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A NEW LOOK AT THE CROSSED PRODUCTS OF PRO-C^*-ALGEBRAS

MARIA JOIŢA

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Abstract. We give a new definition for the full crossed product, respectively reduced crossed product, of a pro-C^*-algebra $A[\tau_\Gamma]$ by an action α and, using these new definitions, we investigate some of their properties.

1. Introduction

Given a C^*-algebra A and a continuous action α of a locally compact group G on A, we can construct a new C^*-algebra, called the crossed product of A by α, usually denoted by $G \times_\alpha A$, and which contains, in some subtle sense, A and G. The origin of this construction goes back to Murray and von Neumann and their group measure space construction by which they associated a von Neumann algebra to a countable group acting on a measure space. The analog of this construction for the case of C^*-algebras is due to Gelfand with co-authors Naimark and Fomin. There is a vast literature on crossed products of C^*-algebras (see, for example, [W]), but the corresponding theory in the context of non-normed topological $*$-algebras has still a long way to go.

Crossed product of pro-C^*-algebras by inverse limit actions of locally compact groups were considered by Phillips [P2] and Joiţa [J2, J3, J4]. If $A[\tau_\Gamma]$ is a pro-C^*-algebra with topology given by the family of C^*-seminorms $\Gamma = \{p_\lambda\}_{\lambda\in\Lambda}$, then $A[\tau_\Gamma]$ can be identified with an inverse limit of C^*-algebras $\lim\limits_{\leftarrow\lambda} A_\lambda$ (the Arens–Michael decomposition of $A[\tau_\Gamma]$), and if α is an inverse limit action of a locally compact group G on $A[\tau_\Gamma]$, then $\alpha_t = \lim\limits_{\leftarrow\lambda} \alpha_t^\lambda$ for all $t \in G$, where for each $\lambda \in \Lambda$, α^λ is an action of G on the C^*-algebra A_λ. In [P2], the full (reduced) crossed product of $A[\tau_\Gamma]$ by α is defined as inverse limit of the full (reduced) crossed

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products of A_λ by α^λ, $\lambda \in \Lambda$. In particular, for a given inverse limit automorphism α of a pro-C^*-algebra $A\left[\tau_\Gamma\right]$, we can associate to the pair $\left(A\left[\tau_\Gamma\right], \alpha\right)$ a pro-C^*-algebra by the above crossed product construction, but if α is not an inverse limit automorphism, this construction is not possible. In the case of C^*-algebras, the crossed product of a C^*-algebra A by an action α is isomorphic to the enveloping C^*-algebra of the covariance algebra $L^1(G, \alpha, A)$. If α is an inverse limit action of G on $A\left[\tau_\Gamma\right]$, then the covariance algebra $L^1(G, \alpha, A\left[\tau_\Gamma\right])$ has a structure of locally m-convex $*$-algebra with topology given by the family of submultiplicative seminorms $\{N_{p_\lambda}\}_{\lambda\in\Lambda}$, where

$$N_{p_\lambda}\left(f\right) = \int\limits_G p_\lambda\left(f\left(g\right)\right) dg,$$

and the enveloping pro-C^*-algebra of $L^1(G, \alpha, A\left[\tau_\Gamma\right])$ can be identified with the inverse limit of the enveloping C^*-algebras of the covariance algebras $L^1(G, \alpha^\lambda, A_\lambda)$. Therefore, the full crossed product of $A\left[\tau_\Gamma\right]$ by α is isomorphic to the enveloping pro-C^*-algebra of the covariance algebra $L^1(G, \alpha, A\left[\tau_\Gamma\right])$. If α is not an inverse limit action, then the covariance algebra has not a structure of locally m-convex $*$-algebra (N_{p_λ} is not a submultiplicative $*$-seminorm). We remark that the above definition of the full crossed product of a pro-C^*-algebra $A\left[\tau_\Gamma\right]$ by an inverse limit action depends of the Arens–Michael decomposition of $A\left[\tau_\Gamma\right]$, and so it is not good to define the notion of full crossed product of a pro-C^*-algebra $A\left[\tau_\Gamma\right]$ by an action which is not an inverse limit action. It is well known that the full crossed product of C^*-algebras is a universal object for nondegenerate covariant representations (see, for example, [R]). The full crossed product of pro-C^*-algebras by inverse limit actions has also the universal property with respect to the nondegenerate covariant representations [J3]. In this paper, we define the full crossed product of a pro-C^*-algebra $A\left[\tau_\Gamma\right]$ by an action α of a locally compact group G as a universal object for nondegenerate covariant representations and we show that the full crossed product of pro-C^*-algebras exists for strong bounded actions. Strong boundless of the action α is essential to prove the existence of a covariant representation. Unfortunately, if the action α of G on $A\left[\tau_\Gamma\right]$ is strongly bounded, then there is another family of C^*-seminorms on $A\left[\tau_\Gamma\right]$ which induces the same topology on A, and α is an inverse limit action with respect to this family of C^*-seminorms.

The organization of this paper is as follows. After preliminaries in Section 2, we present some examples of group actions on pro-C^*-algebras in Section 3. In Section 4, we show that for a strong bounded action α of a locally compact group G on a pro-C^*-algebra $A\left[\tau_\Gamma\right]$ there is an injective covariant morphism from $A\left[\tau_\Gamma\right]$ to the pro-C^*-algebra $\mathcal{L}(\mathcal{H})$ for some locally Hilbert space $\mathcal{H}$. In Section 5, the full pro-C^*-crossed product of $A\left[\tau_\Gamma\right]$ by α is defined to be the pro-C^*-algebra $G \times_\alpha A\left[\tau_\Gamma\right]$ generated by the images of ι_A and ι_G, where (ι_A, ι_G) is a universal covariant morphism of $A\left[\tau_\Gamma\right]$, in the sense that for any covariant morphism (j_A, j_G) from $A\left[\tau_\Gamma\right]$ to a pro-C^*-algebra $B\left[\tau_{\Gamma'}\right]$, there is a unique pro-$C^*$-morphism $\Phi : G \times_\alpha A\left[\tau_\Gamma\right] \to B\left[\tau_{\Gamma'}\right]$ such that $\Phi \circ \iota_A = j_A$ and $\Phi \circ \iota_G = j_G$. For inverse limit actions, this definition coincides with the definition from

[P2, J2]. We show that the full pro-C^*-crossed product of $A[\tau_\Gamma]$ by α exists if α is strongly bounded and it is invariant under the conjugacy of the actions. In Section 6, the reduced pro-C^*-crossed product of a pro-C^*-algebra $A[\tau_\Gamma]$ by a strong bounded action α is defined to be the pro-C^*-subalgebra of the multiplier algebra $M(A[\tau_\Gamma]\otimes_{\min}\mathcal{K}(L^2(G)))$ of the minimal tensor product of $A[\tau_\Gamma]$ and $\mathcal{K}(L^2(G))$ generated by $\{\widetilde{\alpha}(a)(1\otimes\lambda_G(f));a\in A,\ f\in C_c(G)\}$, where $\widetilde{\alpha}$ is the pro-C^*-morphism from $A[\tau_\Gamma]$ to $M(A[\tau_\Gamma]\otimes_{\min}C_0(G))$ induced by α. We show that, for inverse limit actions, this definition coincides with the definition from [P2, J2]. Also, we show that the reduced pro-C^*-crossed product is invariant under the conjugacy of the actions, and if G is amenable, then the full pro-C^*-crossed product of $A[\tau_\Gamma]$ by α is isomorphic to the reduced pro-C^*-crossed product of $A[\tau_\Gamma]$ by α. Section 7 is dedicated the relation between the full pro-C^*-crossed product and the maximal tensor product of pro-C^*-algebras, respectively the reduced pro-C^*-crossed product and the minimal tensor product of pro-C^*-algebras. We show that there is a property of "associativity" between $\times_\alpha$ and $\otimes_{\max}$, respectively $\times_{\alpha,r}$ and $\otimes_{\min}$.

2. Preliminaries

A seminorm p on a topological $*$-algebra A satisfies the C^*-condition (or is a C^*-seminorm) if $p(a^*a)=p(a)^2$ for all $a\in A$. It is known that such a seminorm must be submultiplicative ($p(ab)\leq p(a)p(b)$ for all $a,b\in A$) and $*$-preserving ($p(a^*)=p(a)$ for all $a\in A$).

A pro-C^*-algebra is a complete Hausdorff topological $*$-algebra A whose topology is given by a directed family of C^*-seminorms $\{p_\lambda\}_{\lambda\in\Lambda}$. Other terms used for pro-C^*-algebras are: locally C^*-algebras (A. Inoue, M. Fragoulopoulou, A. Mallios, etc.), LMC^*-algebras (G. Lassner, K. Schmüdgen), b^*-algebras (C. Apostol).

Let $A[\tau_\Gamma]$ be a pro-C^*-algebra with topology given by $\Gamma=\{p_\lambda\}_{\lambda\in\Lambda}$ and let $B[\tau_{\Gamma'}]$ be a pro-C^*-algebra with topology given by $\Gamma'=\{q_\delta\}_{\delta\in\Delta}$. A continuous $*$-morphism $\varphi:A[\tau_\Gamma]\to B[\tau_{\Gamma'}]$ (that is, φ is linear, $\varphi(ab)=\varphi(a)\varphi(b)$ for all $a,b\in A$, $\varphi(a^*)=\varphi(a)^*$ for all $a\in A$ and for each $q_\delta\in\Gamma'$, there is $p_\lambda\in\Gamma$ such that $q_\delta(\varphi(a))\leq p_\lambda(a)$ for all $a\in A$) is called a pro-C^*-morphism. Two pro-C^*-algebras $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$ are isomorphic if there is a pro-C^*-isomorphism $\varphi:A[\tau_\Gamma]\to B[\tau_{\Gamma'}]$ (that is, φ is invertible, φ and φ^{-1} are pro-C^*-morphisms).

If $\{A_\lambda;\pi_{\lambda\mu}\}_{\lambda\geq\mu,\lambda,\mu\in\Lambda}$ is an inverse system of C^*-algebras, then $\lim\limits_{\leftarrow\lambda}A_\lambda$ with topology given by the family of C^*-seminorms $\{p_\lambda\}_{\lambda\in\Lambda}$, with $p_\lambda\left((a_\mu)_{\mu\in\Lambda}\right)=\|a_\lambda\|_{A_\lambda}$ for all $\lambda\in\Lambda$, is a pro-C^*-algebra.

Let $A[\tau_\Gamma]$ be a pro-C^*-algebra with topology given by $\Gamma=\{p_\lambda\}_{\lambda\in\Lambda}$. For $\lambda\in\Lambda$, $\ker p_\lambda$ is a closed $*$-bilateral ideal and $A_\lambda=A/\ker p_\lambda$ is a C^*-algebra in the C^*-norm $\|\cdot\|_{p_\lambda}$ induced by p_λ (that is, $\|a\|_{p_\lambda}=p_\lambda(a)$, for all $a\in A$). The canonical map from A to A_λ is denoted by π_λ^A, $\pi_\lambda^A(a)=a+\ker p_\lambda$ for all $a\in A$. For $\lambda,\mu\in\Lambda$ with $\mu\leq\lambda$ there is a unique surjective C^*-morphism $\pi_{\lambda\mu}^A:A_\lambda\to A_\mu$ such that $\pi_{\lambda\mu}^A(a+\ker p_\lambda)=a+\ker p_\mu$, and then $\{A_\lambda;\pi_{\lambda\mu}^A\}_{\lambda,\mu\in\Lambda}$ is an inverse system of

C^*-algebras. Moreover, pro-C^*-algebras $A[\tau_\Gamma]$ and $\lim\limits_{\leftarrow\lambda} A_\lambda$ are isomorphic (the Arens–Michael decomposition of $A[\tau_\Gamma]$).

Let $\{(\mathcal{H}_\lambda, \langle\cdot,\cdot\rangle_\lambda)\}_{\lambda\in\Lambda}$ be a family of Hilbert spaces such that $\mathcal{H}_\mu \subseteq \mathcal{H}_\lambda$ and $\langle\cdot,\cdot\rangle_\lambda |_{\mathcal{H}_\mu} = \langle\cdot,\cdot\rangle_\mu$ for all $\lambda,\mu\in\Lambda$ with $\mu\le\lambda$. $\mathcal{H} = \lim\limits_{\lambda\rightarrow}\mathcal{H}_\lambda$ with inductive limit topology is called a locally Hilbert space.

Let $\mathcal{L}(\mathcal{H}) = \{T : \mathcal{H}\to\mathcal{H}; T_\lambda = T|_{\mathcal{H}_\lambda} \in L(\mathcal{H}_\lambda)$ and $P_{\lambda\mu}T_\lambda = T_\lambda P_{\lambda\mu}$ for all $\lambda,\mu\in\Lambda$ with $\mu\le\lambda\}$, where $P_{\lambda\mu}$ is the projection of $\mathcal{H}_\lambda$ on $\mathcal{H}_\mu$. Clearly, $\mathcal{L}(\mathcal{H})$ is an algebra in an obvious way, and $T\to T^*$ with $T^*|_{\mathcal{H}_\lambda} = (T_\lambda)^*$ for all $\lambda\in\Lambda$ is an involution.

