Convex measures on locally convex spaces

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1. Introduction

The purpose of this paper is not to give a complete treatment of so-called convex measures but merely to point out a new technique. It will be proved that a Gauss measure is a convex measure, and using the inequality (I_s) below, we get very neat proofs of results already known for Gauss measures. On the other hand our results apply to all convex measures.

Throughout this paper E denotes a locally convex Hausdorff vector space, and $\mathfrak{B}(E)$ is the set of all Borel sets in E. Given a Borel probability measure μ on E we define the inner μ-measure $\mu_*(A)$ of a subset A of E by

$$\mu_*(A) = \sup \{\mu(K) | K \text{ compact} \subseteq A\}.$$

We shall say that μ is a Radon probability measure if $\mu_*(A) = \mu(A)$ for all $A \in \mathfrak{B}(E)$. It is known that all Borel probability measures on E are Radon probability measures, if E is a Souslin space. (See e.g. [3, p. 132].) Further, we define $M_s^\lambda(u, v) = (\lambda u^s + (1-\lambda)v^s)^{1/s}$, $-\infty < s < 0$, $= \min(u, v)$, $s = -\infty$, and $= u^\lambda v^{1-\lambda}$, $s = 0$, for $u, v \geq 0$. Here $0^0 = 0$.

Definition 1.1. Let μ be a Radon probability measure on E and assume $s \in [-\infty, 0]$. Then we shall say that μ belongs to the class $\mathfrak{M}_s(E)$ if the inequality

$$\mu_*(\lambda A + (1-\lambda)B) \geq M_s^\lambda(\mu(A), \mu(B)) \tag{I$_s$}$$

holds for all $0 \leq \lambda \leq 1$, and all $A, B \in \mathfrak{B}(E)$.

A measure belonging to the class $\mathfrak{M}_{-\infty}(E)$ will be called a convex measure.

By definition, $\lambda A + (1-\lambda)B = \{\lambda x + (1-\lambda)y | x \in A, y \in B\}$. Note that this set need not be a Borel set even if A and B are so. (See [7].) Note also that $\mathfrak{M}_{s_1}(E) \subseteq \mathfrak{M}_{s_2}(E)$, if $s_1 \geq s_2$, and therefore the class $\mathfrak{M}_{-\infty}(E)$ is the largest one.

In a recent paper [4] the author gives a complete description of the classes $\mathfrak{M}_s(\mathbf{R}^n)(-\infty \leq s \leq 0)$ and for the readers conveniency we recapitulate the result here

THEOREM 1.1. *A Radon probability measure μ on $\mathbf{R}^n$ belongs to the class $\mathfrak{M}_s(\mathbf{R}^n)$ if and only if there are an integer k, $0 \leq k \leq n$, a Radon probability measure ν on $\mathbf{R}^k$, absolutely continuous with respect to Lebesgue measure m_k, and an affine mapping $h: \mathbf{R}^k \to \mathbf{R}^n$ such that $\mu = \nu h^{-1}$ and such that the function $e_{s,k}(d\nu/dm_k)$ is convex.*

Here $e_{s,k}(u) = u^{-1/k}$, $s = -\infty$, $= u^{\frac{s}{1-sk}}$, $-\infty < s < 0$, and $= -\log u$, $s = 0$, for $u \geq 0$, $k > 0$. We recall that a continuous mapping $h: E \to F$, for each Borel probability measure ν on E, induces a Borel probability measure νh^{-1} on F (a topological space), defined by $\nu h^{-1}(C) = \nu(h^{-1}(C))$, when $C \in \mathfrak{B}(F)$. Note that νh^{-1} is a Radon probability measure if ν is so.

The "if" part of Theorem 1.1 is rather easy to prove and let us now briefly indicate how this can be done.

Proof of Theorem 1.1. Using Lemma 2.1 below, it is readily seen that we only need to consider the case when $\mu = fm_n$ and $e_{s,n}(f)$ is a convex function on $\mathbf{R}^n$. An application of Hölder's inequality (several times) then yields

$$f(\lambda x + (1-\lambda)y) \prod_1^n (\lambda a_i + (1-\lambda)b_i) \geq M_s^\lambda(f(x) \prod_1^n a_i, f(y) \prod_1^n b_i) \tag{1.1}$$

for all $x, y \in \mathbf{R}^N$, $a_1, \ldots, a_n, b_1, \ldots, b_n > 0$, and $0 < \lambda < 1$. Let $\mathfrak{J}$ be the family of all finite disjoint unions of compact non-degenerated n-dimensional intervals, with sides parallel to the coordinate axes, and which are included in the interior of supp (f). It is enough to show the inequality (I_s) when $A, B \in \mathfrak{J}$ and $0 < \lambda < 1$. Let $n(A)$ be the number of disjoint intervals defining $A \in \mathfrak{J}$. Suppose first that the inequality (I_s) is true when $n(A) + n(B) \leq p$, where $p \geq 2$ is a fixed natural number. We will then prove that the inequality (I_s) is still true when $n(A) + n(B) = p + 1$. To see this we can assume that $n(A) \geq 2$. We then choose a hyperplane $x_i = c$, with a normal vector parallel to the i:th basis vector, so that $n(A') < n(A)$ and $n(A'') < n(A)$, if $A' = A \cap \{x_i \leq c\}$ and $A'' = A \cap \{x_i \geq c\}$ respectively.

