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LOCAL OPERATORS AND A CHARACTERIZATION OF THE VOLTERRA OPERATOR

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ABSTRACT. We consider locally defined operators of the form $D^n \circ K$ where D is the operator of differentiation and K maps the space of continuous functions into the space of n-times differentiable functions. As a corollary we obtain a characterization of the Volterra operator. Locally defined operators acting in the space of analytic functions are also discussed.

1. Introduction

In the present paper we examine the operators K mapping the continuous functions into the set of differentiable functions such that the composition $D^n \circ K$ is a locally defined operator (or an operator with memory); here D^n denotes the nth iterate of the operator of differentiation. As a corollary we obtain a characterization of the Volterra operator.

To clarify the meaning of a *locally defined* or *locally determined operator* (cf. [1, p. 10-11]), take a topological space X and an arbitrary set Y. Let $\mathcal{F}_1(X,Y)$ and $\mathcal{F}_2(X,Y)$ be two families of functions $\varphi : X \to Y$. An operator $K : \mathcal{F}_1(X,Y) \to \mathcal{F}_2(X,Y)$ is called locally defined if, for any open subset $U \subset X$, and any functions $\varphi, \psi \in \mathcal{F}_1$,

$$\varphi|_U = \psi|_U \Longrightarrow K(\varphi)|_U = K(\psi)|_U,$$

where $\varphi|_U$ denotes the restriction of φ to U.

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The form of locally defined operators strongly depends on both the function spaces $\mathcal{F}_1(X,Y)$ and $\mathcal{F}_2(X,Y)$. To illustrate this fact take an interval $I \subset \mathbb{R}$, put $X := I$, $Y := \mathbb{R}$, and consider$\mathcal{F}_1(X,Y) = C^m(I)$, $\mathcal{F}_2(X,Y) = C^n(I)$ where m, n are nonnegative integers and $C^m(I)$ denotes the space of all n-times continuously differentiable real functions defined on I. In [2] the following (still open) conjecture is presented.

If $K : C^m(I) \to C^n(I)$ is locally defined, then for all $\varphi \in C^m(I)$,

$$K(\varphi)(x) = h(x, \varphi(x), \varphi'(x), \cdots, \varphi^{(m-n)}(x)) \qquad (x \in I) \tag{1.1}$$

for some function $h : I \times \mathbb{R}^{m-n+1} \to \mathbb{R}$ that in the case $m < n$ reduces to a single variable function $h \in C^n(I)$. Let us note that, under some additional assumption, it has been recently proved by Wróbel [7].

In [2] it was proved that this conjecture holds true if $n = 0$ or $n = 1$. We apply this result in the present paper.

In section 2, assuming that $I = [a, b]$, we give the form of any operator $K : C^0(I) \to C^m(I)$ such that $D^m \circ K$ is locally defined. Hence we conclude that $K : C^0(I) \to C^1(I)$ is the Volterra operator iff $D \circ K$ is locally defined and, for all $\varphi \in C^0(I)$,

$$(D \circ K)(\varphi)(a) = 0.$$

In [2] it was also shown that the counterpart of formula (1.1) for locally defined operators $K : C^\infty(I) \to C^0(I)$ holds also true. The situation strikingly changes for locally defined operators defined on the space $\mathcal{A}(I) \subset C^\infty(I)$ of all analytic functions. Let $\mathcal{F}(I, \mathbb{R})$ denote the set of all real functions defined on an interval I. In section 3 we observe that every locally defined operator $K : \mathcal{A}(I) \to \mathcal{F}(I, \mathbb{R})$ is locally defined.

2. Some definitions and auxiliary results

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and let $I \subset \mathbb{R}$ be an interval. By $\mathcal{F}(I)$ denote the set of all functions $\varphi : I \to \mathbb{R}$. For $m \in \mathbb{N}_0$, by $C^m(I)$ denote the set of all m-times continuously differentiable functions $\varphi : I \to \mathbb{R}$ and put

$$C^\infty(I) := \bigcap_{m=1}^{\infty} C^m(I).$$

Let us introduce the following definitions.

