Annals of Functional Analysis
Ann. Funct. Anal. 6 (2015), no. 2, 104–115
http://doi.org/10.15352/afa/06-2-10
ISSN: 2008-8752 (electronic)
http://projecteuclid.org/afa

INNER FUNCTIONS AND WEIGHTED COMPOSITION OPERATORS ON THE HARDY-HILBERT SPACE WITH THE UNBOUNDED WEIGHTS

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Communicated by J. Esterle

ABSTRACT. Let φ be an analytic self-map of the open unit disk. It is given several sufficient conditions on φ for which there is $u \in H^2 \setminus H^\infty$ such that the weighted composition operator M_uC_φ on H^2 is bounded.

1. INTRODUCTION

Let $\mathbb{D}$ be the open unit disk and m be the normalized Lebesgue measure on $\partial\mathbb{D}$. We denote by $L^2(\partial\mathbb{D})$ the space of square integrable functions on $\partial\mathbb{D}$ with respect to m. For $1 \le p < \infty$, let H^p be the space of analytic functions f on $\mathbb{D}$ satisfying

$$\|f\|_p^p := \lim_{r\to 1} \int_{\partial\mathbb{D}} |f(re^{i\theta})|^p \, dm(e^{i\theta}) < \infty.$$

The space H^p is called the Hardy space. We denote by H^∞ the space of bounded analytic functions on $\mathbb{D}$ with the supremum norm $\|f\|_\infty$. For each $f \in H^2$, there is the boundary function f^* of f defined by $f^*(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta})$ a.e. on $\partial\mathbb{D}$. We have $f^* \in L^2(\partial\mathbb{D})$ (see [2, 3, 4]).

We denote by $\mathcal{S}$ the set of analytic self-maps of $\mathbb{D}$. For each $\varphi \in \mathcal{S}$, we may define the composition operator C_φ by $C_\varphi f = f \circ \varphi$ for $f \in H^2$. By the Littlewood subordination theorem [6], C_φ is a bounded linear operator on H^2. Recently there are many researches on composition operators on various spaces of analytic functions. For $u \in H^\infty$, we may define the weighted composition

Date: Received: Jun. 1, 2014; Accepted: Jul. 13, 2014.
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2010 *Mathematics Subject Classification.* Primary 47B38; Secondary 30H10.
Key words and phrases. Hardy space, inner function, weighted composition operator, boundedness.

operator $M_uC_\varphi : H^2 \to H^2$ by $(M_uC_\varphi)f = u(f \circ \varphi)$. Of course, M_uC_φ is bounded on H^2. See [1, 9] for the basic properties of (weighted) composition operators.

Let $u \in H^2$ and $\varphi \in \mathcal{S}$. For each $f \in H^2$, we have

$$\|M_uC_\varphi f\|_1 \leq \|u\|_2\|C_\varphi f\|_2 \leq \|u\|_2\|C_\varphi\|\|f\|_2.$$

Hence $M_uC_\varphi : H^2 \to H^1$ is a bounded linear map. If $\|\varphi\|_\infty < 1$, then it is not difficult to see that $M_uC_\varphi : H^2 \to H^2$ is bounded.

In this paper, for $\varphi \in \mathcal{S}$ with $\|\varphi\|_\infty = 1$ we shall study the boundedness of $M_uC_\varphi : H^2 \to H^2$ (see [5, 8]). More precisely, we consider the following problem.

Problem 1.1. For which $\varphi \in \mathcal{S}$, is there $u \in H^2 \backslash H^\infty$ such that $M_uC_\varphi : H^2 \to H^2$ is bounded?

A function $\psi \in H^\infty$ is called inner if $|\psi^*| = 1$ a.e. on $\partial\mathbb{D}$. In [5, Corollary 2.2], Nguyen, Ohno and the first author showed that if $\varphi \in \mathcal{S}$ is not inner, then there is $u \in H^2 \setminus H^\infty$ such that $M_uC_\varphi : H^2 \to H^2$ is bounded. So mainly we assume that φ is an inner function. We denote by $supp(\varphi)$ the set of $e^{i\theta} \in \partial\mathbb{D}$ at which φ does not have a continuous extension. Then $supp(\varphi)$ is a closed subset of $\partial\mathbb{D}$. It is known that $supp(\varphi) = \emptyset$ if and only if φ is a finite Blaschke product. It is not difficult to see that if $supp(\varphi) = \emptyset$, then $M_uC_\varphi : H^2 \to H^2$ is unbounded for every $u \in H^2 \setminus H^\infty$ (see [5, p. 1335]).

It is known that φ may be extended to a non-vanishing analytic function on some neighborhood of each $e^{i\theta} \in \partial\mathbb{D} \setminus supp(\varphi)$. Hence we may think that φ^* is differentiable on $\partial\mathbb{D} \setminus supp(\varphi)$. In [5, Proposition 2.9], Nguyen, Ohno and the first author essentially proved that if $\sup_{z\in\partial\mathbb{D}\setminus supp(\varphi)} |\varphi^{*\prime}(z)| = \infty$, then there is $u \in H^2 \setminus H^\infty$ such that $M_uC_\varphi : H^2 \to H^2$ is bounded.