Set

$$\mu(A') = \theta\mu(A), \quad \mu(A'') = (1-\theta)\mu(A), \tag{1.2}$$

where $0 < \theta < 1$.

Since $\mu \ll m_n$, we can find a hyperplane $x_i = d$ so that

$$\mu(B') = \theta\mu(B), \quad \mu(B'') = (1-\theta)\mu(B), \tag{1.3}$$

where $B' = B \cap \{x_i \leq d\}$ and $B'' = B \cap \{x_i \geq d\}$.

Using that the inequality (I_s) is true whenever $A, B \in \mathfrak{J}$ and $n(A) + n(B) \leq p$, we get

$$\mu(\lambda A + (1-\lambda)B) \geq \mu(\lambda A' + (1-\lambda)B') \cup (\lambda A'' + (1-\lambda)B'') =$$

$$\mu(\lambda A' + (1-\lambda)B') + \mu(\lambda A'' + (1-\lambda)B'') \geq M_s^\lambda(\mu(A'), \mu(B')) + M_s^\lambda(\mu(A''), \mu(B'')) =$$

$$\theta M_s^\lambda(\mu(A), \mu(B)) + (1-\theta)M_s^\lambda(\mu(A), \mu(B) = M_s^\lambda(\mu(A), \mu(B)), \ 0 < \lambda < 1.$$

From now on let $\lambda, 0 < \lambda < 1$, be fixed. It only remains to be proved that the inequality (I_s) is true when $A, B \in \mathfrak{J}$ and $n(A) = n(B) = 1$. If not there is a positive number ε such that

$$\mu_\varepsilon(\lambda A + (1 - \lambda)B) < M_s^\lambda(\mu(A), \mu(B)), \tag{1.4}$$

where $\mu_\varepsilon = (f + \omega(\varepsilon))m_n$, and

$$\omega(\varepsilon) = \sup_{\substack{|x-y| < \varepsilon \\ x, y \in \lambda A + (1-\lambda)B}} |f(x) - f(y)|.$$

Now let us divide A into two congruent intervals $A' = A \cap \{x_1 \leq c\}$ and $A'' = A \cap \{x_1 \geq c\}$, respectively, and choose θ so that (1.2) holds. In the next step we choose d such that (1.3) holds with $B' = B \cap \{x_1 \leq d\}$ and $B'' = B \cap \{x_1 \geq d\}$. Using the same technique as above, we conclude that the inequality (1.4) cannot be wrong for both the pairs (A', B') and (A'', B''). By repetition we conclude that

$$\mu_\varepsilon(\lambda C + (1 - \lambda)D) < M_s^\lambda(\mu(C); \mu(D))$$

where $C \subseteq A$, $D \subseteq B$, and where diam (C) and diam (D) can be made arbitrarily small. Setting

$$C = \prod_1^n [u_i - (1/2)a_i, u_i + (1/2)a_i], \quad D = \prod_1^n [v - (1/2)b_i, v_i + (1/2)b_i], \quad (a_i, b_i > 0),$$

we conclude that there are points $x \in C$, $y \in D$, and $z \in \lambda C + (1 - \lambda)D$ such that

$$(f(z) + \omega(\varepsilon)) \prod_1^n (\lambda a_i + (1 - \lambda)b_i) < M_s^\lambda(f(x) \prod_1^n a_i, f(y) \prod_1^n b_i).$$

By choosing C and D small enough, we have $f(z) + \omega(\varepsilon) \geq f(\lambda x + (1 - \lambda)y)$, and we have got an inequality opposite to (1.1). This contradiction proves that the inequality (I_s) must be true when $A, B \in \mathfrak{J}$ and $n(A) + n(B) = 2$. This proves the "if" part of Theorem 1.1.