For each $\lambda\in\Lambda$, the map $p_{\lambda,\mathcal{L}(\mathcal{H})} : \mathcal{L}(\mathcal{H}) \to [0,\infty)$ given by $p_{\lambda,\mathcal{L}(\mathcal{H})}(T) = \|T|_{\mathcal{H}_\lambda}\|_{L(\mathcal{H}_\lambda)}$ is a C^*-seminorm on $\mathcal{L}(\mathcal{H})$, and with topology given by the family of C^*-seminorms $\{p_{\lambda,\mathcal{L}(\mathcal{H})}\}_{\lambda\in\Lambda}$, $\mathcal{L}(\mathcal{H})$ becomes a pro-C^*-algebra.

Since $\mathcal{L}(\mathcal{H})$ is a pro-C^*-algebra, it has an Arens–Michael decomposition, given by the C^*-algebras $\mathcal{L}(\mathcal{H})_\lambda = \mathcal{L}(\mathcal{H})/\ker p_{\lambda,\mathcal{L}(\mathcal{H})}$, $\lambda\in\Lambda$. Moreover, for each $\lambda\in\Lambda$, the map $\varphi_\lambda : \mathcal{L}(\mathcal{H})_\lambda \to L(\mathcal{H}_\lambda)$ given by $\varphi_\lambda\left(T+\ker p_{\lambda,\mathcal{L}(\mathcal{H})}\right) = T|_{\mathcal{H}_\lambda}$ is an isometric $*$-morphism. The canonical maps from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H})_\lambda, \lambda\in\Lambda$ are denoted by $\pi_\lambda^{\mathcal{H}}, \lambda\in\Lambda$, and $\pi_\lambda^{\mathcal{H}}(T) = T|_{\mathcal{H}_\lambda}$. For a given pro-$C^*$-algebra $A[\tau_\Gamma]$ there is a locally Hilbert space $\mathcal{H}$ such that $A[\tau_\Gamma]$ is isomorphic to a pro-C^*-subalgebra of $\mathcal{L}(\mathcal{H})$ (see [I, Theorem 5.1]).

A multiplier of $A[\tau_\Gamma]$ is a pair (l,r) of linear maps $l,r : A[\tau_\Gamma]\to A[\tau_\Gamma]$ such that are respectively left and right A-module homomorphisms and $r(a)b = al(b)$ for all $a,b\in A$. The set $M(A[\tau_\Gamma])$ of all multipliers of $A[\tau_\Gamma]$ is a pro-C^*-algebra with multiplication given by $(l_1,r_1)(l_2,r_2) = (l_1l_2, r_2r_1)$, the involution given by $(l,r)^* = (r^*,l^*)$, where $r^*(a) = r(a^*)^*$ and $l^*(a) = l(a^*)^*$ for all $a\in A$, and the topology given by the family of C^*-seminorms $\{p_{\lambda,M(A[\tau_\Gamma])}\}_{\lambda\in\Lambda}$, where $p_{\lambda,M(A[\tau_\Gamma])}(l,r) = \sup\{p_\lambda(l(a)); p_\lambda(a)\le 1\}$. Moreover, for each $p_\lambda\in\Gamma$, the C^*-algebras $(M(A[\tau_\Gamma]))_\lambda$ and $M(A_\lambda)$ are isomorphic. The strict topology on $M(A[\tau_\Gamma])$ is given by the family of seminorms $\{p_{\lambda,a}\}_{(\lambda,a)\in\Lambda\times A}$, where $p_{\lambda,a}(l,r) = p_\lambda(l(a)) + p_\lambda(r(a))$, $M(A[\tau_\Gamma])$ is complete with respect to the strict topology and $A[\tau_\Gamma]$ is dense in $M(A[\tau_\Gamma])$ (see [P1] and [J1, Proposition 3.4]).

A pro-C^*-morphism $\varphi : A[\tau_\Gamma]\to M(B[\tau_{\Gamma'}])$ is nondegenerate if $[\varphi(A)B] = B[\tau_{\Gamma'}]$, where $[\varphi(A)B]$ denotes the closed subspace of $B[\tau_{\Gamma'}]$ generated by $\{\varphi(a)b; a\in A, b\in B\}$. A nondegenerate pro-$C^*$-morphism $\varphi : A[\tau_\Gamma]\to M(B[\tau_{\Gamma'}])$ extends to a unique pro-C^*-morphism $\overline{\varphi} : M(A[\tau_\Gamma])\to M(B[\tau_{\Gamma'}])$.

3. Group actions on pro-C^*-algebras

Throughout this paper, $A[\tau_\Gamma]$ is a pro-C^*-algebra with topology given by the family of C^*-seminorms $\Gamma = \{p_\lambda\}_{\lambda\in\Lambda}$ and G is a locally compact group.

Definition 3.1. (1) *An action* of G on $A[\tau_\Gamma]$ is a group morphism α from G to $\mathrm{Aut}(A[\tau_\Gamma])$, the group of all automorphisms of $A[\tau_\Gamma]$, such that the map $t\mapsto\alpha_t(a)$ from G to $A[\tau_\Gamma]$ is continuous for each $a\in A$.

(2) An action α of G on $A[\tau_\Gamma]$ is *strongly bounded*, if for each $\lambda\in\Lambda$ there is $\mu\in\Lambda$ such that

$$p_\lambda(\alpha_t(a)) \le p_\mu(a)$$

for all $t \in G$ and for all $a \in A$.
(3) An action α is *an inverse limit action*, if $p_\lambda(\alpha_t(a)) = p_\lambda(a)$ for all $a \in A$, for all $t \in G$ and for all $\lambda \in \Lambda$.

Remark 3.2. (1) If α is an inverse limit action of G on $A[\tau_\Gamma]$, then for each $\lambda \in \Lambda$, there is an action α^λ of G on A_λ such that $\alpha_t^\lambda \circ \pi_\lambda^A = \pi_\lambda^A \circ \alpha_t$ for all $t \in G$, and $\alpha_t = \lim\limits_{\leftarrow\lambda} \alpha_t^\lambda$ for all $t \in G$.
(2) Any inverse limit action of G on $A[\tau_\Gamma]$ is strongly bounded.
(3) If A is a C^*-algebra, then any action of G on A is strongly bounded.
(4) If G is a compact group, then any action of G on $A[\tau_\Gamma]$ is strongly bounded.

Let X be a compactly countably generated Hausdorff topological space (that is, X is a direct limit of a countable family $\{K_n\}_n$ of compact spaces). The $*$-algebra $C(X)$ of all continuous complex valued functions on X is a pro-C^*-algebra with topology given by the family of C^*-seminorms $\{p_{K_n}\}_n$, where $p_{K_n}(f) = \sup\{|f(x)|; x \in K_n\}$.

Example 3.3. Let (G, X) be a transformation group (that is, there is a continuous map $(t, x) \mapsto t \cdot x$ from $G \times X$ to X such that $e \cdot x = x$ and $s \cdot (t \cdot x) = (st) \cdot x$ for all $s, t \in G$ and for all $x \in X$) with $X = \lim\limits_{n\rightarrow} K_n$ a compactly countably generated Hausdorff topological space. Then there is an action α of G on the pro-C^*-algebra $C(X)$, given by

$$\alpha_t(f)(x) = f\left(t^{-1} \cdot x\right).$$

If for any positive integer n, there is a positive integer m such that $G \cdot K_n \subseteq K_m$, the action α is strongly bounded, since for each n, there is m such that

$$p_{K_n}(\alpha_t(f)) = \sup\{\left|f\left(t^{-1} \cdot x\right)\right|; x \in K_n\} \le \sup\{|f(y)|; y \in K_m\} = p_{K_m}(f)$$

for all $f \in C(X)$ and for all $t \in G$. If $G \cdot K_n = K_n$ for all n, then α is an inverse limit action. Take, for instance, $\mathbb{R} = \lim\limits_{n\rightarrow}[-n, n]$. Suppose that $\mathbb{Z}_2$ actions on $\mathbb{R}$ by $\widehat{0} \cdot x = x$ and $\widehat{1} \cdot x = 2 - x$ for all $x \in \mathbb{R}$. Then $(\mathbb{Z}_2, \mathbb{R})$ is a transformation group such that for each positive integer n, $\mathbb{Z}_2 \cdot [-n, n] \subseteq [-n-2, n+2]$.

Example 3.4. Let $X = \lim\limits_{n\rightarrow} K_n$ be a compactly countably generated Hausdorff topological space and $h : X \rightarrow X$ a homeomorphism with the property that for each positive integer n, there is a positive integer m such that $h^k(K_n) \subseteq K_m$ for all integers k. Then the map $n \mapsto \alpha_n$ from $\mathbb{Z}$ to $\operatorname{Aut}(C(X))$, where $\alpha_n(f) = f \circ h^n$, is a strong bounded action of $\mathbb{Z}$ to $C(X)$. If $h(K_n) = K_n$ for all n, then α is an inverse limit action. Take, for instance, $\mathbb{R} = \lim\limits_{n\rightarrow}[-n, n]$. The map $h : \mathbb{R} \rightarrow \mathbb{R}$ defined by $h(x) = 1 - x$ is a homeomorphism such that for each positive integer n, $h^k([-n, n]) \subseteq [-n-1, n+1]$ for all integers k.

Example 3.5. The $*$-algebra $C[0, 1]$ equipped with the topology 'cc' of uniform convergence on countable compact subsets is a pro-C^*-algebra denoted by $C_{cc}[0, 1]$ (see, for example, [F, p. 104]). The action of $\mathbb{Z}_2$ on $C_{cc}[0, 1]$ given by $\alpha_{\widehat{0}} = \mathrm{id}_{C_{cc}[0,1]}$ and $\alpha_{\widehat{1}}(f)(x) = f(1 - x)$ for all $f \in C_{cc}[0, 1]$ and for all $x \in [0, 1]$ is strongly bounded.

Remark 3.6. (1) Let α be a strong bounded action of G on $A[\tau_\Gamma]$. Then, for each $\lambda \in \Lambda$, the map $p^\lambda : A \to [0, \infty)$ given by

$$p^\lambda(a) = \sup\{p_\lambda(\alpha_t(a)); t \in G\}$$

is a continuous C^*-seminorm on $A[\tau_\Gamma]$. Let $\Gamma^G = \{p^\lambda\}_{\lambda\in\Lambda}$. Since, for each $\lambda \in \Lambda$, there is $\mu \in \Lambda$ such that

$$p_\lambda \leq p^\lambda \leq p_\mu,$$

Γ^G defines on A a structure of pro-C^*-algebra, and moreover, the pro-C^*-algebras $A[\tau_\Gamma]$ and $A\left[\tau_\Gamma^G\right]$ are isomorphic.

(2) If the action α of G on $A[\tau_\Gamma]$ is strongly bounded,, then α is an inverse limit action of G on $A\left[\tau_\Gamma^G\right]$.

4. Covariant representations

Definition 4.1. A pro-C^*-dynamical system is a triple $(G, \alpha, A[\tau_\Gamma])$, where G is a locally compact group, $A[\tau_\Gamma]$ is a pro-C^*-algebra and α is an action of G on $A[\tau_\Gamma]$.

A representation of a pro-C^*-algebra $A[\tau_\Gamma]$ on a Hilbert space $\mathcal{H}$ is a continuous $*$-morphism $\varphi : A[\tau_\Gamma] \to L(\mathcal{H})$. A representation $(\varphi, \mathcal{H})$ of $A[\tau_\Gamma]$ is *nondegenerate* if $[\varphi(A)\mathcal{H}] = \mathcal{H}$.

Definition 4.2. A covariant representation of $(G, \alpha, A[\tau_\Gamma])$ on a Hilbert space $\mathcal{H}$ is a triple $(\varphi, u, \mathcal{H})$ consisting of a representation $(\varphi, \mathcal{H})$ of $A[\tau_\Gamma]$ on $\mathcal{H}$ and a unitary $*$-representation $(u, \mathcal{H})$ of G on $\mathcal{H}$ such that

$$\varphi(\alpha_t(a)) = u_t\varphi(a)u_t^*$$

for all $a \in A$ and for all $t \in G$. A covariant representation $(\varphi, u, \mathcal{H})$ is nondegenerate if $(\varphi, \mathcal{H})$ is nondegenerate.

Two representations $(\varphi, u, \mathcal{H})$ and $(\psi, v, \mathcal{K})$ of $(G, \alpha, A[\tau_\Gamma])$ are unitarily equivalent if there is a unitary operator $U : \mathcal{H} \to \mathcal{K}$ such that $U\varphi(a) = \psi(a)U$ for all $a \in A$ and $Uu_t = v_tU$ for all $t \in G$.