In Section 2, we shall prove that if there is an open subarc U of $\partial\mathbb{D}$ such that $U \cap supp(\varphi) \neq \emptyset$ and $U \cap supp(\varphi)$ does not contain any interior points, then $\sup_{z\in\partial\mathbb{D}\setminus supp(\varphi)} |\varphi^{*\prime}(z)| = \infty$, so there exists a function $u \in H^2 \setminus H^\infty$ such that $M_uC_\varphi : H^2 \to H^2$ is bounded.

For an inner function φ, we consider the following two conditions.

(α) There is a sequence of mutually disjoint measurable subsets $\{C_n\}_{n\geq1}$ of $\partial\mathbb{D}$ and a sequence of positive numbers $\{\delta_n\}_{n\geq1}$ satisfying $\sum_{n=1}^\infty \delta_n < \infty$ such that $m(C_n) > 0$ and $m(C_n \cap \varphi^{*(-1)}(E)) \leq \delta_n m(E)$ for every measurable subset E of $\partial\mathbb{D}$ and for every $n \geq 1$.

(β) There is a sequence of mutually disjoint measurable subsets $\{E_n\}_{n\geq1}$ of $\partial\mathbb{D}$ such that $m(E_n \cap \varphi^{*(-1)}(E)) > 0$ for every measurable subset E of $\partial\mathbb{D}$ satisfying $m(E) > 0$ and for every $n \geq 1$.

We do not know whether conditions (α) and (β) hold or not for every inner function φ satisfying $m(supp(\varphi)) > 0$. In Section 3, we shall prove that if an inner function φ satisfies condition (α), then there is $u \in H^2 \setminus H^\infty$ such that $M_uC_\varphi : H^2 \to H^2$ is bounded. We also show that if φ satisfies condition (β), then φ satisfies condition (α).

The techniques used here will give us some light on further study of Problem 1.1.

2. Bounded weighted composition operators

The following proposition was proven in [5, Corollary 2.2]. We shall give its another proof.

Proposition 2.1. *Let $\varphi \in \mathcal{S}$. If φ is not inner, then there is $u \in H^2 \setminus H^\infty$ such that $M_u C_\varphi : H^2 \to H^2$ is bounded.*

Proof. For $0 < r < 1$, we write $\{|\varphi^*| < r\} = \{e^{it} \in \partial\mathbb{D} : |\varphi^*(e^{it})| < r\}$. Take $0 < r < 1$ satisfying $m(\{|\varphi^*| < r\}) > 0$. Let η be a positive unbounded function in $L^2(\partial\mathbb{D})$ such that $\eta = 1$ on $\partial\mathbb{D} \setminus \{|\varphi^*| < r\}$ and $\eta \geq 1$ a.e. on $\{|\varphi^*| < r\}$. By [4, p. 53], there is $u \in H^2$ satisfying $|u^*| = \eta$ a.e. on $\partial\mathbb{D}$. We have $u \notin H^\infty$. For $f \in H^2$, by [2, p. 36] we have

$$|f(z)| \leq \frac{\sqrt{2}\|f\|_2}{\sqrt{1-r}}, \quad |z| \leq r.$$

Hence

$$|f(\varphi^*(e^{i\theta}))| \leq \frac{\sqrt{2}\|f\|_2}{\sqrt{1-r}}, \quad e^{i\theta} \in \{|\varphi^*| < r\}.$$

Therefore

$$\begin{aligned}
&\|M_u C_\varphi f\|_2^2 \\
&= \int_{\partial\mathbb{D}} |u^*|^2 |(f\circ\varphi)^*|^2 \, dm \\
&= \int_{\{|\varphi^*|<r\}} |u^*|^2 |(f\circ\varphi)^*|^2 \, dm + \int_{\partial\mathbb{D}\setminus\{|\varphi^*|<r\}} |u^*|^2 |(f\circ\varphi)^*|^2 \, dm \\
&\leq \frac{2\|f\|_2^2}{1-r} \int_{\partial\mathbb{D}} |u^*|^2 \, dm + \int_{\partial\mathbb{D}} |(f\circ\varphi)^*|^2 \, dm \\
&\leq \Big(\frac{2\|\eta\|_2^2}{1-r} + \|C_\varphi\|^2 \Big) \|f\|_2^2.
\end{aligned}$$

Thus $M_u C_\varphi : H^2 \to H^2$ is bounded. □

Hereafter, to study Problem 1.1 we assume that φ is an inner function satisfying $supp(\varphi) \neq \emptyset$. In [5, Proposition 2.9], Nguyen, Ohno and the first author proved the following essentially.

Lemma 2.2. *Let φ be an inner function. If $\sup_{z\in\partial\mathbb{D}\setminus supp(\varphi)} |\varphi^{*\prime}(z)| = \infty$, then there exists a function $u \in H^2 \setminus H^\infty$ such that $M_u C_\varphi : H^2 \to H^2$ is bounded.*

Let φ be an inner function and $I = \{e^{it} : t_1 < t < t_2\}$ satisfy $I \cap supp(\varphi) = \emptyset$. Then there is a real valued differentiable function $\sigma(t)$ such that $\varphi^*(e^{it}) = e^{i\sigma(t)}$ and $\sigma'(t) > 0$ on (t_1, t_2). Admitting the values $\infty, -\infty$, we may define $s_k = \lim_{t\to t_k} \sigma(t)$ for $k = 1, 2$. Putting $\sigma(t_k) = s_k$, we think $\sigma(t)$ of an extended real valued continuous function on $[t_1, t_2]$ and $\varphi^*(I) = \{e^{is} : s_1 < s < s_1\}$.