2. Characterization of the classes $\mathfrak{M}_s(E)(-\infty \leq s \leq 0)$

Suppose $\xi_1, \ldots, \xi_n \in E'$, the topological dual of E. We shall write $\mu h^{-1} = \mu_{\xi_1 \ldots \xi_n}$, when

$$h(x) = (\xi_1(x), \ldots, \xi_n(x)), \quad x \in E. \tag{2.1}$$

THEOREM 2.1. *A Radon probability measure μ on E belongs to the class $\mathfrak{M}_s(E)$ if and only if $\mu_{\xi_1 \ldots \xi_n} \in \mathfrak{M}_s(\mathbf{R}^n)$ for all $\xi_1, \ldots, \xi_n \in E'$, and all positive integers n.*

We need

LEMMA 2.1. *Let E and F be locally convex Hausdorff vector spaces and let $h\colon E \to F$ be a continuous linear mapping. Then $\mu h^{-1} \in \mathfrak{M}_s(F)$, if $\mu \in \mathfrak{M}_s(E)$.*

Proof of Lemma 2.1. We have

$$\lambda h^{-1}(C) + (1-\lambda)h^{-1}(D) = h^{-1}[\lambda(C \cap h(E)) + (1-\lambda)(D \cap h(E))] \tag{2.2}$$

for all $0 \leq \lambda \leq 1$, and all $C, D \subseteq F$.

Using that $(\mu h^{-1})_*(C) = \mu_*(h^{-1}(C))$, $C \subseteq F$, the proof of Lemma 2.1 follows at once from the inequality (I_s).

Proof of Theorem 2.1. Lemma 2.1 proves the "only if" part. To prove the "if" part let h be of the form (2.1). The assumptions and the identity (2.2) then yield

$$\mu_*(\lambda h^{-1}(C) + (1-\lambda)h^{-1}(D)) \geq M_s^\lambda(\mu(h^{-1}(C)), \mu(h^{-1}(D))) \tag{2.3}$$

for all $0 \leq \lambda \leq 1$, and all $C, D \in \mathfrak{B}(\mathbf{R}^n)$.

Now choose $\lambda \in [0, 1]$, and compact sets A and B in E. Holding λ, A, and B fixed, we shall prove the inequality (I_s), which will prove the theorem. To this end let O be an open set containing $\lambda A + (1-\lambda)B$. Since $\lambda A + (1-\lambda)B$ is compact there is an open convex neighbourhood V of the origin so that

$$O \supseteq \lambda A + (1-\lambda)B + 2V.$$

Further, choose $x_1, \ldots, x_m \in A$ and $y_1, \ldots, y_n \in B$ such that

$$\bigcup_1^m (x_i + V) \supseteq A, \quad \bigcup_1^n (y_j + V) \supseteq B, \tag{2.4}$$

and set

$$F = \bigcup_{i,j} (\lambda x_i + (1-\lambda)y_j + \bar{V}).$$

Note that $O \supseteq F$. For each $z \notin F$, $i \in \{1, \ldots, m\}$, and $j \in \{1, \ldots, n\}$, the Hahn-Banach separation theorem gives us a $k_{ijz} \in \mathbf{R}$, and a $\xi_{ijz} \in E'$ such that

$$z \notin \lambda x_i + (1-\lambda)y_j + \xi_{ijz}^{-1}([k_{ijz}, +\infty[), \quad \xi_{ijz}^{-1}([k_{ijz}, +\infty[) \supseteq \bar{V}. \tag{2.5}$$

Set

$$F_z = \bigcup_{i,j} (\lambda x_i + (1-\lambda)y_j + \xi_{ijz}^{-1}([k_{ijz}, +\infty[)).$$

Clearly, $F = \bigcap_{z \notin F} F_z$, and since μ is a Radon probability measure we have

$$\mu(F) = \inf_{\substack{z_1, \ldots, z_p \notin F \\ p \in \mathbf{Z}_+}} \mu(F_{z_1} \cap \ldots \cap F_{z_p}). \tag{2.6}$$

Furthermore, it holds that

$$F_{z_1} \cap \ldots \cap F_{z_p} \supseteq \bigcup_{i,j} [\lambda x_i + (1-\lambda)\lambda_j + \bigcap_{i,j,r} \xi_{ijz_r}^{-1}([k_{ijz_r}, +\infty[)] =$$
$$= \lambda[\bigcup_i (x_i + \bigcap_{i,j,r} \xi_{ijz_r}^{-1}([k_{ijz_r}, +\infty[)] + (1-\lambda)[\bigcup_j (y_j + \bigcap_{i,j,r} \xi_{ijz_r}^{-1}([k_{ijz_r}, +\infty[)].$$

Here the right-hand side is of the form

$$\lambda h^{-1}(C) + (1-\lambda)h^{-1}(D)$$

for suitable $C, D \in \mathbf{R}^{mnp}$, and $h \in (E')^{mnp}$. The equations (2.3)—(2.6) therefore yield

$$\mu(O) \geq M_s(\mu(A), \mu(B)).$$

Since O is an arbitrary open set containing $\lambda A + (1-\lambda)B$, we get the inequality (I_s). This proves Theorem 2.1.