For each $p_\lambda \in \Gamma$, we denote by $\mathcal{R}_\lambda(G, \alpha, A[\tau_\Gamma])$ the collection of all unitary equivalence classes of nondegenerate covariant representations $(\varphi, u, \mathcal{H})$ of $(G, \alpha, A[\tau_\Gamma])$ with the property that $\|\varphi(a)\| \leq p_\lambda(a)$ for all $a \in A$. Clearly,

$$\bigcup_\lambda \mathcal{R}_\lambda(G, \alpha, A[\tau_\Gamma]) = \mathcal{R}(G, \alpha, A[\tau_\Gamma]),$$

where $\mathcal{R}(G, \alpha, A[\tau_\Gamma])$ denotes the collection of all unitary equivalence classes of nondegenerate covariant representations of $(G, \alpha, A[\tau_\Gamma])$.

Remark 4.3. If α is an inverse limit action, then the map

$$(\varphi_\lambda, u, \mathcal{H}) \to \left(\varphi_\lambda \circ \pi_\lambda^A, u, \mathcal{H}\right)$$

is a bijection between $\mathcal{R}\left(G, \alpha^\lambda, A_\lambda\right)$ and $\mathcal{R}_\lambda(G, \alpha, A[\tau_\Gamma])$ (see, [J2]).

By [J2], if α is an inverse limit action, then $\mathcal{R}\left(G,\alpha,A\left[\tau_{\Gamma}\right]\right)$ is non empty. From this result and Remark 3.6, we conclude that if α is strongly bounded, then $\mathcal{R}\left(G,\alpha,A\left[\tau_{\Gamma}\right]\right)$ is non empty too. In the following proposition we give another proof for this result.

Proposition 4.4. *Let $\left(G,\alpha,A\left[\tau_{\Gamma}\right]\right)$ be a pro-C^{*}-dynamical system such that α is strongly bounded. Then there is a covariant representation of $\left(G,\alpha,A\left[\tau_{\Gamma}\right]\right)$.*

Proof. Let $(\varphi,\mathcal{H})$ be a representation of $A\left[\tau_{\Gamma}\right]$. Then there is $\lambda\in\Lambda$ such that $\left\|\varphi\left(a\right)\right\|\leq p_{\lambda}\left(a\right)$ for all $a\in A$. Let $a\in A$ and $\xi\in L^{2}(G,\mathcal{H})$. Since, there is $p_{\mu}\in\Gamma$ such that

$$\begin{aligned}\int_{G}\left\|\varphi\left(\alpha_{s^{-1}}\left(a\right)\right)\left(\xi\left(s\right)\right)\right\|^{2}ds &\leq \int_{G}\left\|\varphi\left(\alpha_{s^{-1}}\left(a\right)\right)\right\|^{2}\left\|\xi\left(s\right)\right\|^{2}ds\\ &\leq \int_{G}p_{\lambda}\left(\alpha_{s^{-1}}\left(a\right)\right)^{2}\left\|\xi\left(s\right)\right\|^{2}ds\leq p_{\mu}\left(a\right)^{2}\left\|\xi\right\|^{2},\end{aligned}$$

the map $s\mapsto\varphi\left(\alpha_{s^{-1}}\left(a\right)\right)\left(\xi\left(s\right)\right)$ defines an element in $L^{2}(G,\mathcal{H})$. Therefore, there is $\widetilde{\varphi}\left(a\right)\in L(L^{2}(G,\mathcal{H}))$ such that

$$\widetilde{\varphi}\left(a\right)\left(\xi\right)\left(s\right)=\varphi\left(\alpha_{s^{-1}}\left(a\right)\right)\left(\xi\left(s\right)\right).$$

In this way, we obtain a map $\widetilde{\varphi}:A\rightarrow L(L^{2}(G,\mathcal{H}))$. Moreover, $\widetilde{\varphi}$ is a continuous $*$-morphism, and then $\left(\widetilde{\varphi},L^{2}\left(G,\mathcal{H}\right)\right)$ is a representation of $A\left[\tau_{\Gamma}\right]$.

Let $\left(\lambda_{G}^{\mathcal{H}},L^{2}\left(G,\mathcal{H}\right)\right)$ be the unitary $*$-representation of G on $L^{2}\left(G,\mathcal{H}\right)$ given by $\left(\lambda_{G}^{\mathcal{H}}\right)_{t}\left(\xi\right)\left(s\right)=\xi\left(t^{-1}s\right)$. It is easy to verify that $\left(\widetilde{\varphi},\lambda_{G}^{\mathcal{H}},L^{2}\left(G,\mathcal{H}\right)\right)$ is a covariant representation of $\left(G,\alpha,A\left[\tau_{\Gamma}\right]\right)$. □

Remark 4.5. Let $\left(G,\alpha,A\left[\tau_{\Gamma}\right]\right)$ be a pro-C^{*}-dynamical system. Suppose that α is strongly bounded. Then, for each representation $(\varphi,\mathcal{H})$ of $A\left[\tau_{\Gamma}\right]$, $\ker\widetilde{\varphi}\subseteq\ker\varphi$. Indeed, if $\widetilde{\varphi}\left(a\right)=0$, then $\varphi\left(\alpha_{s}\left(a\right)\right)\left(\xi\left(s\right)\right)=0$ for all $s\in G$ and for all $\xi\in L^{2}\left(G,\mathcal{H}\right)$, whence $\varphi\left(a\right)\left(\xi\left(e\right)\right)=0$ for all $\xi\in L^{2}\left(G,\mathcal{H}\right)$ and so $\varphi\left(a\right)=0$.

Definition 4.6. A covariant pro-C^{*}-morphism from $\left(G,\alpha,A\left[\tau_{\Gamma}\right]\right)$ to a pro-C^{*}-algebra $B\left[\tau_{\Gamma'}\right]$ is a pair (φ,u) consisting of a pro-C^{*}-morphism $\varphi:A\left[\tau_{\Gamma}\right]\rightarrow M(B\left[\tau_{\Gamma'}\right])$ and a strict continuous group morphism $u:G\rightarrow\mathcal{U}(M(B\left[\tau_{\Gamma'}\right]))$, the group of all unitaries of $M(B\left[\tau_{\Gamma'}\right])$, such that

$$\varphi\left(\alpha_{t}\left(a\right)\right)=u_{t}\varphi\left(a\right)u_{t}^{*}$$

for all $t\in G$ and for all $a\in A$. A covariant pro-C^{*}-morphism (φ,u) from $\left(G,\alpha,A\left[\tau_{\Gamma}\right]\right)$ to $B\left[\tau_{\Gamma'}\right]$ is nondegenerate if $\left[\varphi\left(A\right)B\right]=B\left[\tau_{\Gamma'}\right]$.

Theorem 4.7. *Let $\left(G,\alpha,A\left[\tau_{\Gamma}\right]\right)$ be a pro-C^{*}-dynamical system. If α is strongly bounded, then there is a locally Hilbert space $\mathcal{H}$ and a covariant pro-C^{*}-morphism (i_{A},i_{G}) from $\left(G,\alpha,A\left[\tau_{\Gamma}\right]\right)$ to $\mathcal{L}(\mathcal{H})$. Moreover, i_{A} and i_{G} are injective.*

Proof. Let $\lambda\in\Lambda$. By Proposition 4.4, $\mathcal{R}_{\lambda}\left(G,\alpha,A\left[\tau_{\Gamma}\right]\right)$ is non empty. Let $\left(\varphi^{\lambda},u^{\lambda},H_{\lambda}\right)$ be the direct sum of one representative $(\varphi,u,H_{\varphi,u})$ of each unitary equivalence class of nondegenerate covariant representations of $\left(G,\alpha,A\left[\tau_{\Gamma}\right]\right)$ from

$\mathcal{R}_\lambda(G,\alpha,A[\tau_\Gamma])$. Then $(\varphi^\lambda, u^\lambda, H_\lambda)$ is a nondegenerate covariant representation of $(G,\alpha,A[\tau_\Gamma])$ such that $\|\varphi^\lambda(a)\| \leq p_\lambda(a)$ for all $a \in A$**.**

Let $\mathcal{H}_\lambda = \oplus_{\mu\leq\lambda} H_\mu$. Then $\mathcal{H} = \lim\limits_{\lambda\to} \mathcal{H}_\lambda$ is a locally Hilbert space. For $a \in A$, the map $i_A^\lambda(a) : \mathcal{H}_\lambda \to \mathcal{H}_\lambda$ defined by

$$i_A^\lambda(a)(\oplus_{\mu\leq\lambda}\xi_\mu) = \oplus_{\mu\leq\lambda}\varphi^\mu(a)\,\xi_\mu$$

is an element in $L(\mathcal{H}_\lambda)$ and $\|i_A^\lambda(a)\| \leq p_\lambda(a)$. Moreover, $i_A^\lambda(a^*) = i_A^\lambda(a)^*$ and $i_A^\lambda(ab) = i_A^\lambda(a)\,i_A^\lambda(b)$ for all $a, b \in A$. Clearly, $(i_A^\lambda(a))_\lambda$ is a direct system of bounded linear operators and $i_A(a) = \lim\limits_{\lambda\to} i_A^\lambda(a)$ is an element $\mathcal{L}(\mathcal{H})$ such that $i_A(a^*) = i_A(a)^*$ and $i_A(ab) = i_A(a)\,i_A(b)$ for all $a, b \in A$. Moreover,

$$p_{\lambda,\mathcal{L}(\mathcal{H})}(i_A(a)) = \|i_A^\lambda(a)\| \leq p_\lambda(a)$$

for all $a \in A$ and for all $\lambda \in \Lambda$. Therefore, i_A is a pro-C^*-morphism.

For $t \in G$, the map $i_G^\lambda(t) : \mathcal{H}_\lambda \to \mathcal{H}_\lambda$ defined by

$$i_G^\lambda(t)(\oplus_{\mu\leq\lambda}\xi_\mu) = \oplus_{\mu\leq\lambda} u^\mu(t)\,\xi_\mu$$

is a unitary element in $L(\mathcal{H}_\lambda)$. Moreover, the map $t \mapsto i_G^\lambda(t)$ is a unitary $*$-representation of G on $\mathcal{H}_\lambda$. Clearly, $(i_G^\lambda(t))_\lambda$ is a direct system of unitary operators, and then $i_G(t) = \lim\limits_{\lambda\to} i_G^\lambda(t)$ is a unitary element $\mathcal{L}(\mathcal{H})$. Moreover, $t \mapsto i_G(t)$ is a group morphism from G to the group of unitary operators on $\mathcal{H}$, and since for each $\xi \in \mathcal{H}$, the map $t \mapsto i_G(t)\,\xi$ from G to $\mathcal{H}$ is continuous, $t \mapsto i_G(t)$ is a unitary $*$-representation of G on $\mathcal{H}$. We have

$$\begin{aligned} i_A(\alpha_t(a))(\oplus_{\mu\leq\lambda}\xi_\mu) &= i_A^\lambda(\alpha_t(a))(\oplus_{\mu\leq\lambda}\xi_\mu) = \oplus_{\mu\leq\lambda}\varphi^\mu(\alpha_t(a))(\xi_\mu) \\ &= \oplus_{\mu\leq\lambda} u^\mu(t)\,\varphi^\mu(a)\,u^\mu(t)^*(\xi_\mu) \\ &= i_G(t) i_A(a)\, i_G(t)^*(\oplus_{\mu\leq\lambda}\xi_\mu) \end{aligned}$$

for all $a \in A$, for all $t \in G$ and for all $\oplus_{\mu\leq\lambda}\xi_\mu \in \mathcal{H}_\lambda$, $\lambda \in \Lambda$, and so

$$i_A(\alpha_t(a)) = i_G(t) i_A(a)\, i_G(t)^*$$

for all $a \in A$ and for all $t \in G$.

Suppose that $i_A(a) = 0$. Then $i_A^\lambda(a) = 0$ for all $\lambda \in \Lambda$ and so $\varphi(a) = 0$ for all nondegenerate covariant representation $(\varphi, u, H_{\varphi,u})$ of $(G,\alpha,A[\tau_\Gamma])$. By Proposition 4.4 and Remark 4.5, $\psi(a) = 0$ for all representations ψ of A. Therefore, $p_\lambda(a) = 0$ for all $\lambda \in \Lambda$, and then $a = 0$.

Suppose that $i_G(t) = \mathrm{id}_{\mathcal{H}}$. Then $i_G^\lambda(t) = \mathrm{id}_{\mathcal{H}_\lambda}$ for all $\lambda \in \Lambda$, and so $u(t) = \mathrm{id}_{H_{\varphi,u}}$ for all nondegenerate covariant representation $(\varphi, u, H_{\varphi,u})$ of $(G,\alpha,A[\tau_\Gamma])$, whence we deduce that $t = e$. □

The following proposition gives a characterization of inverse limit actions.