An operator $K : C^m(I) \to \mathcal{F}(I)$ is said to be

- *left-defined*, if for any real a and any $\varphi, \psi \in C^m(I)$,

$$\varphi|_{(-\infty,a)\cap I} = \psi|_{(-\infty,a)\cap I} \Longrightarrow K(\varphi)|_{(-\infty,a)\cap I} = K(\psi)|_{(-\infty,a)\cap I};$$

- *right-defined*, if for any real a and any $\varphi, \psi \in C^m(I)$,

$$\varphi|_{(a,\infty)\cap I} = \psi|_{(a,\infty)\cap I} \Longrightarrow K(\varphi)|_{(a,\infty)\cap I} = K(\psi)|_{(a,\infty)\cap I};$$

- *locally defined,* if for any nonempty open subinterval $J \subset I$, and any $\varphi, \psi \in C^m(I)$,

$$\varphi|_J = \psi|_J \Longrightarrow K(\varphi)|_J = K(\psi)|_J.$$

Remark 2.1. It is easy to check that an operator $K : C^m(I) \to C^n(I)$ is locally defined iff it is *left-defined* and right-defined (cf. [2]).

In [2] the following results have been proved.

Theorem 2.2. *Let $m \in \mathbb{N}_0$. An operator $K : C^m(I) \to C^0(I)$ is locally defined if, and only if, there exists a function $h : I \times \mathbb{R}^{m+1} \to \mathbb{R}$ such that*

$$K(\varphi)(x) = h(x, \varphi(x), \varphi'(x), \cdots, \varphi^{(m)}(x)) \quad (\varphi \in C^m(I), x \in I).$$

Theorem 2.3. *Let $m \in \mathbb{N}$. An operator $K : C^m(I) \to C^1(I)$ is locally defined if, and only if, there exists a function $h : I \times \mathbb{R}^m \to \mathbb{R}$ such that*

$$K(\varphi)(x) = h(x, \varphi(x), \varphi'(x), \cdots, \varphi^{(m-1)}(x)) \quad (\varphi \in C^m(I), x \in I).$$

Theorem 2.4. *An operator $K : C^\infty(I) \to C^0(I)$ is locally defined if, and only if, there exists a function $h : I \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R}$ such that*

$$K(\varphi)(x) = h(x, \varphi(x), \varphi'(x), \varphi''(x), \cdots) \quad (\varphi \in C^m(I), x \in I).$$

In particular, by Theorem 2.2 with $m = 0$, an operator $K : C^0(I) \to C^0(I)$ is locally defined iff, there exists a function $h : I \times \mathbb{R} \to \mathbb{R}$ such that

$$K(\varphi)(x) = h(x, \varphi(x)) \quad (\varphi \in C^0(I), x \in I).$$

Thus, in this case, K is a Nemytskij composition operator and the function h must be continuous (cf. [1, p. 167, Theorem 6.3]).

Note that, by Theorem 2.3 for $m = 1$, an operator $K : C^1(I) \to C^1(I)$ is a locally defined iff, there exists a function $h : I \times \mathbb{R} \to \mathbb{R}$ such that

$$K(\varphi)(x) = h(x, \varphi(x)) \quad (\varphi \in C^1(I), x \in I).$$

The present author has proved (cf. [1, p. 224]), the following surprisingly enough

Remark 2.5. There are discontinuous functions $h : I \times \mathbb{R} \to \mathbb{R}$ such that the Nemytskij operator K maps $C^1(I)$ into $C^1(I)$!

Note also that the above three theorems remain true for the function spaces defined in the suitable subsets of $\mathbb{R}^k$ (cf. [6]) as well as for the class of the Whitney differentiable functions ([4, 5]).

Let us mention that applicability of some contractive fixed point theorems in some problems involving the Nemytskij composition operators (substitution operators) is discussed in [3].

In the sequel D stands for the operator of differentiation; more precisely, thus $D : C^1(I) \to C^0(I)$ is defined by

$$D(f) := f'.$$

Moreover, denoting by D^0 the identity map, for $k \in \mathbb{N}$, we define recursively $D^k : C^k(I) \to C^{k-1}(I)$ by

$$D^k := D \circ D^{k-1} \qquad (k \in \mathbb{N}).$$

Thus D^k is the kth iterate of D.

