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ON THE CONVEXITY OF CERTAIN INTEGRAL OPERATORS

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ABSTRACT. In this paper we consider the classes of starlike functions of order α, convex functions of order α and we study the convexity and α-order convexity for some general integral operators. Several corollaries of the main results are also considered.

1. Introduction and preliminaries

Let $U = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk of the complex plane and denote by $H(U)$ the class of the olomorphic functions in U. Consider $\mathcal{A} = \{f \in H(U) : f(z) = z + a_2z^2 + a_3z^3 + \cdots ,\ z \in U\}$ be the class of analytic functions in U and $S = \{f \in \mathcal{A} :\ f \text{ is univalent in } U\}$.

Consider S^* the class of starlike functions in unit disk, defined by

$$S^* = \left\{ f \in \mathcal{A} : \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in U \right\}.$$

Definition 1.1. A function $f \in S$ is a *starlike function of order* α, $0 \le \alpha < 1$ and denote this class by $S^*(\alpha)$ if f verifies the inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad (z \in U).$$

Denote with K the class of convex functions in U, defined by

$$K = \left\{ f \in \mathcal{A} :\ \operatorname{Re}\left(\frac{zf''(z)}{f'(z)} + 1\right) > 0,\ z \in U \right\}.$$

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Definition 1.2. A function $f \in S$ is *convex function of order* α, $0 \le \alpha < 1$ and denote this class by $K(\alpha)$ if f verifies the inequality

$$\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}+1\right) > \alpha, \quad (z \in U).$$

Definition 1.3. A function $f \in \mathcal{A}$ is said to be in the class $R(\alpha)$ if $\mathrm{Re}(f'(z)) > \alpha$, $(z \in U)$.

It is well known that $K(\alpha) \subset S^*(\alpha) \subset S$.

Frasin and Jahangiri defined in [1] the family $B(\mu, \alpha)$, $\mu \ge 0$, $0 \le \alpha < 1$ so that it consists of functions $f \in \mathcal{A}$ satisfying the condition

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1\right| < 1-\alpha, \quad (z \in U). \tag{1.1}$$

For $\mu = 0$ we have $B(0,\alpha) \equiv R(\alpha)$ and for $\mu = 1$ we have $B(1,\alpha) \equiv S^*(\alpha)$.

In this paper we will obtain the order of convexity of the following integral operators:

$$H_n(z) = \int_0^z \prod_{i=1}^{n} \left(te^{f_i(t)}\right)^{\gamma_i} \mathrm{d}t, \tag{1.2}$$

$$H(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\beta}} t^{n(\beta-1)}\mathrm{d}t, \tag{1.3}$$

$$G(z) = \int_0^z \prod_{i=1}^{n} (f_i(t))^{\beta-1}\mathrm{d}t, \tag{1.4}$$

where the functions $f_i(t)$ are in $B(\mu_i, \alpha_i)$ for all $i = 1, 2, \cdots, n$.

Lemma 1.4. [2]*(General Schwarz lemma). Let the function f be regular in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \le \frac{M}{R^m} \cdot |z|^m \quad (z \in U_R).$$

The equality can hold only if

$$f(z) = e^{i\theta} \cdot \frac{M}{R^m} \cdot z^m,$$

where θ is constant.

2. Main results

Theorem 2.1. *Let $f_i(z) \in \mathcal{A}$ be in the class $B(\mu_i, \alpha_i)$, $\mu_i \ge 0$, $0 \le \alpha_i < 1$ for all $i = 1, 2, \cdots, n$. If $|f_i(z)| \le M_i$ ($M_i \ge 1$, $z \in U$) for all $i = 1, 2, \cdots, n$, then the integral operator*

$$H_n(z) = \int_0^z \prod_{i=1}^{n} \left(te^{f_i(t)}\right)^{\gamma_i} \mathrm{d}t$$

is in $K(\delta)$*, where*

$$\delta = 1 - \sum_{i=1}^{n} |\gamma_i| \cdot [1 + (2 - \alpha_i) M_i^{\mu_i}]$$

and $\sum\limits_{i=1}^{n} |\gamma_i| \cdot [1 + (2 - \alpha_i) M_i^{\mu_i}] < 1,\ \gamma_i \in \mathbb{C}$ for all $i = 1, 2, \cdots n.$.

Proof. Let $f_i \in \mathcal{A}$ be in the class $B(\mu_i, \alpha_i),\ \mu_i \geq 0,\ 0 \leq \alpha_i < 1$. It follows from (1.2) that

$$H_n(z) = \int_0^z t^{\sum\limits_{i=1}^{n} \gamma_i} e^{\sum\limits_{i=1}^{n} \gamma_i f_i(t)} \mathrm{d}t \quad \text{and} \quad H_n'(z) = z^{\sum\limits_{i=1}^{n} \gamma_i} e^{\sum\limits_{i=1}^{n} \gamma_i f_i(z)}.$$

Also

$$H_n''(z) = z^{\sum\limits_{i=1}^{n} \gamma_i - 1} \cdot e^{\sum\limits_{i=1}^{n} \gamma_i f_i(z)} \left[\sum_{i=1}^{n} \gamma_i + z \sum_{i=1}^{n} \gamma_i f_i'(z) \right]$$

