Singular integrals and multiplier operators

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Contents

Preface

In 1952, A. P. Calderon and A. Zygmund [6] showed the boundedness of singular integral operators in the L^p spaces. One of the main tools in their proof was an extension to m dimensions of a form of F. Riesz's sun rising lemma. Later F. Jones [15], and E. B. Fabes and N. M. Rivière [9] further extended the lemma to prove the boundedness of parabolic singular integrals and singular integrals with mixed homogeneity. However in both cases the lemma kept its geometric form and its extension was obtained by a simple change of parameters in each coordinate axis.

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In this paper the covering lemma is substantially changed (see Theorems (2.1), (3.1) and (3.2), Chapter I), eliminating its geometric form. This permits, among other things, the study of singular integrals in locally compact groups (see Section 6, Chapter I). It also allows the study of singular integrals, in R^m, where the homogeneity is given by a one parameter group of transformations (see Section 7, Chapter I).

The paper is divided into two chapters and contains essentially the results announced in [25] and [26]. The first chapter is devoted to the study of singular integrals, while the second shows some of its applications to the theory of multipliers.

In the fourth section of the first chapter, we introduce the notion of singular kernels for a family of neighbourhoods of the origin. As we shall see later in section 7, this contains the singular kernels studied in [6], [9], and [7]. No explicit use is made of the homogeneity (in the style of [12], [2] and [16]). In the fifth section we show that the maximal operator of such singular integrals is bounded in L^p $(1 < p < \infty)$ and it is of weak type L^1. Theorems on pointwise convergence are immediate consequences without making any extra assumptions of smoothness on the kernels. This result is new even in the elliptic case studied by Calderon, M. Weiss and Zygmund (see [5]).

In the second chapter we make use of the theory of vector valued singular integrals developed in the first chapter and of the Riesz theory of interpolation to improve the multiplier theorems obtained in [12], [9], [27] and [18]. Some of the applications of these results yield the boundedness, in spaces of mixed norms, of multipliers such as the characteristic function of a convex set (in particular the characteristic function of the disc) and the bounded ratio of two polynomials (see Theorems (2.2), (2.3) and (2.5) of Chapter II).

This paper is essentially self-contained. Necessary prerequisites are some basic properties of the Fourier transformation, the Riesz theory of interpolation and vector valued integration.

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Chapter I

1. Notation and preliminaries

Throughout the paper C will denote a constant and a subscript will be added when we wish to make clear its dependance on the parameter in the subscript.

For a given set E, E' will denote its complement and χ_E its characteristic function.

For the Banach spaces B_1 and B_2, $\mathcal{L}(B_1, B_2)$ will denote the space of continuous linear operators from B_1 into B_2.

$\mathcal{F}(X, B, \mu)$ denotes the class of μ-measurable functions from X into the Banach space B.

$L^p(X, B)$ will denote the space of functions, $f \in \mathcal{F}(X, B, \mu)$, such that

$$\|f\|_{L^p(X, B)} = \left(\int_X |f(x)|_B^p \, d\mu\right)^{1/p} < \infty$$

where $|\cdot|_B$ denotes the norm on the space B. More generally if $\Phi(t)$ is a non-negative Borel function of $R_+ = \{t, t > 0\}$, we define

$$L_\Phi(X, B, \mu) = \left\{f\,; f \in \mathcal{F}(X, B, \mu) \text{ and } \int_X \Phi(|f(x)|_B) d\mu < \infty\right\}$$

The operator T from $L_\Phi(X, B_1, \mu)$ into $\mathcal{F}(Y, B_2, \nu)$ is said to be sublinear if when f, g and $f + g \in L_\Phi(X, B_1, \mu)$

$$|T(f+g)(x)|_{B_2} \leq |T(f)(x)|_{B_2} + |T(g)(x)|_{B_2} \text{ a.e.}$$

Let $\lambda_{f,\mu}$ denote the distribution function of the function f, i.e. $\lambda_{f,\mu}(t) = \mu(\{x, |f(x)|_B \geq t\})$.

The following form of the Marcinkiewicz interpolation theorem will be used later in the chapter. (See also [20] and [29].)

Definition (1.1). The sublinear operator is of weak type (Φ, Φ) (or weak type L_Φ) if and only if there exists a constant C, such that

$$\nu(\{x, |T(f)(x)|_{B_2} \geq t\}) \leq C \int_X \Phi\left(\left|\frac{f(x)}{t}\right|_{B_1}\right) d\mu\,.$$

THEOREM (1.1). *Let T be a sublinear operator of weak types (Φ, Φ) and (ψ, ψ). Set*

$$b_p = \max\left\{\int_0^1 t^{p-1}\, \psi(1/t)dt\,; \int_1^\infty t^{p-1}\, \Phi(1/t)dt\right\}.$$

If $b_p < \infty$, then for $f \in L^p(X, B_1)$, $\|T(f)\|_{L^p(X, B_2)}^p \leq b_p\, C^p \|f\|_{L^p(X, B_1)}^p$.

Proof. Let C be the larger of both constants used in the definition of weak types (Φ, Φ) and (ψ, ψ). For $f \in L^p(X, B_1)$ and $t > 0$, set $f_t(x) = f(x)$ when $|f(x)|_{B_1} \leq t$, $f_t(x) = 0$ otherwise and set $f^t(x) = f(x) - f_t(x)$. Since T is a sublinear operator of weak types (Φ, Φ) and (ψ, ψ)

$$\lambda_{T(f),\nu}(t) \leq \lambda_{T(f^t),\nu}(t/2) + \lambda_{T(f_t),\nu}(t/2) \leq$$

$$\leq C\left[\int_{\{x, |f(x)|_{B_1} \leq t\}} \psi(2|f(x)|_{B_1}/t)d\mu + \int_{\{x, |f(x)|_{B_1} > t\}} \Phi(2|f(x)|_{B_1}/t)d\mu\right]$$

From this estimate and interchanging the order of integration

$$\|T(f)\|^p_{L^p(X,B_2)} = p \int_0^\infty t^{p-1} \lambda_{T(f),\nu}(t) dt \leq$$

$$\leq p\, C \int_X \left\{ \int_0^{|f(x)|_{B_1}} t^{p-1} \psi(2|f(x)|_{B_1}/t) dt + \int_{|f(x)|_{B_1}}^\infty t^{p-1} \Phi(2|f(x)|_{B_1}/t) dt \right\} d\mu \leq$$

$$\leq p\, C\, 2^p b_p \int |f(x)|^p_{B_1} d\mu .$$

The theorem follows.

When $\Phi(t) = t^p(ln(2+t))^n$ and $\psi(t) = t^q[ln(2+t)]^m$ ($m = 0$ when $q = \infty$), representing the spaces $L^p(lnL)^n$ and $L^q(lnL)^m$, the constant C_r of Theorem (1.1) is bounded by $C^r(|r-p|^{-n-1} + |r-q|^{-m-1})$. ($C_r \leq C^r|r-p|^{-n-1}$ when $q = \infty$).

For the multi-index $P = (p_1, \ldots, p_m)$ we define the Banach space $X^P(R^m, B) = X_1^{p_1} X_2^{p_2} \ldots X_m^{p_m}(R^m, B)$ (R^m denotes the m-dimensional real space) as follows: $X_1^{p_1}(R, B) = L^{p_1}(R, B)$, and if $x' \in R^{m-1}$, $P' = (p_1, \ldots, p_{m-1})$, $X^P(R^m, B) = L^{p_m}(R, X^{P'}(R^{m-1}, B))$.

Finally for $f \in L^1(R^m, H)$, H a Hilbert space, we define $\mathcal{F}(f)(x) = \int_{R^m} f(y) \exp(2\pi i\langle x, y\rangle) dy$ and $\mathcal{F}^{-1}(f)(x) = \int_{R^m} f(y) \exp(-2\pi i\langle x, y\rangle) dy$.

2. The maximal function

Let $\{U_\alpha, \alpha \in R_+\}$ be a family of open sets of R^m whose closure is compact.

Definition (2.1). $\{U_\alpha, \alpha > 0\}$ is a Vitali family if and only if:

1) For $\alpha < \beta$, $U_\alpha \subset U_\beta$ and $\bigcap_\alpha \bar{U}_\alpha = \{0\}$.

2) $m(U_\alpha - U_\alpha) \leq A m(U_\alpha)$ where $m(.)$ denotes the Lebesgue measure and $U_\alpha - U_\alpha = \{z, z = x - y$ and $x, y \in U_\alpha\}$.

3) $m(U_\alpha)$ is a left continuous function of α. In other words if $\alpha_n \nearrow \alpha$ then $m(U_{\alpha_n}) \nearrow m(U_\alpha)$.

Remark. Condition (3) is not essential. If the family $\{U_\alpha\}$ satisfies (1) and (2), the family $\{U_\alpha^*\}$; where $U_\alpha^* = \bigcup_{\beta<\alpha} U_\beta$; satisfies (1), (2) and (3).

The following is a covering theorem of the type studied by A. P. Calderón [3], and R. E. Edwards and E. Hewitt [8].

THEOREM (2.1) (Covering theorem). *Let E be a measurable set in R^m and $\alpha : E \to R_+$ be a mapping satisfying:*

(a) *$\alpha(x)$ is bounded and for every $\alpha_0 > 0$, $\{x, x \in E, \alpha(x) > \alpha_0\}$ is a bounded set of R^m.*

(b) *If* $x_n \to x$ *and* $\alpha(x_n) \nearrow \alpha$, *then* $x \in E$ *and* $\alpha(x) \geq \alpha$.

Under these conditions, for each Vitali family $\{U_\alpha\}$ *there exists a sequence* $\{x_n\} \subset E$, *such that:*

i) $\{x_n + U_{\alpha(x_n)}\}$ *is a disjoint family*

(ii) $E \subset \bigcup_{n=1}^{\infty} [x_n + (U_{\alpha(x_n)} - U_{\alpha(x_n)})]$

(iii) $m(E) < A \sum_{n=1}^{\infty} m(U_{\alpha(x_n)})$.

Proof. Let $\alpha_1 = \sup_{x \in E} \alpha(x) < \infty$ and choose $\alpha(y_n) \nearrow \alpha_1$. By (a) $\{y_n\}$ is a bounded sequence. Choose a subsequence converging to a limit x_1; by (b) $x_1 \in E$ and $\alpha(x_1) = \alpha_1$. For $k \geq 1$ set $A_k = E \cap (\bigcup_{n=1}^{k} [x_n + (U_{\alpha(x_n)} - U_{\alpha(x_n)})])$. If $A_1 = \emptyset$ the theorem follows. Observe that A_k satisfies properties (a) and (b). Hence with the above argument, set by induction $\alpha(x_k) = \alpha_k = \sup_{x \in A_{k-1}} \alpha(x)$. Since the sets A_k form a decreasing sequence, the α_k also form a decreasing sequence. Therefore if $j < k$, $\alpha_j \geq \alpha_k$, and $(x_j + U_{\alpha_j}) \cap (x_k + U_{\alpha_k}) \neq \emptyset$ there exist $u \in U_{\alpha_j}$ and $v \in U_{\alpha_k} \subset U_{\alpha_j}$ such that $x_j + u = x_k + v$. In other words $x_k \in x_j + (U_{\alpha_j} - U_{\alpha_k})$ but this is impossible by construction. Hence the family $\{x_k + U_{\alpha_k}\}$ is disjoint.

To prove (ii) it is enough to show that $\bigcap_{n=1}^{\infty} A_n = \emptyset$, or in other words that $\alpha_n \to 0$ as $n \to \infty$. If this is not true, then $\alpha_n = \alpha(x_n) > \varepsilon > 0$. Hence $\{x_n\}$ is a bounded sequence and therefore $F = \bigcup_{n=1}^{\infty} (x_n + U_{\alpha_n})$ is a bounded set. But $m(F) = \sum_{n=1}^{\infty} m(U_{\alpha_n}) = \infty$.

Property (iii) is a clear consequence of (ii) and of the definition of Vitali families.

THEOREM (2.2) (The maximal function). *Let* $\{U_\alpha\}$ *be a Vitali family,* $f \in L^1(R^m, B)$. *Define*

$$M(f)(x) = \sup_{\alpha} \frac{1}{m(U_\alpha)} \int_{x+U_\alpha} |f(y)|_B dy .$$

Then

$$m(\{x, M(f)(x) > 1\}) \leq A \int_{R^m} |f(y)|_B dy .$$

In other words M *is of weak type* L^1.

Proof. Since $\dfrac{1}{m(U_\alpha)} \displaystyle\int_{x+U_\alpha} |f(y)|_B dy$ is a continuous function of x, $M(f)$ is lower semi-continuous and hence measurable. Let

$$E = \{x, \int_{x+U_\alpha} |f(y)|_B dy \geq m(U_\alpha) , \text{ for some } \alpha\} .$$

For $x \in E$ set $\alpha(x) = \max\left\{\alpha\,;\, \frac{1}{m(U_\alpha)}\int_{x+U_\alpha} |f(y)|_B dy \geq 1\right\}$. It is easy to see that the set E and the mapping $\alpha(x)$ satisfy the conditions of Theorem (2.1). Hence there exists a disjoint sequence $\{x_n + U_{\alpha_n}\}$ such that

$$m(\{x,\, M(f)(x) > 1\}) \leq m(E) \leq A\sum_{n=1}^{\infty} m(U_{\alpha_n}) \leq A\sum_{n=1}^{\infty}\int_{x_n+U_{\alpha_n}} |f(y)|_B dy \leq A\int_{R^m} |f(y)|_B dy\,.$$

Theorem (2.2) has as an immediate consequence a differentiation theorem for the family $\{U_\alpha\}$.