For $e^{it_0} \in \partial\mathbb{D}$ and $\varepsilon > 0$, we write $I_\varepsilon(e^{it_0}) = \{e^{it} : t_0 - \varepsilon < t < t_0 + \varepsilon\}$.

Lemma 2.3. *Let φ be an inner function. If $supp(\varphi) = \{e^{it_0}\}$, then*

$$\sup_{z\in I_\varepsilon(e^{it_0})\setminus\{e^{it_0}\}} |\varphi^{*\prime}(z)| = \infty$$

for every $\varepsilon > 0$*.*

Proof. There is a real valued differentiable function $\sigma(t)$ on $(t_0, t_0 + 2\pi)$ such that $\varphi^*(e^{it}) = e^{i\sigma(t)}$ and $\sigma'(t) > 0$ for every $t_0 < t < t_0 + 2\pi$. Then either $\lim_{t\to t_0} \sigma(t) = -\infty$ or $\lim_{t\to t_0+2\pi} \sigma(t) = \infty$ (see [3, p. 90–91]). Hence we get the assertion. $\square$

Lemma 2.4. *Let* φ_1, φ_2 *be inner functions and* I *be an open subarc of* $\partial\mathbb{D}$ *such that* $I \cap supp(\varphi_1\varphi_2) = \emptyset$*. Then* $|(\varphi_1\varphi_2)^{*\prime}| = |\varphi_1^{*\prime}| + |\varphi_2^{*\prime}|$ *on* I*.*

Proof. Let $I = \{e^{it} : t_1 < t < t_2\}$. There are real valued differentiable functions $\sigma_1(t), \sigma_2(t)$ on (t_1, t_2) such that $\varphi_1^*(e^{it}) = e^{i\sigma_1(t)}$, $\varphi_2^*(e^{it}) = e^{i\sigma_2(t)}$, $\sigma_1'(t) > 0$ and $\sigma_2'(t) > 0$ for every $t_1 < t < t_2$. We have $(\varphi_1\varphi_2)^*(e^{it}) = e^{i(\sigma_1(t)+\sigma_2(t))}$. Hence

$$(\varphi_1\varphi_2)^{*\prime}(e^{it}) = -ie^{-it}\frac{d}{dt}(\varphi_1\varphi_2)^*(e^{it}) = e^{-it}(\sigma_1'(t) + \sigma_2'(t))e^{i(\sigma_1(t)+\sigma_2(t))}$$

for $t_1 < t < t_2$. Therefore

$$|(\varphi_1\varphi_2)^{*\prime}(e^{it})| = \sigma_1'(t) + \sigma_2'(t) = |\varphi_1^{*\prime}(e^{it})| + |\varphi_2^{*\prime}(e^{it})|.$$

$\square$

For a subset E of $\partial\mathbb{D}$, we denote by *int* E the interior of E in $\partial\mathbb{D}$.

Theorem 2.5. *Let* φ *be an inner function. If* $supp(\varphi) \neq \overline{int\, supp(\varphi)}$*, then* $\sup_{z\in\partial\mathbb{D}\setminus supp(\varphi)} |\varphi^{*\prime}(z)| = \infty$*.*

Proof. Take $e^{it_0} \in supp(\varphi) \setminus \overline{int\, supp(\varphi)}$ and then take an open subarc I of $\partial\mathbb{D}$ such that $e^{it_0} \in I$ and $I \cap \overline{int\, supp(\varphi)} = \emptyset$. For each $\lambda \in \mathbb{D}$, let $\tau_\lambda(z) = (z - \lambda)/(1 - \overline{\lambda}z), z \in \mathbb{D}$. By Frostman's theorem (see [3, p. 79]), there is $\lambda \in \mathbb{D}$ such that $\psi := \tau_\lambda \circ \varphi$ is a Blaschke product. We have $supp(\psi) = supp(\varphi)$. Then there is a sequence $\{a_k\}_{k\geq 1}$ in $\mathbb{D}$ such that $\psi(a_k) = 0$ for every $k \geq 1$ and $a_k \to e^{it_0}$ as $k \to \infty$. Let ψ_1 be the Blaschke subproduct of ψ with zeros $\{a_k\}_{k\geq 1}$. Then $supp(\psi_1) = \{e^{it_0}\}$. Let $\psi_2 = \psi/\psi_1$. Retaking a further subsequence of $\{a_k\}_{k\geq 1}$, we may assume that $supp(\psi_2) = supp(\psi)$. Since $e^{it_0} \in I$, we may take $\varepsilon > 0$ satisfying

$$I_\varepsilon(e^{it_0}) = \{e^{it} : t_0 - \varepsilon < t < t_0 + \varepsilon\} \subset I.$$

