilon, w(t)^p)\,dt/t<\infty$. We shall see that this contradicts (3). Let us choose θ sufficiently small so that $p>1/(1-\theta)$. We also introduce a second positive number α chosen to ensure that

(4) (i) $p>1/(1-\theta)(1-\alpha)>1/(1-\theta)$

(ii) $r=p(1-\theta)(1-\alpha)/\alpha$ is a rational number greater than 1.

Let $g=\{g_n\}\in(A_0, A_1)_{\theta,p(1-\alpha)}$. Then $w(t)=\left(t^{-\theta}K(t, g;\ A_0, A_1)\right)^{1-\alpha}$ satisfies $\int_0^\infty w(t)^p\,dt/t<\infty$ and $K(t, g)=w(t)\,t^{\theta(1-\alpha)}(K(t, g))^\alpha$.

Our next step will be to show that $t^{\theta(1-\alpha)}(K(t, g))^\alpha \leqq K(t, f)$ for some $f\in[A_0, A_1]_\theta$. On the assumption that the sharpened version of Theorem 1 is true, $K(t, g)\leqq \leqq w(t)K(t, f)$ then implies that $g\in[A_0, A_1]_\theta$. But g is an arbitrary element of $(A_0, A_1)_{\theta,p(1-\alpha)}$ and so (3) will be contradicted.

As a non-decreasing concave function of t, $K(t, g)$ must be absolutely continuous on every compact subinterval of $(0, \infty)$. Thus it is differentiable almost everywhere and the derivative $K'(t, g)$ must coincide almost everywhere with a non-increasing non-negative function. We introduce the function $h(t)$,

$$h(t) = [\theta(1-\alpha)\,t^{\theta(1-\alpha)-1}(K(t^{1/r}, g))^{\alpha r} + \alpha t^{\theta(1-\alpha)+1/r-1}(K'(t^{1/r}, g))^{\alpha r-1}]^{1/r}.$$

From (4) and the fact that $K(t, g)/t$ is non-increasing we see that $h(t)$ is a non-increasing function such that $h(t)^r=(d/dt)\left[t^{\theta(1-\alpha)}(K(t^{1/r}, g))^{\alpha r}\right]$ almost everywhere. But $t^{\theta(1-\alpha)}(K(t^{1/r}, g))^{\alpha r}$ is also absolutely continuous on every compact subinterval of $(0, \infty)$ and tends to zero as t tends to zero. It follows that

$$t^{\theta(1-\alpha)}(K(t^{1/r}, g))^{\alpha r} = \int_0^t h(s)^r\,ds$$

and so

$$t^{\theta(1-\alpha)}(K(t, g))^\alpha = \left(\int_0^{t^r} h(s)^r\,ds\right)^{1/r} \leqq K(t, h;\ L^r(\mathbf{R}_+), L^\infty(\mathbf{R}_+))$$

(as in [10] p. 159). Since r is rational $r=r_m$ for some m and if $f=\{f_n\}$ is a sequence in A_0+A_1 which is zero for all $n\neq m$ and has $f_m=h$, then $K(t, f;\ A_0, A_1)=K(t, h;\ L^r, L^\infty)$. It remains only to show that $f\in[A_0, A_1]_\theta$ which amounts to showing that $h\in L^{r/(1-\theta)}$. But

$$h(t)^r \leqq \frac{1}{t}\int_0^t h(s)^r\,ds = t^{\theta(1-\alpha)-1}(K(t^{1/r}, g))^{\alpha r},$$

and so

$$\int_0^\infty h(t)^{r/(1-\theta)}\,dt \leqq \int_0^\infty [t^{-\theta/r}K(t^{1/r}, g)]^{p(1-\alpha)}\,dt/t$$

$$= (r\|g\|_{(A_0, A_1)_{\theta,p(1-\alpha)}})^{p(1-\alpha)} < \infty.$$

Example 2. Let $C(\mathbf{T})$ be the space of continuous functions on $\mathbf{T}$ with the supremum norm. In the light of section 2 we can immediately see that $(L^1(\mathbf{T}), C(\mathbf{T}))$

is not a Calderón pair and that

$$K(t,f;\,L^1,C) = K(t,f;\,L^1,L^\infty) = \int_0^t f^*(s)\,ds \quad \text{for} \quad 0 < t \leqq 2\pi,$$
$$= K(2\pi,f) = \|f\|_{L^1} \quad \text{for} \quad t \geqq 2\pi.$$

Theorem 2. *Let f and g belong to $L^1(\mathbf{T})+C(\mathbf{T})$ such that $K(t,g)\leqq w(t)K(t,f)$ for all t, $0<t\leqq 2\pi$, where $0\leqq w(t)\leqq 1$ and $\lim_{t\to 0} w(t)=0$. Then for any $\varepsilon>0$ there exists an operator $S\in\mathscr{L}_{1+\varepsilon}(L^1)\cap\mathscr{L}_{1+\varepsilon}(C)$ such that $Sf=g$.*

Proof. Let φ be a C^∞ function on $[-\pi,\pi]$ with $\varphi(\pi)=\varphi(-\pi)=0$ and $\int_{-\pi}^{\pi}\varphi(x)\,dx=1$. For each positive integer n let

$$\varphi_n(x) = n\varphi(nx) \quad \text{for} \quad |x| \leqq \pi/n$$
$$= 0 \qquad \text{for} \quad \pi/n \leqq |x| \leqq \pi.$$

We shall first prove the theorem under the extra assumption that g is bounded. From Calderón's study of (L^1, L^∞) in [5] there exists an operator U in $\mathscr{L}_1(L^1)\cap\mathscr{L}_1(L^\infty)$ with $Uf=g$. Let $(m_n)_{n=1}^\infty$ be an increasing subsequence of the positive integers, and $(a_n)_{n=1}^\infty$ a sequence of positive numbers tending to zero. Let $E_n=\{x\,|\,|f(x)|>f^*(a_n)\}$ and let $a_n'=|E_n|$. Then $a_n'\leqq a_n$.

The required operator S will have the form:

$$Sh = S_1h+S_2h$$
$$= Uh*\varphi_{m_1}+\sum_{n=1}^\infty \big(g*(\varphi_{m_{n+1}}-\varphi_{m_n})\big)\frac{\int_{E_n} h|f|/f\,dx}{\int_{E_n}|f|\,dx}$$

where "$*$" denotes convolution of functions on $\mathbf{T}$.