THEOREM (2.3). *Let $\{U_\alpha\}$ be a Vitali family, $f \in L^1(R^m, B)$. Then for almost every $x \in R^m$*

$$\lim_{\alpha\to 0}\frac{1}{m(U_\alpha)}\int_{x+U_\alpha} |f(y) - f(x)|_B dy = 0\,.$$

As a consequence for any such x,

$$\lim_{\alpha\to 0}\frac{1}{m(U_\alpha)}\int_{x+U_\alpha} f(y)dy = f(x)\,.$$

Proof. For $f \in L^1(R^m, B)$ set

$$M'(f)(x) = \overline{\lim_{\alpha\to 0}}\,\frac{1}{m(U_\alpha)}\int_{x+U_\alpha} |f(y) - f(x)|_B dy\,.$$

$M'(f)$ is well-defined almost everywhere and $M'(f)(x) < M(f)(x) + |f(x)|_B$ a.e. Consider a sequence $\{f_n\}$ of continuous functions with compact support such that $\|f_n - f\|_{L^1(R^m,B)} \to 0$, as $n \to \infty$. Note that the regularity of U_α's implies that $M'(f_n)(x) = 0$ everywhere. Set $g_n = f - f_n$. Then

$$m(\{x,\, M'(f)(x) > t\}) = m(\{x,\, M'(g_n)(x) > t\}) <$$
$$< m(\{x,\, M(g_n)(x) > t/2\}) + m(\{x,\, |g_n(x)|_B > t/2\}) < \frac{2A}{t}\|f - f_n\|_{L^1(R^m,B)}$$

Since n and t are arbitrary, it follows that $m(\{x,\, M'(f)(x) > 0\}) = 0$ and the theorem is proved.

Theorem (2.3) can be extended to all finite measures showing that when μ is a finite singular measure with respect to the Lebesgue measure,

$$\lim_{\alpha\to 0}\frac{|\mu(x + U_\alpha)|_B}{m(U_\alpha)} = 0 \quad \text{a.e.}$$

If the U_α's are convex and symmetric sets ($U_\alpha = -U_\alpha$), then $U_\alpha - U_\alpha \subset 2U_\alpha$ and hence $m(U_\alpha - U_\alpha) < 2^m m(U_\alpha)$. In particular when the U_α's are m-dimensional intervals Theorem (2.3) implies Theorem 6 of [13]. Related results are also obtained in [22] and [23].

3. The weak type estimate

Definition (3.1). $\{U_\alpha, \Phi\}$ is a regular Vitali family if

(1) $\{U_\alpha\}$ is a Vitali family

(2) $\Phi : R_+ \to R_+$ is a continuous and onto mapping with the property that $U_\alpha - U_\alpha \subset U_{\Phi(\alpha)}$ and $m(U_{\Phi(\alpha)}) \leq Am(U_\alpha)$.

It is immediate from definition (3.1) that $\Phi(\alpha) > \alpha$ and that Φ can be chosen to be non-decreasing. Observe also that $\bigcup_\alpha U_\alpha = R^m$.

THEOREM (3.1). *For $f \in L^1(R^m, B)$ and a regular family $\{U_\alpha, \Phi\}$, there exists a disjoint sequence $\{x_n + U_{\alpha_n}\}$ such that*

(i) $$\frac{1}{m(U_{\Phi(\alpha_n)})}\int_{x_n+U_{\Phi(\alpha_n)}} |f(y)|_B dy < 1 \leq \frac{1}{m(U_{\alpha_n})}\int_{x_n+U_{\alpha_n}} |f(y)|_B dy$$

(ii) *For* $x \notin \bigcup_{n=1}^\infty (x_n + U_{\Phi(\alpha_n)})$, $|f(x)|_B \leq 1$ *a.e.*

Proof. Let $E = \left\{x, \frac{1}{m(U_\alpha)}\int_{x+U_\alpha} |f(y)|_B dy \geq 1 \text{ for some } \alpha\right\}$. For $x \in E$ set

$$\alpha(x) = \sup\left\{\alpha ; \frac{1}{m(U_\alpha)}\int_{x+U_\alpha} |f(y)|_B dy \geq 1\right\}.$$

In virtue of Theorem (2.1) there exists a sequence $\{x_n\} \subset E$ such that if $\alpha_n = \alpha(x_n)$, $\{x_n + U_{\alpha_n}\}$ is disjoint and $E \subset \bigcup_{n=1}^\infty [x_n + (U_{\alpha_n} - U_{\alpha_n})] \subset \bigcup_{n=1}^\infty (x_n + U_{\Phi(\alpha_n)})$. $|f(x)|_B \leq 1$ a.e. outside E.

From Theorem (3.1) we can deduce a decomposition theorem for functions of $L^1(R^m, B)$.

THEOREM (3.2). *For $f \in L^1(R^m, B)$, and a regular Vitali family $\{U_\alpha, \Phi\}$, we can write $f = g + h$, where*

(i) *$g \in L^1(R^m, B) \cap L^\infty(R^m, B)$ with $\|g\|_{L^\infty(R^m,B)} \leq A$ and $\|g\|^p_{L^p(R^m,B)} \leq A^{p-1}\|f\|_{L^1(R^m,B)}$. ($A$ is the constant of Definition (3.1).)*

(ii) *$h = \sum_{n=1}^\infty h_n$, the support of h_n is contained in $x_n + U_{\beta_n}$, $\int_{x_n+U_{\beta_n}} h_n(y)dy = 0$, and $\sum_{n=1}^\infty m(U_{\beta_n}) \leq A\|f\|_{L^1(R^m,B)}$. Moreover the supports of the h_n's are disjoint.*

Proof. Consider the sequences $\{x_n\}$, $\{\alpha_n\}$ of Theorem (3.1) and write $\beta_n = \Phi(\alpha_n)$. Let V_n be a sequence of measurable disjoint sets, such that

(a) $x_n + U_{\alpha_n} \subset V_n \subset x_n + U_{\beta_n}$

(b) $\bigcup_{n=1}^{\infty} V_n = \bigcup_{n=1}^{\infty} (x_n + U_{\beta_n})$.

Set

$$g(x) = \begin{cases} \dfrac{1}{m(V_n)} \displaystyle\int_{V_n} f(y)dy & \text{for } x \in V_n \\ f(x) & \text{for } x \notin \displaystyle\bigcup_{n=1}^{\infty} V_n . \end{cases}$$

Let $h(x) = f(x) - g(x) = \sum_{n=1}^{\infty} (f(x) - g(x)) \chi_{V_n}(x) = \sum_{n=1}^{\infty} h_n(x)$, where $h_n = (f - g)\chi_{V_n}$. In virtue of (i), Theorem (3.1),

$$\|g\|_\infty \le \sup_n \left(\frac{1}{m(V_n)} \int_{V_n} |f(y)|_B \, dy \right) \le A \sup_n \left(\frac{1}{m(U_{\beta n})} \right) \int_{x_n + U_{\beta_n}} |f(y)|_B \, dy \le A .$$

Similarly

$$\|g\|_p^p = \sum_{n=1}^{\infty} \int_{V_n} \left(\frac{1}{m(V_n)} \int_{V_n} |f(y)|_B \, dy \right)^p dx + \int_{(\bigcup_{n=1}^{\infty} V_n)'} |f(x)|_B^p \, dx \le A^{p-1} \int_{R_m} |f(x)|_B \, dx .$$

Clearly the support of h_n is contained in $x_n + U_{\beta n}$ and $\int_{x_n + U_{\beta_n}} h_n(y)dy = 0$. Moreover from (i), Theorem (3.1), we have,

$$\sum_{n=1}^{\infty} m(U_{\beta n}) \le A \sum_{n=1}^{\infty} m(U_{\alpha n}) \le A \sum_{n=1}^{\infty} \int_{x_n + U_{\alpha_n}} |f(y)|_B \, dy \le A \int_{R^m} |f(y)|_B \, dy ;$$

and the theorem follows.

THEOREM (3.3). *Let T be a sublinear operator satisfying:*

(i) *For* $f \in L^p(R^m, B_0) \cap L^\infty(R^m, B_0)$, $0 < p \le \infty$,

$$|T(f)(x)|_{B_1} \le |S(f)(x)|_{B_1} + C\|f\|_{L^\infty(R^m, B_0)} \quad a.e.,$$

where S is a sublinear operator of weak type L^p.

(ii) *There exists a regular Vitali family $\{U_\alpha, \Phi\}$ such that if $f \in L^1(R^m, B_0)$ and $f = g + h$ as in Theorem* (3.2), *then for every positive t,*

$$m(\{x, |T(th)(x)|_{B_1} > t\}) \le C\|f\|_{L^1(R^m, B_0)} .$$

Under assumption (i) *and* (ii) *we conclude that T is an operator of weak types L^1 and L^p, and hence bounded in L^q for $1 < q < p$ (see Theorem* (1.1)).

Proof. Let $f \in L^p(R^m, B_0)$ and set $f_t = f\chi_{\{x, |f(x)|_{B_0} \leq t\}}$ and $f_t + f^t = f$. Then $f_t \in L^p(R^m, B_0) \cap L^\infty(R^m, B_0)$ and $|f_t(x)|_{B_0} \leq t$, therefore $|T(f_t)(x)|_{B_1} \leq |S(f_t)(x)|_{B_1} < Ct$ a.e., and

$$m(\{x, |T(f_t)(x)|_{B_1} > (C+1)t\}) \leq m(\{x, |S(f_t)(x)|_{B_1} > t\}) \leq \quad (3.3.1)$$
$$\leq C \int_{R^m} \left| \frac{1}{t} f_t(x) \right|_{B_0}^p dx \leq \frac{C}{t^p} \int_{R^m} |f(x)|_{B_0}^p \, dx \, .$$

$f^t \in L^1(R^m, B_0)$ and we set $f^t/t = h + g$ as in Theorem (3.2). Since $|g(x)|_{B_0} \leq A$ the argument shows that

$$m(\{x, |T(tg)(x)|_{B_1} > t\}) \leq \frac{C}{t^p} \int_{R^m} |g(x)|_{B_1}^p \, dx \, . \quad (3.3.2)$$

Finally our assumption (ii) gives us

$$m(\{x, |T(th)(x)|_{B_1} > t\}) \leq C \int_{R^m} \left| \frac{1}{t} f^t(x) \right|_{B_0}^p dx \leq C \int_{R^m} \left| \frac{1}{t} f(x) \right|_{B_0}^p dx \, . \quad (3.3.3)$$

Putting (3.3.1), (3.3.2) and (3.3.3) together,

$$m(\{x, |T(f)(x)|_{B_1} > 2(CA+1)t\} \leq \frac{C'}{t^p} \int |f(x)|_{B_0}^p \, dx \, .$$

A similar argument applies to the case $f \in L^1(R^m, B_0)$ and the theorem follows.

Remark (3.1). When T is countably subadditive,

$$\text{i.e. } |T(\sum_{n=1}^{\infty} f_n)(x)|_{B_1} \leq \sum_{n=1}^{\infty} |T(f_n)(x)|_{B_1} \text{ a.e. } (\sum_{n=1}^{\infty} |f_n| \in L^1(R^m, B_0)) \, ,$$

condition (2) of Theorem (3.3) can be replaced by the stronger condition:

(2′) If the support of f is contained in $x + U_\alpha$ and f has mean value zero

$$\int_{(x+U_{\Phi(\alpha)})'} |T(f)(x)|_{B_1} \, dx \leq C \int_{R^m} |f(x)|_{B_0} \, dx \, .$$

Proof.

$$m(\{x, |T(th)(x)|_{B_1} > t\}) \leq m(\{x, |T(th)(x)|_{B_1} > t\} \cap$$
$$\cap \{ \bigcup_n [x_n + U_{\Phi(\alpha_n)}]\}') + m(\bigcup_n [x_n + U_{\Phi(\alpha_n)}]) \leq$$
$$\leq \frac{1}{t} \int_{(\bigcup_n (x_n + U_{\Phi(\alpha_n)}))'} |T(th)(x)|_{B_1} \, dx + \sum_{n=1}^{\infty} m(U_{\Phi(\alpha_n)}) \leq$$

$$\leq \frac{1}{t} \sum_n \int_{(x_n + U_{\Phi(\alpha_n)})'} |T(th_n)(x)|_{B_1}\, dx + A^2 \int_{R^m} |f(x)|_{B_0}\, dx \leq$$

$$\leq C \int_{R^m} |f(x)|_{B_1}\, dx \,.$$

4. Singular integrals

To avoid notational difficulties, throughout this section we will work with complex valued functions. The extension of the results to vector valued functions will be briefly discussed in Section 7.

Definition (4.1). The function $k(x)$ is a singular kernel for the regular Vitali family $\{U_\alpha, \Phi\}$ when

1) For every Ω compact, $\Omega \subset R^m \cap \{0\}'$, $k \in L^1(\Omega)$. Moreover the integral $\int_{U'_\alpha \cap U_\gamma} k(x)dx$ is uniformly bounded independently of α and γ, and its limit exists for fixed γ as α tends to zero.

2) The integral, $\int_{U'_\alpha \cap U_{\Phi(\alpha)}} |k(x)|dx$, is uniformly bounded independently of α.

Set $k_{\alpha,\gamma}(x) = k(x)\chi_{U'_\alpha \cap U_\gamma}(x)$. For $f \in L^p(R^m)$, $1 \leq p \leq \infty$, let

$$K_{\alpha,\gamma}(f)(x) = (k_{\alpha,\gamma} * f)(x) = \int_{R^m} k_{\alpha,\gamma}(x-y) f(y)dy \,.$$

The convolution is well-defined almost everywhere and it belongs to $L^p(R^m)$.

THEOREM (4.1). *Let $k(x)$ be a singular kernel for the family $\{U_\alpha, \Phi\}$, where $k(x)$ satisfies:*

(4.1.1) *If $y \in U_\beta$, the integral, $\int_{U'_{\Phi(\beta)}} |k(x-y) - k(x)|dx$, is uniformly bounded independently of β.*

Under condition (4.1.1) *we conclude*

(i) *If $f \in L^1(R^m)$, $m(\{x, |K_{\alpha,\gamma}(f)(x)| > 1\}) \leq C\|f\|_1$*

(ii) *If $f \in L^p(R^m)$, $1 < p < \infty$, $\|K_{\alpha,\gamma}(f)\|_p \leq C_p\|f\|_p$. Here $C_p \leq C\left(\frac{1}{(p-1)} + p\right)$ and C depends only on the uniform bounds of Definition* (4.1) *and condition* (4.1.1).