A Radon probability measure μ on E is said to be a Gauss measure, if μ_ξ is a Gauss measure on $\mathbf{R}$ for all $\xi \in E'$. The set of all Gauss measures on E will be denoted by $\mathfrak{G}(E)$. Assuming that μ is a Gauss measure, it is well known that $\mu_{\xi_1 \ldots \xi_n} \in \mathfrak{G}(\mathbf{R}^n)$ for all $\xi_1, \ldots, \xi_n \in E'$, and all positive integers n.

Theorems 1.1 and 2.1 thus give

Corollary 2.1. $\mathfrak{G}(E) \subseteq \mathfrak{M}_0(E)$.

In particular, the Wiener measure W on $\mathbf{C}[0, 1]$, equipped with the uniform topology, satisfies the inequality (I_0).

Nowadays there is no real problem to construct Radon probability measures on our most important vector spaces. We recall the fundamental theorems due to Kolmogorov, Minlos, and Sazonov. (See e.g. [16], [15], and [21].) The so-called Radonifying mappings, introduced by L. Schwartz (see e.g. [2]), also make important contributions to this area.

To check that a given measure μ belongs to $\mathfrak{M}_s(E)$, it is not necessary to prove that $\mu_{\xi_1 \ldots \xi_n}$ belongs to $\mathfrak{M}_s(\mathbf{R}^n)$ for all $\xi_1 \ldots \xi_n \in E'$, and all $n \in \mathbf{Z}_+$. Rather than giving a general theorem, we will illustrate this in a few examples. Suppose μ is a Radon probability measure on $\mathbf{R}^{\mathbf{Z}_+}$, equipped with the product topology, and let $\Pi_n\colon \mathbf{R}^{\mathbf{Z}_+} \to \mathbf{R}^n$ be the natural projection. We claim that $\mu \in \mathfrak{M}_s(\mathbf{R}^{\mathbf{Z}_+})$, if $\mu\Pi_n^{-1} \in \mathfrak{M}_s(\mathbf{R}^n)$ for all $n \in \mathbf{Z}_+$. To see this, let $e_n(x) = x_n$, $x = \{x_n\}_1^\infty$, and note that each bounded linear functional on $\mathbf{R}^{\mathbf{Z}_+}$ is a finite linear combination of the e_n. Therefore, given $\xi_1, \ldots, \xi_n \in E'$, there are $m \in \mathbf{Z}_+$, and a linear mapping $h\colon \mathbf{R}^m \to \mathbf{R}^n$ such that

$$\mu_{\xi_1 \ldots \xi_n} = \mu_{e_1 \ldots e_m} h^{-1} = (\mu\Pi_m^{-1})h^{-1}.$$

Using Lemma 2.1, we conclude that $\mu_{\xi_1 \ldots \xi_n} \in \mathfrak{M}_s(\mathbf{R}^n)$. This proves the assertion.

Now let μ be a Radon probability measure on a separable Hilbert space H, and let $\{e_n\}_1^\infty$, be an orthonormal basis in H. We claim that $\mu \in \mathfrak{M}_s(E)$, if

$\mu_{e_1 \ldots e_n} \in \mathfrak{M}_s(\mathbf{R}^n)$ for all positive integers n. As above, we conclude that $\mu_{\xi_1 \ldots \xi_n} \in \mathfrak{M}_s(\mathbf{R}^n)$, if each ξ_k is a finite linear combination of the e_n. Hence it suffices to prove that the class $\mathfrak{M}_s(\mathbf{R}^n)$ is weakly closed.

THEOREM 2.2. *Let* E *be metrizable*[1]) *and let* $\mu_n, n \in \mathbf{N}$, *be Radon probability measures on* E *such that* $\mu_n \Rightarrow \mu_0$, *as* $n \to +\infty$. *Then* $\mu_0 \in \mathfrak{M}_s(E)$, *if* $\mu_n \in \mathfrak{M}_s(E)$ *for each* $n \in \mathbf{Z}_+$.

Proof. Let d be a translation-invariant metric on E. For $\varepsilon > 0$, and $C \subseteq E$, let $C_\varepsilon = \{x | d(x, C) \leq \varepsilon\}$, and let φ_C^ε be a continuous function $= 1$ on C, and $= 0$ on $E \setminus C_\varepsilon$. Now choose $\lambda \in [0, 1]$, and compact subsets A and B of E. We have

$$\int \varphi^\varepsilon_{(\lambda A + (1-\lambda)B)_\varepsilon} d\mu_n \geq \mu_n((\lambda A + (1 - \lambda)B)_\varepsilon) \geq \mu_{n_*}(\lambda A_\varepsilon + (1 - \lambda)B_\varepsilon) \geq$$

$$\geq M_s^\lambda(\mu_n(A_\varepsilon), \mu_n(B_\varepsilon)) \geq M_s^\lambda\left(\int \varphi_A^\varepsilon d\mu_n, \int \varphi_B^\varepsilon d\mu_n\right), \quad n \in \mathbf{Z}_+.$$