First note that $Sf=g$ since $Uf=g$ and $\lim_{n\to 0}\|g*\varphi_n-g\|_{L^1}=0$. It is clear that $S_1\in\mathscr{L}_1(L^1)\cap\mathscr{L}_1(C)$ and thus it suffices to show that for suitable choices of the sequences (m_n) and (a_n), $S_2\in\mathscr{L}_\varepsilon(L^1)\cap\mathscr{L}_\varepsilon(C)$.

For any $h\in C(\mathbf{T})$,

$$\|S_2h\|_C \leqq \sum_{n=1}^\infty \|g*(\varphi_{m_{n+1}}-\varphi_{m_n})\|_C a_n'\|h\|_C\Big/\int_{E_n}|f|\,dx$$
$$\leqq 2\|g\|_{L^\infty}\|h\|_C\sum_{n=1}^\infty a_n'\Big/\int_{E_n}|f|\,dx.$$

For each a_n' sufficiently small we have:

$$\|g\|_{L^\infty}a_n' \leqq 2\int_0^{a_n'} g^*(t)\,dt \leqq 2w(a_n')\int_0^{a_n'} f^*(t)\,dt = 2w(a_n')\int_{E_n}|f|\,dx.$$

Therefore $\|S_2h\|_C\leqq 4\|h\|_C\sum_{n=1}^\infty w(a_n')$. By choosing a sequence (a_n) of sufficiently small and sufficiently rapidly decreasing numbers we will have $\|S_2h\|_C\leqq\varepsilon\|h\|_C$. Now for any $h\in L^1$,

$$\|S_2h\|_{L^1} \leqq \|h\|_{L^1}\sum_{n=1}^\infty \|g*(\varphi_{m_{n+1}}-\varphi_{m_n})\|_{L^1}\Big/\int_{E_n}|f|\,dx.$$

The choice of the sequence (a_n) which we have made to guarantee that $S_2 \in \mathscr{L}_\varepsilon(C)$ is independent of the sequence (m_n). Since $\lim_{n\to\infty} \|g * \varphi_n - g\|_{L^1} = 0$, we can now choose the sequence (m_n) so as to obtain that $S_2 \in \mathscr{L}_\varepsilon(L^1)$. This completes the proof if g is bounded.

Now the proof will be extended to the case of $g \in L^1 + C = L^1$. Let (k_n) be a positive sequence increasing to infinity. Let $G_n(x) = \min(|g(x)|, k_n) g(x)/|g(x)|$. Thus $G_n \to g$ in L^1 or equivalently $g = \sum_{n=1}^\infty g_n$ where $g_1 = G_1$, $g_n = G_n - G_{n-1}$ for $n \geqq 2$, and the sum converges in L^1. Clearly $\int_0^t g_1^*(s)\,ds \leqq \int_0^t g^*(s)\,ds \leqq w(t) \int_0^t f^*(s)\,ds$ and thus using the boundedness of g_1, we can find an operator $U_1 \in \mathscr{L}_{1+\varepsilon/2}(L^1) \cap \mathscr{L}_{1+\varepsilon/2}(C)$ with $U_1 f = g_1$. For $n \geqq 2$, $|g_n(x)| \leqq |g(x)|$ and $g_n(x)$ vanishes for all x such that $|g(x)| \leqq \leqq k_{n-1}$. Therefore $g_n^*(t) = 0$ for all $t > k_n' = |\{x \mid |g(x)| > k_{n-1}\}|$. It follows that

$$\int_0^t g_n^*(s)\,ds \leqq w_n(t) \int_0^t f^*(s)\,ds$$

where

$$\begin{aligned} w_n(t) &= w(t) \quad \text{for} \quad 0 < t \leqq k_n', \\ &= w(k_n') \quad \text{for} \quad t \geqq k_n'. \end{aligned}$$

If k_n increases rapidly enough, k_n' decreases sufficiently rapidly to give $\sup_{0<t<2\pi} w_n(t) \leqq \varepsilon 2^{-n-2}$. The function g_n is bounded and so, by an obvious rescaling of the first part of the proof, there exists an operator U_n in $\mathscr{L}_{\varepsilon 2^{-n-1}}(L^1) \cap \mathscr{L}_{\varepsilon 2^{-n-1}}(C)$ such that $U_n f = g_n$. Clearly the operator $S = \sum_{n=1}^\infty U_n$ is in $\mathscr{L}_{1+\varepsilon}(L^1) \cap \mathscr{L}_{1+\varepsilon}(C)$ and $Sf = g$.

Remark. Using methods very similar to those in the proof of Theorem 1 we can readily weaken the hypotheses on $w(t)$ in the above theorem to, for example,

$$\lim_{n\to\infty} w(2^{-n}) = 0 \quad \text{or} \quad \lim_{n\to\infty} \int_{r^{-n}}^{r^{-n+1}} \min(\varepsilon, w(t))\,dt/t = 0$$

for some $\varepsilon > 0$ and some $r > 1$. Of course in these cases we will have poorer estimates for the norms of the operator S.

4. Weighted L^p spaces

Let (X, Σ, μ) be a measure space on which are defined two positive weight functions $v(x)$ and $w(x)$. Let p and $q \in [1, \infty]$. We present an alternative proof of Sparr's result that $(L_v^p(u), L_w^q(u))$ is a Calderón pair. It seems that both Sparr's and our methods could well provide ideas for the study of further interpolation pairs and for obtaining better estimates of the constants in quantitative versions of these types of theorems. One advantage of Sparr's method is that it can also be applied when p or q take values less than 1.

The main result of this section will be Theorem 4, which, together with the remarks of section 1, shows that (L^p, L^q) is a Calderón pair. We give the proofs for an arbitrary underlying measure space, thus dispensing with some restrictions imposed in earlier studies of (L^1, L^∞) and (L^p, L^∞). The case where L^p and L^q have different weights will be an easy corollary when $p\neq q$. Finally we consider the case $p=q$ and give a brief description of analogues of Theorems 1, 2 and 3 which lead to an alternative proof of the result of Sedaev.

Theorem 1. *Let $p\in[1, \infty)$ and let f, g be non-negative non-increasing simple functions on $\mathbf{R}_+$ such that:*

$$\int_0^t g(s)^p\,ds \leqq \int_0^t f(s)^p\,ds \quad \textit{for all positive } t.$$

Then there exists an operator $S\in\mathscr{L}_1(L^p(\mathbf{R}_+))\cap\mathscr{L}_1(L^\infty(\mathbf{R}_+))$ such that $Sf=g$.

Proof. This is exactly Lemma 4 of [11]. (The case $p=1$ was treated in [5].)