Proof. It is not difficult to see that $k_{\alpha,\gamma}$ also satisfies condition (4.1.1). In fact, if $y \in U_\beta$, $\Phi(\beta_1) = \alpha$ and $\Phi^{(2)}(t) = \Phi(\Phi(t))$,

$$\int_{U'_{\Phi(\beta)}} |k_{\alpha,\gamma}(x-y) - k_{\alpha,\gamma}(x)|dx \leq \int_{U'_{\Phi(\beta)}} |k(x-y) - k(x)|dx +$$

$$+ 2\left(\int_{U'_\alpha \cap U_{\Phi^2(\alpha)}} |k(x)|dx + 2 \int_{U'_{\beta_1} \cap U_{\Phi^2(\beta)}} |k(x)|dx + \int_{U'_\gamma \cap U_{\Phi^2(\gamma)}} |k(x)|dx \right) \leq C\,.$$

To apply the weak type estimates obtained in Section 3 we must have a result in L^p for some p, $1 < p < \infty$. The key argument lies in the L^2 space where we make use of the Fourier transform. More precisely we prove the uniform boundedness of $K_{\alpha,\gamma}$ in $L^2(R^m)$ showing that $|\mathcal{F}(k_{\alpha,\gamma})(x)| \leq C$, where $\mathcal{F}(.)$ denotes the Fourier transform. The fact that the Fourier transform is an isometry in L^2 finishes the argument.

To simplify the notation call $h(x) = k_{\alpha,\gamma}(x)$, and $X(y) = \exp(2\pi i\langle x, y\rangle)$.

Set $H_x = \{z, |X(z) - 1| \geq \frac{1}{2},\ x, z \in R^m\}$, and $A = \{\alpha, U_\alpha \cap H_x \neq \emptyset\}$.

Since $\bigcup_\alpha U_\alpha = R^m$, $A \neq \emptyset$. On the other hand Φ is continuous and $\Phi(\alpha) > \alpha$, hence there exists $\beta_1 \notin A$ such that $\Phi(\beta_1) = \beta \in A$. Define the sequence $\{\beta_n\}$ by $\Phi(\beta_n) = \beta_{n-1}$. Take $z \in U_\beta \cap H_x$, then

$$\mathcal{F}(h)(x) = \int_{R^m} h(y)X(y)dy = \overline{X(z)} \int_{R^m} h(y-z)X(y)dy\,.$$

Hence

$$(1 - X(z))\,\mathcal{F}(h)(x) = \int_{R^m} [h(y) - h(y-z)]X(y)dy =$$

$$= \int_{U'_{\Phi(\beta)}} [h(y) - h(y-z)]X(y)dy +$$

$$+ \int_{U'_{\beta_2} \cap U_{\Phi(\beta)}} h(y)X(y)dy + \int_{U_{\beta_2}} h(y)[X(y) - 1]dy +$$

$$+ \int_{U_{\beta_2}} h(y)dy - \int_{U_{\Phi(\beta)}} h(y-z)X(y)dy = I_1 + I_2 + I_3 + I_4 - I_5\,.$$

From condition (4.1.1) it follows that $|I_1| \leq C$. Meanwhile $|I_2| \leq C$ and $|I_4| \leq C$ by definition of singular kernels (see Definition (4.1)). On the other hand $|I_3| \leq \sum_{n=2}^{\infty} \int_{U'_{\beta_{2n}} \cap U_{\beta_{2(n-1)}}} |X(y) - 1||k(y)|dy$. Since $2U_{\beta_n} \subset U_{\beta_{n-2}}$, if $y \in U_{\beta_{2n}}$, then $2^j y \in U_{\beta_2}$ for $0 \leq j \leq n-1$. Hence for such y, $|X(y)^{2^j} - 1| < 1/2$ for $0 \leq j \leq n-1$. Therefore $|X(y) - 1| \leq 2^{-n}$. Hence $|I_3| \leq C \sum_{n=2}^{\infty} 2^{-n} \leq C'$. Finally since $U_{\beta_1} \subset z + U_{\Phi(\beta)}$

$$\overline{X(z)}I_5 = \int\limits_{(U_{\Phi(\beta)}+z)} h(y)X(y)dy = \int\limits_{(U_{\Phi(\beta)}+z)\cap U'_{\beta_1}} h(y)X(y)dy +$$

$$+ \int\limits_{U_{\beta_1}} h(y)[1 - X(y)]dy - \int\limits_{U_{\beta_1}} h(y)dy = II_1 + II_2 + II_3 .$$

Observe that $U'_{\beta_1} \cap (z + U_{\Phi(\beta)}) \subset U'_{\beta_1} \cap U_{\Phi(\Phi(\beta))}$. Hence by definition of singular kernels $|II_1| \leq C$ and $|II_3| \leq C$. The integral II_2 is identical to I_3.

Therefore $\|K_{\alpha,\gamma}(f)\|_2 = \|\mathcal{F}(k_{\alpha,\gamma})\,\mathcal{F}(f)\|_2 \leq C\|\mathcal{F}(f)\|_2 = C\|f\|_2$.

To prove part (i), we make use of Remark (3.1) (remark to Theorem (3.3)) with $T = S = K_{\alpha,\gamma}$. Clearly, $f = \sum_{n=1}^{\infty} f_n$, $|K_{\alpha,\gamma}(f)| \leq \sum_{n=1}^{\infty} |K_{\alpha,\gamma}(f_n)|$. On the other hand, if f has support in $x + U_\beta$ and $\int_{x+U_\beta} f(y)dy = 0$,

$$K_{\alpha,\gamma}(f)(z) = \int\limits_{x+U_\beta} k_{\alpha,\gamma}(z-y)f(y)dy = \int\limits_{x+U_\beta} [k_{\alpha,\gamma}(z-y) - k_{\alpha,\gamma}(z-x)]f(y)dy .$$

Therefore

$$\int\limits_{(x+U_{\Phi(\beta)})'} |K_{\alpha,\gamma}(f)(z)|dz \leq \int\limits_{(x+U_{\Phi(\beta)})'} \left\{\int\limits_{x+U_\beta} |k(z-y) - k(z-x)|\ |f(y)|dy\right\} dz =$$

$$= \int\limits_{x+U_\beta} |f(y)| \left\{\int\limits_{U'_{\Phi(\beta)}} |k(z-(y-x)) - k(z)|dz\right\} dy \leq C\|f\|_1$$

since $y - x \in U_\beta$.

Theorem (3.3) also yields that $\|K_{\alpha,\gamma}(f)\|_p \leq C_p\|f\|_p$ for $1 < p < 2$ where $C_p \leq C\left(\dfrac{1}{p-1} + \dfrac{1}{p-2}\right)$. To obtain the same result for $2 < p < \infty$ a classical duality argument is used. $\int_{R^m} K_{\alpha,\gamma}(f)(y)\overline{g(y)}dy = \int_{R^m} f(y)K^*_{\alpha,\gamma}(\bar{g})(y)dy$, where $k^*(x) = \overline{k(-x)}$. k^* is also a singular kernel and satisfies condition (4.1.1). Therefore if $2 < p < \infty$ and $1/p' + 1/p = 1$,

$$\|K_{\alpha,\gamma}(f)\|_p = \sup_{\|g\|_{p'}=1} \left\{\left|\int\limits_{R^m} K_{\alpha,\gamma}(f)(y)\bar{g}(y)dy\right|\right\} = \sup_{\|g\|_{p'}=1} \left\{\left|\int\limits_{R^m} f(y)K^*_{\alpha,\gamma}(\bar{g})(y)dy \leq\right.\right.$$

$$\leq \sup_{\|g\|_{p'}=1} \{\|f\|_p\, \|K^*_{\alpha,\gamma}(\bar{g})\|_{p'}\} \leq C\|f\|_p .$$

The proof of the theorem is now complete.

5. The maximal operator for singular integrals and the pointwise convergence

Let O be a bounded open set whose boundary is of measure zero. If x does not belong to the boundary then in virtue of (1) Definition (4.1) and the boundedness of $O, K_{\beta,\gamma}(\chi_0)(x)$ has a limit as $\alpha \to 0$, $\gamma \to \infty$. Therefore, if $\mathcal{S}_0$ denotes the class of simple functions over bounded open sets whose boundary is of measure zero, and if $f \in \mathcal{S}_0$

$$\lim_{\substack{\alpha \to 0 \\ \gamma \to \infty}} K_{\alpha,\gamma}(f)(x) = K(f)(x) \quad \text{a.e.} \tag{5.1.1}$$

Hence Theorem (4.1) implies that for $f \in \mathcal{S}_0$, $m\{x, |K(f)(x)| \geq t\} \leq \frac{C_p^p}{t^p} \|f\|_p^p$, $1 \leq p < \infty$. From this inequality the operator K can be extended by continuity to the entire class $L^p(R^m)$, $1 \leq p < \infty$. The main question studied in this section is whether the identity (5.1.1) holds for every function $f \in L^p(R^m)$, $1 \leq p < \infty$. To answer the question we have to turn our attention to the corresponding maximal operator.

Remark (5.1). Let $f \in L^p(R^m)$, $1 \leq p \leq \infty$. Set $E_f = \{x\,; (|k_{\alpha,\gamma}| * |f|)(x) = +\infty$, for some $(\alpha, \gamma)\}$. Then $m(E_f) = 0$.

Proof. Consider the sequences $\alpha_n \to 0$, $\gamma_n \to \infty$. Set

$$E_{n,f} = \{x, (|K_{\alpha_n,\gamma_n}| * |f|)(x) = +\infty\}.$$

Clearly $\bigcup_n E_{n,f} = E_f$. Since $m(E_{n,f}) = 0$ the remark follows.

After Remark (5.1) when $x \notin E_f$ we can define the maximal operator of the family $\{K_{\alpha,\gamma}\}$.

$$\mathcal{M}(f)(x) = \sup_{(\alpha,\gamma)} |K_{\alpha,\gamma}(f)(x)|\,.$$

The next theorem shows that $\mathcal{M}$ has the same boundedness properties as the kernel $K_{\alpha,\gamma}$.

THEOREM (5.1) (The maximal operator). *If $k(x)$ is a singular kernel satisfying condition* (4.1.1) *of Theorem* (4.1), *the operator $\mathcal{M}$ is of weak type L^1 and bounded in $L^p(R^m)$ for $1 < p < \infty$.*

Proof. For $f \in \mathcal{S}_0$, set $K_\alpha(f)(x) = \lim_{\gamma \to \infty} K_{\alpha,\gamma}(f)(x)$. If $Q_\alpha(x) = x - U_\alpha$, then

$$|K_\alpha(f)(x)| \leq |K_\alpha(f)(x) - K(f)(x_1) + K(\chi_{Q_\alpha(x)}f)(x_1)| +$$

$$+ |K(f)(x_1)| + |K(\chi_{Q_\alpha(x)}f)(x_1)| = \left| \int_{(x-U_\alpha)'} [k(x-y) - k(x_1-y)] f(y)dy \right| +$$

$$+ |K(f)(x_1)| + |K(\chi_{Q_\alpha(x)}f)(x_1)|\,.$$

Assume, for simplicity, that $m(\bar{U}_\alpha \cap U'_\alpha) = 0$. Then the above inequality is well-defined for almost every x_1. The average of the inequality over $x_1 \in x - U_\beta$ with $\Phi(\beta) = \alpha$ and the supremum of all such β's yield

$$|K_\alpha(f)(x)| \leq R_1(f)(x) + S(f)(x) ,$$

where

$$R_1(f)(x) = \sup_{\alpha,\, \Phi(\beta)=\alpha} \left[\frac{1}{m(U_\beta)} \int_{(x-U_\beta)} \left\{\int_{(x-U_\alpha)'} |k(x-y) - k(x_1 - y)|\, |f(y)|dy\right\} dx_1\right]$$

and

$$S(f)(x) = \sup_\beta \left[\frac{1}{m(U_\beta)} \int_{x-U_\beta} |K(f)(x_1)|dx_1\right] + \sup_\beta \left[\frac{1}{m(U_\beta)} \int_{x-U_\beta} |K(\chi_{Q_\alpha(x)} f)(x_1)|dx_1\right].$$

In virtue of condition (4.1.1) of Theorem (4.1) it follows that $\|R_1(f)\|_\infty \leq C\|f\|_\infty$. On the other hand from Hölder's inequality and the continuity of K in $L^p(R^m)$, $1 < p < \infty$,

$$\frac{1}{m(U_\beta)} \int_{x-U_\beta} |K(\chi_{Q_\alpha(x)} f)(x_1)|dx_1 \leq \left(\frac{1}{m(U_\beta)} \int_{R^m} |K(\chi_{Q_\alpha(x)} f)(x_1)|^p dx_1\right)^{1/p} \leq$$

$$\leq B_q \left(\frac{1}{m(U_\beta)} \int_{x-U_\alpha} |f(y)|^p dy\right)^{1/p}$$

Therefore if $M(f)$ denotes as in section 2 the maximal function of f,

$$S(f)(x) \leq 2A\{M(K(f))(x) + [M(|f|^p)(x)]^{1/p}\}.$$

With such an estimate, it is easy to see that S is an operator of weak type L^p and bounded in $L^q(R^m)$ for $p < q < \infty$.

If the argument used for K_α is repeated for $K_{\alpha,\gamma}$, it shows that

$$|K_{\alpha,\gamma}(f)(x)| \leq \int_{(x-U_\alpha)' \cap (x-U_\gamma)} |k(x-y) - k(x_1 - y)||f(y)|dy +$$

$$+ \int_{(x-U_\alpha)'} |k(x-y) - k(x_1 - y)||f(y)|dy + |K_\alpha(f)(x_1)| + |K(\chi_{Q_\alpha(x)} f)(x_1)| .$$

The averaging argument now yields $|K_{\alpha,\gamma}(f)(x)| \leq R(f)(x) + 2S(f)(x)$, where R is the sum of R_1 and the supremum over β of the average of the two integrals above. Hence $\|R(f)\|_\infty \leq C\|f\|_\infty$.

To extend the estimate to every function of $L^p(R^m) \cap L^\infty(R^m)$, observe that S is a bounded operator in $L^p(R^m)$; therefore if $f \in L^p(R^m) \cap L^\infty(R^m)$, choose $f_n \in \mathcal{S}_0$ such that $\|f_n - f\|_p \to 0$ as $n \to \infty$ and $\|f_n\|_\infty \to \|f\|_\infty$. Then in virtue of the dominated convergence theorem, $K_{\alpha,\gamma}(f_n)(x)$ converges to $K_{\alpha,\gamma}(f)(x)$ everywhere. Choose an adequate subsequence such that $S(f_n) \to S(f)$ a.e. Then

$$|K_{\alpha,\gamma}(f)(x)| \leq C\|f\|_\infty + 2S(f)(x)$$

except for a set of measure zero which does not depend on (α, γ). Hence

$$\mathcal{M}(f)(x) \leq C\|f\|_\infty + 2S(f)(x) \quad \text{a.e.},$$

and the first condition of Theorem (3.3) is satisfied.