Proposition 4.8. *Let $(G,\alpha,A[\tau_\Gamma])$ be a pro-C^*-dynamical system. Then the following statements are equivalent.*

1. *α is an inverse limit action.*
2. *There is a locally Hilbert space $\mathcal{H}$ and a covariant pro-C^*-morphism (i_A, i_G) from $(G,\alpha,A[\tau_\Gamma])$ to $\mathcal{L}(\mathcal{H})$ such that $p_{\lambda,\mathcal{L}(\mathcal{H})}(i_A(a)) = p_\lambda(a)$ for all $\lambda \in \Lambda$ and $a \in A$.*

Proof. (1) ⇒ (2) See [J3, Proposition 3.1] and [I, Theorem 5.1].

(2) ⇒ (1) From

$$i_A(\alpha_t(a)) = i_G(t)\, i_A(a)\, i_G(t)^*$$

for all $t \in G$ and for all $a \in A$, and taking into account that $i_G(t)$ is a unitary element in $\mathcal{L}(\mathcal{H})$ for all $t \in G$, we deduce that

$$\begin{aligned} p_\lambda(\alpha_t(a)) &= p_{\lambda,\mathcal{L}(\mathcal{H})}(i_A(\alpha_t(a))) = p_{\lambda,\mathcal{L}(\mathcal{H})}(i_G(t)\, i_A(a)\, i_G(t)^*) \\ &= p_{\lambda,\mathcal{L}(\mathcal{H})}(i_A(a)) = p_\lambda(a) \end{aligned}$$

for all $t \in G$, for all $a \in A$ and for all $t \in G$. Therefore, α is an inverse limit action. □

5. The full pro-C^*-crossed product

Let $(G, \alpha, A[\tau_\Gamma])$ be a pro-C^*-dynamical system and let $B[\tau_{\Gamma'}]$ be a pro-C^*-algebra whose topology is given by the family of C^*-seminorms $\Gamma' = \{q_\delta\}_{\delta\in\Delta}$.

If u is a strict continuous group morphism from G to $\mathcal{U}(M(B[\tau_{\Gamma'}]))$, then there is a $*$-morphism $u : C_c(G) \to M(B[\tau_{\Gamma'}])$ given by $u(f) = \int_G f(s)u_s ds$, where ds denotes the Haar measure on G (see [J2]).

Definition 5.1. Let $(G, \alpha, A[\tau_\Gamma])$ be a pro-C^*-dynamical system. A pro-C^*-algebra, denoted by $G \times_\alpha A[\tau_\Gamma]$, together with a covariant pro-C^*-morphism (ι_A, ι_G) from $(G, \alpha, A[\tau_\Gamma])$ to $G \times_\alpha A[\tau_\Gamma]$ which verifies the following:

1. for each nondegenerate covariant representation $(\varphi, u, \mathcal{H})$ of $(G, \alpha, A[\tau_\Gamma])$, there is a nondegenerate representation $(\Phi, \mathcal{H})$ of $G \times_\alpha A[\tau_\Gamma]$ such that $\overline{\Phi} \circ \iota_A = \varphi$ and $\overline{\Phi} \circ \iota_G = u$;
2. $\overline{\operatorname{span}\{\iota_A(a)\,\iota_G(f)\,; a \in A, f \in C_c(G)\}} = G \times_\alpha A[\tau_\Gamma]$;

is called the full pro-C^*-crossed product of $A[\tau_\Gamma]$ by α.

Remark 5.2. The covariant morphism (ι_A, ι_G) from the above definition is nondegenerate.

Proposition 5.3. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro-C^*-dynamical system such that there is a full pro-C^*-crossed product of $A[\tau_\Gamma]$ by α and (φ, u) a nondegenerate covariant morphism from $(G, \alpha, A[\tau_\Gamma])$ to a pro-C^*-algebra $B[\tau_{\Gamma'}]$. Then there is a unique nondegenerate pro-C^*-morphism $\varphi \times u : G \times_\alpha A[\tau_\Gamma] \to M(B[\tau_{\Gamma'}])$ such that*

$$\overline{\varphi \times u} \circ \iota_A = \varphi \text{ and } \overline{\varphi \times u} \circ \iota_G = u.$$

Moreover, the map $(\varphi, u) \to \varphi \times u$ is a bijection between nondegenerate covariant morphisms of $(G, \alpha, A[\tau_\Gamma])$ onto nondegenerate morphisms of $G \times_\alpha A[\tau_\Gamma]$.

Proof. Let $q_\delta \in \Gamma'$ and $(\psi_\delta, \mathcal{H})$ a faithful nondegenerate representation of B_δ. Then, $(\overline{\psi_\delta} \circ \overline{\pi_\delta^B} \circ \varphi, \overline{\psi_\delta} \circ \overline{\pi_\delta^B} \circ u, \mathcal{H})$ is a nondegenerate covariant representation of $(G, \alpha, A[\tau_\Gamma])$, and by Definition 5.1, there is a nondegenerate representation $(\phi_\delta, \mathcal{H})$ of $G \times_\alpha A[\tau_\Gamma]$ such that

$$\overline{\phi_\delta} \circ \iota_A = \overline{\psi_\delta} \circ \overline{\pi_\delta^B} \circ \varphi \text{ and } \overline{\phi_\delta} \circ \iota_G = \overline{\psi_\delta} \circ \overline{\pi_\delta^B} \circ u.$$

Let $\Phi_\delta = \overline{\psi_\delta^{-1}} \circ \phi_\delta$. Then Φ_δ is a nondegenerate pro-C^*-morphism from $G \times_\alpha A\,[\tau_\Gamma]$ to $M(B_\delta)$. Moreover, for $q_{\delta_1}, q_{\delta_2} \in \Gamma'$ with $q_{\delta_1} \geq q_{\delta_2}$, we have $\overline{\pi_{\delta_1\delta_2}^B} \circ \Phi_{\delta_1} = \Phi_{\delta_2}$. Therefore, there is a nondegenerate pro-C^*-morphism $\varphi \times u : G \times_\alpha A\,[\tau_\Gamma] \to M(B\,[\tau_{\Gamma'}])$ such that

$$\overline{\pi_\delta^B} \circ \varphi \times u = \Phi_\delta$$

for all $q_\delta \in \Gamma'$. Moreover, $\overline{\varphi \times u} \circ \iota_A = \varphi$ and $\overline{\varphi \times u} \circ \iota_G = u$, and since $\{\iota_A(a)\,\iota_G(f)\,; a \in A,\ f \in C_c(G)\}$ generates $G \times_\alpha A\,[\tau_\Gamma]$, $\varphi \times u$ is unique with the above properties.

Let $\Phi : G \times_\alpha A\,[\tau_\Gamma] \to M(\,B\,[\tau_{\Gamma'}])$ be a nondegenerate pro-C^*-morphism. Then $\varphi = \overline{\Phi} \circ \iota_A$ is a nondegenerate pro-C^*-morphism from $A\,[\tau_\Gamma]$ to $M(B\,[\tau_{\Gamma'}])$ and $u = \overline{\Phi} \circ \iota_G$ is a strict continuous morphism from G to $\mathcal{U}(M(B\,[\tau_{\Gamma'}]))$, since ι_G is a strict continuous morphism from G to $M(G \times_\alpha A\,[\tau_\Gamma])$ and $\overline{\Phi}$ is strongly continuous on the bounded subsets of $M(G \times_\alpha A\,[\tau_\Gamma])$. Moreover, (φ, u) is a nondegenerate covariant morphism from $A\,[\tau_\Gamma]$ to $B\,[\tau_{\Gamma'}]$, and $\varphi \times u = \Phi$. If (ψ, v) is another nondegenerate covariant morphism from $A\,[\tau_\Gamma]$ to $B\,[\tau_{\Gamma'}]$ such that $\psi \times v = \Phi$, then $\psi = \overline{\Phi} \circ \iota_A = \varphi$ and $v = \overline{\Phi} \circ \iota_G = u$. □

The following corollary provides uniqueness of the full pro-C^*-crossed product.

Corollary 5.4. *Let $(G, \alpha, A\,[\tau_\Gamma])$ be a pro-C^*-dynamical system such that there is a full pro-C^*-crossed product of $A\,[\tau_\Gamma]$ by α. Then the full pro-C^*-crossed product of $A\,[\tau_\Gamma]$ by α is unique up to a pro-C^*-isomorphism.*

Proof. Let $B\,[\tau_{\Gamma'}]$ be a pro-C^*-algebra and (j_A, j_G) a covariant pro-C^*-morphism from $(G, \alpha, A\,[\tau_\Gamma])$ to $B\,[\tau_{\Gamma'}]$ which satisfy the relations $(1) - (2)$ from Definition 5.1. Then, by Proposition 5.3, there is a nondegenerate pro-C^*-morphism $\Phi : G \times_\alpha A\,[\tau_\Gamma] \to M(B\,[\tau_{\Gamma'}])$ such that $\overline{\Phi} \circ \iota_A = j_A$ and $\overline{\Phi} \circ \iota_G = j_G$. Since $\{\iota_A(a)\,\iota_G(f)\,; a \in A,\ f \in C_c(G)\}$ generates $G \times_\alpha A\,[\tau_\Gamma]$ and $\{j_A(a)\,j_G(f)\,; a \in A,$ $f \in C_c(G)\}$ generates $B\,[\tau_{\Gamma'}]$, $\Phi\,(G \times_\alpha A\,[\tau_\Gamma]) \subseteq B$.

In the same way, there is a pro-C^*-morphism $\Psi : B\,[\tau_{\Gamma'}] \to G \times_\alpha A\,[\tau_\Gamma]$ such that $\overline{\Psi} \circ j_A = \iota_A$ and $\overline{\Psi} \circ j_G = \iota_G$. From these facts and Definition 5.1 (2), we deduce that $\Phi \circ \Psi = \mathrm{id}_B$ and $\Psi \circ \Phi = \mathrm{id}_{G \times_\alpha A[\tau_\Gamma]}$, and so Φ is a pro-C^*-isomorphism. □

The following proposition relates the nondegenerate covariant representations of a pro-C^*-dynamical system $(G, \alpha, A\,[\tau_\Gamma])$ with the nondegenerate representations of the full pro-C^*-crossed product of $A\,[\tau_\Gamma]$ by α.

Proposition 5.5. *Let $(G, \alpha, A\,[\tau_\Gamma])$ be a pro-C^*-dynamical system such that there is the full pro-C^*-crossed product of $A\,[\tau_\Gamma]$ by α. Then there is a bijective correspondence between nondegenerate covariant representations of $(G, \alpha, A\,[\tau_\Gamma])$ and nondegenerate representations of $G \times_\alpha A\,[\tau_\Gamma]$.*

Proof. Let $(\varphi, u, \mathcal{H})$ be a nondegenerate covariant representation of $(G, \alpha, A\,[\tau_\Gamma])$. Then, by Definition 5.1, there is a nondegenerate representation $(\varphi \times u, \mathcal{H})$ of $G \times_\alpha A\,[\tau_\Gamma]$ such that $\overline{\varphi \times u} \circ \iota_A = \varphi$ and $\overline{\varphi \times u} \circ \iota_G = u$. Moreover, by Definition 5.1(2), $(\varphi \times u, \mathcal{H})$ is unique, and since φ is nondegenerate, it is nondegenerate too.

Let $(\Phi,\mathcal{H})$ be a nondegenerate representation of $G\times_{\alpha}A\left[\tau_{\Gamma}\right]$. Then $(\overline{\Phi}\circ\iota_{A}, \overline{\Phi}\circ\iota_{G},\mathcal{H})$ is a covariant representation of $(G,\alpha,A\left[\tau_{\Gamma}\right])$, and moreover, $\left(\overline{\Phi}\circ\iota_{A}\right)\times\left(\overline{\Phi}\circ\iota_{G}\right)=\Phi$. Since ι_{A} and Φ are nondegenerate, the net $\{\overline{\Phi}\left(\iota_{A}\left(e_{i}\right)\right)\}_{i}$, where $\{e_{i}\}_{i}$ is an approximate unit of $A\left[\tau_{\Gamma}\right]$, converges strictly to $\mathrm{id}_{\mathcal{H}}$, and so $\overline{\Phi}\circ\iota_{A}$ is nondegenerate.