Theorem 2. *Let $q\in(1, \infty)$ and let f, g be non-negative non-increasing simple functions on $\mathbf{R}_+$ such that:*

$$\int_t^\infty g(s)^q\,ds \leqq \int_t^\infty f(s)^q\,ds \quad \textit{for all positive } t. \tag{1}$$

Then there exists an operator $U\in\mathscr{L}_1(L^1(\mathbf{R}_+))\cap\mathscr{L}_1(L^q(\mathbf{R}_+))$ such that $Uf=g$.

Proof. We proceed via two lemmas.

Lemma 2A. *Let φ, ψ be two measurable functions on a finite measure space such that φ is a constant and let $q>1$. Then $\|\psi\|_{L^q}\leqq\|\varphi\|_{L^q}$ implies $\|\psi\|_{L^1}\leqq\|\varphi\|_{L^1}$.*

Proof. Simple application of Hölder's inequality.

Lemma 2B. *Let f be a non-negative non-increasing simple function on $\mathbf{R}_+$ taking a constant value α on an interval $[a, b)$. Then for any a', $0<a'\leqq a$, there exists an operator $S\in\mathscr{L}_1(L^1(\mathbf{R}_+))\cap\mathscr{L}_1(L^q(\mathbf{R}_+))$ such that:*

(i) *Sf is non-negative and non-increasing*

(ii) *$Sf=\alpha$ on $[a', b)$*

(iii) *$\int_t^\infty (Sf)^q\,ds = \int_t^\infty f^q\,ds$ for all $0\leqq t\leqq a''$*

where $[a'', a')$ is the interval of constancy of Sf preceding $[a', b)$

(iv) *The number of different values taken by Sf on $[0, a')$ does not exceed the number of different values taken by f on $[0, a)$.*

Proof. Let $f=\sum_{j=1}^N \alpha_j\chi_{[a_{j-1}, a_j)}+\alpha\chi_{[a,b)}+f\chi_{[b,\infty)}$ where $0=a_0<a_1<\ldots<a_N=a$, and $\alpha_1>\alpha_2>\ldots>\alpha_N>\alpha$. For each $u\in[a_{N-1}, a_N)$ define the function f_u to equal α on $[u, a_N)$ and to equal $\lambda(u)\alpha_N$ on $[a_N, u)$, where $\lambda(u)>1$ is chosen to give

$$\int_{a_{N-1}}^{a_N} f_u^q\,ds = \int_{a_{N-1}}^{a_N} f^q\,ds.$$

By Lemma 2A,

$$\int_{a_{N-1}}^{a_N} f_u\,ds \leqq \int_{a_{N-1}}^{a_N} f\,ds.$$

Clearly $\lambda(u)$ is a continuous decreasing function of u. Let u_N be the smallest value of u in $[a_{N-1}, a_N)$ for which $\lambda(u)\alpha_N \leqq \alpha_{N-1}$, and for all $u\in[u_N, a_N)$ define the operator S_u by:

$$S_u h = \frac{f_u}{(a_N - a_{N-1})\alpha_N}\int_{a_{N-1}}^{a_N} h\,ds \quad \text{on} \quad [a_{N-1}, a_N)$$
$$= h \quad \text{elsewhere,}$$

for all $h\in L^1+L^q$.

It is easy to see that $S_u\in\mathscr{L}_1(L^1)\cap\mathscr{L}_1(L^q)$, and that $S_u f=f_u$ on $[a_{N-1}, a_N)$ and equals f elsewhere. Thus $S_u f$ satisfies (i), (ii), (iii) and (iv) with $a'=u$ and $[a'', a')=$ $=[a_{N-1}, u)$. If the given number a' satisfies $a'\geqq u_N$ this completes the proof of the lemma. If instead $a'<u_N$ the process must be reapplied as follows. Let us redefine a_{N-1} to be u_N. Then

$$S_{u_N} f = \sum_{j=1}^{N-1}\alpha_j\chi_{[a_{j-1}, a_j)} + \alpha\chi_{[a_{N-1}, b)} + f\chi_{[b,\infty)}.$$

We may apply the preceding argument to the function $S_{u_N} f$ and construct a new function $S_u(S_{u_N} f)$ which equals α on the interval $[u, b)$. This construction will be valid for all $u\in[u_{N-1}, u_N)$ where u_{N-1} is determined by conditions analogous to those above which fix u_N. Again S_u will be an operator in the class $\mathscr{L}_1(L^1)\cap\mathscr{L}_1(L^q)$ and consequently the composed operator $S_u S_{u_N}$ will also be in this class. Reiterating this argument as many times as necessary we can, so to speak, move the point u back to any point $a'>0$ by an operator $S=S_{a'} S_{u_M} S_{u_{M+1}} \dots S_{u_N}$, such that $S\in\mathscr{L}_1(L^1)\cap\mathscr{L}_1(L^q)$ and Sf satisfies (i), (ii), (iii) and (iv).

Proof of Theorem 2. Let F and g be functions satisfying the hypotheses of the theorem. Let $f=\sum_{j=1}^{N}\alpha_j\chi_{[c_{j-1}, c_j)}$, with $0=c_0<c_1<c_2<\dots<c_N$ and $\alpha_1>\alpha_2>\dots>\alpha_N$. We shall perform induction on N. If $N=1$, $f=\alpha_1\chi_{[0,c_1)}$, g must vanish outside $[0, c_1)$ and so $\int_0^{c_1} g^q\,ds \leqq \int_0^{c_1} f^q\,ds$. By Lemma 2A we then have $\int_0^{c_1} g\,ds\leqq\int_0^{c_1} f\,ds$ and the desired operator U is given by $Uh=\left((\alpha_1 c_1)^{-1}\int_0^{c_1} h\,ds\right)g$ for all $h\in L^1+L^q$.

Now suppose the theorem is proven in the case where f has $N-1$ different positive values and consider $f=\sum_{j=1}^{N}\alpha_j\chi_{[c_{j-1}, c_j)}$ and g as above such that (1) holds for all $t>0$. It follows that $g(s)$ must vanish for $s>c_N$ and so:

$$\int_{c_{N-1}}^{c_N} g^q\,ds \leqq \int_{c_{N-1}}^{c_N} f^q\,ds = \alpha_N^q(c_N - c_{N-1}). \tag{2}$$

At this point we must consider two possible cases.

Case 1. Suppose that $\int_0^{c_N} g^q\,ds\leqq\alpha_N^q c_N$. Then, by Lemma 2A, $\int_0^{c_N} g\,ds\leqq\alpha_N c_N$ and the operator U can be obtained in the form $Uh=\left(c_N^{-1}\int_0^{c_N} h/f\,ds\right)g$.