Let $h = \sum_{n=1}^\infty h_n$ be defined as in the second condition of Theorem (3.3). The support of h_n is contained in $x_n + U_{\beta_n}$ and its mean value is zero. We will divide the $\sum_{n=1}^\infty h_n(x)$ into three functions according to fixed values of x, α and γ. Let $\Phi(\alpha_1) = \alpha$, $\Phi(\gamma_1) = \gamma$, set $I_1(x, \alpha, \gamma) = \{n\,;$ where either $\alpha < \beta_n < \gamma$ and $x \in x_n + U_{\gamma_1}$, or $\beta_n \leq \alpha$ and $x \in (x_n + U_{\gamma_1}) \cap (x_n + U'_{\Phi(\alpha)})\}$, $I_2(x, \alpha, \gamma) = \{n\,; \alpha \geq \beta_n$ and $x_n \in (x + U'_{\gamma_1}) \cap (x_n + U_{\Phi(\alpha)})\}$, $I_3(x, \alpha, \gamma) = \{n;$ where either $\alpha < \beta_n < \gamma$ and $x \in (x_n + U'_{\gamma_1}) \cap (x_n + U_{\Phi(\gamma)})$, or $\beta_n \leq \alpha$ and $x \in (x_n + U'_{\Phi(\alpha)}) \cap (x_n + U_{\Phi(\gamma)}) \cap (x_n + U'_{\gamma_1})\}$. Let $h^{(i)}(y) = \sum_{n \in I_i(x,\alpha,\gamma)} h_n(y)$, $1 \leq i \leq 3$, and set $F = \bigcup_{n=1}^\infty (x_n + \bigcup_{\Phi(\Phi(\beta_n))})$. Then when $x \notin F$, $K_{\alpha,\gamma}(h)(x) = \sum_{i=1}^3 K_{\alpha,\gamma}(h^{(i)})(x)$. Now we set

$$\mathcal{M}_i(h)(x) = \sup_{\alpha,\gamma} |K_{\alpha,\gamma}(h)^{(i)}(x)|, \quad i = 1, 2, 3.$$

Observe from Theorem (3.2) that

$$m(\{x, \mathcal{M}(h)(x) > C\}) \leq m(\{x, \mathcal{M}(h)(x) > t\} \cap F) +$$
$$+ m(\{x, \mathcal{M}(h)(x) < C\} \cap F') \leq C \int_{R^m} |f(x)|dx + \sum_{i=1}^3 m(\{x, \mathcal{M}_i(h)(x) > C/3\} \cap F').$$

Hence it suffices to prove that

$$m(\{x, \mathcal{M}_i(h)(x) > C\} \cap F') \leq C \int_{R^m} |f(x)|dx$$

where C is a fixed constant depending only on the bounds of the kernel k given in definition 4.1 and the bound given by condition (4.1.1).

When $x \notin E_h$ (see Remark (5.1))

$$|K_{\alpha,\gamma}(h^{(1)})(x)| \leq \sum_{n \in I_1(x,\alpha,\gamma)} \left| \int_{(x_n + U_{\beta_n})} k(x-y)h(y)dy \right| =$$

$$= \sum_{n\in I_1}\left[\int_{(x_n+U_{\beta_n})} |k(x-y)-k(x-x_n)|\,|h(y)|dy\right] \le$$

$$\le \sum_{n=1}^{\infty} \int_{(x_n+U_{\beta_n})} |k(x-y)-k(x-x_n)|\,|h(y)|dy\,.$$

Therefore if $E_1 = \{x, \mathcal{M}_1(f)(x) \ge 1\} \cap F' \cap E'_h$,

$$m(E_1) \le \int_{F'} \sum_{n=1}^{\infty}\left\{\int_{(x_n+U_{\beta_n})} |k(x-y)-k(x-x_n)|\,|h(y)|\right\} dy \le C\|h\|_1 \le 2C\|f\|_1$$

in virtue of condition (4.1.1).

To study $\mathcal{M}_2(h)(x)$, set $E(x,\alpha) = (x+U_\alpha)'$ and let V_n denote the support of h_n. $V_n \cap V_m = \emptyset$ for $n \ne m$ and $V_n \subset x_n + U_{\beta_n}$. Set

$$c_n(x,\alpha) = \frac{1}{m(V_n)}\int_{V_n} \chi_{E(x,\alpha)} h(y)dy\,.$$

Clearly $|c_n(x,\alpha)| \le 2$. Then for $x \notin E_h$,

$$|K_{\alpha,\gamma}(h^{(2)})(x)| \le \sum_{n\in I_2(x,\alpha)} \left|\int_{V_n} k(x-y)\chi_{E(x,\alpha)} h(y)dy\right| \le$$

$$\le \sum_{n\in I_2(x,\alpha)} \left|\int_{V_n} k(x-y)[\chi_{E(x,\alpha)}(y)h(y) - c_n(x,\alpha)]dy\right| + \sum_{n\in I_2(x,\alpha)} |c_n(x,\alpha)| \int_{V_n} |k(x-y)|dy \le$$

$$\le \sum_n \left|\int_{V_n} \{k(x-y)-k(x-x_n)\}[\chi_{E(x,\alpha)}h(y) - c_n(x,\alpha)]dy\right| + 2\int_{x-y\in U'_{\Phi^{-2}(\alpha)}\cap U_{\Phi(\alpha)}} |k(x-y)|dy$$

$$\le \sum_n \int_{V_n} |k(x-y)-k(x-x_n)|(|h(y)|+2)dy + C\,.$$

Therefore

$$m(\{x, \mathcal{M}_2(f)(x) \ge C+1\}\cap F' \cap E'_h) \le \int_{F'}\left[\sum_n \int_{V_n} |k(x-y)-k(x-x_n)|(|h(y)|+2)dy\right] dx$$

$$\le C' \sum_n \int_{V_n} (|h(y)|+2)dy \le C'(\|h\|_1 + 2\sum_n m(V_n)) \le C''\|f\|_1\,.$$

The operator $\mathcal{M}_3$ is handled with a similar argument, and the theorem is proved.

The pointwise convergence of the singular integrals is an easy consequence of Theorem (5.1).

THEOREM (5.2). *Let* $f \in L^p(R^m)$, $1 \le p < \infty$. *Then*

$$\lim_{\substack{\alpha \to 0 \\ \gamma \to \infty}} K_{\alpha,\gamma}(f)(x) = K(f)(x) \quad a.e.$$

Proof. For $f \in L^p(R^m)$, $1 \le p < \infty$, set $\mathcal{M}'(f)(x) = \overline{\lim_{\substack{\alpha \to 0 \\ \gamma \to \infty}}} |K_{\alpha,\gamma}(f)(x) - K(f)(x)|$. $\mathcal{M}'$ is well-defined almost everywhere, and

$$\mathcal{M}'(f)(x) \le \mathcal{M}(f)(x) + |K(f)(x)| \quad \text{a.e.}$$

Consider a sequence $\{f_n\}$, $f_n \in \mathcal{S}_0$, such that $\|f_n - f\|_p \to 0$. Let $g_n = f - f_n$. Since $\mathcal{M}'(f_n)(x) = 0$ a.e.

$$\begin{aligned} & m(\{x\,; \mathcal{M}'(f)(x) > t\}) = m(\{x\,; \mathcal{M}'(f - f_n) > t\} \le \\ & \le m(\{x\,; \mathcal{M}(g_n)(x) > t/2\}) + m(\{x\,; |K(g_n)(x)| > t/2\}) \le \\ & \le \left(\frac{2C_p}{t}\right)^p \int_{R^m} |f(x) - f_n(x)|^p dx\,. \end{aligned}$$

Since n and t are arbitrary, $\mathcal{M}'(f)(x) = 0$ a.e. and the theorem follows.

6. Singular integrals on locally compact groups

It is plain from the proofs of most of the theorems of Sections 2 through 5 that the results remain valid if R^m is replaced by a locally compact group G and the functions take values in a Hilbert space. In this section we outline briefly the modifications necessary to extend the results.

The Lebesgue measure R^m, must be replaced by a Haar measure of the group G; and the absolute values by the appropriate norms. The definition of regular Vitali families does not imply that either $\Phi(\alpha) > \alpha$ or $\bigcup_\alpha U_\alpha = G$ for a general group G. Hence these conditions must be added. With such modifications Theorems (3.1), (3.2), (3.3), Remark (3.1) and Theorems (4.1), (5.1), (5.2) remain valid over any locally compact Abelian group G, for functions taking values in a Hilbert space. The proofs are identical.

The definition of singular kernel is

Definition (6.1). For each $x \in G$, let $k(x)$ be a bounded linear operator from the Hilbert space H_1 into the Hilbert space H_2. $k(x)$ is a singular kernel for the regular Vitali family $\{U_\alpha, \Phi\}$ when

1) For every Ω compact, $\Omega \subset G \cap \{e\}'$, $k \in L^1(\Omega)$. Moreover the integral, $\int_{U'_\alpha \cap U_\gamma} k(x)dx$, is an operator, uniformly bounded in α and γ, and its limit exists for every fixed γ as α tends to zero.

2) If $h \in H_1$ with $|h|_{H_1} \leq 1$, the integral $\int_{U'_\alpha \cap U_{\emptyset(\alpha)}} |k(x)[h]|_{H_2} dx$ is uniformly bounded in α and h.

Set $\mathcal{M}(f)(x) = \sup_{\alpha, \gamma} |K_{\alpha,\gamma}(f)(x)|_{H_2}$

We reannounce Theorem (5.1), since the conclusions are slightly different.

THEOREM (6.1). *Let G be a locally compact Abelian group. $k : G \to \mathcal{L}(H_1, H_2)$, a singular kernel for the regular Vitali family $\{U_\alpha, \Phi\}$. Assume*

$$\int_{U_{\Phi(\beta)}} |(k(x-y) - k(x))[h]|_{H_2} dx \tag{6.1.1}$$

is uniformly bounded in β and h whenever $y \in U_\beta$ and $|h|_{H_1} \leq 1$.

Then

1) $m(\{x, \mathcal{M}|(f)(x)|_{H_2} \geq 1\}) \leq C \int_G |f(x)|_{H_1} dx,$

2) *for* $1 < p \leq 2$; $|\mathcal{M}(f)|_{L^p(G, H_2)} \leq C_p |f|_{L^p(G, H_1)}$.

THEOREM (6.2). *When $k(x)$ and $k^*(x)$ (the adjoint operator) are singular kernels for the family $\{U_\alpha, \Phi\}$, and both satisfy condition* (6.1.1); *then conclusion* (2) *of Theorem* (6.1) *is valid for all p, $1 < p < \infty$.*

Theorem (5.2) should also be reannounced accordingly.

To prove the L^2-boundedness of the kernels it suffices to change, in Theorem (4.1), $\exp(2\pi i \langle x, y\rangle)$ by $\sigma(y)$ (the characters of the group). Observe that Parseval's identity remains valid on $L^2(G, H)$, the space of square summable functions from the Abelian group G to the Hilbert space H.

Actually with the exception of the L^2-boundedness of the singular integrals the rest of the theory developed in Sections 2, 3, 4 and 5 is valid whether the group G is Abelian or not. Therefore, if the boundedness in L^2 is assumed, all the other theorems follow for the non-Abelian case as well (see also [17])[1].

In the non-Abelian case sometimes it is impossible to construct a regular Vitali family for all values of α. In those cases where the construction is possible for $\alpha \leq \alpha_0$, and provided that the L^2-boundedness holds, the results of the previous sections remain valid for the local truncations k_{α, α_0}.

7. Homogeneous kernels

Let $\pi : R_+ \to \mathcal{L}(R^m, R^m)$ be a continuous mapping with the properties

(1) $\pi(\lambda\mu) = \pi(\lambda) \cdot \pi(\mu)$, $\pi(1) = I$, the identity-matrix.

[1] See also A. Knapp and E. Stein, Intertwining operators for semisimple groups. *Ann. of Math.*, 93 (1971), 489–578.

(2) For $0 < \lambda \leq 1$, $\|\pi(\lambda)\| \leq \lambda$, ($\|.\|$ denotes the norm in $\mathcal{L}(R^m, R^m)$).

For simplicity, we let $\pi(\lambda) = T_\lambda$.

THEOREM (7.1). *Set* $F(x, r) = \|T_{r^{-1}}(x)\|$. *There exists a unique solution* $r = r(x)$ *for* $x \neq 0$, *such that* $F(x, r(x)) = 1$, *and moreover the solution has the following properties:*

(i) $r(x + y) \leq r(x) + r(y)$ *and* $r(x) = 0$ *if and only if* $x = 0$. *(In other words* $r(.)$ *is a metric.)*

(ii) $r(T_\lambda(x)) = \lambda r(x)$

(iii) *If* $\|x\| \leq 1$, $r(x) \geq \|x\|$. *If* $\|x\| \geq 1$, $r(x) \leq \|x\|$.

Proof. The existence of a unique solution $r = r(x)$, such that $F(r, x) = 1$, is a consequence of the following properties of the function F, for fixed $x \in R^m \cap \{0\}'$, $(r(0) = 0)$.

(1) $F(x, r) \to 0$, as $r \to \infty$,

(2) $F(x, r) \to \infty$, as $r \to 0$,

(3) $F(x, r)$ is a continuous decreasing function of r.

If $r \leq 1$, $F(x, r) = \|T_{r^{-1}}(x)\| \leq \|T_{r^{-1}}\|\|x\| \leq r^{-1}\|x\|$, and (1) follows.

On the other hand if $r \leq 1$ and $x = T_r(y)$, then $\|x\| \leq \|T_r\|\|y\| \leq rF(x, r)$. Hence $F(x, r) \to \infty$, as $r \to 0$.

Finally if $r_1 \leq r_2$, $F(x, r_2) = \|T_{r_2^{-1}}(x)\| = \|T_{r_1 r_2^{-1}}(T_{r_1^{-1}}(x))\| \leq r_1 r_2^{-1} F(x, r_1) \leq F(x, r_1)$. The continuity of the function $F(x, .)$ is clear.

It is easy to see that $r(x) = 0$ if and only if $x = 0$.