By letting $n \to +\infty$, and making use of trivial inequalities, we get

$$\mu_0((\lambda A + (1 - \lambda)B)_{2\varepsilon}) \geq M_s^\lambda(\mu_0(A), \mu_0(B)).$$

If ε tends to zero, we obtain the inequality (I_s) and the proof is clear

Finally we shall say a few words about the space $\mathbf{C}[0, 1]$, equipped with the sup-norm topology. Given a Radon probability measure μ on $\mathbf{C}[0, 1]$, we have that $\mu \in \mathfrak{M}_s(\mathbf{C}[0, 1])$, if $\mu_{\delta_{t_1} \ldots \delta_{t_n}} \in \mathfrak{M}_s(\mathbf{R}^n)$ for all $0 \leq t_1 < t_2 < \ldots < t_n \leq 1$, and all positive integers n. Here δ_t is the Dirac measure at the point t. For example, set $S_\alpha = (\sigma_\alpha \otimes W)h^{-1}$, where $h(\theta, x) = \theta x, \theta \geq 0, x \in \mathbf{C}[0, 1]$, and where σ_α is the probability distribution of a real-valued random variable X, such that α/X^2 has a chi-square distribution with α degrees of freedom. A straight-forward computation shows that $(S_\alpha)_{\delta_{t_1} \ldots \delta_{t_n}} \in \mathfrak{M}_{-1/\alpha}(\mathbf{R}^n)$ and so $S_\alpha \in \mathfrak{M}_{-1/\alpha}(\mathbf{C}[0, 1])$.

The measure S_α will be called a Student measure on $\mathbf{C}[0, 1]$, and in the special case $\alpha = 1$ we have the Cauchy measure on $\mathbf{C}[0, 1]$. It is not hard to prove that $S_\alpha \Rightarrow W$, when $\alpha \to +\infty$.

3. Integrability for certain functions of seminorms

For a long time it was an open question whether the norm in a separable Banach space E must be L^p-integrable for each $1 \leq p < \infty$, with respect to an arbitrary Gauss measure. The question was solved in the affirmative by Vakhania in case $E = l^q, 1 \leq q < +\infty$. (See e.g. [3, p. 184].) Later, a much stronger result was

[1]) This condition can easily be omitted.

given by Fernique [8][1]). Given a Gauss measure μ on E, which now may be arbitrary, and a μ-measurable seminorm φ, which is finite a.e. $[\mu]$, Fernique proves that $\exp(\varepsilon\varphi^2) \in L^1(\mu)$, if $\varepsilon > 0$ is small enough. Other, less precise results, may be found in [17] and [3, p. 187].

We have

THEOREM 3.1. *Let $\mu \in \mathfrak{M}_s(E)$ and assume that φ is a μ-measurable seminorm on E, which is finite a.e. $[\mu]$. Then in case $s = 0$, the function $\exp(\varepsilon\varphi) \in L^1(\mu)$, if $\varepsilon > 0$ is small enough, and in case $s > 0$, the function $\varphi \in L^p(\mu)$ for all $0 < p < -1/s$.*

We need

LEMMA 3.1. *Let $\mu \in \mathfrak{M}_s(E)$ and let A be a convex, μ-measurable, subset of E, symmetric about the origin. Assume $\mu(A) = \theta > 1/2$. Then*

$$\text{in case } s = 0\text{:} \quad \mu_*(E \setminus tA) \le \theta\left(\frac{1-\theta}{\theta}\right)^{\frac{1+t}{2}}, \quad t \ge 1,$$

$$\text{in case } -\infty < s < 0\text{:} \quad \mu_*(E \setminus tA) \le \left\{\frac{t+1}{2}\,[(1-\theta)^s - \theta^s] + \theta^s\right\}^{1/s}, \quad t \ge 1.$$

Proof of Lemma 3.1. We have

$$E \setminus A \supseteq \frac{2}{t+1}\,(E \setminus tA) + \frac{t-1}{t+1}\,A, \quad t \ge 1. \tag{3.1}$$

In fact, assume $a' = \frac{2}{t+1}\,x + \frac{t-1}{t+1}\,a''$, where $a', a'' \in A$. This yields

$$x = t\left(\frac{t+1}{2t}\,a' + \frac{t-1}{2t}\,(-a'')\right) \in tA,$$

which proves (3.1).

Now let $-\infty < s < 0$. The inequality (I_s) gives

$$\mu^s(E \setminus A) \le \frac{2}{(t+1)}\,\mu_*^s(E \setminus tA) + \frac{t-1}{t+1}\,\mu^s(A),$$

and an easy computation yields the desired estimate. The case $s = 0$ can be treated similarly.