Suppose that there is another nondegenerate covariant representation $(\varphi,u,\mathcal{H})$ of $(G,\alpha,A\left[\tau_{\Gamma}\right])$ such that $\varphi\times u=\Phi$. Then $\varphi=\overline{\varphi\times u}\circ\iota_{A}=\overline{\Phi}\circ\iota_{A}$ and $u=\overline{\varphi\times u}\circ\iota_{G}=\overline{\Phi}\circ\iota_{G}$. Therefore, the map $(\varphi,u,\mathcal{H})\mapsto(\varphi\times u,\mathcal{H})$ is bijective. $\square$

Theorem 5.6. *Let $(G,\alpha,A\left[\tau_{\Gamma}\right])$ be a pro-C^{*}-dynamical system such that α is strongly bounded. Then, there is the full pro-C^{*}-crossed product of $A\left[\tau_{\Gamma}\right]$ by α.*

Proof. By Theorem 4.7**,** there is a locally Hilbert space $\mathcal{H}$ and a covariant pro-C^{*}-morphism (i_{A},i_{G}) from $A[\tau_{\Gamma}]$ to $\mathcal{L}(\mathcal{H})$.

Let $B=\overline{\mathrm{span}\{i_{A}\left(a\right)i_{G}\left(f\right);a\in A,f\in C_{c}\left(G\right)\}}\subseteq\mathcal{L}(\mathcal{H})$. To show that B is a pro-C^{*}-algebra, we must show that B is closed under taking adjoints and multiplication. For this, since $B=\lim\limits_{\longleftarrow\lambda}\overline{\pi_{\lambda}^{\mathcal{H}}\left(B\right)}$ ([M, Chapter III, Theorem 3.1]), it is sufficient to show that for each $\lambda\in\Lambda$, $\pi_{\lambda}^{\mathcal{H}}\left(i_{A}\left(b\right)i_{G}\left(f\right)i_{A}\left(a\right)i_{G}(h)\right)$ and $\pi_{\lambda}^{\mathcal{H}}\left(i_{G}\left(f\right)i_{A}\left(a\right)\right)$ are elements in the closure of $\pi_{\lambda}^{\mathcal{H}}\left(B\right)$ in $L\left(\mathcal{H}_{\lambda}\right)$ for all $a,b\in A$ and for all $f,h\in C_{c}(G)$.

The map $s\rightarrow\pi_{\lambda}^{A}\left(f\left(s\right)\alpha_{s}(a)\right)$ from G to A_{λ} defines an element in $C_{c}(G,A_{\lambda})$, and so there is a net $\{\pi_{\lambda}^{A}(a_{j})\otimes f_{j}\}_{j\in J}$ in $A_{\lambda}\otimes_{\mathrm{alg}}C_{c}(G)$ with $\mathrm{supp}f_{j},\mathrm{supp}f\subseteq K$ for some compact subset K, which converges uniformly to this map.

By [J4, Lemma 3.7],

$$\begin{aligned}\pi_{\lambda}^{\mathcal{H}}\left(i_{G}\left(f\right)i_{A}\left(a\right)\right) &= \int_{G}f\left(s\right)i^{\lambda}\left(s\right)ds\pi_{\lambda}^{\mathcal{H}}\left(i_{A}(a)\right)=\int_{G}f\left(s\right)\pi_{\lambda}^{\mathcal{H}}\left(i_{G}(s)i_{A}(a)\right)ds\\ &= \int_{G}f\left(s\right)\pi_{\lambda}^{\mathcal{H}}\left(i_{A}(\alpha_{s}\left(a\right))i_{G}(s)\right)ds\\ &= \int_{G}f\left(s\right)i_{A}^{\lambda}(\alpha_{s}(a))i_{G}^{\lambda}(s)ds\end{aligned}$$

and

$$\pi_{\lambda}^{\mathcal{H}}\left(i_{A}\left(a_{j}\right)i_{G}\left(f_{j}\right)\right)=\pi_{\lambda}^{\mathcal{H}}\left(i_{A}\left(a_{j}\right)\right)\int_{G}f_{j}\left(s\right)i_{G}^{\lambda}(s)ds=\int_{G}i_{A}^{\lambda}(a_{j})f_{j}\left(s\right)i_{G}^{\lambda}(s)ds$$

for each $j \in J$. Then, we have

$$\begin{aligned}
& \left\| \pi_\lambda^{\mathcal{H}} \left(i_G(f) \, i_A(a) \right) - \pi_\lambda^{\mathcal{H}} \left(i_A(a_j) \, i_G(f_j) \right) \right\|_{L(\mathcal{H}_\lambda)} \\
\leq & \int_G \left\| f(s) \, i_A^\lambda(\alpha_s(a)) i_G^\lambda(s) - i_A^\lambda(a_j) f_j(s) \, i_G^\lambda(s) \right\|_{L(\mathcal{H}_\lambda)} ds \\
\leq & M \sup\{ \left\| f(s) \, i_A^\lambda(\alpha_s(a)) i_G^\lambda(s) - i_A^\lambda(a_j) f_j(s) \, i_G^\lambda(s) \right\|_{L(\mathcal{H}_\lambda)}, s \in K \} \\
= & M \sup\{ \left\| i_A^\lambda(f(s) \, \alpha_s(a) - f_j(s) \, a_j) \right\|_{L(\mathcal{H}_\lambda)} \left\| i_G^\lambda(s) \right\|_{L(\mathcal{H}_\lambda)}, s \in K \} \\
\leq & M \sup\{ p_\lambda(f(s) \, \alpha_s(a) - f_j(s) \, a_j), s \in K \} \\
= & M \sup\{ \left\| \pi_\lambda^A(f(s) \, \alpha_s(a)) - f_j(s) \, \pi_\lambda^A(a_j) \right\|_{A_\lambda}, s \in K \}
\end{aligned}$$

for all $j \in J$, where $M = \int_K dg$, and so $\pi_\lambda^{\mathcal{H}} \left(i_G(f) \, i_A(a) \right) \in \overline{\pi_\lambda^{\mathcal{H}}(B)}$.

On the other hand,

$$\begin{aligned}
& \left\| \pi_\lambda^{\mathcal{H}} \left(i_A(b) \, i_G(f) \, i_A(a) \, i_G(h) \right) - \pi_\lambda^{\mathcal{H}} \left(i_A(ba_j) \, i_G(f_j * h) \right) \right\|_{L(\mathcal{H}_\lambda)} \\
= & \left\| \pi_\lambda^{\mathcal{H}} \left(i_A(b) \, i_G(f) \, i_A(a) \, i_G(h) - i_A(b) i_A(a_j) \, i_G(f_j) \, i_G(h) \right) \right\|_{L(\mathcal{H}_\lambda)} \\
\leq & \left\| \pi_\lambda^{\mathcal{H}} \left(i_A(b) \right) \pi_\lambda^{\mathcal{H}} \left(i_G(f) \, i_A(a) - i_A(a_j) \, i_G(f_j) \right) \pi_\lambda^{\mathcal{H}} \left(i_G(h) \right) \right\|_{L(\mathcal{H}_\lambda)} \\
\leq & \left\| \pi_\lambda^{\mathcal{H}} \left(i_A(b) \right) \right\|_{L(\mathcal{H}_\lambda)} \left\| \pi_\lambda^{\mathcal{H}} \left(i_G(h) \right) \right\|_{L(\mathcal{H}_\lambda)} \\
& \left\| \pi_\lambda^{\mathcal{H}} \left(i_G(f) \, i_A(a) \right) - \pi_\lambda^{\mathcal{H}} \left(i_A(a_j) \, i_G(f_j) \right) \right\|_{L(\mathcal{H}_\lambda)}
\end{aligned}$$

whence, we deduce that $\pi_\lambda^{\mathcal{H}} \left(i_A(b) \, i_G(f) \, i_A(a) \, i_G(h) \right) \in \overline{\pi_\lambda^{\mathcal{H}}(B)}$. Thus, we showed that $\pi_\lambda^{\mathcal{H}} \left(i_G(f) \, i_A(a) \right), \pi_\lambda^{\mathcal{H}} \left(i_A(b) \, i_G(f) \, i_A(a) \, i_G(h) \right) \in \overline{\pi_\lambda^{\mathcal{H}}(B)}$ for each $\lambda \in \Lambda$, and so $i_G(f) \, i_A(a)$, $i_A(b) \, i_G(f) \, i_A(a) \, i_G(h) \in B$. Therefore, B is a pro-C^*-algebra.

In the same manner, we show that for each $a \in A, i_A(a) \, i_A(b) \, i_G(f) \in B$ and $i_A(b) \, i_G(f) i_A(a) \in B$ for all $b \in A$ and for all $f \in C_c(G)$, and so $i_A(a) \in M(B)$.

From,

$$i_G(t) i_A(a) \, i_G(f) = \int_G f(s) i_A(\alpha_t(a)) \, i_G(ts) ds \in B$$

and

$$i_G(f) i_A(a) \, i_G(t) = \int_G f(s) i_A(\alpha_s(a)) \, i_G(st) ds \in B$$

for all $a \in A$, for all $f \in C_c(G)$ and for all $t \in G$, we deduce that $i_G(t) \in M(B)$ for all $t \in G$.

Let $(\psi, v, H_{\psi,v})$ be a nondegenerate covariant representation of $(G, \alpha, A[\tau_\Gamma])$. Then there is $(\varphi, u, H_{\varphi,u}) \in \mathcal{R}_\lambda(G, \alpha, A[\tau_\Gamma])$ such that $(\psi, v, H_{\psi,v})$ and $(\varphi, u, H_{\varphi,u})$ are unitarily equivalent. So there is a unitary operator $U : H_{\psi,v} \to H_{\varphi,u}$ such that $\psi(a) = U^* \varphi(a) U$ for all $a \in A$ and $v_t = U^* u_t U$ for all $t \in G$. The map $\Psi : \mathcal{L}(\mathcal{H}) \to L(H_\lambda)$ given by

$$\Psi(T) = \pi_\lambda^{\mathcal{H}}(T) \, |_{H_\lambda}$$

is a representation of $\mathcal{L}(\mathcal{H})$ on H_λ (see the proof of Theorem 4.7). From

$$\Psi(i_A(a)) \, (H_{\varphi,u}) = i_A^\lambda(a) \, (H_{\varphi,u}) \subseteq H_{\varphi,u}$$

for all $a\in A$ and

$$\Psi\left(i_{G}(t)\right)\left(H_{\varphi,u}\right)=i_{G}^{\lambda}(t)\left(H_{\varphi,u}\right)\subseteq H_{\varphi,u}$$

for all $t\in G$, and taking into account that B is generated by $\{i_{A}\left(a\right)i_{G}\left(f\right);a\in A,f\in C_{c}\left(G\right)\}$, we deduce that $\Psi\left(B\right)\left(H_{\varphi,u}\right)\subseteq H_{\varphi,u}$. Let $\Phi:B\rightarrow L(H_{\psi,v})$ given by

$$\Phi\left(b\right)=U^{*}\Psi\left(b\right)|_{H_{\varphi,u}}U.$$

Clearly, Φ is a nondegenerate representation of B on $H_{\psi,v}$,

$$\overline{\Phi}\left(i_{A}(a)\right)=U^{*}\Psi\left(i_{A}(a)\right)|_{H_{\varphi,u}}U=U^{*}i_{A}^{\lambda}(a)|_{H_{\varphi,u}}U=U^{*}\varphi(a)U=\psi\left(a\right)$$

for all $a\in A$, and

$$\overline{\Phi}\left(i_{G}(t)\right)=U^{*}\Psi\left(i_{G}(t)\right)|_{H_{\varphi,u}}U=U^{*}i_{G}^{\lambda}(t)|_{H_{\varphi,u}}U=U^{*}u_{t}U=v_{t}$$

for all $t\in G$. □

Remark 5.7. The index of the family of seminorms which gives the topology on the full pro-C^{*}-crossed product of $A\left[\tau_{\Gamma}\right]$ by α is the same with the index of the family of seminorms which gives the topology on $A\left[\tau_{\Gamma}\right]$.