Set $\lambda_1 = r(x)$, $\lambda_2 = r(y)$. To prove that $r(x + y) \leq r(x) + r(y)$, it is enough to show that $\|T_{(\lambda_1+\lambda_2)^{-1}}(x + y)\| \leq 1$. But

$$\begin{aligned}\|T_{(\lambda_1+\lambda_2)^{-1}}(x + y)\| &\leq \|T_{(\lambda_1+\lambda_2)^{-1}\lambda_1}(T_{\lambda_1^{-1}}(x))\| + \\ &+ \|T_{(\lambda_1+\lambda_2)^{-1}\lambda_2}(T_{\lambda_2^{-1}}(y))\| \leq \lambda_1(\lambda_1 + \lambda_2)^{-1}\|T_{\lambda_1^{-1}}(x)\| + \\ &+ \lambda_2(\lambda_1 + \lambda_2)^{-1}\|T_{\lambda_2^{-1}}(y)\| \leq 1 .\end{aligned}$$

It is immediate from the definition that $r(T_\lambda(x)) = \lambda r(x)$. If $\|x\| \geq 1$, then $\|T_{\|x\|^{-1}}(x)\| \leq \|x\|^{-1}\|x\| \leq 1$, in other words $r(x) \leq \|x\|$. The reciprocal inequality follows similarly.

Consider the following change of variables induced by the metric $r(x)$, $x \to (x', r)$, where $x' \in S^{m-1}$ $(S^{m-1} = \{x, \|x\| = 1\})$ and $T_r(x') = x$. If x' is expressed in a coordinate system of S^{m-1}

$$\begin{aligned} x'_1 &= \cos(\Phi_1) \ldots \cos(\Phi_{m-1}) \\ &\vdots \\ x'_m &= \sin(\Phi_1) . \end{aligned}$$

The change of variables is expressed by, $x \to (r, \Phi_1, \ldots, \Phi_{m-1})$. To compute the Jacobian of the transformation, we observe that $T_\lambda = \exp(P \ln \lambda)$, where P is a $m \times m$ real matrix (the infinitesimal generator of the group).

Hence

$$\frac{\partial x}{\partial r} = r^{-1}PT_r(x') = r^{-1}T_r(P(x')), \quad \text{and} \quad \frac{\partial x}{\partial \Phi_i} = T_r\left(\frac{\partial x'}{\partial \Phi_i}\right), \ 1 \leq i \leq m-1.$$

Then $J(x; r, \Phi_1, \ldots, \Phi_{m-1}) = \det(r^{-1}T_r(P(x'))$,

$$\left(T_r\left(\frac{\partial x'}{\partial \Phi_1}\right), \ldots, T_r\left(\frac{\partial x'}{\partial \Phi_{m-1}}\right)\right) = r^{tr(P)-1}H(\Phi),$$

where $\operatorname{tr}(P)$ denotes the trace of P and

$$H(\Phi) = \left(P(x'), \frac{\partial x'}{\partial \Phi_1}, \ldots, \frac{\partial x'}{\partial \Phi_{m-1}}\right).$$

It is not hard to see that if $d\sigma$ denotes the classical measure over S^{m-1}, then $H(\Phi)d\Phi = (P(x'), x')d\sigma$; where the identity is understood as measures over S^{m-1}.

Set $U_\alpha = \{x; r(x) < \alpha\}$ and $\Phi(\alpha) = 2\alpha$. Then $U_\alpha - U_\alpha \subset U_{2\alpha}$, and $m(U_{2\alpha}) = 2^{tr(P)-1}m(U_\alpha)$. To prove the last identity it suffices to integrate the characteristic function of $U_{2\alpha}$ and to change variables into the polar-like coordinates discussed above.

Definition (7.1). A function $k(x)$ is a homogeneous kernel with respect to the group $\{T_\lambda\}$ when

(1) $k(x)$ is defined in $R^m \cap \{0\}'$, $k \in L^1(S^{m-1})$, and

$$\int_{S^{m-1}} k(x')H(\Phi)d\Phi = \int_{S^{m-1}} k(x')(P(x'), x')d\sigma = 0.$$

(2) $k(T_\lambda(x)) = \lambda^{-tr(P)}k(x)$, for $x \neq 0$.

It is clear, in virtue of the properties observed above, that a homogenous kernel is a singular kernel for the regular Vitali family $\{\{x, r(x) < \alpha\}, 2\alpha\}$. Now set

$$k_{\varepsilon,R}(x) = \begin{cases} k(x) & \text{when } \varepsilon \leq r(x) \leq R \\ 0 & \text{otherwise} \end{cases}$$

and $K_{\varepsilon,R}(f)(x) = (k_{\varepsilon,R} * f)(x)$ for $f \in L^p(R^m)$, $1 \leq p < \infty$.

THEOREM (7.1). *Let $k(x)$ be a homogenous kernel for the group $\{T_\lambda\}$, satisfying*

$$\int_{\{x,\, r(x) \geq 2r(y)\}} |k(x-y) - k(x)|dx \leq C. \tag{7.1.1}$$

Then, if $\mathcal{M}(f)(x) = \sup_{\varepsilon,R} |K_{\varepsilon,R}(f)(x)|$, $\mathcal{M}$ *is of weak type* L^1 *and bounded in* $L^p(R^m)$ *for* $1 < p < \infty$. *Moreover*

$$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} K_{\varepsilon,R}(f)(x) = K(f)(x) \quad \text{a.e., for every } f \in L^p(R^m), \quad 1 \leq p < \infty.$$

Theorem (7.1) is a consequence of Theorems (5.1), (6.1) and (6.2).

Condition (7.1.1) is easily implied by a Dini condition of the kernel over the unit sphere. More precisely, set $\omega(t) = \sup_{x\in S^{m-1};\, \|h\|\leq t} \{|k(x-h) - k(x)|\}$.

THEOREM (7.2). *If* $\int_0^{1/2} \omega(t)t^{-1}dt \leq C$, *condition* (7.1.1) *is satisfied.*

Proof. Let $x', y' \in S^{m-1}$

$$\int_{r(x)\geq 2r(y)} |k(x-y) - k(x)|dx = \int_{2r(y)}^{\infty} r^{-1}\left\{\int_{S^{m-1}} |k(x' - T_{r(y)r(x)^{-1}}(y')) - k(x')|H(\Phi)d\Phi\right\} dr \leq$$

$$\leq C\int_{2r(y)}^{\infty} \omega(|T_{r(y)r^{-1}}(y')|)r^{-1}dr \leq C\int_{2r(y)}^{\infty} \omega(r(y)r^{-1})r^{-1}dr \leq C\int_0^{1/2} \omega(r)r^{-1}dr \leq C' .$$

When $P = I$, $T_\lambda(x) = \lambda x$, $r(x) = \|x\|$, and $H(\Phi)d\Phi = d\sigma$. In such case the corresponding homogeneous kernels are the elliptic kernels studied by A. P. Calderón and A. Zygmund in [6]. For those kernels the pointwise convergence was studied by Calderón, M. Weiss and Zygmund in [5]; however theorems (5.1), (5.2) and (7.1) yield a better result even in this case.

If P is a diagonal matrix, the kernels obtained are the parabolic and the semi-elliptic kernels or kernels of mixed homogeneities studied by F. Jones in [15] and E. B. Fabes and N. M. Rivière in [9].

The general treatment over a group $\{T_\lambda\}$ was first considered by M. de Guzman in [7], where he proves the continuity of the kernels in $L^2(R^m)$.

Chapter II

1. Vector valued multipliers

In this section we study some vector valued applications of results obtained in section 6, Chapter I, when the group G is R^m.

Let $\mathcal{F} = \mathcal{F}_H$ denote the Fourier transform in the space $L^2(R^m, H)$. For $a \in L^\infty(R^m, \mathcal{L}(H_1, H_2))$, $f \in L^2(R^m, H_1)$, set $T_a(f) = \mathcal{F}_{H_2}^{-1}(a\mathcal{F}_{H_1}(f))$. More explicitly T_a is defined by the composition

$$\begin{array}{ccc} L^2(R^m, H_1) & \xrightarrow{\mathcal{F}_{H_1}} & L^2(R^m, H_1) \\ \Big\downarrow T_a & & \Big\downarrow a \\ L^2(R^m, H_2) & \xleftarrow{\mathcal{F}_{H_2}^{-1}} & L^2(R^m, H_2) . \end{array}$$

Since the Fourier transform is an isometry in $L^2(R^m, H)$ it follows that

$$\|T_a(f)\|_{L^2(R^m, H_2)} \leq \|a\|_{L^\infty(R^m, \mathcal{L}(H_1, H_2))} \|f\|_{L^2(R^m, H_1)} .$$

The operator T_a so defined is called a multiplier. Our aim is to study multiplier operators that are bounded in the spaces L^p.

In [12] L. Hörmander showed that theorems of the type of Theorem (3.3) of Chapter I could be used to prove the boundedness in L^p of multiplier operators. Basically the idea is to require enough smoothness on the Fourier transform of the kernel (i.e. the multiplier) so that the kernel will satisfy the integrability condition imposed in Theorem (4.1), Chapter I.

We will prove a vector valued multiplier theorem when the regularity conditions of the multiplier are imposed over a regular Vitali family of convex symmetric sets. First we have to set up the adequate notation to state the multiplier theorem.

LEMMA (1.1). *Let C be a bounded convex, symmetric, open set in R^m. There exists a linear invertible operator from R^m into R^m such that*

$$S_1 = \{x, |x| \leq 1\} \subset T^{-1}(C) \subset S_m = \{x, |x| \leq m\} .$$

The proof of the lemma may be found in [14]. We will briefly outline a very simple argument showing the existence of such mapping if we allow $T^{-1}(C) \subset S_\tau$ where τ depends on m only. In any case such a result will suffice for our applications.

Let E be the symmetric ellipsoid of maximal measure with $E \subset C$. Let T_E be any invertible linear operator over R^m such that $T_E(E) = S_1$. It suffices to prove that $T_E(C) \subset \{x, |x| \leq \tau\}$ for some fixed number τ independent of C. We argue by contradiction. If it is not true, there exist points $\{x, -x\} \subset T_E(C)$ very far away. But then the convex symmetric set generated by S_1 and $\{x, -x\}$ would contain a symmetric ellipsoid E' such that $m(S_1) < m(E')$. On the other hand, since $E' \subset T_E(C)$, it follows that $T_E^{-1}(E') \subset C$ and $m(T_E^{-1}(E')) > m(E)$ contradicting our assumption that the measure of E is maximal.

Throughout this chapter the sets U_α of a regular Vitali family will be assumed symmetric and convex. We will say that $\{U_{\alpha_n}, n \in Z\}$ is a lacunary sequence of the family $\{U_\alpha, \Phi\}$ when $\alpha_{n+1} = \Phi(\alpha_n)$. Lemma (1.1) allows us to choose a sequence of invertible operators, $T_n \in \mathcal{L}(R^m, R^m)$ such that

$$S_1 \subset T_n^{-1}(U_{\alpha_n}) \subset S_\tau . \tag{1.1.1}$$

We will fix the family $\{T_n, n \in Z\}$ and we will consider, from now on, the pair (U_{α_n}, T_n). It is useful to point out that

$$\text{for } r > 0, \ 2^r U_{\alpha_{n-r}} \subset U_{\alpha_n}, \text{ or equivalently } 2^{-r} U_{\alpha_{n+r}} \supset U_{\alpha_n}. \tag{1.1.2}$$

Let a be a measurable function defined from R^m into $\mathcal{L}(H_1, H_2)$. We say that

$$a \in \mathcal{S}'(R^m, \mathcal{L}(H_1, H_2)) \text{ if } \langle a(x)[h_1], \ h_2 \rangle \in \mathcal{S}'(R^m),$$

the class of temperate distributions, for each $h_1 \in H_1$, $h_2 \in H_2$. When $a \in \mathcal{S}'(R^m, \mathcal{L}(H_1, H_2))$ we define $D^\alpha a$ (D^α denotes the α-partial derivative with respect to x) by the identity

$$\langle D^\alpha a(x)[h_1], h_2 \rangle = D^\alpha(\langle a(x)[h_1], h_2 \rangle) .$$

The following is one of our main multiplier theorems.

THEOREM (1.1). *Let β_j, $1 \leq j \leq m$, be positive integers such that $\sum_{j=1}^m 1/\beta_j < 2$, $\{(U_{\alpha_n}, T_n), \ n \in Z\}$ be a lacunary sequence of the family $\{U_\alpha, \Phi\}$, and let $a \in L^\infty(R^m, \mathcal{L}(H_1, H_2))$. Set $a_n(x) = a(T_n(x))$ and assume that*

$$\sup_{n,j} \int_{1/4 \leq |x| \leq 2\tau} |D_{x_j}^{\beta_j}(a_n)(x)[h]|_{H_2}^2 dx \leq C_0 |h|_{H_1}^2 . \tag{1.1.3}$$

Then the multiplier operator $T_a(f) = \mathcal{F}^{-1}(a\mathcal{F}(f))$ satisfies:

(i) $m(\{x, |T_a(f)|_{H_2} \geq 1\}) \leq C \int_{R^m} |f(x)|_{H_1} dx$

(ii) *when* $1 < p \leq 2$, $\|T_a(f)\|_{L^p(R^m, H_2)} \leq C_p \|f\|_{L^p(R^m, H_1)}$

where C and C_p depend only on C_0, the family $\{U_\alpha, \Phi\}$ and the L^∞ norm of a. (C_p, of course, depends also on p.)

Proof. To simplify our notation we set $U_{\alpha_n} = V_n$. Let $\psi \in C^\infty(R^m)$ be a non-negative function equal to one in $1/2 \leq |x| \leq \tau$ and having its support in $1/4 \leq |x| \leq 2\tau$. Set $\varphi(x) = \sum_{n=-\infty}^{\prime\infty} \psi(T_n^{-1}(x))$. From properties (1.1.1) and (1.1.2) it follows that when $x \in V_n \cap V'_{n-1}$

(a) $T_n^{-1}(x) \in \{y, 1/2 \leq |y| \leq \tau\}$ and

(b) for $r \geq 0$, $|T_{n+r}^{-1}(x)| \leq 2^{-r}\tau$ and $|T_{n-r}^{-1}(x)| \geq 2^r$.

Hence $1 \leq \varphi(x) = \sum_{(4\tau)^{-1} \leq 2^r \leq 2\tau} \psi(T_{n+r}^{-1}(x))$, and φ is a bounded, infinitely differentiable function. Set $b(x) = a(x)/\varphi(x)$, $b_n(x) = b(T_n(x))$ and $a^{(n)}(x) = b(x)\psi(T_n^{-1}(x))$. Observe that $\sum_{n \in Z} a^{(n)} = a$.