By Lemma 2.3, we have $\sup_{z\in I_\varepsilon(e^{it_0})\setminus\{e^{it_0}\}} |\psi_1^{*\prime}(z)| = \infty$. Since $I \cap \overline{int\, supp(\psi)} = \emptyset$, $I_\varepsilon(e^{it_0}) \setminus supp(\psi)$ is dense in $I_\varepsilon(e^{it_0})$. Hence

$$\sup_{z\in I_\varepsilon(e^{it_0})\setminus supp(\psi)} |\psi_1^{*\prime}(z)| = \infty.$$

Therefore by Lemma 2.4, we have

$$\begin{aligned}\sup_{z\in I_\varepsilon(e^{it_0})\setminus supp(\psi)} |\psi^{*\prime}(z)| &= \sup_{z\in I_\varepsilon(e^{it_0})\setminus supp(\psi)} (|\psi_1^{*\prime}(z)| + |\psi_2^{*\prime}(z)|) \\ &\geq \sup_{z\in I_\varepsilon(e^{it_0})\setminus supp(\psi)} |\psi_1^{*\prime}(z)| = \infty.\end{aligned}$$

We have $\varphi = \tau_{-\lambda} \circ \psi$ and $\varphi^{*\prime} = \psi^{*\prime}(\tau_{-\lambda}' \circ \psi^*)$ on $I_\varepsilon(e^{it_0}) \setminus supp(\psi)$. Since $\inf_{z\in\partial\mathbb{D}} |\tau_{-\lambda}'(z)| > 0$, we have $\sup_{z\in I_\varepsilon(e^{it_0})\setminus supp(\psi)} |\varphi^{*\prime}(z)| = \infty$. Since $supp(\varphi) = supp(\psi)$, we get the assertion. $\square$

By Lemma 2.2 and Theorem 2.5, we have the following theorem.

Theorem 2.6. *Let φ be an inner function. If $supp(\varphi) \neq \overline{int\, supp(\varphi)}$, then there exists a function $u \in H^2 \setminus H^\infty$ such that $M_u C_\varphi : H^2 \to H^2$ is bounded.*

There are many examples of inner functions φ such that $supp(\varphi) = \overline{int\, supp(\varphi)}$ and there exists a function $u \in H^2 \setminus H^\infty$ such that $M_u C_\varphi : H^2 \to H^2$ is bounded. For example, let φ_1 be an inner function satisfying $supp(\varphi_1) = \{e^{it} : 0 \leq t \leq \pi\}$. Let

$$\varphi_2(z) = \exp\Big(\frac{z+1}{z-1} + \frac{z-1}{z+1}\Big).$$

Then φ_2 is a singular inner function satisfying $supp(\varphi_2) = \{1, -1\}$. Put $\varphi = \varphi_1\varphi_2$. Then we have that $supp(\varphi) = \{e^{it} : 0 \leq t \leq \pi\}$ and

$$\sup_{\pi < t < 2\pi} |\varphi^{*\prime}(e^{it})| = \infty.$$

Hence there exists a function $u \in H^2 \setminus H^\infty$ such that $M_u C_\varphi : H^2 \to H^2$ is bounded.

Let ψ be an inner function with $supp(\psi) = \partial\mathbb{D}$. By the above fact, $C_\psi(M_u C_\varphi) = M_{u\circ\psi} C_{\varphi\circ\psi}$ is bounded on H^2. We have that $supp(\varphi\circ\psi) = \partial\mathbb{D}$ and $u\circ\psi \in H^2\setminus H^\infty$. Hence there are an inner function η with $supp(\eta) = \partial\mathbb{D}$ and $v \in H^2 \setminus H^\infty$ such that $M_v C_\eta : H^2 \to H^2$ is bounded.

We shall give another sufficient condition. One may check the following easily.

Lemma 2.7. *Let φ_1, φ_2 be inner functions and I be an open subarc of $\partial\mathbb{D}$ such that $I \cap supp(\varphi_1\varphi_2) = \emptyset$. Then $m(\varphi_1^*(I)) \leq m((\varphi_1\varphi_2)^*(I))$.*

Lemma 2.8. *Let φ be an inner function and I be an open subarc of $\partial\mathbb{D}$ such that $I \cap supp(\varphi) = \emptyset$. We write $I = \{e^{it} : t_1 < t < t_2\}$. Let $\sigma(t)$ be an extended real valued continuous function on $[t_1, t_2]$ such that $\varphi^*(e^{it}) = e^{i\sigma(t)}$, $\sigma(t)$ is differentiable and $\sigma'(t) > 0$ on (t_1, t_2). If $\sigma(t_2) - \sigma(t_1) < \infty$, then for each $\varepsilon > 0$, there is an inner function ψ such that φ/ψ is inner, $supp(\psi) = supp(\varphi)$ and $m(\psi^*(I)) < \varepsilon$.*