Case 2. Alternatively we have:

$$\int_0^{c_N} g^q\,ds > \alpha_N^q c_N. \tag{3}$$

From (2) and (3) and the fact that g is non-increasing we deduce that there exists a number $a' \in (0, c_{N-1}]$ for which

$$\int_{a'}^{c_N} g^q\,ds = \alpha_N^q (c_N - a') = \int_{a'}^{c_N} (Sf)^q\,ds, \tag{4}$$

where $S \in \mathscr{L}_1(L^1) \cap \mathscr{L}_1(L^q)$ is an operator of the type constructed in Lemma 2B, chosen to give $Sf = \alpha_N$ on $[a', c_N)$. Furthermore Sf is a non-negative non-increasing simple function vanishing on (c_N, ∞) and

$$\int_t^\infty g^q\,ds \leqq \int_t^\infty (Sf)^q\,ds \quad \text{for all} \quad t \leqq a'' \tag{5}$$

where $[a'', a')$ is the interval of constancy of Sf preceding a'. In fact (5) will be shown to hold for all positive t. If $t \geqq c_N$

$$\int_t^\infty g^q\,ds = \int_t^\infty (Sf)^q\,ds = 0.$$

If $t \in [a', c_N)$

$$\int_t^\infty (Sf)^q\,ds = \alpha_N^q (c_N - t) \geqq \int_t^\infty g^q\,ds$$

from (4) and the fact that g is non-increasing. It remains to consider $t \in [a'', a')$. On this interval $\int_t^\infty (Sf)^q ds$ is a linear function and $\int_t^\infty g^q ds$ is a convex function since its gradient is increasing (becoming less negative). The inequality (5) holds for $t = a''$, $t = a'$, and so holds for all $t \in [a'', a']$.

Using (4) and the constancy of Sf on $[a', c_N)$ we see that the operator W, defined by

$$Wh = \chi_{[0, a')} h + \left(\frac{1}{\alpha_N (c_N - a')} \int_{a'}^{c_N} h\,ds \right) \chi_{[a', c_N)} g$$

is in $\mathscr{L}_1(L^1) \cap \mathscr{L}_1(L^q)$ and $WSf = \chi_{[0,a')} Sf + \chi_{[a', c_N)} g$. $\chi_{[0,a')} Sf$ is a non-increasing simple function taking no more than $N-1$ different non-zero values (by (iv) in Lemma 2A) and from (4) and (5),

$$\int_t^\infty [\chi_{[0,a')} g]^q\,ds \leqq \int_t^\infty [\chi_{[0,a')} Sf]^q\,ds \quad \text{for all} \quad t \geqq 0.$$

By the inductive hypothesis there exists an operator $V \in \mathscr{L}_1(L^1) \cap \mathscr{L}_1(L^q)$ with $V(\chi_{[0,a')} Sf) = \chi_{[0,a')} g$.

Let U be the operator

$$Uh = \chi_{[0,a')} V[\chi_{[0,a')} Sh] + \chi_{[a', c_N)} W[\chi_{[a', c_N)} Sh]$$

for all $h \in L^1 + L^q$. Then $U \in \mathscr{L}_1(L^1) \cap \mathscr{L}_1(L^q)$ and $Uf = g$, proving Theorem 2.

Theorem 3. *Let $1 \leqq p < q < \infty$ and let the number α be given by $1/\alpha = 1/p - 1/q$. Let f and g be non-negative non-increasing simple functions on $\mathbf{R}_+$ such that*

$$\left(\int_0^{t^\alpha} g^p\,ds\right)^{1/p} + t\left(\int_{t^\alpha}^\infty g^q\,ds\right)^{1/q} \leqq \left(\int_0^{t^\alpha} f^p\,ds\right)^{1/p} + t\left(\int_{t^\alpha}^\infty f^q\,ds\right)^{1/q} \tag{6}$$

for all positive t. Then there exists an operator $W\in\mathcal{L}_{2^{1/p}}(L^p(\mathbf{R}_+))\cap\mathcal{L}_{2^{1/q}}(L^q(\mathbf{R}_+))$ *such that* $Wf=g$.

Proof. Let $P(t)=\int_0^t(f^p-g^p)ds$ and $Q(t)=\int_t^\infty(f^q-g^q)ds$. Let $A=\{t\in\mathbf{R}_+|P(t)\geqq 0\}$, $B=\{t\in\mathbf{R}_+|Q(t)\geqq 0\}$. By (6) $A\cup B=\mathbf{R}_+$. A is a union of disjoint intervals A_i $i=1,\dots,n$ with $P(t)=0$ at each end point. Similarly $B=\bigcup_{i=1}^m B_i$ where the B_i's are disjoint intervals with $Q(t)=0$ at the end points.

In the following it will be convenient to use a second copy of $\mathbf{R}_+$ which we shall denote $\mathbf{R}_+^0$. $\mathbf{R}_+\cup\mathbf{R}_+^0$ will denote the measure space consisting of the disjoint union of $\mathbf{R}_+$ and $\mathbf{R}_+^0$ each equipped with Lebesgue measure. Let φ be the measure preserving map of $\mathbf{R}_+\cup\mathbf{R}_+^0$ onto itself which interchanges each point t of $\mathbf{R}_+$ with the corresponding point t^0 of $\mathbf{R}_+^0$.

The operator W will be constructed as the composition of three operators $W=W_3W_2W_1$, where

(7) $$W_1\in\mathcal{L}_{2^{1/p}}(L^p(\mathbf{R}_+),L^p(\mathbf{R}_+\cup\mathbf{R}_+^0))\cap\mathcal{L}_{2^{1/q}}(L^q(\mathbf{R}_+),L^q(\mathbf{R}_+\cup\mathbf{R}_+^0))$$

(8) $$W_2\in\mathcal{L}_1(L^p(\mathbf{R}_+\cup\mathbf{R}_+^0),L^p(\mathbf{R}_+\cup\mathbf{R}_+^0))\cap\mathcal{L}_1(L^q(\mathbf{R}_+\cup\mathbf{R}_+^0),L^q(\mathbf{R}_+\cup\mathbf{R}_+^0))$$

(9) $$W_3\in\mathcal{L}_1(L^p(\mathbf{R}_+\cup\mathbf{R}_+^0),L^p(\mathbf{R}_+))\cap\mathcal{L}_1(L^q(\mathbf{R}_+\cup\mathbf{R}_+^0),L^q(\mathbf{R}_+)).$$

From this it follows of course that $W\in\mathcal{L}_{2^{1/p}}(L^p(\mathbf{R}_+))\cap\mathcal{L}_{2^{1/q}}(L^q(\mathbf{R}_+))$. For each $h\in L^p(\mathbf{R}_+)+L^q(\mathbf{R}_+)$, W_1 puts a copy of h onto both $\mathbf{R}_+$ and $\mathbf{R}_+^0$, that is:

$$W_1h(t)=\chi_{\mathbf{R}_+}(t)h(t)+\chi_{\mathbf{R}_+^0}(t)h(\varphi t)\quad\text{for all}\quad t\in\mathbf{R}_+\cup\mathbf{R}_+^0.$$

Then (7) is obvious.