We will sketch the main steps in the proof.

We show that $\mathcal{F}(a^{(n)}) \in L^1(R^m, \mathcal{L}(H_1, H_2))$ and that the kernels $k_N = \sum_{n=-N}^N \mathcal{F}(a^{(n)})$ satisfy condition (4.1.1) of Chapter I, uniformly in N, for another family $\{\mathcal{U}_\alpha, \Phi'\}$ related to the original family $\{U_\alpha, \Phi\}$. Theorem (3.3) of Chapter I then completes the proof.

When $\gamma_j \leq \beta_j$

$$\int_{1/4 \leq |x| \leq 2\tau} |D_{x_j}^{\gamma_j}(a_n)(x)[h]|_{H_2}^2 dx \leq C\left[\int_{1/4 \leq |x| \leq 2\tau} |D_{x_j}^{\beta_j}(a_n)(x)[h]|_{H_2}^2 dx + \sup_x |a_n(x)[h]|_{H_2}^2\right].$$

Therefore, $D_{x_j}^{\gamma_j}(a_n)$ also verifies property (1.1.3). It then follows that

$$\sup_{n \in Z,\, \gamma_j \leq \beta_j} \int_{1/4 \leq |x| \leq 2\tau} |D_{x_j}^{\gamma_j}(b_n)(x)[h]|_{H_2}^2 dx \leq C|h|_{H_1}^2 \tag{1.1.4}$$

Since $a^{(n)}(T_n x) = b_n(x)\psi(x)$, from (1.1.4) it follows that

$$\sup_{n \in Z,\, \gamma_j \leq \beta_j} \int_{R^m} |D_{x_j}^{\gamma_j}(a^{(n)}(T_n x))[h]|_{H_2}^2 dx \leq C|h|_{H_1}^2 .$$

If we Fourier transform the left side we see that

$$\int_{R^m} |(x_j)^{\gamma_j} \mathcal{F}(a^{(n)})((T_n^*)^{-1}x)[h]|_{H_2}^2 (\det T_n)^{-2} dx =$$

$$= \int_{R_m} |(T_n^* x)_j^{\gamma_j} \mathcal{F}(a^{(n)})(x)[h]|_{H_2}^2 (\det T_n)^{-1} dx \leq C|h|_{H_1}^2 ,$$

where T_n^* denotes the adjoint β_j operator of T_n and $\det(T_n)$ the determinant of T_n. Hence if $P(x) = \sum_{j=1}^{\prime m} |x_j|^{\beta_j} + 1$,

$$\int_{R^m} |P(T_n^* x)\mathcal{F}(a^{(n)})(x)[h]|_{H_2}^2 dx \leq C \det(T_n)|h|_{H_1}^2 . \tag{1.1.5}$$

Without loss of generality we may assume that $\beta_1 \leq \beta_2 \leq \ldots \leq \beta_m$. Set $S_\lambda = (\lambda^{\beta_m/\beta_i}\delta_{ij})$ and let $r(x)$ be the metric associated to the group $\{S_\lambda, \lambda \in R_+\}$ (see section 7, Chapter I). Then $P(x) \geq C(1 + r(x))^{\beta_m}$.

If V_n^0 denotes the polar set of V_n, i.e. $V_n^0 = \{x\,;\, \langle x, y\rangle \leq 1,$ for every $y \in V_n\}$, set $W_n = V_{-n}^0$ and $\mathcal{U}_\alpha = W_n$ when $\alpha \in (2^{n-1}, 2^n]$. We will show that $\{\mathcal{U}_\alpha, 2\alpha\}$ is a regular Vitali family.

We point out that

(I) $A \subset B$ iff $B^0 \subset A^0$

and

(II) if T is an invertible operator, $T \in \mathcal{L}(R^m, R^m)$, $(T(A))^0 = (T^*)^{-1}(A^0)$.
To show that $(\mathcal{U}_\alpha, 2\alpha)$ is a regular Vitali family it suffices to show that

$$W_n - W_n = 2W_n \subset W_{n+1},$$

but since $2V_{-n-1} \subset V_{-n}$ the property is an immediate consequence of (II).

We also point out that $S_{\tau-1} \subset T_n^*(W_{-x}) \subset S_1$ and therefore

$$\text{if } k = n + r,\ r > 0, \text{ then } W_{-n+r} \supset 2^r W_{-n} \text{ and hence } T_n^*(W_k) \supset 2^r S_\tau - 1. \quad (1.1.6)$$

$$\text{If } k = -n + r,\ r < 0, \text{ then } W_{-n+r} \subset 2^r W_{-n} \text{ and hence } T_n^*(W_k) \subset 2^r S_1. \quad (1.1.7)$$

We are now ready to complete the proof. We go back to inequality (1.1.5). Observe that

$$\int_{R^m} P(x)^{-2} dx \le \int_{R^m} \frac{1}{(1 + r(x))^{2\beta_m}}\, dx = C_m \int_0^\infty \frac{r^{\beta_m(\Sigma 1/\beta_j)}}{(1 + r)^{2\beta_m}} \frac{dr}{r} < \infty$$

since $\sum_{j=1}^m 1/\beta_j < 2$. Using Cauchy's inequality and (1.1.5),

$$\int_{R^m} |\mathcal{F}(a^{(n)})(x)[h]|_{H_2} dx \le C|h|_{H_1} \quad (1.1.8)$$

and

$$\int_{(W_k)'} |\mathcal{F}(a^{(n)})(x)[h]|_{H_2} dx \le C|h|_{H_1} \Bigl(\int_{(T_n^*(W_k))'} P(x)^{-2} dx\Bigr)^{1/2}.$$

From (1.1.6) it follows that when $k + n = r > 0$, $T_n^*(W_k) \supset \{x, |x| \ge 2^r \tau^{-1}\}$. On the other hand when $|x| \ge 1$, $|x| \le r(x)^{\beta_m}$ and so

$$\int_{(W_k)'} |\mathcal{F}(a^{(n)})(x)[h]|_{H_2} dx \le C 2^{-\varepsilon r} |h|_{H_1}, \quad (1.1.9)$$

where

$$\varepsilon = \frac{1}{2}\,(2 - \sum_{j=1}^m 1/\beta_j) > 0.$$

To study the case $k + n = r < 0$, we set

$$h_y^{(n)}(x) = [\exp(2\pi i\langle x, y\rangle) - 1] a^{(n)}(x).$$

Then

$$h_y^{(n)}(T_n(x)) = [\exp(2\pi i\langle x, T_n^*(y)\rangle) - 1] b_n(x)\psi(x).$$

Hence

$$\sup_{|\alpha| \le M,\, n \in \mathbf{Z}} \int_{R^m} |D^\alpha(h_y^{(n)}(T(\cdot)))(x)[h]|_{H_2}^2 dx \le C|h|_{H_1}^2 |T_n^*(y)|.$$

Observe that $\mathcal{F}(h_y^{(n)})(x) = \mathcal{F}(a^{(n)})(x-y) - \mathcal{F}(a^{(n)})(x)$. Repeating the above argument, instead of (1.1.4) we obtain

$$\int_{R^m} |[\mathcal{F}(a^{(n)})(x-y) - \mathcal{F}(a^{(n)})(x)](h)|_{H_2} dx \leq C|h|_{H_1} |T_n^*(y)| . \tag{1.1.10}$$

When $r = k + n < 0$ and $y \in W_k$, (1.1.7) shows that $|T_n^*(y)| \leq 2^r \tau$. Therefore putting (1.1.9) and (1.1.10) together, when $y \in W_{k-1}$

$$\sum_{n \in Z} \int_{(W_k)'} |[\mathcal{F}(a^{(n)})(x-y) - \mathcal{F}(a^{(n)})(x)](h)| dx \leq C|h|_{H_1} . \tag{1.1.11}$$

Hence if $A_N(x) = \sum_{|n| \leq N} a^{(n)}(x)$, it follows from (1.1.8) that $\mathcal{F}(A_N)$ is an integrable kernel.

Since (1.1.11) implies

$$\int_{(W_k)'} |[\mathcal{F}(A_N)(x-y) - \mathcal{F}(A_N)(x)](h)|_{H_2} dx \leq C|h|_{H_1}$$

uniformly in N whenever $y \in W_{k-1}$, $T_{A_N}(f) = \mathcal{F}(A_N) * f$ satisfies the conditions of Remark (3.1) (remark to Theorem (3.3)), Chapter I, for the regular Vitali family $\{\mathcal{U}_\alpha, 2\alpha\}$.

Hence

$$m(\{x, |T_{A_N}(f)(x)|_{H_2} \geq 1\}) \leq C \int_{R_m} |f(x)|_{H_1} dx$$

where C is independent of N. To finish the proof it suffices to observe that for a good function f,

$$|T_{A_N}(f)(x) - T_a(f)(x)|_{H_2} \to 0 \text{ a.e., as } N \to \infty.$$

Note. The functions $\mathcal{F}(A_n)$ are actually singular kernels for the family $\{\mathcal{U}_\alpha, 2\alpha\}$ according to Definition (6.1) with bounds independent of N.

Let $a^*(x)$ denote the adjoint operator of $a(x)$.

THEOREM (1.2). *Let* $a(x)$ *and* $a^*(x)$ *satisfy condition* (1.1.1), *or the stronger condition*

$$\sup_{\beta_j, n \in Z} \int_{1/2 \leq |x| \leq 2\tau} \|D_{x_j}^{\beta_j}(a_n)(x)\|^2_{\mathcal{L}(H_1, H_2)} dx \leq C .$$

Then both T_a *and* T_{a^*} *are of weak type* L^1 *and bounded in* L^p *for all* p *such that* $1 < p < \infty$.

Proof. The proof follows by a simple duality argument. See Theorems (4.1) and (6.2), Chapter I.

Theorems (1.1) and (1.2) are an extension of the multiplier theorems presented in [9] and [12]. The reason for choosing a sequence $\beta_1, \ldots, \beta_m$ is to allow for multipliers which are not very smooth with respect to some variable provided, they are sufficiently smooth with respect to the others. In fact the theorems could be stated in terms of fractional derivatives. In such cases we may allow $\beta_1 = 1/2 + \varepsilon$ provided the rest of the β_j's are sufficiently large.

THEOREM (1.3). *Let* $H_1 = H_2 = L^2(R^k)$ *and* $g(x, \xi) \in L^\infty(R^m \times R^k)$. *For* $h \in L^2(R^k)$, *set* $a(x)[h] = \mathcal{F}_\xi^{-1}(g(x, \cdot)\mathcal{F}(h))$. *Assume that*

$$\sup_{n\in \mathbf{Z}, \beta_j, \xi \in R^k} \int_{1/4 \leq |x| \leq 2r} |D_{x_j}^{\beta_j}(g(T_n x, \xi))|^2 dx \leq C_0 ,$$

where T_n *and* β_j *are as in Theorem* (1.1). *For* $f \in L^p(R^m, H)$ *set* $T_a(f) = \mathcal{F}^{-1}(a\mathcal{F}(f))$.

Then T_a is of weak type L^1 and bounded in $L^p(R^m, H)$ for any $p, 1 < p < \infty$.

The theorem is an immediate consequence of Theorem (1.2).

Observe that if we identify $f \in L^p(R^m, H)$ with $\bar{f} \in X^p\xi^2(R^{m+k})$, then $T_a(\bar{f}) = \mathcal{F}_{x,\xi}(g\mathcal{F}_{x,\xi}(\bar{f}))$. We will present another theorem of a similar nature later in section 2 (Theorem (2.1)).

It has been pointed out in previous papers (see [2], [12] and [27]) that the Littlewood-Paley inequalities may be regarded as expressing the continuity of vector valued kernels. We will prove extended forms of those inequalities using Theorem (1.2). The advantage of the multiplier theorem is the fact that the operators are defined explicitely in terms of multiplier functions.

Let $r(x)$ be the homogeneous metric associated to a one parameter group of operators $\{L_\lambda, \lambda \in R_+\}$ (see section 7, Chapter I). Define $a(x) \in \mathcal{L}(C, H)$, where

$$H = \left\{ h; \; |h|_H = \left(\int_0^\infty |h(t)|^2 \frac{dt}{t} \right)^{1/2} < \infty \right\},$$

by $a(x)[c] = t \cdot r(x) \exp(-1/4t \cdot r(x)) \cdot c$. For $f \in L^p(R^m)$ set $T_a(f) = \mathcal{F}^{-1}(a\mathcal{F}(f))$ and $g(f) = |T_a(f)|_H$.

THEOREM (1.4). *There exists a constant* C_p *such that*

$$C_p^{-1}\|f\|_p \leq \|g(f)\|_p \leq C_p\|f\|_p \quad \textit{for every } p, 1 < p < \infty .$$

Proof. Choose $U_\alpha = L_\alpha(S_1)$, $\Phi(\alpha) = 2\alpha$, $T_n = L_{2^n}$ and $V_n = U_{2^n}$. A straightforward calculation shows that $a_n(x) = a(T_n x)$ verifies the condition of Theorem (1.2). Hence

$$\|T_a(f)\|_{L^p(R^m, H)} = \|g(f)\|_p \leq C_p\|f\|_p, \quad 1 < p < \infty .$$

To reverse the inequality it suffices to observe that

a) when $F \in L^p(R^m, H)$, $1 < p < \infty$, $\|T_{a*}(F)\|_{L^p(R^m, \mathbf{C})} \leq C_p\|F\|_{L^p(R^m, H)}$
b) when $f \in L^p(R^m)$, $T_{a*}(T_a(f)) = f$.

When $r(x) = |x|$, $g(f)$ is the classical g-function of a half space. With a similar technique we will derive inequalities concerning lacunary partitions of the spectra of L^p functions.