Proof. By the assumption, $-\infty < \sigma(t_1) < \sigma(t_2) < \infty$. Take a positive integer n satisfying $(\sigma(t_2) - \sigma(t_1))/2n\pi < \varepsilon$. It is not difficult to see the existence of inner functions $\varphi_1, \varphi_2, \cdots, \varphi_n$ such that $\varphi = \varphi_1\varphi_2\cdots\varphi_n$ and $supp(\varphi_j) = supp(\varphi)$ for every $1 \leq j \leq n$. For each $1 \leq j \leq n$, there is a real valued continuous function $\sigma_j(t)$ on $[t_1, t_2]$ such that $\varphi_j^*(e^{it}) = e^{i\sigma_j(t)}$, $\sigma_j(t)$ is differentiable and $\sigma_j'(t) > 0$ on (t_1, t_2). We have

$$\sigma(t_2) - \sigma(t_1) = \sum_{j=1}^{n} (\sigma_j(t_2) - \sigma_j(t_1)).$$

Then $(\sigma_{j_0}(t_2) - \sigma_{j_0}(t_1))/2\pi < \varepsilon$ for some $1 \leq j_0 \leq n$. Hence $m(\varphi_{j_0}^*(I)) < \varepsilon$. Put $\psi = \varphi_{j_0}$. Then φ/ψ is inner, $supp(\psi) = supp(\varphi)$ and $m(\psi^*(I)) < \varepsilon$. $\square$

For an inner function φ and a measurable subset $E \subset \partial\mathbb{D}$, we put

$$\varphi^{*(-1)}(E) = \big\{e^{i\theta} \in \partial\mathbb{D} : \varphi^*(e^{i\theta}) \in E\big\}.$$

If $\varphi(0) = 0$, then it is known that

$$m(\varphi^{*(-1)}(E)) = m(E)$$

for any measurable subset E of $\partial\mathbb{D}$.

Theorem 2.9. *Let φ be an inner function. Suppose that there is a sequence of mutually disjoint open subarcs $\{I_n\}_{n\geq 1}$ of $\partial\mathbb{D}$ such that $\bigcup_{n=1}^{\infty} I_n = \partial\mathbb{D} \setminus supp(\varphi)$. For each $n \geq 1$, let $I_n = \{e^{it} : t_{n,1} < t < t_{n,2}\}$ and $\sigma_n(t)$ be an extended real valued continuous function on $[t_{n,1}, t_{n,2}]$ such that $\varphi_n^*(e^{it}) = e^{i\sigma_n(t)}$, $\sigma_n(t)$ is differentiable and $\sigma_n'(t) > 0$ on $(t_{n,1}, t_{n,2})$. Then we have the following.*

(i) *If $\sigma_n(t_{n,2}) - \sigma_n(t_{n,1}) = \infty$, then $\sup_{z\in I_n} |\varphi^{*\prime}(z)| = \infty$.*

(ii) *Suppose that $\sigma_n(t_{n,2}) - \sigma_n(t_{n,1}) < \infty$ for every $n \geq 1$. If $\sum_{n=1}^{\infty}(\sigma_n(t_{n,2}) - \sigma_n(t_{n,1})) = \infty$, then*

$$\sup_{z\in\partial\mathbb{D}\setminus supp(\varphi)} |\varphi^{*\prime}(z)| = \infty.$$

(iii) *Suppose that $\sigma_n(t_{n,2}) - \sigma_n(t_{n,1}) < \infty$ for every $n \geq 1$. If $m(supp(\varphi)) = 0$, then $\sum_{n=1}^{\infty}(\sigma_n(t_{n,2}) - \sigma_n(t_{n,1})) = \infty$.*

If one of the assumptions of (i), (ii) and (iii) holds, then there exists $u \in H^2 \setminus H^\infty$ such that $M_u C_\varphi : H^2 \to H^2$ is bounded.

Proof. (i) follows from the mean valued theorem.

(ii) For each positive integer j, there is n_j such that

$$j \leq \frac{\sigma_{n_j}(t_{n_j,2}) - \sigma_{n_j}(t_{n_j,1})}{t_{n_j,2} - t_{n_j,1}}.$$

For, if not, then there is j_0 such that

$$\frac{\sigma_n(t_{n,2}) - \sigma_n(t_{n,1})}{t_{n,2} - t_{n,1}} < j_0$$

for every $n \geq 1$. Then we have

$$\infty = \sum_{n=1}^{\infty}(\sigma_n(t_{n,2}) - \sigma_n(t_{n,1})) < j_0 \sum_{n=1}^{\infty}(t_{n,2} - t_{n,1}) \leq 2\pi j_0.$$

This is a contradiction.

By the mean valued theorem, there is $t_{n_j,1} < \theta_j < t_{n_j,2}$ satisfying $j \leq \sigma_{n_j}'(\theta_j) = |\varphi^{*\prime}(e^{i\theta_j})|$ for every j. Therefore we get

$$\sup_{z\in\partial\mathbb{D}\setminus supp(\varphi)} |\varphi^{*\prime}(z)| = \sup_{n\geq 1} \sup_{z\in I_n} |\varphi^{*\prime}(z)| = \infty.$$

(iii) To prove (iii), suppose that $\sum_{n=1}^{\infty}(\sigma_n(t_{n,2}) - \sigma_n(t_{n,1})) < \infty$. We shall lead a contradiction. By the assumption, there is n_0 such that $\sum_{n=n_0}^{\infty}(\sigma_n(t_{n,2}) - \sigma_n(t_{n,1})) < 1$. By Lemmas 2.7 and 2.8, there is an inner function ψ such that φ/ψ is inner, $supp(\psi) = supp(\varphi)$ and $\sum_{n=1}^{\infty} m(\psi^*(I_n)) < 1$. Therefore

$$m\Big(\psi^*\Big(\bigcup_{n=1}^{\infty} I_n\Big)\Big) = m\Big(\bigcup_{n=1}^{\infty} \psi^*(I_n)\Big) \leq \sum_{n=1}^{\infty} m(\psi^*(I_n)) < 1.$$