Proposition 5.8. *Let $(G,\alpha,A\left[\tau_{\Gamma}\right])$ be a pro-C^{*}-dynamical system such that α is an inverse limit action. Then for each $\lambda\in\Lambda$, the C^{*}-algebra $\left(G\times_{\alpha}A\left[\tau_{\Gamma}\right]\right)_{\lambda}$ is isomorphic to the full C^{*}-crossed product of A_{λ} by α^{λ}.*

Proof. By Theorem 4.7, Proposition 4.8 and Corollary 5.4, there is a C^{*}-morphism $i_{A_{\lambda}}:A_{\lambda}\rightarrow M\left(\left(G\times_{\alpha}A\left[\tau_{\Gamma}\right]\right)_{\lambda}\right)$ such that $i_{A_{\lambda}}\circ\pi_{\lambda}^{A}=\overline{\pi_{\lambda}^{G\times_{\alpha}A[\tau_{\Gamma}]}}\circ i_{A}$. Using the fact that α is an inverse limit action, it is easy to check that $\left(i_{A_{\lambda}},\overline{\pi_{\lambda}^{G\times_{\alpha}A[\tau_{\Gamma}]}}\circ i_{G}\right)$ is a covariant C^{*}-morphism from $\left(G,\alpha^{\lambda},A_{\lambda}\right)$ to $\left(G\times_{\alpha}A\left[\tau_{\Gamma}\right]\right)_{\lambda}$. Moreover,

$$\begin{aligned}
&\overline{\operatorname{span}\{i_{A_{\lambda}}\left(\pi_{\lambda}^{A}\left(a\right)\right)\overline{\pi_{\lambda}^{G\times_{\alpha}A[\tau_{\Gamma}]}}\left(i_{G}\left(f\right)\right);a\in A,f\in C_{c}\left(G\right)\}}\\
=&\ \overline{\operatorname{span}\{\pi_{\lambda}^{G\times_{\alpha}A[\tau_{\Gamma}]}\left(i_{A}\left(a\right)i_{G}\left(f\right)\right);a\in A,f\in C_{c}\left(G\right)\}}\\
=&\ \pi_{\lambda}^{G\times_{\alpha}A[\tau_{\Gamma}]}\left(G\times_{\alpha}A\left[\tau_{\Gamma}\right]\right)=\left(G\times_{\alpha}A\left[\tau_{\Gamma}\right]\right)_{\lambda}.
\end{aligned}$$

Let $(\varphi,u,\mathcal{H})$ be a nondegenerate covariant representation of $\left(G,\alpha^{\lambda},A_{\lambda}\right)$. Then $\left(\varphi\circ\pi_{\lambda}^{A},u,\mathcal{H}\right)$ is a nondegenerate covariant representation of $(G,\alpha,A\left[\tau_{\Gamma}\right])$ and by Definition 5.1, there is a nondegenerate representation $(\Phi,\mathcal{H})$ of $G\times_{\alpha}A\left[\tau_{\Gamma}\right]$ such that $\overline{\Phi}\circ i_{A}=\varphi\circ\pi_{\lambda}^{A}$ and $\overline{\Phi}\circ i_{G}=u$. Moreover, by the proof of Theorem 5.6,

$$\left\|\Phi\left(b\right)\right\|\leq p_{\lambda,G\times_{\alpha}A[\tau_{\Gamma}]}\left(b\right)$$

for all $b\in G\times_{\alpha}A\left[\tau_{\Gamma}\right]$. Therefore, there is the C^{*}-morphism $\Phi_{\lambda}:\left(G\times_{\alpha}A\left[\tau_{\Gamma}\right]\right)_{\lambda}\rightarrow L(\mathcal{H})$ such that $\Phi_{\lambda}\circ\overline{\pi_{\lambda}^{G\times_{\alpha}A[\tau_{\Gamma}]}}=\Phi$. Moreover, $(\Phi_{\lambda},\mathcal{H})$ is a nondegenerate representation of $\left(G\times_{\alpha}A\left[\tau_{\Gamma}\right]\right)_{\lambda}$ such that

$$\overline{\Phi_{\lambda}}\circ i_{A_{\lambda}}=\varphi\text{ and }\overline{\Phi_{\lambda}}\circ\left(\overline{\pi_{\lambda}^{G\times_{\alpha}A[\tau_{\Gamma}]}}\circ i_{G}\right)=u.$$

Thus, we showed that $\left(G\times_{\alpha}A\left[\tau_{\Gamma}\right]\right)_{\lambda}$ is isomorphic to $G\times_{\alpha^{\lambda}}A_{\lambda}$. □

Corollary 5.9. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro-C^*-dynamical system such that α is an inverse limit action. Then the pro-C^*-algebras $G \times_\alpha A[\tau_\Gamma]$ and $\lim\limits_{\leftarrow\lambda} G \times_{\alpha^\lambda} A_\lambda$ are isomorphic.*

Remark 5.10. If $(G, \alpha, A[\tau_\Gamma])$ is a pro-C^*-dynamical system such that α is an inverse limit action, then the notion of full pro-C^*-crossed product in the sense of Definition 5.1 coincides to the notion of full crossed product introduced by [P2, J4].

Definition 5.11. We say that $(G, \alpha, A[\tau_\Gamma])$ and $(G, \beta, B[\tau_{\Gamma'}])$ are conjugate if there is a pro-C^*-isomorphism $\varphi : A[\tau_\Gamma] \to B[\tau_{\Gamma'}]$ such that $\varphi \circ \alpha_t = \beta_t \circ \varphi$ for all $t \in G$.

Remark 5.12. If $(G, \alpha, A[\tau_\Gamma])$ and $(G, \beta, B[\tau_{\Gamma'}])$ are conjugate and α is strongly bounded, then β is strongly bounded too.

Proposition 5.13. *Let $(G, \alpha, A[\tau_\Gamma])$ and $(G, \beta, B[\tau_{\Gamma'}])$ be two pro-C^*-dynamical systems such that α and β are strongly bounded. If $(G, \alpha, A[\tau_\Gamma])$ and $(G, \beta, B[\tau_{\Gamma'}])$ are conjugate, then the full pro-C^*-crossed products associated to these pro-C^*-dynamical systems are isomorphic.*

Proof. Let $\varphi : A[\tau_\Gamma] \to B[\tau_{\Gamma'}]$ be a pro-C^*-isomorphism such that $\varphi \circ \alpha_t = \beta_t \circ \varphi$ for all $t \in G$. It is easy to check that $(\iota_B \circ \varphi, \iota_{G,B})$ is a nondegenerate covariant morphism from $(G, \alpha, A[\tau_\Gamma])$ to $G \times_\beta B[\tau_{\Gamma'}]$, where $(\iota_B, \iota_{G,B})$ is the covariant morphism from $(G, \beta, B[\tau_{\Gamma'}])$ to $G \times_\beta B[\tau_{\Gamma'}]$ which defines the full pro-C^*-crossed product of $B[\tau_{\Gamma'}]$ by β. Then, by Proposition 5.3, there is a nondegenerate pro-C^*-morphism $\Phi : G \times_\alpha A[\tau_\Gamma] \to M(G \times_\beta B[\tau_{\Gamma'}])$ such that $\overline{\Phi} \circ \iota_A = \iota_B \circ \varphi$ and $\overline{\Phi} \circ \iota_{G,A} = \iota_{G,B}$. Moreover, using Definition 5.1, it is easy to check that $\Phi(G \times_\alpha A[\tau_\Gamma]) \subseteq G \times_\beta B[\tau_{\Gamma'}]$. In the same manner, we obtain a nondegenerate pro-C^*-morphism $\Psi : G \times_\beta B[\tau_{\Gamma'}] \to M(G \times_\alpha A[\tau_\Gamma])$ such that $\overline{\Psi} \circ \iota_B = \iota_A \circ \varphi^{-1}$ and $\overline{\Psi} \circ \iota_{G,B} = \iota_{G,A}$.

From

$$(\Phi \circ \Psi)(\iota_B(b)\, \iota_{G,B}(f)) = \Phi\left(\iota_A \circ \varphi^{-1}(b)\, \iota_{G,A}(f)\right) = \iota_B(b)\, \iota_{G,B}(f)$$

and

$$(\Psi \circ \Phi)(\iota_A(a)\, \iota_{G,A}(f)) = \Psi(\iota_B \circ \varphi(a)\, \iota_{G,B}(f)) = \iota_A(a)\, \iota_{G,A}(f)$$

for all $b \in B[\tau_{\Gamma'}]$, $a \in A[\tau_\Gamma]$ and $f \in C_c(G)$ and Definition 5.1, we deduce that Φ and Ψ are pro-C^*-isomorphisms. □

Corollary 5.14. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro-C^*-dynamical system such that α is strongly bounded.*

1. *Pro-C^*-algebras $G \times_\alpha A[\tau_\Gamma]$ and $G \times_\alpha A[\tau_{\Gamma^G}]$ are isomorphic.*
2. *$A[\tau_\Gamma]$ is isomorphic to a pro-C^*-subalgebra of $M(G \times_\alpha A[\tau_\Gamma])$.*

6. The reduced pro-C^*-crossed product

Let $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$ be two pro-C^*-algebras. For each $p_\lambda \in \Gamma$ and $q_\delta \in \Gamma'$, the map $\vartheta_{p_\lambda, q_\delta} : A[\tau_\Gamma] \otimes_{\text{alg}} B[\tau_{\Gamma'}] \to [0, \infty)$ given by

$$\vartheta_{p_\lambda, q_\delta}(z) = \sup\{\|(\varphi \otimes \psi)(z)\|\, ; \varphi \in \mathcal{R}_\lambda(A[\tau_\Gamma]),\, \psi \in \mathcal{R}_\delta(B[\tau_{\Gamma'}])\}$$

defines a C^*-seminorm on the algebraic tensor product $A[\tau_\Gamma] \otimes_{\text{alg}} B[\tau_{\Gamma'}]$. The completion of $A[\tau_\Gamma] \otimes_{\text{alg}} B[\tau_{\Gamma'}]$ with respect to the topology given by the family of C^*-seminorms $\{\vartheta_{p_\lambda, q_\delta}; p_\lambda \in \Gamma, q_\delta \in \Gamma'\}$ is a pro-C^*-algebra, denoted by $A[\tau_\Gamma] \otimes_{\min} B[\tau_{\Gamma'}]$, and called the minimal or injective tensor product of the pro-C^*-algebras $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$ (see [F, Chapter VII]). Moreover, for each $p_\lambda \in \Gamma$ and $q_\delta \in \Gamma'$, the C^*-algebras $(A[\tau_\Gamma] \otimes_{\min} B[\tau_{\Gamma'}])_{(\lambda,\delta)}$ and $A_\lambda \otimes_{\min} B_\delta$ are isomorphic.

Let $(G, \alpha, A[\tau_\Gamma])$ be a pro-C^*-dynamical system such that α is strongly bounded. Since α is strongly bounded, for each $a \in A$, the map $t \mapsto \alpha_{t^{-1}}(a)$ defines an element in $C_b(G, A[\tau_\Gamma])$, the pro-C^*-algebra of all bounded continuous functions from G to $A[\tau_\Gamma]$, and so there is a map $\widetilde{\alpha} : A[\tau_\Gamma] \to C_b(G, A[\tau_\Gamma])$ given by $\widetilde{\alpha}(a)(t) = \alpha_{t^{-1}}(a)$.

Lemma 6.1. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro-C^*-dynamical system such that α is strongly bounded. Then $\widetilde{\alpha}$ is a nondegenerate faithful pro-C^*-morphism from $A[\tau_\Gamma]$ to $M(A[\tau_\Gamma] \otimes_{\min} C_0(G))$ with closed range. Moreover, if α is an inverse limit action, then $\widetilde{\alpha}$ is an inverse limit pro-C^*-morphism.*

Proof. Clearly, $\widetilde{\alpha}$ is a $*$-morphism. For each $p_\lambda \in \Gamma$, there is $p_\mu \in \Gamma$ such that

$$p_\lambda(a) = p_\lambda(\alpha_e(a)) \le \sup\{p_\lambda(\alpha_t(a)); t \in G\} = p_{\lambda, C_b(G, A[\tau_\Gamma])}(\widetilde{\alpha}(a)) \le p_\mu(a)$$

for all $a \in A$. Therefore, $\widetilde{\alpha}$ is an injective pro-C^*-morphism with closed range. By [J2, p. 76], $C_b(G, A[\tau_\Gamma])$ can be identified to a pro-C^*-subalgebra of $M(A[\tau_\Gamma] \otimes_{\min} C_0(G))$, and then $\widetilde{\alpha}$ can be regarded as a pro-C^*-morphism from $A[\tau_\Gamma]$ to $M(A[\tau_\Gamma] \otimes_{\min} C_0(G))$.

To show that $\widetilde{\alpha}$ is nondegenerate, let $\{e_i\}_{i \in I}$ be an approximate unit for $A[\tau_\Gamma]$. In the same manner as in [V, Proposition 5.1.5], we show that $\{\widetilde{\alpha}(e_i)\}_{i \in I}$ is strictly convergent. Indeed, let $a \in A$, $f \in C_c(G)$ and $p_\lambda \in \Gamma$. Then

$$\begin{aligned}
& p_{\lambda, C_b(G, A[\tau_\Gamma])}(\widetilde{\alpha}(e_i)(a \otimes f) - a \otimes f) \\
= \; & \sup\{p_\lambda(\alpha_{t^{-1}}(e_i)af(t) - af(t)); t \in G\} \\
\le \; & \|f\|_\infty \sup\{p_\lambda(\alpha_{t^{-1}}(e_i\alpha_t(a) - \alpha_t(a))); t \in \operatorname{supp}(f)\} \\
\le \; & \|f\|_\infty \sup\{p_\mu(e_i\alpha_t(a) - \alpha_t(a)); t \in \operatorname{supp}(f)\}
\end{aligned}$$

for some $p_\mu \in \Gamma$. For each $i \in I$, consider the function $f_i : G \to \mathbb{C}$, $f_i(t) = p_\mu(e_i\alpha_t(a) - \alpha_t(a))$. Clearly, $\{f_i\}_{i \in I}$ is a net of continuous functions on G which is uniformly bounded and equicontinuous. Then, by Arzelà–Ascoli's theorem, it is uniformly convergent on compact subsets of G. Therefore, $\{\widetilde{\alpha}(e_i)\}_{i \in I}$ is strictly convergent, and so the pro-C^*-morphism $\widetilde{\alpha}$ is nondegenerate.