Let (V_n, T_n) be a lacunary sequence of a regular Vitali family and φ an infinitely differentiable function with compact support away from the origin. Let $l^2(H)$ denote the Hilbert space of square summable sequences with values in the Hilbert space H. Define $a(x) \in \mathcal{L}(H, l^2(H))$ by

$$a(x)[h] = \{\varphi(T_n^{-1}x)h, n \in \mathbf{Z}\}, \quad a \in L^\infty(R^m, \mathcal{L}(H, l^2(H))) .$$

THEOREM (1.5). *For $f \in L^p(R^m, H)$, set $T_a(f) = \mathcal{F}^{-1}(a\mathcal{F}(f))$. Then T_a is of weak type L^1 and bounded in L^p for $1 < p < \infty$. More explicitely if*

$$f_n = \mathcal{F}^{-1}(\varphi(T_n^{-1}x)\mathcal{F}(f)), \quad 1 < p < \infty ,$$
$$\|T_a(f)\|_{L^p(R^m, l^2(H))} = \|(\sum_{n\in\mathbf{Z}} |f_n(x)|_H^2)^{1/2}\|_p \leq C_p\|f\|_{L^p(R^m, H)} . \tag{1.5.1}$$

The proof is a straightforward consequence of Theorem (1.2).

Inequality (1.5.1) is an extension to R^m of a well-known inequality of Littlewood and Paley (see also [18] and [27]).

A different m-dimensional result can be obtained from iteration of the one dimensional form of inequality (1.5.1). (See also [18].)

Let $N = (n_1, \ldots, n_m)$, $n_j \in \mathbf{Z}$. Let φ be a differentiable function on the real line with compact support away from the origin. For $f \in L^p(R^m, H)$ set

$$f_N = \mathcal{F}^{-1}(\varphi(2^{n_1}x_1)\varphi(2^{n_2}x_2) \ldots \varphi(2^{n_m}x_m)\mathcal{F}(f)) .$$

THEOREM (1.6). *For all p with $1 < p < \infty$,*

$$\|\{f_N\}\|_{L^p(R^m, l^2(H))} \leq C\|f\|_{L^p(R^m, H)} .$$

Proof. The result follows by induction over the number of variables using the inequality (1.5.1). Assume the theorem is true in R^{m-1}. Set $\mathcal{H} = l^2(H)$, the Hilbert space of $(m - 1)$ tuples of square summable sequences with values in H. Let $\bar{N} = (n_1, \ldots, n_{m-1})$, then $f_N = (f_{\bar{N}})_{n_m}$. If we use the inequality (1.5.1) on $f_{\bar{N}} \in L^p(R^m, \mathcal{H})$ the theorem follows.

We can further extend Theorem (1.6) with the use of interpolation theory.

Let $P = (p_1, \ldots, p_m)$, $p_j \geq 1$. $X^P(R^m, H)$ will denote the Banach space of measurable function from R^m to the Hilbert space H, such that

$$\|f\|_{X^P(R^m, H)} = \left(\int_R \left(\cdots \left(\int_R |f(x)|_H^{p_m} dx_m\right)^{p_{m-1}/p_m} \cdots\right)^{p_1/p_2} dx_1\right)^{1/p_1} < \infty .$$

(See definition in Section I, Chapter I.)

THEOREM (1.7). *For* $f \in X^P(R^m, H)$, *set* f_N *as in Theorem* (1.3). *Then for* $1 < p_j < \infty$, $1 \leq j \leq m$

$$\|\{f_N\}\|_{X^P(R^m, l^2(H))} \leq C_P \|f\|_{X^P(R^m, H)} .$$

Proof. Let $P = (p, q)$. For $f \in X^P(R^2, H)$ set $T_i(f) = \{\mathcal{F}(\varphi(2^n x_i)\mathcal{F}(f)\}$, $i = 1, 2$. To prove the theorem, for this case, it suffices to show that T_i is a bounded operator from $X^P(R^2, H)$ into $X^P(R^2, l^2(H))$. The boundedness of T_2 is an immediate consequence of Theorem (1.5).

For the operator T_1 we argue as follows. Write

$$T_1(f)(x_1) = \int_R k(x_1 - y_1) f(y_1) dy_1 \quad \text{where } f(y_1) \in L^q(R, H)$$

and $k(x) \in \mathcal{L}(L^q(R, H), L^q(R, H))$ with $\|k(x)\|_{\mathcal{L}(L^q, L^q)} = \|k(x)\|_{\mathcal{L}(H, H)}$.

Observe that Theorem (1.1) shows that

$$\int_{|x_1| > 2\alpha} \|k(x_1 - y_1) - k(x_1)\|_{\mathcal{L}(H, H)} dx_1 \leq C \quad \text{when } |y_1| < \alpha .$$

Observe also that Theorem (1.5) implies the boundedness of the operator T_1 when $p = q$. Therefore; using Remark (3.1) (remark to Theorem (3.3)) of Chapter I, with $B_0 = B_1 = L^q(R, H)$, the boundedness of T_1 follows for all p, $1 < p \leq q$. To obtain the range $q \leq p < \infty$, it suffices to observe that, repeating the above argument, T_1^* is bounded from $X^P(R^2, l^2(H))$ into $X^P(R^2, H)$ for $1 < p \leq q$. The general case follows by induction.

J. Marcinkiewicz noted in [20] that inequalities of the type of Theorems (1.5) and (1.7) are useful in the study of multipliers that satisfy a variational condition. We will apply this principle to obtain some multiplier theorems.

Observe that if $a \in \mathcal{L}(H, H)$, and a is normal, i.e. $aa^* = a^*a$, then $a = u(aa^*)^{1/2} = (aa^*)^{1/2}u$ where u is a unitary transformation (see [24]). We denote $(aa^*)^{1/4}$ by $a^{\#}$.

Definition (1.1). Let $g(x)$ be a function from R^m into $\mathcal{L}(H, H)$. The variation of g is finite if there exist a sequence $\{g_n(x)\}$ and a constant M satisfying

1) $g_n(x) = \sum_{r=-\infty}^{\infty} a_{n,r}\chi_{Q_{n,r}}(x)$, where $a_{n,r}$ is a normal bounded operator in H and $Q_{n,r} = \{x, x_j \geq \lambda, \lambda$ depending on (j, n, r), $1 \leq j \leq m\}$.
2) For $h_r \in H$,

$$|\sum_{r=-\infty}^{\infty} a_{n,r}^{\#}(h_r)|_H^2 \le M \sum_{r=-\infty}^{\infty} |h_r|_H^2 .$$

3) $\|g_n(x) - g(x)\|_{\mathcal{L}(H,H)} \to 0$ as $n \to \infty$, for almost every $x \in R^m$.

We define $V_H(g)$ as the infimum of the constants M assumed in Definition (1.1). It is clear from the definition that $g \in L^\infty(R^m, \mathcal{L}(H_1, H_2))$ and $\|g\|_\infty \le V_H(g)$. The discussion of the variation will be left for the next section.

LEMMA (1.2). *If $\{a_r\}$ satisfies condition (2) of Definition (1.1), and $h \in H$, then*

$$\sum_{r=-\infty}^{\infty} |a_r^{\#}(h)|_H^2 \le M |h|_H^2 .$$

Proof.

$$\sum_{r=-\infty}^{\infty} |a_r^{\#}(h)|_H^2 = \langle \sum_{r=-\infty}^{\infty} a_r^{\#}(a_r^{\#}(h)), h\rangle \le |\sum_{r=-\infty}^{\infty} a_r^{\#}(a_r^{\#}(h))|_H |h|_H \le M^{1/2} (\sum_{r=-\infty}^{\infty} |a_r^{\#}(h)|_H^2)^{1/2} |h|_H ,$$

and the lemma follows.

Let P be a multi-index as in Theorem (1.7).

THEOREM (1.8). *Consider $g \in L^\infty(R^m, \mathcal{L}(H, H))$ such that $V_H(g) < \infty$. Define $T : X^P(R^m, H) \to X^P(R^m, H)$, by $T(f) = \mathcal{F}^{-1}(g\mathcal{F}(f))$. Then for $1 < p_i < \infty$, $1 \le i \le k$, T is a bounded operator and $\|T\| \le C_P V_H(g)$.*

The proof of the theorem makes use of the following lemma.

LEMMA (1.3). *If $\{f_r\} \in X^P(R^m, l^2(H))$ and $U(\{f_r\})$ is defined by $[U(\{f_r\})]_r = U_r(f_r) = \mathcal{F}^{-1}(\chi_{Q_r}\mathcal{F}(f_r))$, where Q_r is a set of the type used in Definition (1.1), then U is a bounded operator in $X^P(R^m, l^2(H))$ for $1 < p_j < \infty$, $1 \le j \le m$. Moreover, the norm of the operator U depends only on P.*

Proof. Assume by induction that the lemma is true in R^{m-1}. Let λ_r be the real number corresponding to x_1 in the definition of Q_r. Let $J = \{x, x_1 \ge 0\}$ and Q_r be the set defined by all the remaining variables $(x_2, \ldots, x_m)$.

Observe that if $g_r(x) = \exp(-2\pi i x_1\lambda_r) f_r(x)$, then

$$U_r(f_r)(x) = \exp(2\pi i x_1 \lambda_r)\mathcal{F}^{-1}(\chi_J \chi_{\tilde{Q}_r} \mathcal{F}(g_r))(x) .$$

Set $\mathcal{H}(f) = \mathcal{F}^{-1}(\chi_J \mathcal{F}(f))$ and $\tilde{U}_r(f) = \mathcal{F}^{-1}(\chi_{\tilde{Q}_r}\mathcal{F}(f))$. Observe that $\mathcal{H} = 1/2(I + \mathcal{K})$ where I is the identity operator and $\mathcal{K}$ is the vector valued Hilbert transform in the x_1 variable. More explicitely

$$\mathcal{K}(f)(x) = \lim_{\substack{\varepsilon \to 0 \\ R \to \infty}} \int_{\varepsilon \le |t| \le R} \frac{1}{t} f(x_1 - t, \ x_2, \ldots, x_m) dt .$$

Hence, using Theorem (6.2) of Chapter I, $\mathcal{K}$ and therefore $\mathcal{H}$ are bounded operators in $L^p(R^m, H)$. The interpolation argument of Theorem (1.7) shows that $\mathcal{H}$ is also bounded in $X^P(R^m, H)$. Therefore using the inductive hypothesis,

$$\|U(\{f_r\})\|_{X^P(R^m, l^2(H))} = \|(\sum_{r=-\infty}^{\infty} |U_r(f_r)(x)|_H^2)^{1/2}\|_{X^P(R^m)} = \|\mathcal{U}(\{\tilde{U}_r(g_r)\})\|_{X^P(R^m, l^2(H))} \leq$$
$$\leq C_p\|\{\tilde{U}_r(g_r)\}\|_{X^P(R^m, l^2(H))} \leq C_p'\|\{g_r\}\|_{X^P(R^m, l^2(H))} = C_p'\|\{f_r\}\|_{X^P(R^m, l^2(H))}.$$

Proof of Theorem (1.8). Let $g_n(x)$ be an element of the sequence approaching $g(x)$ as in Definition (1.1). $g_n(x) \sum a_{n,r}\chi_{Q_{n,r}}(x)$, and $a_{n,r} = a^{\#}_{n,r}a^{\#}_{n,r}u_r = a^{\#}_{n,r}u_r a^{\#}_{n,r}$. In virtue of condition (2) of Definition (1.1)

$$|\mathcal{F}^{-1}(g_n\mathcal{F}(f))|_H^2 = |\sum_{r=-\infty}^{\infty} a^{\#}_{n,r}(\mathcal{F}^{-1}(\chi_{Q_{n,r}}\mathcal{F}(u_r a^{\#}_{n,r}f)))|_H^2 \leq M\sum_{r=-\infty}^{\infty} |\mathcal{F}^{-1}(\chi_{Q_{n,r}}(a^{\#}_{n,r}f))|_H^2.$$

Let $[U_n(\{f_r\})]_r = \mathcal{F}^{-1}(\chi_{Q_{n,r}}\mathcal{F}(f_r))$. It follows from Lemmas (1.2) and (1.1) that

$$\|\mathcal{F}^{-1}(g_n\mathcal{F}(f))\|_{X^P(R^m, H)} \leq M^{1/2}\|U_n(\{a^{\#}_{n,r}(f)\})\|_{X^P(R^m, l^2(H))} \leq$$
$$\leq C_p M^{1/2}\|\{a^{\#}_{n,r}(f)\}\|_{X^P(R^m, l^2(H))} \leq C_P M\|f\|_{X^P(R^m, H)}.$$

To complete the proof we observe that $\|\mathcal{F}^{-1}((g_n - g)\mathcal{F}(f)\|_{X^P(R^m, H)} \to 0$ as $n \to \infty$ for good functions f.

Using Theorem (1.8) we can extend the inequality of Theorem (1.7) to the case $\varphi = \chi_I$, where $I = \{x, 1 \leq |x| < 2\}$. Set $I_N = \{x, 2^{n_j} \leq |x_j| < 2^{n_j+1}, n_j \in Z, 1 \leq j \leq m\}$ and $f_N = \mathcal{F}^{-1}(\chi_{I_N}\mathcal{F}(f))$.

THEOREM (1.9) (Littlewood-Paley). *For* $f \in X^P(R^m, H)$, $1 < p_j < \infty$,

$$\|f\|_{X^P(R^m, H)} \leq C_P\|\{f_N\}\|_{X^P(R^m, l^2(H))} \leq C_P'\|f\|_{X^P(R^m, H)}.$$

Proof. Let φ be any differentiable function with compact support away from the origin such that $\varphi(x) = 1$ for $x \in I$. For $f \in X^P(R^m, H)$, set

$$g_N = \mathcal{F}^{-1}(\varphi(2^{n_1}x_1)\ldots\varphi(2^{n_m}x_m)\mathfrak{F}(f))$$

and $T(f) = \{g_N\}$. Theorem (1.7) shows that T is a continuous mapping from $X^P(R^m, H)$ into $X^P(R^m, l^2(H))$. Let $\{h_N\} \in X^P(R^m, l^2(H))$ and set

$$S(\{h_N\}) = \{\mathcal{F}^{-1}(\chi_{I_N}\mathcal{F}(h_N))\} = \mathcal{F}^{-1}(a\mathcal{F}(\{h_N\})),$$

where $a(x) \in \mathcal{L}(l^2(H), l^2(H))$ is defined by the identity

$$a(x)(\{C_N\}) = \{\chi_{IN}(x)C_N\}.$$

It is easy to verify that $V_{l^2(H)}(a) = 2^m$. Therefore Theorem (1.8) shows that S is a bounded operator in $X^P(R^m, l^2(H))$. Hence $T \circ S$ is a continuous operator from $X^P(R^m, H)$ into $X^P(R^m, l^2(H))$. But $T \circ S(f) = S(\{g_N\}) = \{f_N\}$, that is

$$\|\{f_N\}\|_{X^P(R^m, l^2(H))} \leq C_P\|f\|_{X^P(R^m, H)}$$

On the other hand, if T^* denotes the adjoint operator of T, T^* is a continuous operator from $X^P(R^m, l^2(H))$ into $X^P(R^m, H)$ and $T^*(\{f_N\}) = f$. Therefore the theorem follows.