Since $P(t)=0$ at the left end point a_i of the interval A_i it follows that

$$\int_{a_i}^t(\chi_{A_i}g)^p\,ds\leqq\int_{a_i}^t(\chi_{A_i}f)^p\,ds\quad\text{for all}\quad t\geqq a_i.$$

Thus, using Theorem 1 and an obvious translation, there exists an operator $U_i\in\mathcal{L}_1(L^p(\mathbf{R}_+))\cap\mathcal{L}_1(L^\infty(\mathbf{R}_+))$ such that

$$U_i(\chi_{A_i}f)=\chi_{A_i}g.$$

Then the operator U given by

$$Uh=\sum_{i=1}^n\chi_{A_i}U_i(\chi_{A_i}h)$$

is also in $\mathcal{L}_1(L^p(\mathbf{R}_+))\cap\mathcal{L}_1(L^\infty(\mathbf{R}_+))$ and $U(\chi_Af)=\chi_Ag$. Since $Q(t)=0$ at the right end point of the interval B_i we also have $\int_t^\infty(\chi_{B_i}g)^q dx\leqq\int_t^\infty(\chi_{B_i}f)^q dx$ for all t, and a translation of Theorem 2 gives us an operator $V_i\in\mathcal{L}_1(L^1(\mathbf{R}_+))\cap\mathcal{L}_1(L^q(\mathbf{R}_+))$ such that $V_i(\chi_{B_i}f)=\chi_{B_i}g$. Then $Vh=\sum_{i=1}^m\chi_{B_i}V_i(\chi_{B_i}h)$ defines an operator in $\mathcal{L}_1(L^1(\mathbf{R}_+))\cap\mathcal{L}_1(L^q(\mathbf{R}_+))$.

Let V^0 denote the operator which is a copy of V acting on functions defined on $\mathbf{R}_+^0$ instead of on $\mathbf{R}_+$. Then W_2 is defined by:

$$W_2h=U(\chi_Ah)+V^0(\chi_{\varphi(B)}h).$$

(8) can readily be deduced with the help of the Riesz—Thorin theorem ([20] Chapter XII). Finally W_3 collects up pieces of function on $\mathbf{R}_+$ and $\mathbf{R}_+^0$ and patches them together on $\mathbf{R}_+$:

$$W_3h(t) = \chi_{A\setminus B}(t)h(t)+\chi_B(t)h(\varphi t) \quad \text{for all} \quad h\in L^p(\mathbf{R}_+\cup\mathbf{R}_+^0)+L^q(\mathbf{R}_+\cup\mathbf{R}_+^0)$$
$$\text{and all} \quad t\in\mathbf{R}_+.$$

Clearly (9) holds and $Wf=W_3W_2W_1f=g$, completing the proof of the theorem.

Remark. This proof of Theorem 3 does not seem to use the full strength of condition (6). Possibly a more refined proof would enable the sharpened conclusion $W\in\mathscr{L}_1(L^p)\cap\mathscr{L}_1(L^q)$.

Theorem 4. *Let $p, q\in[1, \infty]$ and let f and g be complex valued functions in L^p+L^q on a measure space (X, Σ, μ) such that*

$$(10)\qquad K(t, g; L^p(\mu), L^q(\mu)) \leqq K(t, f; L^p(\mu), L^q(\mu)) \quad \textit{for all} \quad t>0.$$

Then there exists an operator $S\in\mathscr{L}_\xi(L^p(\mu))\cap\mathscr{L}_\eta(L^q(\mu))$, where ξ and η are constants depending only on p and q, such that $Sf=g$.

Proof. The operator Φ, $\Phi h=\varphi h$ where $\|\varphi\|_{L^\infty}\leqq 1$, is in the class $\mathscr{L}_1(L^p)\cap\mathscr{L}_1(L^q)$ and so it suffices to treat the case where f and g are non-negative. Also, since $K(t, a; A_0, A_1)=tK(1/t, a; A_1, A_0)$ we can suppose without loss of generality that $p<q$. In view of Holmstedt's estimate for $K(t, f; L^p, L^q)$ (see section 1) there exists a constant λ depending only on p and q, such that:

$$(11)\quad \Big(\int_0^{t^\alpha}(g^*)^p ds\Big)^{1/p}+t\Big(\int_{t^\alpha}^\infty (g^*)^q ds\Big)^{1/q}\leqq\Big(\int_0^{t^\alpha}(\lambda f^*)^p ds\Big)^{1/p}+t\Big(\int_{t^\alpha}^\infty(\lambda f^*)^q ds\Big)^{1/q}$$

for all $t>0$, where $1/\alpha=1/p-1/q$ and the L^q integrals are understood to be zero if $q=\infty$.

Step 1. If f and g are simple functions then Theorems 1 and 3 together with (11) give an operator S_2 in $\mathscr{L}_\xi(L^p(\mathbf{R}_+))\cap\mathscr{L}_\eta(L^q(\mathbf{R}_+))$ which maps f^* to g^*. ξ and η depend only on p and q (for example $\xi=2^{1/p}\lambda$, $\eta=2^{1/q}\lambda$ if $q<\infty$). One can easily find an operator S_1 in $\mathscr{L}_1(L^1(\mu), L^1(\mathbf{R}_+))\cap\mathscr{L}_1(L^\infty(\mu), L^\infty(\mathbf{R}_+))$ taking f to f^* and another, S_3 in $\mathscr{L}_1(L^1(\mathbf{R}_+), L^1(\mu))\cap\mathscr{L}_1(L^\infty(\mathbf{R}_+), L^\infty(\mu))$ taking g^* to g. (Cf. Lemma 2 in [5].) Using the Riesz—Thorin theorem we obtain that $S=S_3S_2S_1\in\mathscr{L}_\xi(L^p(\mu))\cap$ $\cap\mathscr{L}_\eta(L^q(\mu))$ and of course $Sf=g$.