Proof of Theorem 3.1. We only consider the case $-\infty < s < 0$. By definition,

[1]) See also Landau & Shepp, On the supremum of a Gaussian process, Sankhyā, Ser. A. 32, 369—378 (1971).

$$\int \varphi^p(x)d\mu(x) = p \int_0^\infty t^{p-1}\mu\{x|\varphi(x) \geq t\}dt.$$

Since $\varphi < +\infty$ a.e. $[\mu]$, there is a $\lambda > 0$ such that $\mu\{x|\varphi(x) < \lambda\} > 1/2$. Lemma 3.1 implies that $\mu\{x|\varphi(x) \geq \lambda t\} = 0(t^{1/s})$, $t \to +\infty$, and this proves the theorem.

4. A zero-one law for convex measures

Let μ be a Gauss measure on E, and assume that G is an additive, μ-measurable, subgroup of E. Then Kallianpur [11][1]) proves that G is μ-trivial, that is $\mu(G) = 0$ or 1. Kallianpur's proof has been simplified by Le Page [14]. There are several other papers in the literature, proving less precise zero-one laws, and the interested reader may consult [5], [20], and [10].

We have

THEOREM 4.1. *Let μ be a convex measure on E and G an additive subgroup of E. Then $\mu_*(G) = 0$ or 1.*

Proof. Suppose $\mu_*(G) > 0$, and let K_0 be a compact subset of G such that $\mu(K_0) > 0$. Set $K = K_0 \cup (-K_0)$, and let H be the least additive subgroup of E containing K, that is

$$H = \bigcup_{n \in \mathbf{Z}_+} \underbrace{(K + K + \ldots + K)}_{n \text{ terms}}.$$

We will prove that $\mu_*(G) = 1$ and therefore it suffices to prove $\mu(H) = 1$. Suppose to the contrary that $\mu(H) < 1$, and choose $\varepsilon > 0$ so that

$$\varepsilon < \min (1 - \mu(H), \mu(K)).$$

Further, choose a compact subset L of $E \setminus H$ such that

$$\mu(L) > 1 - \mu(H) - \varepsilon.$$

Now observe that

$$E \setminus (H \cup L) \supseteq \frac{1}{n} \{E \setminus [H \cup ((n-1)K + nL)]\} + \left(1 - \frac{1}{n}\right) K, \quad n \in \mathbf{Z}_+.$$

This relation can be proved in the same manner as (3.1). Using the inequality $(\mathrm{I}_{-\infty})$, we get

$$\mu(E \setminus (H \cup L)) \geq \min (\mu(E \setminus [H \cup ((n-1)K + nL)]), \ \mu(K)).$$

[1]) See also footnote [1]), p. 245.

But

$$\mu(E \setminus (H \cup L)) = 1 - \mu(H) - \mu(L) < \varepsilon < \mu(K).$$

Hence

$$\varepsilon > \mu(E \setminus (H \cup L)) \geq \mu(E \setminus [H \cup ((n-1)K + nL)]), \quad n \in \mathbf{Z}_+.$$

This yields

$$\mu((n-1)K + nL) \geq 1 - \mu(H) - \varepsilon > 0, \quad n \in \mathbf{Z}_+. \tag{4.1}$$

Now choose a compact subset A of E such that

$$\mu(E \setminus A) < \tfrac{1}{2}(1 - \mu(H) - \varepsilon). \tag{4.2}$$

Since K and L are compact sets, and $0 \notin K + L$, we can find a positive integer n so that

$$(n-1)K + nL \subseteq E \setminus A. \tag{4.3}$$

Clearly, (4.1)—(4.3) give a contradiction. Hence $\mu(H) = 1$, which proves the theorem.

Given a convex measure μ, and an additive, μ-measurable, subgroup G of E, it is interesting to know whether G is of probability zero or one. Rather than giving a general theorem of this kind, we prefer to illustrate the method in a simple case.

Let $D_0 = [0, 1]$, and $D_n = \{(t, u) | t, u \geq 0, t + nu \leq 1\}$, $n \in \mathbf{Z}_+$. For $x \in \mathbf{R}^{[0,1]}$, set $\Delta_0 x(t) = x(t)$, $t \in D_0$, $\Delta_1 x(t, u) = x(t+u) - x(t)$, $(t, u) \in D_1$, and $\Delta_n x(t, u) = \Delta_1(\Delta_{n-1}x(\cdot, u))(t, u)$, $(t, u) \in D_n$, $n \geq 2$.