With the aid of Theorem (1.9) we can now improve Theorem (1.8).

THEOREM (1.10). (The multiplier theorem.)
Let $T : X^P(R^m, H) \to X^P(R^m, H)$ be defined as $T(f) = \mathcal{F}^{-1}(g\mathcal{F}(f))$. For the multi-index N, let I_N be defined as in Theorem (1.9). Then if $1 < p_j < \infty$, $1 \le j \le m$ and $\|T\|$ denotes the norm of T as an operator in $X^P(R^m, H)$; we have

$$\|T\| \le C_P \sup_N V_H(\chi_{I_N} g) .$$

Proof. Define $G : R^m \to \mathcal{L}(l^2(H), l^2(H))$ by $G(x)(\{h_N\}) = \{\chi_{I_N} g(x)h_N\}$, and for $F \in X^P(R^m, l^2(H))$ set $S(F) = \mathcal{F}^{-1}(G\mathcal{F}(F))$. In virtue of Theorems (1.8) and (1.9), if $F = \{\mathcal{F}^{-1}(\chi_{I_N}\mathcal{F}(F))\}$, then

$$\begin{aligned}\|T(f)\|_{X^P(R^m, H)} &\le C_P \|(\sum_N |\mathcal{F}^{-1}(\chi_{I_N} g\mathcal{F}(f))|_H^2)^{1/2}\|_{X^P(R^m)} = \\ &= C_P\|S(F)\|_{X^P(R^m, l^2(H))} \le C'_P V_{l^2(H)}(G)\|F\|_{X^P(R^m, l^2(H))} \le \\ &\le C''_P V_{l^2(H)}(G)\|f\|_{X^P(R^m, H)} .\end{aligned}$$

But it is easy to verify that $V_{l^2(H)}(G) \le \sup_N V_H(\chi_{I_N} g)$, hence the theorem follows.

2. Multipliers in R^m and applications

In this section we consider some applications of Theorem (1.7) when $H = C$ and when $H = L^2(R^k)$.

For the case $H = \mathbf{C}$, the second condition of the definition of functions of finite variation becomes $\sum_{r=-\infty}^{\infty} |a_{n,r}| \le M$. Therefore it coincides with the definition of variation given in [18]. Theorem (1.10) extends Theorem (3.3) of [18] to the X^P spaces. We point out that in this case the notion of finite variation is equivalent to the classical notion of bounded variation in R^m. More precisely if g is a function of bounded support, then $V_C(g) = V(g) < \infty$ if and only if

$$\frac{\partial^m g}{\partial x_1 \ldots \partial x_m} = \mu ,$$

where μ is a finite measure. (The identity should be understood in the sense of distributions) also $|\mu|(R^m) = V(g)$. For details see [16].

When $H = L^2(R^k)$, we restrict our attention to mappings $g : R^m \to \mathcal{L}(L^2(R^k), L^2(R^k))$ defined by a function $g(x, \xi)$, $\xi \in R^k$ as follows.

For $h \in L^2(R^m)$, $g(x)(h) = \mathcal{F}^{-1}(g(x, \cdot)\mathcal{F}(h))$. $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier and the inverse Fourier transforms in $L^2(R^k)$.

LEMMA (2.1). *Let g be defined as above $(H = L^2(R^k))$. Then*

$$V_H(g) \le \sup_{\xi} V_C(g(\cdot, \xi)) .$$

Proof. If $V_C(g(\cdot, \xi)) < M$, for almost every ξ, a uniform approximation argument shows that there exists a sequence $\{g_n(x, \xi)\}$ such that

1) $g_n(x, \xi) = \sum_{r=-\infty}^{\infty} a_{n,r}(\xi)\chi_{Q_{n,r}}(x)$,
2) $\sum_{r=-\infty}^{\infty} |a_{n,r}(\xi)| < M$,
3) $g_n(x, \xi) \to g(x, \xi)$ as $n \to \infty$ almost everywhere in R^{m+k}.

For $h \in L^2(R^k)$ define $A_{n,r}(h) = \mathcal{F}^{-1}(a_{n,r}\mathcal{F}(h))$. Clearly $A_{n,r}$ is a normal operator and $A^{\#}_{n,r}(h) = \mathcal{F}^{-1}(|a_{n,r}|^{1/2}\mathcal{F}(h))$. Set $g_n(x) = \sum_{r=-\infty}^{\infty} A_{n,r}\chi_{Q_{n,r}}(x)$. Observe that

$$\| \sum_{r=-\infty}^{\infty} A^{\#}_{n,r}(h_r)\|^2_{L^2(R^k)} = \| \sum_{r=-\infty}^{\infty} |a_{n,r}(\cdot)|^{1/2}\mathcal{F}(h_r)(\cdot)\|^2_{L^2(R^k)} \leq$$

$$\leq \| \sum_{r=-\infty}^{\infty} |a_{n,r}(\cdot)|^{1/2}\|^2_{L^\infty(R^k)} \cdot \|(\sum_{r=-\infty}^{\infty} |\mathcal{F}(h_r)|^2)^{1/2}\|^2_{L^2(R^k)} \leq M \sum_{r=-\infty}^{\infty} \|h_r\|^2_{L^2(R^k)} .$$

Therefore to complete the proof of the lemma it suffices to show that $\|g_n(x) - g(x)\|_{\mathcal{L}(H,H)} \to 0$, as $n \to \infty$, for almost every $x \in R^m$. But

$$\|g_n\|_{L^\infty(R^{m+k})} \leq \sup_{\xi} V_C(g_n(\cdot, \xi)) \leq M ,$$

and property (3) implies that for almost each $x \in R^m$, $g_n(x, \xi) \to g(x, \xi)$ a.e. in R^k. Hence $\|g_n(x) - g(x)\|_{\mathcal{L}(H,H)} \to 0$, and the lemma follows.

In virtue of Lemma (2.1) we can reannounce Theorem (1.10) for this case as follows. Let N, I_N be defined as in Theorem (1.10), $x \in R^m$, $\xi \in R^k$.

THEOREM (2.1). *For $f \in X^P(R^m, L^2(R^k))$ define*

$$T(f) = \mathcal{F}^{-1}(g\mathcal{F}(f)) \text{ where } g(x)(h) = \mathcal{F}_{\xi}^{-1}(g(x, .)\mathcal{F}_{\xi}(h)),\ (h \in L^2(R^k)) .$$

In other words $T(f) = \mathcal{F}_x^{-1}\mathcal{F}_{\xi}^{-1}(g(. , .)\mathcal{F}_x\mathcal{F}_{\xi}(f)$. Then for $1 < p_j < \infty$, if $\|T\|$ denotes the norm of the operator T in $X^P(R^m, L^2(R^k))$,

$$\|T\| \leq C_P . \sup_{\xi, N} V(\chi_{I_N} g(. , \xi)) .$$

When $m = 1$, we obtain the following applications of Theorem (2.1).

THEOREM (2.2). *Let S be a convex set of $R \times R^k$ $(x \in R, \xi \in R^k)$. Define $T(f) = \mathcal{F}^{-1}(\chi_S\mathcal{F}(f))$, where the Fourier transforms are taken in both variables. Then T is a bounded operator in $X^p\xi^2(R \times R^k)$ for every p, such that $1 < p < \infty$.*

Proof. In virtue of Theorem (2.1) it is enough to observe that $V(g(. , \xi)) \leq 2$.

THEOREM (2.3). *Let $P(x, \xi)$, $Q(x, \xi)$ be two polynomials in the variable x $(x \in R$, $\xi \in R^k)$ with degress n_1 and n_2 respectively. Assume that $|P(x, \xi)/Q(x, \xi)| \leq M$ for every (x, ξ). Set $T(f) = \mathcal{F}^{-1}(P/Q\mathcal{F}(f))$ (the Fourier transforms are taken in* *both variables). Then T is a bounded operator in $X^p\xi^2(R\times R^k)$ for every p such that $1 < p < \infty$. Moreover $\|T\| \leq C_p(n_1 + n_2)M$.*

Proof. Assume P and Q are real valued. Then

$$V(P(\,.\,,\xi)/Q(\,.\,,\xi))\chi_{[a,b]}) \leq \int_a^b \left|\left(\frac{P}{Q}\right)'(x,\xi)\right| dx\,.$$

Note that $(P/Q)'$ changes sign only when $P'Q + PQ' = 0$. Let $x_j(\xi)$ $(1 \leq j \leq n_1 + n_2 - 1)$ be the zeros of $P'Q + PQ'$. Then

$$\int_a^b \left|\left(\frac{P}{Q}\right)'(x,\xi)\right| dx \leq \left|\left(\frac{P}{Q}\right)(a,\xi)\right| + \left|\left(\frac{P}{Q}\right)(b,\xi)\right| + 2\sum_{j=1}^{n_1+n_2-1}\left|\frac{P}{Q}(x_j(\xi),\xi)\right| \leq 3(n_1+n_2)M\,.$$

When P and Q are complex valued, taking real and imaginary parts reduces the problem of the previous case.

Theorems (2.2) and (2.3) are the best possible of their kind. More precisely, when $S = \{(x,\xi)\,;\,x^2 + |\xi|^2 \leq 1\}$ and $T(f) = \mathcal{F}^{-1}(\chi_S\mathcal{F}(f))$, if T is a bounded operator in $X^p\xi^q(R\times R^k)$ for all p, such that $1 < p < \infty$, then $q = 2$. (See [11].) Similarly if $Q(x,\xi) = |\xi|^2 - x + i$ and $T(f) = \mathcal{F}^{-1}(1/Q\mathcal{F}(f))$ the same observation is valid. (See [19].) In both cases the papers quoted prove that T is not a bounded operator in $L^p(R^{k+1})$ for either $p < 2(k+1)/k$; however the main estimates obtained in [11] and [19] imply our remark.

With the use of interpolation theory, Theorems (2.2) and (2.3) can be extended to classes of functions related to $L^p(R^{k+1})$ where p ranges over $(2(k+1)/k + 2, 2(k+1)/k)$. (See Theorem (2.5).)

In our use of the Riesz theory of interpolation we follow the notation introduced in [4].

Let P be the multi-index $(p_1, \ldots, p_m)$. With $X^P(jk)(R^m)$ we denote the space of mixed norms where the j^{th} and the k^{th} variables with their p_j and p_k norms have been permuted.

LEMMA (2.2). *Assume that $j \leq k$. If $1 \leq p_j \leq p_k$, then $X^P(R^m) \supset X^P(jk)(R^m)$ and moreover $\|f\|_{X^P} \leq \|f\|_{X^{P}(jk)}$.*

Proof. When $p_j = p_k$ the lemma is trivial. When $p_k = 1$ the lemma is a consequence of Minkowski's inequality. The general case follows by interpolation.

Given two Banach spaces B_0 and B_1 we denote $[B_0, B_1]_\alpha$, $0 \leq \alpha \leq 1$, the α-intermediate space for the Riesz interpolator of the pair (B_0, B_1).

Set $B_1^{(p)}(R^m) = X_1^pX_2^2\ldots X_m^2(R^m)$, and by induction

$$B_{j+1}^{(p)}(R^m) = [B_j^{(p)}(R^m)\,;\,X_{j+1}^pX_1^2\ldots X_m^2(R^m)]_{j/(j+1)}\,.$$

THEOREM (2.4). *Define* $Q = (q_1, \ldots, q_m)$ *by*

$$j/q_k = 1/p + (j-1)/2 \text{ when } 1 \le k \le j, \text{ and } q_k = 2 \text{ for } j < k \le m .$$

Then if

$$1 \le p \le 2 \quad (2j/j+1 \le q_k \le 2, \text{ for } 1 \le k \le j)$$

we have $X^Q(R^m) \supset B_j^{(p)}(R^m)$. *Moreover* $\|f\|_{X^Q} \le \|f\|_{B_j^{(p)}}$. *If*

$$2 \le p \le \infty \, (2 \le q_k \le 2j/j-1, \text{ for } 1 \le k \le j) ,$$

then $X^Q(R^m) \subset B_j^{(p)}(R^m)$ *and* $\|f\|_{B_j^{(p)}} \le \|f\|_{X^Q}$.

Proof. We argue inductively. The result is trivial when $j = 1$. If, say, $1 \le p \le 2$, set $q_j = 2jp/(j+1)$. In virtue of Lemma (2.2) and interpolation theory

$$\begin{aligned} &B_{j+1}^{(p)}(R^m) = [B_j^{(p)}(R^m) \,;\, X_{j+1}^p X_1^2 \ldots X_m^2(R^m)]_{j/(j+1)} \subset \\ &\subset [X_1^{q_j} \ldots X_j^{q_j} X_{j+1}^2 \ldots X_m^2(R^m) \,;\, X_1^2 \ldots X_j^2 X_{j+1}^p \ldots X_m^2(R^m)]_{j/(j+1)} = \\ &= X_1^{q_{j+1}} \ldots X_{j+1}^{q_{j+1}} X_{j+2}^2 \ldots X_m^2(R^m) . \end{aligned}$$

The second part of the theorem follows with a similar argument.

To make the notation more adequate with the results of Theorem (2.4) set $B_p(R^m) = B_{m+1}^{(q)}(R^m)$, where $1/p = m/2 - (m+1)/q$. With such notation, Theorems (2.2), (2.3) and (2.4) imply

THEOREM (2.5). *Let* $T(f) = \mathcal{F}(g\mathcal{F}(f))$, *where* $g(x)$ *is the characteristic function of a convex set (as in Theorem* (2.2)*) or the bounded ratio of two polynomials (as in Theorem* (2.3)*). Then*

(i) *For* $\dfrac{2m}{m+1} < p \le 2$, $\|T(f)\|_{L^p(R^m)} \le \|T(f)\|_{B_p(R^m)} \le C_p \|f\|_{B_p(R^m)}$,

(ii) *For* $2 \le p < \dfrac{2m}{m-1}$, $\|T(f)\|_{B_p(R^m)} \le C_p \|f\|_{B_p(R^m)} \le C_p \|f\|_{L^p(R^m)}$.

The theorem is a simple application of the theory of interpolation.

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