Step 2. If only g is simple then, given any ε, $0<\varepsilon<1$, we shall construct $S\in\mathscr{L}_\xi(L^p)\cap\mathscr{L}_\eta(L^q)$ with $Sf=(1-\varepsilon)g$ where ξ and η are as estimated in step 1. If $q=\infty$ it is easy to see that there exists a simple function $f_\varepsilon\leqq f$ such that $\int_0^t[(1-\varepsilon)g^*]^p ds\leqq\int_0^t(f_\varepsilon^*)^p ds$ for all $t>0$. Thus the desired operator is obtained by first multiplying by $(f_\varepsilon/f)\chi_{\{x|f(x)>0\}}$ and then applying the operator in $\mathscr{L}_\xi(L^p)\cap\mathscr{L}_\eta(L^\infty)$

which maps f_ε to $(1-\varepsilon)g$. For $q<\infty$ more care is needed. We must first examine the behaviour of the function $K(t, g)=K(t, g; L^p, L^q)$ near $t=0$ and $t=\infty$. For each $t>0$, there exist functions u_t, v_t such that $u_t+v_t=g$, $0\leqq u_t\leqq g$, $0\leqq v_t\leqq g$ and

$$(12)\qquad \|u_t\|_{L^p}+t\|v_t\|_{L^q}-\min(t^2, 1/t^2)\leqq K(t, g)\leqq \min(\|g\|_{L^p}, t\|g\|_{L^q}).$$

Consequently $\lim_{t\to\infty}\|v_t\|_{L^q}=\lim_{t\to 0}\|u_t\|_{L^p}=0$. Thus there are subsequences $\{v_{t(n)}\}_{n=1}^\infty$, $\{u_{s(n)}\}_{n=1}^\infty$ which tend to zero almost everywhere. ($\lim_{n\to\infty} t(n)=\infty$, $\lim_{n\to\infty} s(n)=0$.) By dominated convergence $u_{t(n)}\to g$ in L^p and $v_{s(n)}\to g$ in L^q. $K(t, g)$ and $K(t, g)/t=K(1/t, g; L^q, L^p)$ are each continuous monotone functions. So using (12) again we deduce that

$$\lim_{t\to\infty} K(t, g)=\|g\|_{L^p} \quad\text{and}\quad \lim_{t\to 0}\frac{1}{t}K(t, g)=\|g\|_{L^q}.$$

In particular, given ε, $0<\varepsilon<1$, there exist positive numbers a_0 and a_∞ such that:

$$K(t, (1-\varepsilon)g)<(1-\varepsilon/2)\|g\|_{L^q}t<K(t, g)\quad\text{for all}\quad t\leqq a_0$$

and

$$K(t, (1-\varepsilon)g)<(1-\varepsilon/2)\|g\|_{L^p}<K(t, g)\quad\text{for all}\quad t\geqq a_\infty.$$

We seek to construct a continuous piecewise linear function $H(t)$ with finitely many vertices such that $K(t, (1-\varepsilon)g)<H(t)<K(t, g)$ for all $t>0$. From the above estimates we may take $H(t)=(1-\varepsilon/2)\|g\|_{L^q}t$ on $(0, a_0]$ and $H(t)=(1-\varepsilon/2)\|g\|_{L^p}$ on $[a_\infty, \infty)$. Since $K(t, g)$ is continuous and strictly positive on the compact interval $[a_0, a_\infty]$ it is easy to extend the definition of $H(t)$ to the whole of $(0, \infty)$ using only finitely many linear segments.

Let $(f_n)_{n=1}^\infty$ be an increasing sequence of simple functions, $0\leqq f_n\leqq f_{n+1}\leqq f$ with $\lim_{n\to\infty} f_n=f$ a.e. Since $f\in L^p+L^q$, f_n tends to f in L^p+L^q norm also and thus $\lim_{n\to\infty} K(t, f_n)=K(t, f)$ for each positive t. Also $K(t, f_n)\leqq K(t, f_{n+1})\leqq\ldots\leqq K(t, f)$ since multiplication by the function $(f_n/f_{n+1})\chi_{\{x|f_{n+1}(x)>0\}}$ is an operator in $\mathscr{L}_1(L^p)\cap\cap\mathscr{L}_1(L^q)$. Let $v_1, v_2, \ldots, v_M$ be the values of t where $H(t)$ has its vertices. For some sufficiently large n we have $K(v_i, f_n)>H(v_i)$ for $i=1, 2, \ldots, M$. But $K(t, f_n)$ is concave and so for all $t>0$ $K(t, f_n)>H(t)>K(t, (1-\varepsilon)g)$. It follows that

$$\Big(\int_0^{t^\alpha}((1-\varepsilon)g^*)^p\,ds\Big)^{1/p}+t\Big(\int_{t^\alpha}^\infty((1-\varepsilon)g^*)^q\,ds\Big)^{1/q}\leqq\Big(\int_0^{t^\alpha}(\lambda f_n^*)^p\,ds\Big)^{1/p}+t\Big(\int_{t^\alpha}^\infty(\lambda f_n^*)^q\,ds\Big)^{1/q}$$

for all positive t, and so, as for $q=\infty$, we have an operator in $\mathscr{L}_\xi(L^p)\cap\mathscr{L}_\eta(L^q)$ taking f to $(1-\varepsilon)g$.

Step 3. Proof of the theorem under the assumption that the measure space is σ-finite: Let $(g_n)_{n=1}^\infty$ be a sequence of simple functions which tend monotonically almost everywhere to g from below. Then using step 2, let S_n be an operator in $\mathscr{L}_\xi(L^p)\cap\mathscr{L}_\eta(L^q)$ with $S_n f=(1-1/n)g_n$. Let ω be a continuous linear functional of norm one on l^∞ such that $\omega(\{a_n\})=\lim_{n\to+\infty} a_n$ for every convergent sequence

$\{a_n\}$. Define the bilinear functional τ acting on pairs of simple functions, by

$$\tau(\varphi, \psi) = \omega\left(\left\{\int (S_n\varphi)\psi\, d\mu\right\}\right).$$

Of course $\tau(\varphi, \psi)$ is defined also for φ and ψ ranging over larger classes of functions. In particular

$$|\tau(\varphi, \psi)| \leqq \xi \|\varphi\|_{L^p} \|\psi\|_{L^{p'}} \quad \text{for all} \quad \varphi \in L^p, \quad \psi \in L^{p'}$$

and

$$|\tau(\varphi, \psi)| \leqq \eta \|\varphi\|_{L^q} \|\psi\|_{L^{q'}} \quad \text{for all} \quad \varphi \in L^q, \quad \psi \in L^{q'}.$$