We have

THEOREM 4.2. *Let σ be a σ-finite positive Borel measure on D_n, and let $\mu \in \mathfrak{M}_s(\mathbf{C}[0, 1])$, $0 \geq s > -1/p$, where $1 \leq p < +\infty$ is a fixed real number. Then*

$$\mu\{x \in \mathbf{C}[0, 1] | \Delta_n x \in L^p(\sigma)\} = 1 \quad \textit{iff} \quad \varrho \in L^p(\sigma),$$

where

$$\varrho(\cdot) = \left(\int |\Delta_n x(\cdot)|^p d\mu(x)\right)^{1/p}.$$

Shepp [18, Section 19] proves the special case $n = 0$, $\sigma \ll m_1$, $p = 2$, $\mu = W$, and Varberg [22, Theorem 3] the special case $n = 0$, $\sigma \ll m_1$, $p = 2$, $\mu \in \mathfrak{G}(\mathbf{C}[0, 1])$.

Proof. Fubini's theorem gives

$$\int \varrho^p d\sigma = \int \left(\int |\Delta_n x|^p d\sigma\right) d\mu(x). \tag{4.4}$$

The "if" part is trivial. To prove the other direction, set

$$\varphi(x) = \left(\int |\Delta_n x|^p d\sigma\right)^{1/p}, \quad x \in \mathbf{C}[0, 1].$$

Then φ is a μ-measurable seminorm, and $\varphi < +\infty$ a.e. $[\mu]$, by assumption. Theorem 3.1 now tells us that $\varphi \in L^p(\mu)$, and (4.4) gives the result.

For example, let S_α be the Student measure introduced at the very end of Section 2, and assume $\alpha > p, n \geq 1$. Then a straight-forward computation shows that $S_\alpha\{x \in \mathbf{C}[0, 1] | \Delta_n x \in L^p(|t|^{-a} dt du)\} = 1$ if and only if $a < p/2 + 1$.

Now let $\mu \in \mathfrak{M}_s(\mathbf{R}^{\mathbf{Z}_+})$, $0 \geq s > -1/p$, where $1 \leq p < +\infty$ is a fixed real number, and let $\theta_k \geq 0$, $k \in \mathbf{Z}_+$. The same technique as in the proof of Theorem 4.2. then yields

$$\mu\{x | \sum_1^\infty \theta_k |x_k|^p < +\infty\} = 1 \text{ iff } \sum_1^\infty \theta_k \int |x_k|^p d\mu(x) < +\infty. \tag{4.5}$$

For instance, let $\lambda_k \geq 0$, $k \in \mathbf{Z}_+$, $\Sigma \lambda_k < +\infty$, and define a linear operator $S\colon l^2 \to l^2$, by setting

$$S\{x_k\} = \{\lambda_k x_k\}, \quad \{x_k\} \in l^2.$$

By Sazonov's theorem [16, p. 160], there is a Gauss measure μ on l^2 such that

$$\int e^{i<x, y>} d\mu(x) = e^{-<Sy, y>}, \quad y \in l^2.$$

Using (4.5), we conclude that

$$\mu\{x | \sum \theta_k |x_k|^p < +\infty\} = 1 \quad \text{iff} \quad \sum \theta_k \lambda_k^{p/2} < +\infty.$$

Finally we shall discuss Hölder conditions. Let $\omega\colon]0, 1] \to]0, +\infty[$ be a continuous function. We shall say that a continuous function $x\colon [0, 1] \to \mathbf{R}$ belongs to $H_n(\omega)$, $n \geq 1$, if

$$\sup_{(t, u) \in \mathring{D}_n} \frac{|\Delta_n x(t, u)|}{\omega(u)} < +\infty.$$

THEOREM 4.3. *Let μ, p, and ϱ be as in Theorem 4.2. Then*

$$\sup_{(t, u) \in \mathring{D}_n} \frac{\varrho(t, u)}{\omega(u)} = +\infty \quad \textit{implies} \quad \mu(H_n(\omega)) = 0.$$

Proof. Let y be a nonnegative Lebesgue integrable function on D_n, and set

$$\varphi(x) = \left[\int \left|\frac{\Delta_n x(t, u)}{\omega(u)}\right|^p y(t, u) dt\, du\right]^{1/p}, \quad x \in \mathbf{C}[0, 1].$$

Suppose $\mu(H_n(\omega)) > 0$, that is $\mu(H_n(\omega)) = 1$. Then $\varphi < +\infty$ a.e. $[\mu]$, and Theorem 3.1 gives $\varphi \in L^p(\mu)$. Hence, by Fubini's theorem,

$$\int ((\omega(u))^{-p} \int |\Delta_n x(t, u)|^p d\mu(x)) y(t, u) dt du < +\infty.$$

Since $y \geq 0$ is an arbitrary Lebesgue integrable function, we have

$$\text{ess sup } (\omega(u))^{-1} \varrho(t, u) < +\infty.$$

Theorem 3.1 implies that ϱ is continuous, and therefore we have a contradiction. Hence $\mu(H_n(\omega)) = 0$, which proves the theorem.