Let $\lambda = \psi(0)$ and $\tau_\lambda(z) = (z-\lambda)/(1-\overline{\lambda}z)$. Set $\eta(z) = \tau_\lambda \circ \psi$. Since τ_λ is an automorphism, $m(E) = 0$ if and only if $m(\tau_\lambda^*(E)) = 0$ for every measurable subset E of $\partial\mathbb{D}$. Then we have

$$m\Big(\eta^*\Big(\bigcup_{n=1}^\infty I_n\Big)\Big) = m\Big(\tau_\lambda^*\Big(\psi^*\Big(\bigcup_{n=1}^\infty I_n\Big)\Big)\Big) < 1.$$

Since η is an inner function and $\eta(0) = 0$, we have

$$m\Big(\bigcup_{n=1}^\infty I_n\Big) \le m\Big(\eta^{*(-1)}\Big(\eta^*\Big(\bigcup_{n=1}^\infty I_n\Big)\Big)\Big) = m\Big(\eta^*\Big(\bigcup_{n=1}^\infty I_n\Big)\Big) < 1.$$

Since $m(supp(\varphi)) = 0$, we have $m(\bigcup_{n=1}^\infty I_n) = 1$. Thus we get a contradiction.

The last part of the assertion follows from Lemma 2.2. □

3. Other sufficient conditions

For an inner function φ, first we consider the following condition.

(α) There is a sequence of mutually disjoint measurable subsets $\{C_n\}_{n\ge1}$ of $\partial\mathbb{D}$ and a sequence of positive numbers $\{\delta_n\}_{n\ge1}$ satisfying $\sum_{n=1}^\infty \delta_n < \infty$ such that $m(C_n) > 0$ and $m(C_n \cap \varphi^{*(-1)}(E)) \le \delta_n m(E)$ for every measurable subset E of $\partial\mathbb{D}$ and for every $n \ge 1$.

We do not know whether condition (α) holds or not for any inner function φ satisfying $m(supp(\varphi)) > 0$. We shall show the following theorem.

Theorem 3.1. *Let φ be an inner function satisfying condition (α). Then there is $u \in H^2 \setminus H^\infty$ such that $M_u C_\varphi : H^2 \to H^2$ is bounded.*

Proof. Since $\{C_n\}_{n\ge1}$ is a sequence of mutually disjoint measurable subsets of $\partial\mathbb{D}$, we have $\sum_{n=1}^\infty m(C_n) \le m(\partial\mathbb{D}) = 1$. Then there is a sequence of positive numbers $\{a_n\}_{n\ge1}$ such that $a_n \ge 1$ for every n,

$$\sum_{n=1}^\infty a_n m(C_n) < \infty$$

and $a_n \to \infty$ as $n \to \infty$. Since $\sum_{n=1}^\infty \delta_n < \infty$, moreover we may assume that

$$\sum_{n=1}^\infty a_n \delta_n < \infty.$$

Put $a_0 = 1$ and $C_0 = \partial\mathbb{D} \setminus \bigcup_{n=1}^\infty C_n$. Let η be the function on $\partial\mathbb{D}$ defined by $\eta = a_n$ on C_n for every $n \ge 0$. Then $\eta \ge 1$ on $\partial\mathbb{D}$ and

$$\int_{\partial\mathbb{D}} \eta \, dm = \sum_{n=0}^\infty a_n m(C_n) < \infty.$$

By [4, p. 53], there exists $u \in H^2$ such that $|u|^2 = \eta$ a.e. on $\partial\mathbb{D}$. Since $a_n \to \infty$ as $n \to \infty$ and $m(C_n) > 0$ for every $n \ge 1$, we have $u \notin H^\infty$.

Let $\mathcal{L}$ be the set of measurable simple functions on $\partial\mathbb{D}$. Let $f \in \mathcal{L}$. We may write

$$f = \sum_{i=1}^{\ell} c_i \chi_{\Lambda_i},$$

where $m(\Lambda_i) > 0$ for every i and $\Lambda_i \cap \Lambda_j = \emptyset$ for $i \neq j$. We have

$$\begin{aligned}
\|M_u C_\varphi f\|_2^2 &= \int_{\partial\mathbb{D}} |u|^2 |f \circ \varphi^*|^2 \, dm = \sum_{n=0}^{\infty} a_n \int_{C_n} |f \circ \varphi^*|^2 \, dm \\
&= \int_{C_0} |f \circ \varphi^*|^2 \, dm + \sum_{n=1}^{\infty} a_n \sum_{i=1}^{\ell} |c_i|^2 \int_{C_n} \chi_{\Lambda_i} \circ \varphi^* \, dm \\
&\leq \int_{\partial\mathbb{D}} |f \circ \varphi^*|^2 \, dm + \sum_{n=1}^{\infty} a_n \sum_{i=1}^{\ell} |c_i|^2 m(C_n \cap \varphi^{*(-1)}(\Lambda_i)) \\
&\leq \|C_\varphi\|^2 \|f\|_2^2 + \sum_{n=1}^{\infty} a_n \delta_n \sum_{i=1}^{\ell} |c_i|^2 m(\Lambda_i) \qquad \text{by condition } (\alpha) \\
&= \Big(\|C_\varphi\|^2 + \sum_{n=1}^{\infty} a_n \delta_n \Big) \|f\|_2^2.
\end{aligned}$$