Thus for a fixed $\varphi \in L^q$ $\tau(\varphi, \psi)$ is a continuous linear functional on $L^{q'}$ and so, since $q>1$, there exists a function $h_\varphi \in L^q$ determined by φ uniquely to within a set of zero measure, such that $\tau(\varphi, \psi)=\int h_\varphi \psi\, d\mu$ for all $\psi \in L^{q'}$. The above estimates for τ imply that $\|h_\varphi\|_{L^q} \leqq \eta \|\varphi\|_{L^q}$ and if $\varphi \in L^p \cap L^q$ we also have $\|h_\varphi\|_{L^p} \leqq \xi \|\varphi\|_{L^p}$. The operator S, $S\varphi = h_\varphi$ is thus in $\mathscr{L}_\eta(L^q)$ and its restriction to $L^p \cap L^q$ extends uniquely to an operator in $\mathscr{L}_\xi(L^p)$ which we may also denote by S. If $\psi \in L^p \cap L^q$, $\tau(\varphi, \psi)$ is defined for $\varphi \in L^p + L^q$ and $\tau(\varphi, \psi) = \int (S\varphi)\psi\, d\mu$. In particular

$$\int (Sf)\psi\, d\mu = \tau(f, \psi) = \omega\left(\left\{\int (S_n f)\psi\, d\mu\right\}\right) = \omega\left(\left\{(1-1/n)\int g_n \psi\, d\mu\right\}\right) = \int g\psi\, d\mu$$

and since this is true for all $\psi \in L^p \cap L^q$ it follows that $Sf = g$.

Step 4. Proof of the theorem for an arbitrary measure space: If $q<\infty$ then the subset of the measure space where f and g are non zero is σ-finite and the methods of step 3 apply immediately. Thus we need only consider the case $q=\infty$. Given positive functions $f, g \in L^p + L^\infty$ which satisfy (10), it follows that $\int_0^t (g^*)^p\, ds \leqq \int_0^t (\lambda f^*)^p\, ds$ for all $t>0$.

Let

$$\alpha = \lim_{t\to\infty} g^*(t) = \lim_{t\to\infty} \left(\frac{1}{t}\int_0^t (g^*)^p\, ds\right)^{1/p}.$$

Then $G = \{x | g(x) > \alpha\}$ is σ-finite and $(g\chi_G)^*(t) \leqq g^*(t)$ for all positive t.

Let

$$\beta = \lim_{t\to\infty} f^*(t) = \lim_{t\to\infty} \left(\frac{1}{t}\int_0^t (f^*)^p\, ds\right)^{1/p}.$$

Then $F_0 = \{x | f(x) > \beta\}$ is σ-finite.

Case 1. If $\beta=0$, then $\alpha=0$ and both f and g have σ-finite support. Step 3 is immediately applicable.

Case 2. $\beta>0$. *Case 2A.* If $\mu(F_0)=\infty$ then $(f\chi_{F_0})^*(t)=f^*(t)$ and there exists an operator $S_0 \in \mathscr{L}_\lambda(L^p) \cap \mathscr{L}_\lambda(L^\infty)$ which maps $f\chi_{F_0}$ to $g\chi_G$. Let $F_n = \{x | f(x) > \beta + 1/n\}$

and let ω be the functional introduced in step 3. Define the operator S_1 by

$$S_1 h = \frac{\omega\left(\left\{\frac{1}{\mu(F_n)}\int_{F_n} h\,d\mu\right\}\right)}{\omega\left(\left\{\frac{1}{\mu(F_n)}\int_{F_n} f\,d\mu\right\}\right)} g\chi_{X\setminus G}.$$

Then S_1 maps L^p to $\{0\}$ and maps L^∞ into itself with norm bounded by $\alpha/\beta \leqq \lambda$. The operator S, $Sh = \chi_G S_0(\chi_F h) + S_1 h$, is in $\mathscr{L}_\lambda(L^p) \cap \mathscr{L}_\lambda(L^\infty)$ and $Sf = g$.

Case 2B. $\mu(F_0) < \infty$. Then for each n, the set $E_n = \{x | \beta \geqq f(x) > \beta - 1/n\}$ has infinite measure.

Case 2B (i). Suppose that each measurable subset E of E_n with $\mu(E) = \infty$ has a subset of finite positive measure. Then each E_n has a subset D_n, $n \leqq \mu(D_n) < \infty$. Let $F = F_0 \cup \bigcup_{n=1}^\infty D_n$. F is σ-finite and $(f\chi_F)^*(t) = f^*(t)$. Much as before we can obtain $S_0 \in \mathscr{L}_\lambda(L^p) \cap \mathscr{L}_\lambda(L^\infty)$ which maps $f\chi_F$ to $g\chi_G$, and S_1, given by

$$S_1 h = \frac{\omega\left(\frac{1}{\mu(D_n)}\int_{D_n} h\,d\mu\right)}{\omega\left(\frac{1}{\mu(D_n)}\int_{D_n} f\,d\mu\right)} g\chi_{X\setminus G}$$

and S, $Sh = \chi_G S_0(\chi_F h) + S_1 h$ is the required operator.

Case 2B (ii). The only remaining possibility is that the above defined sets E_n for each integer bigger than some integer m contain measurable subsets C_n such that every measurable subset of C_n has either zero or infinite measure. Let $L^\infty(C_n)$ be the subspace of $L^\infty(\mu)$ consisting of functions which are a.e. zero on $X \setminus C_n$.

Let l_n be a continuous linear functional of norm 1 on $L^\infty(C_n)$ such that $l_n(\chi_{C_n} f) = \|\chi_{C_n} f\|_{L^\infty}$. Let $(Y, \mathscr{S}, \nu)$ be a measure space consisting of the disjoint union of F_0 equipped with μ-measure together with a sequence $(\mathbf{R}_n)_{n=m}^\infty$ of disjoint copies of the real line, each equipped with Lebesgue measure. We define an operator

$$Q \in \mathscr{L}_1(L^p(\mu), L^p(\nu)) \cap \mathscr{L}_1(L^\infty(\mu), L^\infty(\nu)) \quad \text{by} \quad Qh = \chi_{F_0} h + \sum_{n=m}^\infty \chi_{\mathbf{R}_n} l_n(\chi_{C_n} h).$$

In fact $\chi_{C_n} h = 0$ a.e. for any $h \in L^p(\mu)$. Q has the further property that $(Qf)^*(t) = f^*(t)$, and since $(Y, \mathscr{S}, \nu)$ is σ-finite we may use the arguments of case 2B(i) to construct an operator $S \in \mathscr{L}_\lambda(L^p(\nu), L^p(\mu)) \cap \mathscr{L}_\lambda(L^\infty(\nu), L^\infty(\mu))$ which maps Qf to g. SQ is then required operator and the proof of theorem 4 is complete.