The converse of Theorem 4.3 is, of course, wrong. Take $\mu = W$, $\omega(u) = u^{1/2}$, and $n = 1$. It is well known that $W(H_1(u^{1/2})) = 0$.

5. The support of a convex measure

The support $\text{supp}(\mu)$ of a Radon probability measure μ is, by definition, the least closed set which carries the total mass one. It is well known that the support of a Gauss measure on E is a closed linear subvariety, at least if the space E is not too complicated. (See [9], [12], and [13].)

We have

THEOREM 5.1. *Let* μ *be a convex measure on* E *and assume that*

$$\text{supp}(\mu_\xi) = \textit{singleton set or } \mathbf{R}, \quad \textit{all } \xi \in E'. \tag{5.1}$$

Then $\text{supp}(\mu)$ *is a closed linear subvariety of* E. *Especially, it holds that*

$$\text{supp}(\mu) = \bigcap \{H | H \in \mathfrak{F}\}, \tag{5.2}$$

where $\mathfrak{F}$ *is the family of all closed hyperplanes in* E *with* μ*-measure one.*

Proof. The inequality $(\mathrm{I}_{-\infty})$ implies that $\text{supp}(\mu)$ is convex. Furthermore, we have

$$\text{supp}(\mu) \subseteq \bigcap \{H | H \in \mathfrak{F}\}$$

since μ is a Randon probability measure.

Suppose

$$x_0 \in [\bigcap \{H | H \in \mathfrak{F}\}] \setminus \text{supp}(\mu). \tag{5.3}$$

The Hahn-Banach separation theorem gives us a $\xi \in E'$ such that

$$k := \sup \xi(\text{supp}(\mu)) < \xi(x_0).$$

But $\mu_\xi(]k, +\infty[) = \mu(\xi^{-1}(]k, +\infty[)) = 0$, and therefore (5.1) says that ξ is constant: $= l$ a.e. $[\mu]$. Clearly, $l \leq k$ and $\xi^{-1}(\{l\}) \in \mathfrak{F}$. From (5.3) we have

$\xi(x_0) = l$, which implies that $l > k$. This contradiction proves (5.2) and the theorem.

Note that a Gauss measure μ satisfies (5.1).

6. Some properties of convolutions

We shall conclude this paper with a brief discussion of convolutions. First, we give a generalization of the important inequality due to Anderson [1]. (See also [19].)

A measure μ is said to be symmetric if $\mu(A) = \mu(-A)$ for all $A \in \mathfrak{B}(E)$.

THEOREM 6.1. *Let μ be a symmetric and convex measure on E, and let f, g be nonnegative Borel functions such that the sets*

$$\{x | f(x) \geq t\} \text{ and } \{x | g(x) \geq t\}$$

are convex and symmetric about the origin for all $t \geq 0$. Then

$$(f * g)_\mu(\lambda x) \geq (f * g)_\mu(x), \quad |\lambda| \leq 1, \quad x \in E,$$

where

$$(f * g)_\mu(x) = \int f(x - y) g(y) d\mu(y), \quad x \in E.$$

Proof. Without loss of generality it can be assumed that f and g are characteristic functions of convex sets A and B, respectively, both symmetric about the origin. Then

$$(A + \lambda x) \cap B \supseteq \frac{1 - \lambda}{2} [(A - x) \cap B] + \frac{1 + \lambda}{2} [(A + x) \cap B], \quad |\lambda| \leq 1, \quad x \in E.$$

Hence

$$\mu((A + \lambda x) \cap B) \geq \mu((A + x) \cap B), \quad |\lambda| \leq 1, \quad x \in E,$$

which proves the theorem.

THEOREM 6.2. *Let $\mu \in \mathfrak{M}_0(E)$ and $\nu \in \mathfrak{M}_0(F)$. Then the product measure $\mu \otimes \nu \in \mathfrak{M}_0(E \otimes F)$. Especially, the convolution $\mu * \nu \in \mathfrak{M}_0(E)$ in case $F = E$.*

Note here that $u \otimes \nu$ extends to a Radon probability measure on $E \otimes F$, equipped with the product topology. (See [3, p. 94].)

Proof. The product measure $\mu \otimes \nu$ clearly satisfies the inequality (I_0) when A and B are rectangles in $E \otimes F$. The first assertion therefore follows from Theorem

2.1. and [4, Cor. 3.1]. The second assertion follows from the first and Lemma 2.1, since $\mu * \nu = (\mu \otimes \nu)h^{-1}$, where $h(x, y) = x + y$, $x, y \in E$. This proves the theorem.

From Theorem 6.2 it is not hard to prove that $(f * g)_\mu$ is a logarithmically concave function on E, whenever $\mu \in \mathfrak{M}_0(E)$, and f, g are nonnegative, logarithmically concave, Borel measurable functions. This extends [6], which proves the result when $E = \mathbf{R}^n$ and $\mu = m_n$.

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