Since $\sum_{n=1}^{\infty} a_n \delta_n < \infty$, $M_u C_\varphi : \mathcal{L} \to L^2(\partial\mathbb{D})$ is a bounded linear map. Since $\mathcal{L}$ is dense in $L^2(\partial\mathbb{D})$, $M_u C_\varphi$ may be extended boundedly on $L^2(\partial\mathbb{D})$. Thus $M_u C_\varphi : H^2 \to H^2$ is bounded. $\square$

We shall give a sufficient condition on an inner function φ for which satisfies condition (α). We consider the following condition for φ.

(β) There is a sequence of mutually disjoint measurable subsets $\{E_n\}_{n \geq 1}$ of $\partial\mathbb{D}$ such that $m(E_n \cap \varphi^{*(-1)}(E)) > 0$ for every measurable subset E of $\partial\mathbb{D}$ satisfying $m(E) > 0$ and for every $n \geq 1$.

We do not know whether condition (β) holds or not for any inner function φ satisfying $m(supp(\varphi)) > 0$.

Theorem 3.2. *If an inner function φ satisfies condition (β), then φ satisfies condition (α).*

Proof. We divide the proof into two cases.

Case 1. Suppose that $\varphi(0) = 0$. Then it is known that

$$m(\varphi^{*(-1)}(E)) = m(E) \tag{3.1}$$

for any measurable subset E of $\partial\mathbb{D}$. By condition (β), there is a family of mutually disjoint measurable subsets $\{E_{n,j} : 1 \leq n, 1 \leq j \leq N_n\}$ of $\partial\mathbb{D}$ such that

$$m(E_{n,j} \cap \varphi^{*(-1)}(E)) > 0 \tag{3.2}$$

for every measurable subset E of $\partial\mathbb{D}$ satisfying $m(E) > 0$ and for every $n \geq 1, 1 \leq j \leq N_n$. Moreover we may assume that

$$\sum_{n=1}^{\infty} \frac{1}{N_n} < \infty. \tag{3.3}$$

For each $n \geq 1$, let $W_n = \bigcup_{j=1}^{N_n} E_{n,j}$. Then

$$\{W_n\}_{n\geq 1} \text{ is a sequence of mutually disjoint sets.} \tag{3.4}$$

Put

$$\mu_{n,j}(E) = m(E_{n,j} \cap \varphi^{*(-1)}(E))$$

for every measurable subset E of $\partial\mathbb{D}$. Then $\mu_{n,j}$ is a positive measure on $\partial\mathbb{D}$. By (3.1), we have $\mu_{n,j} \ll m$, so there is a nonnegative integrable function $f_{n,j}$ on $\partial\mathbb{D}$ such that

$$\int_E f_{n,j}\, dm = m(E_{n,j} \cap \varphi^{*(-1)}(E)). \tag{3.5}$$

By (3.1) again, we have

$$\int_E \sum_{j=1}^{N_n} f_{n,j}\, dm = m(W_n \cap \varphi^{*(-1)}(E)) \leq m(\varphi^{*(-1)}(E)) = m(E)$$

for every measurable subset E of $\partial\mathbb{D}$. Hence

$$0 \leq \sum_{j=1}^{N_n} f_{n,j} \leq 1 \quad \text{a.e. on } \partial\mathbb{D}. \tag{3.6}$$

Let

$$A_{n,j} = \Big\{e^{i\theta} \in \partial\mathbb{D} : f_{n,j}(e^{i\theta}) \leq \frac{1}{N_n}\Big\}. \tag{3.7}$$

By (3.6) and (3.7), we have

$$m\Big(\partial\mathbb{D} \setminus \bigcup_{j=1}^{N_n} A_{n,j}\Big) = 0.$$

Let

$$B_{n,1} = A_{n,1}, \quad B_{n,j} = A_{n,j} \setminus \bigcup_{i=1}^{j-1} A_{n,i} \ \ (2 \leq j \leq N_n). \tag{3.8}$$

Then

$$\{B_{n,j} : 1 \leq j \leq N_n\} \quad \text{is a set of mutually disjoint sets} \tag{3.9}$$

and

$$m\Big(\bigcup_{j=1}^{N_n} B_{n,j}\Big) = m\Big(\bigcup_{j=1}^{N_n} A_{n,j}\Big) = 1. \tag{3.10}$$

We have that

$$m(B_{n,j}) > 0 \quad \text{for some } 1 \leq j \leq N_n. \tag{3.11}$$

For a measurable subset E of $B_{n,j}$, we have

$$\begin{aligned} m(E_{n,j} \cap \varphi^{*(-1)}(E)) &= \int_E f_{n,j}\, dm \qquad \text{by (3.5)} \\ &\leq \frac{m(E)}{N_n} \qquad \text{by (3.7) and (3.8).} \end{aligned}$$