Corollary 1. *$(L^p(\mu), L^q(\mu))$ is a Calderón pair.*

Corollary 2. $(L_v^p(\mu), L_w^q(\mu))$ *is a Calderón pair when* $p \neq q$.

Proof. Let u be the function $(w^q v^{-p})^{1/(q-p)}$ and let $\gamma = v^p u^{-p}$. It follows that $\gamma = w^q u^{-q}$ also. For any $f \in L_v^p(\mu) + L_w^q(\mu)$

$$K(t, f;\, L_v^p(\mu), L_w^q(\mu)) = \inf_{g+h=f} \Big(\Big(\int |g|^p v^p \, d\mu\Big)^{1/p} + t\Big(\int |h|^q w^q \, d\mu\Big)^{1/q}\Big)$$
$$= \inf_{gu+hu=fu} \Big(\Big(\int |gu|^p \gamma \, d\mu\Big)^{1/p} + t\Big(\int |hu|^q \gamma \, d\mu\Big)^{1/q}\Big)$$
$$= K(t, fu;\, L^p(\gamma\, d\mu), L^q(\gamma\, d\mu)).$$

Thus if $K(t, g;\, L_v^p(\mu), L_w^q(\mu)) \leqq K(t, f;\, L_v^p(\mu), L_w^q(\mu))$ for all positive t, there exists an operator $Q \in \mathscr{L}_\xi(L^p(\gamma\, d\mu)) \cap \mathscr{L}_\eta(L^q(\gamma\, d\mu))$ such that $Q(fu) = gu$. Define the operator S by $Sh = u^{-1}(Q(hu))$. Then $Sf = g$ and one can readily check that $S \in \mathscr{L}_\xi(L_v^p(\mu)) \cap \mathscr{L}_\eta(L_w^q(\mu))$. If $q = \infty$ the argument is virtually the same. We have $K(t, f;\, L_v^p(\mu), L_w^\infty(\mu)) = K(t, fw;\, L^p(v^p w^{-p} \, d\mu), L^\infty(v^p w^{-p} \, d\mu))$.

Remarks about the pair (L_v^p, L_w^p)

The above proof cannot be used for the case $p = q$. Nevertheless it is possible to adapt the ideas of Theorems 1, 2 and 3 to give a fairly simple proof that $(L_v^p(\mu), L_w^p(\mu))$ is Calderón. This result is originally due to Sedaev and Semenov [15], [16]. We shall sketch some details of our alternative proof.

It is not difficult to reduce the problem to the proof of the following theorem:

Theorem 4'. *Let f and g be non-negative step functions of compact support on* $\mathbf{R}$, *which are constant on each interval $[n, n+1)$. Let $w(x) = e^{rx}$ for some positive constant r. Then, if $K(t, g;\, L^p(\mathbf{R}), L_w^p(\mathbf{R})) \leqq K(t, f;\, L^p(\mathbf{R}), L_w^p(\mathbf{R}))$ for all positive t, there exists an operator $S \in \mathscr{L}_a(L^p(\mathbf{R})) \cap \mathscr{L}_a(L_w^p(\mathbf{R}))$ such that $Sf = g$. Moreover, a depends only on p and r and remains bounded as r approaches zero.*

Theorem 4′ will be a consequence of Theorems 1′, 2′ and 3′ and the estimate:

$$K(t, f;\, L^p(\mathbf{R}), L_w^p(\mathbf{R})) \sim \Big(\int_{r^{-1}\log t^{-1}}^{\infty} |f(x)|^p \, dx\Big)^{1/p} + t\Big(\int_{-\infty}^{r^{-1}\log t^{-1}} |f(x)|^p e^{prx} \, dx\Big)^{1/p}.$$

Theorem 1'. *Let f and g be step functions as above on* $\mathbf{R}$ *such that*

$$\int_s^\infty |g(x)|^p \, dx \leqq \int_s^\infty |f(x)|^p \, dx \quad \textit{for all real } s.$$

Then there exists an operator $S \in \mathscr{L}_a(L^p(\mathbf{R})) \cap \mathscr{L}_a(L_w^p(\mathbf{R}))$ such that $Sf = g$, where a is as above.

Theorem 2'. *Let f and g be step functions as above on* $\mathbf{R}$ *such that*

$$\int_{-\infty}^s |g(x)|^p e^{prx} \, dx \leqq \int_{-\infty}^s |f(x)|^p e^{prx} \, dx \quad \textit{for all real } s.$$

Then there exists an operator S exactly as in Theorem 1′.

These two theorems are proved by induction, rather more easily than their analogues above, Theorems 1 and 2. It is convenient to first consider $w(x)$ replaced by the equivalent function $v(x)=e^{rn}$ on the interval $[n, n+1)$ for each n. In that version Theorem 2′ is an immediate consequence of Theorem 1′.

Theorem 3'. *Let f and g be step functions as above such that*

$$\Big(\int_{r^{-1}\log t^{-1}}^{\infty}|g(x)|^p\,dx\Big)^{1/p}+t\Big(\int_{-\infty}^{r^{-1}\log t^{-1}}|g(x)|^p e^{prx}\,dx\Big)^{1/p}\leqq$$
$$\leqq\Big(\int_{r^{-1}\log t^{-1}}^{\infty}|f(x)|^p\,dx\Big)^{1/p}+t\Big(\int_{-\infty}^{r^{-1}\log t^{-1}}|f(x)|^p e^{prx}\,dx\Big)^{1/p}\quad \textit{for all}\quad t>0.$$

Then there exists an operator $S\in\mathscr{L}_{2^{1/p}a}(L^p(\mathbf{R}))\cap\mathscr{L}_{2^{1/p}a}(L^p_w(\mathbf{R}))$ such that $Sf=g$.

The proof of Theorem 3′ using Theorems 1′ and 2′ is almost exactly analogous to that of Theorem 3 from Theorems 1 and 2. We use a second copy $\mathbf{R}_0$ of the real line and the spaces $L^p(\mathbf{R}\cup\mathbf{R}_0)$ and $L^p_{w+w_0}(\mathbf{R}\cup\mathbf{R}_0)$. In fact it is necessary to use slightly modified versions of Theorems 1′ and 2′.

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Received January 5, 1976

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