Suppose that $\alpha_t = \lim\limits_{\leftarrow \lambda} \alpha_t^\lambda$ for each $t \in G$. Then $\left(\widetilde{\alpha^\lambda}\right)_\lambda$ is an inverse system of C^*-morphisms and $\widetilde{\alpha} = \lim\limits_{\leftarrow \lambda} \widetilde{\alpha^\lambda}$. □

Let $\varphi : A[\tau_\Gamma] \to M(B[\tau_{\Gamma'}])$ be a nondegenerate pro-C^*-morphism and let $M : C_0(G) \to L(L^2(G))$ be the representation by multiplication operators. Then there is a nondegenerate pro-C^*-morphism $\varphi \otimes M : A[\tau_\Gamma] \otimes_{\min} C_0(G) \to M(B[\tau_{\Gamma'}] \otimes_{\min} \mathcal{K}(L^2(G))$ such that $(\varphi \otimes M)(a \otimes f) = \varphi(a) \otimes M_f$, were $\mathcal{K}(L^2(G))$ denotes the C^*-algebra of all compact operators on the Hilbert space $L^2(G)$. Since $\widetilde{\alpha}$ is a nondegenerate pro-C^*-morphism from $A[\tau_\Gamma]$ to $M(A[\tau_\Gamma] \otimes_{\min} C_0(G))$, $\widetilde{\varphi} =$

$\overline{\varphi \otimes M} \circ \widetilde{\alpha}$ is a nondegenerate pro-C^*-morphism from $A[\tau_\Gamma]$ to $M(B[\tau_{\Gamma'}] \otimes_{\min} \mathcal{K}(L^2(G))$.

Let $\lambda_G : G \to \mathcal{U}(L^2(G))$ be the left representation of G on $L^2(G)$ given by $(\lambda_G)_t(\xi)(s) = \xi(t^{-1}s)$. Then $1 \otimes \lambda_G : G \to \mathcal{U}(M(B[\tau_{\Gamma'}] \otimes \mathcal{K}(L^2(G)))$, where $(1 \otimes \lambda_G)_t(b \otimes \xi)(s) = b\xi(t^{-1}s)$, is a strict continuous group morphism from G to $\mathcal{U}(M(B[\tau_{\Gamma'}] \otimes_{\min} \mathcal{K}(L^2(G)))$, and $(\widetilde{\varphi}, 1 \otimes \lambda_G)$ is a nondegenerate covariant morphism of $(G, \alpha, A[\tau_\Gamma])$ to $B[\tau_{\Gamma'}] \otimes_{\min} \mathcal{K}(L^2(G)$. By Proposition 5.3, there is a unique nondegenerate pro-C^*-morphism $\widetilde{\varphi} \times (1 \otimes \lambda_G) : G \times_\alpha A[\tau_\Gamma] \to M(B[\tau_{\Gamma'}] \otimes_{\min} \mathcal{K}(L^2(G))$ such that $\widetilde{\varphi} \times (1 \otimes \lambda_G) \circ \iota_A = \widetilde{\varphi}$ and $\widetilde{\varphi} \times (1 \otimes \lambda_G) \circ \iota_G = 1 \otimes \lambda_G$.

If $\varphi = \mathrm{id}_A$, the nondegenerate pro-C^*-morphism $\widetilde{\mathrm{id}}_A \times (1 \otimes \lambda_G) : G \times_\alpha A[\tau_\Gamma] \to M(A[\tau_\Gamma] \otimes_{\min} \mathcal{K}(L^2(G))$ is denoted by Λ_A^G. It is easy to check that $\widetilde{\varphi} \times (1 \otimes \lambda_G) = \overline{\varphi \otimes \mathrm{id}_{\mathcal{K}(L^2(G))}} \circ \Lambda_A^G$.

If α is an inverse limit action, $\alpha_t = \lim\limits_{\leftarrow \lambda} \alpha_t^\lambda$ for each $t \in G$, then it is easy to check that Λ_A^G is an inverse limit pro-C^*-morphism, $\Lambda_A^G = \lim\limits_{\leftarrow \lambda} \Lambda_{A_\lambda}^G$.

Definition 6.2. The reduced pro-C^*-crossed product of $A[\tau_\Gamma]$ by α is the pro-C^*-subalgebra $G \times_{\alpha,r} A[\tau_\Gamma]$ of $M(A[\tau_\Gamma] \otimes_{\min} \mathcal{K}(L^2(G)))$ generated by the range of Λ_A^G.

Remark 6.3. From

$$\Lambda_A^G(\iota_A(a)\,\iota_G(f)) = \left(\overline{\mathrm{id}_A \otimes M} \circ \widetilde{\alpha}\right)(a)\,(1 \otimes \lambda_G)(f) = \widetilde{\alpha}(a)\,(1 \otimes \lambda_G(f))$$

for all $a \in A$ and for all $f \in C_c(G)$, and taking into account that $G \times_\alpha A[\tau_\Gamma]$ is generated by $\{\iota_A(a)\,\iota_G(f); a \in A,\ f \in C_c(G)\}$, we conclude that $G \times_{\alpha,r} A[\tau_\Gamma]$ is the pro-C^*-subalgebra of $M(A[\tau_\Gamma] \otimes_{\min} \mathcal{K}(L^2(G)))$ generated by $\{\widetilde{\alpha}(a)\,(1 \otimes \lambda_G(f)); a \in A,\ f \in C_c(G)\}$.

Remark 6.4. If α is an inverse limit action, $\alpha_t = \lim\limits_{\leftarrow \lambda} \alpha_t^\lambda$ for each $t \in G$, then

$$G \times_{\alpha,r} A[\tau_\Gamma] = \overline{\Lambda_A^G(G \times_\alpha A[\tau_\Gamma])} = \lim_{\leftarrow \lambda} \overline{\Lambda_{A_\lambda}^G(G \times_{\alpha^\lambda} A_\lambda)} = \lim_{\leftarrow \lambda} G \times_{\alpha^\lambda, r} A_\lambda$$

and moreover, for each $p_\lambda \in \Gamma$, the C^*-algebras $(G \times_{\alpha,r} A[\tau_\Gamma])_\lambda$ and $G \times_{\alpha^\lambda, r} A_\lambda$ are isomorphic.

Remark 6.5. Since the trivial action of a locally compact group G on a pro-C^*-algebra $A[\tau_\Gamma]$ is an inverse limit action, the reduced pro-C^*-crossed product of $A[\tau_\Gamma]$ by the trivial action is the inverse limit of the reduced crossed products of A_λ by the trivial action, and so it is isomorphic to the pro-C^*-algebra $A[\tau_\Gamma] \otimes_{\min} C_r^*(G)$, where $C_r^*(G)$ is the reduced group C^*-algebra of G.

Proposition 6.6. *Let $(G, \alpha, A[\tau_\Gamma])$ and $(G, \beta, B[\tau_{\Gamma'}])$ be two pro-C^*-dynamical systems such that α and β are strongly bounded. If $(G, \alpha, A[\tau_\Gamma])$ and $(G, \beta, B[\tau_{\Gamma'}])$ are conjugate, then the reduced pro-C^*-crossed products associated to these pro-C^*-dynamical systems are isomorphic.*

Proof. Let $\varphi : A[\tau_\Gamma] \to B[\tau_{\Gamma'}]$ be a pro-C^*-isomorphism such that $\varphi \circ \alpha_t = \beta_t \circ \varphi$ for all $t \in G$. It is easy to check that $\overline{\varphi \otimes \mathrm{id}_{\mathcal{K}(L^2(G))}} \circ \widetilde{\alpha} = \widetilde{\beta} \circ \varphi$. From

$$\overline{\varphi \otimes \mathrm{id}_{\mathcal{K}(L^2(G))}}\,(\widetilde{\alpha}(a)\,(1 \otimes \lambda_G(f))) = \widetilde{\beta}(\varphi(a))\,(1 \otimes \lambda_G(f))$$

for all $a \in A$ and for all $f \in C_c(G)$, and taking into account that

$$\overline{\operatorname{span}\{\widetilde{\alpha}(a)(1 \otimes \lambda_G(f))\,;a \in A, f \in C_c(G)\}} = G \times_{\alpha,r} A[\tau_\Gamma]$$

and

$$\overline{\operatorname{span}\{\widetilde{\beta}(a)(1 \otimes \lambda_G(f))\,;a \in B, f \in C_c(G)\}} = G \times_{\beta,r} B[\tau_{\Gamma'}],$$

we conclude that $\Phi_1 = \overline{\varphi \otimes \mathrm{id}_{\mathcal{K}(L^2(G))}}|_{G\times_{\alpha,r}A[\tau_\Gamma]}$ is a pro-C^*-morphism from $G \times_{\alpha,r} A[\tau_\Gamma]$ to $G \times_{\beta,r} B[\tau_{\Gamma'}]$.

In the same manner, we conclude that $\Phi_2 = \overline{\varphi^{-1} \otimes \mathrm{id}_{\mathcal{K}(L^2(G))}}|_{G\times_{\beta,r}B[\tau_{\Gamma'}]}$ is a pro-C^*-morphism from $G \times_{\beta,r} B[\tau_{\Gamma'}]$ to $G \times_{\alpha,r} A[\tau_\Gamma]$. Moreover, $\Phi_1 \circ \Phi_2 = \mathrm{id}_{G\times_{\beta,r}B[\tau_{\Gamma'}]}$ and $\Phi_2 \circ \Phi_1 = \mathrm{id}_{G\times_{\alpha,r}A[\tau_\Gamma]}$, since

$$\Phi_1 \circ \Phi_2 \left(\widetilde{\beta}(b)(1 \otimes \lambda_G(f))\right) = \widetilde{\beta}(b)(1 \otimes \lambda_G(f))$$

for all $b \in B$ and for all $f \in C_c(G)$ and

$$\Phi_2 \circ \Phi_1 (\widetilde{\alpha}(a)(1 \otimes \lambda_G(f))) = \widetilde{\alpha}(a)(1 \otimes \lambda_G(f))$$

for all $a \in A$ and for all $f \in C_c(G)$. Therefore, the pro-C^*-algebras $G \times_{\alpha,r} A[\tau_\Gamma]$ and $G \times_{\beta,r} B[\tau_{\Gamma'}]$ are isomorphic. □

Corollary 6.7. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro-C^*-dynamical system such that α is strongly bounded. Then the pro-C^*-algebras $G \times_{\alpha,r} A[\tau_\Gamma]$ and $G \times_{\alpha,r} A[\tau_{\Gamma^G}]$ are isomorphic.*

Remark 6.8. If α is an action of an amenable locally compact group G on a C^*-algebra A, then the C^*-morphism Λ_A^G is injective and the full crossed product A by α is isomorphic to the reduced crossed product of A by α. If α is an inverse limit action of G on a pro-C^*-algebra $A[\tau_\Gamma]$ and G is amenable, then $\Lambda_A^G = \lim\limits_{\leftarrow\lambda} \Lambda_{A_\lambda}^G$, and so Λ_A^G is an injective pro-C^*-morphism with closed range and its inverse is continuous. Therefore, if G is amenable and α is an inverse limit action, then the full pro-C^*-crossed product of $A[\tau_\Gamma]$ by α is isomorphic to the reduced pro-C^*-crossed product of $A[\tau_\Gamma]$ by α.

Proposition 6.9. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro-C^*-dynamical system such that α is strongly bounded. If G is amenable then the full pro-C^*-crossed product of $A[\tau_\Gamma]$ by α is isomorphic to the reduced pro-C^*-crossed product of $A[\tau_\Gamma]$ by α.*

Proof. It follows from Corollaries 5.14 and 6.7, and Remark 6.8. □

7. Pro-C^*-crossed products and tensor products

Let $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$ be two pro-C^*-algebras. For each $p_\lambda \in \Gamma$ and $q_\delta \in \Gamma'$, the map $t_{p_\lambda,q_\delta} : A[\tau_\Gamma] \otimes_{\mathrm{alg}} B[\tau_{\Gamma'}] \to [0,\infty)$ given by

$$t_{p_\lambda,q_\delta}(z) = \sup\{\|\varphi \circ \pi_{p_\lambda,q_\delta}(z)\|\,;\varphi \text{ is a } *\text{-representation of } A_\lambda \otimes_{\mathrm{alg}} B_\delta\},$$

where $\pi_{p_\lambda,q_\delta}(a \otimes b) = \pi_\lambda^A(a) \otimes \pi_\delta^B(b)$, defines a C^*-seminorm on the algebraic tensor product $A[\tau_\Gamma] \otimes_{\mathrm{alg}} B[\tau_{\Gamma'}]$. The completion of $A[\tau_\Gamma] \otimes_{\mathrm{alg}} B[\tau_{\Gamma'}]$ with respect to the topology given by the family of C^*-seminorms $\{t_{p_\lambda,q_\delta}; p_\lambda \in \Gamma, q_\delta \in \Gamma'\}$ is a pro-C^*-algebra, denoted by $A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}]$, and called the maximal or projective tensor product of the pro-C^*-algebras $A[\tau_\Gamma]$ and $B[\tau_{\Gamma'}]$ (see [F, Chapter VII]).