Hence

$$m(E_{n,j} \cap \varphi^{*(-1)}(E)) \leq \frac{m(E)}{N_n} \qquad \text{for every } E \subset B_{n,j}. \tag{3.12}$$

For each $1 \leq j \leq N_n$, let

$$C_{n,j} = E_{n,j} \cap \varphi^{*(-1)}(B_{n,j}) \tag{3.13}$$

and for each $n \geq 1$, set $C_n = \bigcup_{j=1}^{N_n} C_{n,j}$. Then $C_n \subset W_n$ and by (3.4), $\{C_n\}_{n\geq 1}$ is a sequence of mutually disjoint sets. For each $n \geq 1$, we have

$$\begin{aligned} m(C_n) &= \sum_{j=1}^{N_n} m(C_{n,j}) \\ &= \sum_{j=1}^{N_n} m(E_{n,j} \cap \varphi^{*(-1)}(B_{n,j})) \qquad \text{by (3.13)} \\ &> 0 \qquad \text{by (3.2) and (3.11).} \end{aligned}$$

Hence $m(C_n) > 0$ for every $n \geq 1$.

For a measurable subset E of $\partial\mathbb{D}$ and $n \geq 1$, we have

$$\begin{aligned} m(C_n \cap \varphi^{*(-1)}(E)) &= \sum_{j=1}^{N_n} m(C_{n,j} \cap \varphi^{*(-1)}(E)) \\ &= \sum_{j=1}^{N_n} \sum_{k=1}^{N_n} m\big(C_{n,j} \cap \varphi^{*(-1)}(B_{n,k} \cap E)\big) \\ &\qquad \text{by (3.9) and (3.10)} \end{aligned}$$

$$\begin{aligned}
&= \sum_{j=1}^{N_n}\sum_{k=1}^{N_n} m\big(E_{n,j} \cap \varphi^{*(-1)}(B_{n,j}) \\
&\qquad\qquad \cap \varphi^{*(-1)}(B_{n,k} \cap E)\big) \qquad \text{by (3.13)} \\
&= \sum_{j=1}^{N_n} m\big(E_{n,j} \cap \varphi^{*(-1)}(B_{n,j} \cap E)\big) \qquad \text{by (3.9)} \\
&\leq \sum_{j=1}^{N_n} \frac{m(B_{n,j} \cap E)}{N_n} \qquad \text{by (3.12)} \\
&= \frac{m(E)}{N_n} \qquad \text{by (3.9) and (3.10).}
\end{aligned}$$

Putting $\delta_n = 1/N_n > 0$, we have $m(C_n \cap \varphi^{*(-1)}(E)) \leq \delta_n m(E)$. By (3.3), $\sum_{n=1}^{\infty} \delta_n < \infty$. Thus φ satisfies condition (α).

Case 2. Suppose that $\lambda := \varphi(0) \neq 0$. Let $\tau_\lambda(z) = (z - \lambda)/(1 - \overline{\lambda} z)$ and $\psi = \tau_\lambda \circ \varphi$. We have $\psi^* = \tau_\lambda^* \circ \varphi^*$. For a measurable subset E of $\partial\mathbb{D}$ with $m(E) > 0$, $\psi^{*(-1)}(E) = \varphi^{*(-1)}(\tau_\lambda^{*(-1)}(E))$. Since $m(\tau_\lambda^{*(-1)}(E)) > 0$ and φ satisfies condition (β), we have

$$m(E_n \cap \psi^{*(-1)}(E)) = m(E_n \cap \varphi^{*(-1)}(\tau_\lambda^{*(-1)}(E))) > 0$$

for every $n \geq 1$. Hence ψ satisfies condition (β). By Case 1, ψ satisfies condition (α). Then there is a sequence of mutually disjoint measurable subsets $\{D_n\}_{n\geq 1}$ of $\partial\mathbb{D}$ and a sequence of positive numbers $\{\sigma_n\}_{n\geq 1}$ satisfying $\sum_{n=1}^{\infty} \sigma_n < \infty$ such that $m(D_n) > 0$ and

$$m(D_n \cap \psi^{*(-1)}(A)) \leq \sigma_n m(A)$$

for every measurable subset A of $\partial\mathbb{D}$ and for every $n \geq 1$. Since τ_λ is an automorphism, there is $K > 0$ such that $m(\tau_\lambda^*(A)) \leq K m(A)$ for every A. We have $\psi^{*(-1)}(\tau_\lambda^*(A)) = \varphi^{*(-1)}(A)$ and

$$\begin{aligned}
m(D_n \cap \varphi^{*(-1)}(A)) &= m(D_n \cap \psi^{*(-1)}(\tau_\lambda^*(A)) \\
&\leq \sigma_n m(\tau_\lambda^*(A)) \leq \sigma_n K m(A).
\end{aligned}$$

Hence φ satisfies condition (α). $\square$

By Theorems 3.1 and 3.2, we have the following.

Corollary 3.3. *Let φ be an inner function satisfying condition (β). Then there is $u \in H^2 \setminus H^\infty$ such that $M_u C_\varphi : H^2 \to H^2$ is bounded.*

Acknowledgement. The first author is partially supported by Grant-in-Aid for Scientific Research, Japan Society for the Promotion of Science (No.24540164).

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