3. Representation formula for local operators of the form $D^m \circ K$

In this section we prove the following

Theorem 3.1. *Let $I = [a, b]$ for some $a, b \in \mathbb{R}$, $a < b$ and let $m \in \mathbb{N}$. Suppose that $K : C^0(I) \to C^m(I)$. If the operator $D^m \circ K$ is locally defined, then there exists a continuous function $h : I \times \mathbb{R} \to \mathbb{R}$ such that, for all $\varphi \in C^m(I)$ and $x \in I$,*

$$K(\varphi)(x) = \frac{1}{(m-1)!}\int_a^x (x-t)^m h(t,\varphi(t))dt + \sum_{k=0}^{m-1} \frac{\left(D^k \circ K\right)(\varphi)(a)}{k!}(x-a)^k. \tag{3.1}$$

Proof. Since the operator $D^m \circ K$ maps $C^0(I)$into itself and is locally defined, by Theorem 2.2, there exists a function $h : I \times \mathbb{R} \to \mathbb{R}$ such that, for all $\varphi \in C^m(I)$ and $x \in I$,

$$\left(D^m \circ K\right)(\varphi)(x) = h(x, \varphi(x)).$$

Hence, for a fixed $\varphi \in C^m(I)$ and for all $x \in I$, we get

$$[D \circ \left(D^{m-1} \circ K\right)(\varphi)](t) = h(t, \varphi(t)) \qquad (t \in [a, x]),$$

or, equivalently,

$$[\left(D^{m-1} \circ K\right)(\varphi)]'(t) = h(t, \varphi(t)) \qquad (t \in [a, x]).$$

whence, after integration, for all $x \in I$,

$$\left(D^{m-1} \circ K\right)(\varphi)(x) = \int_a^x h(t, \varphi(t))dt + \left(D^{m-1} \circ K\right)(\varphi)(a).$$

Thus,

$$[\left(D^{m-2} \circ K\right)(\varphi)]'(s) = \int_a^s h(t, \varphi(t))dt + \left(D^{m-1} \circ K\right)(\varphi)(a) \qquad (s \in [a, x]).$$

Integrating both sides, we obtain, for all $x \in I$,

$$\begin{aligned}
&(D^{m-2} \circ K)(\varphi)(x) \\
= &\int_a^x \left(\int_a^s h(t, \varphi(t))dt + \left(D^{m-1} \circ K\right)(\varphi)(a)\right) ds + \left(D^{m-2} \circ K\right)(\varphi)(a) \\
= &\frac{1}{1!}\int_a^x (x-t)h(t, \varphi(t))dt + \frac{D^{m-1} \circ K(\varphi)(a)}{1!}(x-a) + \left(D^{m-2} \circ K\right)(\varphi)(a).
\end{aligned}$$

(The last equality can be also verified by differentiation of both sides with respect to x). Repeating this procedure, for all $x \in I$, we get

$$\begin{aligned}
&(D^{m-3} \circ K)(\varphi)(x) \\
= &\frac{1}{2!}\int_a^x (x-t)^2 h(t, \varphi(t))dt + \frac{D^{m-1} \circ K(\varphi)(a)}{2!}(x-a)^2 \\
&+\frac{(D^{m-2} \circ K)(\varphi)(a)}{1!}(x-a) + (D^{m-3} \circ K)(\varphi)(a).
\end{aligned}$$

After $(m-1)$-steps we obtain (3.1). $\square$

As an immediate corollary from Theorem 2.4 we obtain the following characterization of the Volterra operator.

Theorem 3.2. *Let $I = [a, b]$ for some $a, b \in \mathbb{R}$, $a < b$. An operator $K : C^0(I) \to C^1(I)$ is a Volterra operator if, and only if, the operator $D \circ K$ is locally defined and, for all $\varphi \in C^0(I)$,*

$$(D \circ K)(\varphi)(a) = 0.$$

4. Remark on local operators on a class of analytic functions

Let $\mathcal{A}(I)$ denote the set of all real analytic functions defined on an interval I, and $\mathcal{F}(I)$ the set of all real functions defined on I. We have the following

Theorem 4.1. *Any operator $K : \mathcal{A}(I) \to \mathcal{F}(I)$ is locally defined.*

Proof. Let J be a nonempty open subinterval in I. If $\varphi, \psi \in \mathcal{A}(I)$ and $\varphi|_J = \psi|_J$ then, by the analyticity of φ and ψ, we have $\varphi = \psi$. It follows that $K(\varphi) = K(\psi)$ and, consequently, $K(\varphi)|_J = K(\psi)|_J$, and the result is proved. ☐

In the context of Theorem 3, let us observe a striking difference between locally defined operators $K : C^\infty(I) \to C^0(I)$ and $K : \mathcal{A}(I) \to \mathcal{F}(I)$.

Remark 4.2. Obviously, the above result remains true on replacing $\mathcal{A}(I)$ by the space of all complex variable analytic functions $\varphi : U \to \mathbb{C}$ where U is an arbitrary fixed domain in a complex plane $\mathbb{C}$.

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