Moreover, for each $p_\lambda \in \Gamma$ and $q_\delta \in \Gamma'$, the C^*-algebras $(A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}])_{(\lambda,\delta)}$ and $A_\lambda \otimes_{\max} B_\delta$ are isomorphic.

Remark 7.1. The trivial action of G on $A[\tau_\Gamma]$ is an inverse limit action, and so the full pro-C^*-crossed product of $A[\tau_\Gamma]$ by the trivial action is isomorphic to $A[\tau_\Gamma] \otimes_{\max} C^*(G)$, where $C^*(G)$ is the group C^*-algebra of G, [J2, Corollary 1.3.9].

Let $(G, \alpha, A[\tau_\Gamma])$ be a pro-C^*-dynamical system such that α is strongly bounded and let $B[\tau_{\Gamma'}]$ be a pro-C^*-algebra. Then $t \mapsto (\alpha \otimes \mathrm{id})_t$, where $(\alpha \otimes \mathrm{id})_t (a \otimes b) = \alpha_t (a) \otimes b$, is a strong bounded action of G on $A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}]$.

The following theorem gives an "associativity" between $\times_\alpha$ and $\otimes_{\max}$.

Theorem 7.2. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro-C^*-dynamical system such that α is strongly bounded and let $B[\tau_{\Gamma'}]$ be a pro-C^*-algebra. Then the pro-C^*-algebras $G \times_{\alpha\otimes id} (A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}])$ and $(G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}]$ are isomorphic.*

Proof. Let $\rho_{G\times_\alpha A[\tau_\Gamma]} : G \times_\alpha A[\tau_\Gamma] \to M((G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}])$ and $\rho_B : B[\tau_{\Gamma'}] \to M((G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}])$ be the canonical maps. Then $\overline{\rho_{G\times_\alpha A[\tau_\Gamma]}} \circ \iota_A : A[\tau_\Gamma] \to M((G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}])$ and $\rho_B : B[\tau_{\Gamma'}] \to M((G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}])$ are nondegenerate pro-C^*-morphisms with commuting ranges.

Let $j_{G\times_{\alpha\otimes\mathrm{id}} A[\tau_\Gamma]\otimes_{\max} B[\tau_{\Gamma'}]} = \overline{\rho_{G\times_\alpha A[\tau_\Gamma]}} \circ \iota_A \otimes \rho_B$ and $j_G = \overline{\rho_{G\times_\alpha A[\tau_\Gamma]}} \circ \iota_G$. A simple calculus shows that $\left(j_{G\times_{\alpha\otimes\mathrm{id}} A[\tau_\Gamma]\otimes_{\max} B[\tau_{\Gamma'}]}, j_G\right)$ is a nondegenerate covariant pro-C^*-morphism from $(G, \alpha \otimes \mathrm{id}, A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}])$ to $M((G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}])$. Moreover, from

$$
\begin{aligned}
& j_{G\times_{\alpha\otimes\mathrm{id}} A[\tau_\Gamma]\otimes_{\max} B[\tau_{\Gamma'}]} (a \otimes b)\, j_G (f) \\
= \; & \overline{\rho_{G\times_\alpha A[\tau_\Gamma]}} (\iota_A (a))\, \rho_B (b)\, \overline{\rho_{G\times_\alpha A[\tau_\Gamma]}} (\iota_G (f)) \\
= \; & \overline{\rho_{G\times_\alpha A[\tau_\Gamma]}} (\iota_A (a))\, \overline{\rho_{G\times_\alpha A[\tau_\Gamma]}} (\iota_G (f))\, \rho_B (b) \\
= \; & \rho_{G\times_\alpha A[\tau_\Gamma]} (\iota_A (a)\, \iota_G (f))\, \rho_B (b)
\end{aligned}
$$

for all $a \in A$, for all $b \in B$ and for all $f \in C_c(G)$, and taking into account that $\{\iota_A (a)\, \iota_G (f) \,; a \in A, f \in C_c(G)\}$ generates $G \times_\alpha A[\tau_\Gamma]$ and $\{\rho_{G\times_\alpha A[\tau_\Gamma]} (z)\, \rho_B (b) \,; z \in G \times_\alpha A[\tau_\Gamma], b \in B\}$ generates $(G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}]$, we conclude that

$$
\begin{aligned}
& \overline{\mathrm{span}\{j_{G\times_{\alpha\otimes\mathrm{id}} A[\tau_\Gamma]\otimes_{\max} B[\tau_{\Gamma'}]} (a \otimes b)\, j_G (f) \,; a \in A, b \in b, f \in C_c(G)\}} \\
= \; & (G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}] .
\end{aligned}
$$

Let $(\varphi, u, \mathcal{H})$ be a nondegenerate covariant representation of $(G, \alpha\otimes\mathrm{id}, A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}])$. Then $(\varphi, \mathcal{H})$ is a nondegenerate representation of $A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}]$, and so there is a nondegenerate representation $\left(\varphi_{(\lambda,\delta)}, \mathcal{H}\right)$ of $A_\lambda \otimes_{\max} B_\delta$ such that $\varphi_{(\lambda,\delta)} \circ \pi_{(\lambda,\delta)}^{A[\tau_\Gamma]\otimes_{\max} B[\tau_{\Gamma'}]} = \varphi$. Let $(\varphi_\lambda, \mathcal{H})$ and $(\varphi_\delta, \mathcal{H})$ be the nondegenerate representations of A_λ, respectively B_δ with commuting ranges such that $\varphi_{(\lambda,\delta)} (a \otimes b) = \varphi_\lambda (a)\, \varphi_\delta (b)$ for all $a \in A_\lambda$ and $b \in B_\delta$. Then $\left(\varphi_\lambda \circ \pi_\lambda^A, u, \mathcal{H}\right)$ is a nondegenerate covariant representation of $(G, \alpha, A[\tau_\Gamma])$, and so there is a nondegenerate representation $(\Phi_1, \mathcal{H})$ of $G \times_\alpha A[\tau_\Gamma]$ such that $\overline{\Phi_1} \circ \iota_A = \varphi_\lambda \circ \pi_\lambda^A$ and $\overline{\Phi_1} \circ \iota_G = u$. It is easy to check that $(\Phi_1, \mathcal{H})$ and $(\Phi_2, \mathcal{H})$, where $\Phi_2 = \varphi_\delta \circ \pi_\delta^B$, are nondegenerate representations of $G \times_\alpha A[\tau_\Gamma]$ respectively $B[\tau_{\Gamma'}]$ with commuting ranges. Let

$(\Phi, \mathcal{H})$ be the nondegenerate representation of $(G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}]$ given by $\Phi(z \otimes b) = \Phi_1(z)\Phi_2(b)$. Then

$$\begin{aligned}
& \overline{\Phi}\left(j_{G\times_{\alpha\otimes\mathrm{id}}A[\tau_\Gamma]\otimes_{\max}B[\tau_{\Gamma'}]}(a \otimes b)\right) \\
= \; & \overline{\Phi}\left(\overline{\rho_{G\times_\alpha A[\tau_\Gamma]}}(\iota_A(a))\rho_B(b)\right) = \overline{\Phi_1}\left((\iota_A(a))\overline{\Phi_2}(\rho_B(b)\right) \\
= \; & \left(\varphi_\lambda \circ \pi_\lambda^A(a)\right)\left(\varphi_\delta \circ \pi_\delta^B(b)\right) = \varphi_{(\lambda,\delta)}\left(\pi_\lambda^A(a) \otimes \pi_\delta^B(b)\right) \\
= \; & \varphi_{(\lambda,\delta)} \circ \pi_{(\lambda,\delta)}^{A[\tau_\Gamma]\otimes_{\max}B[\tau_{\Gamma'}]}(a \otimes b) = \varphi(a \otimes b)
\end{aligned}$$

for all $a \in A$ and $b \in B$, and

$$\overline{\Phi}(j_G(f)) = \overline{\Phi}\left(\overline{\rho_{G\times_\alpha A[\tau_\Gamma]}} \circ \iota_G(f)\right) = \overline{\Phi_1}(\iota_G(f)) = u(f)$$

for all $f \in C_c(G)$. Therefore, by Definition 5.1 and Corollary 5.4, the pro-C^*-algebras $G \times_{\alpha\otimes\mathrm{id}} A[\tau_\Gamma] \otimes_{\max} B[\tau_{\Gamma'}]$ and $(G \times_\alpha A[\tau_\Gamma]) \otimes_{\max} B[\tau_{\Gamma'}]$ are isomorphic. $\square$

Let $(G, \alpha, A[\tau_\Gamma])$ be a pro-C^*-dynamical system such that α is strongly bounded and let $B[\tau_{\Gamma'}]$ be a pro-C^*-algebra. Then $t \mapsto (\alpha \otimes \mathrm{id})_t$, where $(\alpha \otimes \mathrm{id})_t(a \otimes b) = \alpha_t(a) \otimes b$, is a strong bounded action of G on $A[\tau_\Gamma] \otimes_{\min} B[\tau_{\Gamma'}]$.

The following theorem gives an "associativity" between $\times_{\alpha,r}$ and $\otimes_{\min}$.

Theorem 7.3. *Let $(G, \alpha, A[\tau_\Gamma])$ be a pro-C^*-dynamical system such that α is strongly bounded and let $B[\tau_{\Gamma'}]$ be a pro-C^*-algebra. Then the pro-C^*-algebras $G \times_{\alpha\otimes id,r} (A[\tau_\Gamma] \otimes_{\min} B[\tau_{\Gamma'}])$ and $(G \times_{\alpha,r} A[\tau_\Gamma]) \otimes_{\min} B[\tau_{\Gamma'}]$ are isomorphic.*

Proof. The map $\mathrm{id}_A \otimes \sigma_{B,\mathcal{K}(L^2(G))} : A[\tau_\Gamma] \otimes_{\min} B[\tau_{\Gamma'}] \otimes_{\min} \mathcal{K}(L^2(G)) \to A[\tau_\Gamma] \otimes_{\min} \mathcal{K}(L^2(G)) \otimes_{\min} B[\tau_{\Gamma'}]$ given by

$$\mathrm{id}_A \otimes \sigma_{B,\mathcal{K}(L^2(G))}(a \otimes b \otimes T) = a \otimes T \otimes b$$

is a pro-C^*-isomorphism. Moreover, $\mathrm{id}_A \otimes \sigma_{B,\mathcal{K}(L^2(G))}$ is an inverse limit of C^*-isomorphisms. From

$$\overline{\mathrm{id}_A \otimes \sigma_{B,\mathcal{K}(L^2(G))}}\left(\widetilde{\alpha \otimes \mathrm{id}}(a \otimes b)(1 \otimes \lambda_G(f)\right) = \widetilde{\alpha}(a)\left(1_{M(A[\tau_\Gamma])} \otimes \lambda_G(f) \otimes b\right)$$

for all $a \in A$, for all $b \in B$ and for all $f \in C_c(G)$, and Remark 6.3, we deduce that

$$\overline{\mathrm{id}_A \otimes \sigma_{B,\mathcal{K}(L^2(G))}}|_{G\times_{\alpha\otimes\mathrm{id},r}A[\tau_\Gamma]\otimes_{\min}B[\tau_{\Gamma'}]}$$

is a pro-C^*-isomorphism from $G \times_{\alpha\otimes\mathrm{id},r} (A[\tau_\Gamma] \otimes_{\min} B[\tau_{\Gamma'}])$ onto $(G \times_{\alpha,r} A[\tau_\Gamma]) \otimes_{\min} B[\tau_{\Gamma'}]$. Therefore, the pro-$C^*$-algebras $(G \times_{\alpha,r} A[\tau_\Gamma]) \otimes_{\min} B[\tau_{\Gamma'}]$ and $G \times_{\alpha\otimes\mathrm{id},r} (A[\tau_\Gamma] \otimes_{\min} B[\tau_{\Gamma'}])$ are isomorphic. $\square$

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Department of Mathematics, Faculty of Applied Sciences, University Politehnica of Bucharest, Spl. Independentei nr. 313, Bucharest, 060042, Romania;
Simion Stoilow Institute of Mathematics of the Roumanian Academy, 21 Calea Grivitei Street, 010702 Bucharest, Romania.

E-mail address: mjoita@fmi.unibuc.ro

URL: http://sites.google.com/a/g.unibuc.ro/maria-joita/