Then

$$\frac{H_n''(z)}{H_n'(z)} = \frac{\sum\limits_{i=1}^{n} \gamma_i + z \sum\limits_{i=1}^{n} \gamma_i f_i'(z)}{z}$$

and, hence

$$\begin{aligned} \left| \frac{z H_n''(z)}{H_n'(z)} \right| &= \left| \sum_{i=1}^{n} \gamma_i + z \sum_{i=1}^{n} \gamma_i f_i'(z) \right| \leq \sum_{i=1}^{n} |\gamma_i| + |z| \sum_{i=1}^{n} |\gamma_i| \cdot |f_i'(z)| \\ &\leq \sum_{i=1}^{n} |\gamma_i| + |z| \cdot \sum_{i=1}^{n} |\gamma_i| \cdot \left| f_i'(z) \left(\frac{z}{f_i(z)} \right)^{\mu_i} \right| \cdot \left| \left(\frac{f_i(z)}{z} \right)^{\mu_i} \right| \quad (2.1) \end{aligned}$$

Applying the General Schwarz lemma, we have $\left| \frac{f_i(z)}{z} \right| \leq M_i$, for all $i = 1, 2, \cdots, n$. Therefore, from (2.1), we obtain

$$\left| \frac{z H_n''(z)}{H_n'(z)} \right| \leq \sum_{i=1}^{n} |\gamma_i| + \sum_{i=1}^{n} |\gamma_i| \cdot \left| f_i'(z) \left(\frac{z}{f_i(z)} \right)^{\mu_i} \right| \cdot M_i^{\mu_i},\ (z \in U). \quad (2.2)$$

From (1.1) and (2.2), we see that

$$\left| \frac{z H_n''(z)}{H_n'(z)} \right| \leq \sum_{i=1}^{n} |\gamma_i| \cdot [1 + (2 - \alpha_i) M_i^{\mu_i}] = 1 - \delta.$$

This completes the proof. □

Letting $M_i = M$ for all $i = 1, 2, \cdots, n$ in Theorem 2.1, we have

Corollary 2.2. *Let* $f_i(z) \in \mathcal{A}$ *be in the class* $B(\mu_i, \alpha_i),\ \mu_i \geq 0,\ 0 \leq \alpha_i < 1$ *for all* $i = 1, 2, \cdots, n$*. If* $|f_i(z)| \leq M$ $(M \geq 1,\ z \in U)$ *for all* $i = 1, 2, \cdots, n$*, then the integral operator*

$$H_n(z) = \int_0^z \prod_{i=1}^{n} \left(t e^{f_i(t)} \right)^{\gamma_i} \mathrm{d}t$$

is in $K(\delta)$, *where*

$$\delta = 1 - \sum_{i=1}^{n} |\gamma_i| \cdot [(2-\alpha_i)M^{\mu_i} + 1]$$

and $\sum_{i=1}^{n} |\gamma_i| \cdot [(2-\alpha_i)M^{\mu_i} + 1] < 1,\ \gamma_i \in \mathbb{C}$ for all $i = 1, 2, \cdots n.$.

Letting $\mu_i = 0$ for all $i = 1, 2, \cdots, n$ in Corollary 2.2, we have

Corollary 2.3. *Let* $f_i(z) \in \mathcal{A}$ *be in the class* $R(\alpha_i)$, $0 \le \alpha_i < 1$ *for all* $i = 1, 2, \cdots, n$. *Then the integral operator defined in (1.2) is in* $K(\delta)$, *where*

$$\delta = 1 - \sum_{i=1}^{n} |\gamma_i|(3-\alpha_i)$$

and $\sum_{i=1}^{n} |\gamma_i|(3-\alpha_i) < 1,\ \gamma_i \in \mathbb{C}$ for all $i = 1, 2, \cdots, n$.

Letting $\mu_i = 1$ for all $i = 1, 2, \cdots, n$ in Corollary 2.2, we have

Corollary 2.4. *Let* $f_i \in \mathcal{A}$ *be in the class* $S^*(\alpha_i)$, $0 \le \alpha_i < 1$ *for all* $i = 1, 2, \cdots, n$. *If* $|f_i(z)| \le M$ $(M \ge 1,\ z \in U)$ *for all* $i = 1, 2, \cdots, n$, *then the integral operator defined in (1.2) is in* $K(\delta)$, *where*

$$\delta = 1 - \sum_{i=1}^{n} |\gamma_i| \cdot [1 + (2-\alpha_i)M]$$

and $\sum_{i=1}^{n} |\gamma_i| \cdot [1 + (2-\alpha_i)M] < 1,\ \gamma_i \in \mathbb{C}$ for all $i = 1, 2, \cdots, n$.

Letting $\alpha_i = \delta = 0$ for all $i = 1, 2, \cdots, n$ in Corollary 2.4, we have

Corollary 2.5. *Let* $f_i \in \mathcal{A}$ *be starlike functions in* U *for all* $i = 1, 2, \cdots, n$. *If* $|f_i(z)| \le M$ $(M \ge 1,\ z \in U)$ *for all* $i = 1, 2, \cdots, n$ *then the integral operator defined in (1.2) is convex in* U, *where* $\sum_{i=1}^{n} |\gamma_i| = \frac{1}{2M+1}$, $\gamma_i \in \mathbb{C}$ for all $i = 1, 2, \cdots, n$.

Theorem 2.6. *Let* $f_i(z)$ *be in the class* $B(\mu_i, \alpha_i)$, $\mu_i \ge 1$, $0 \le \alpha_i < 1$ *for all* $i = 1, 2, \cdots, n$. *If* $|f_i(z)| \le M_i$ $(M_i \ge 1,\ z \in U)$ *for all* $i = 1, 2, \cdots, n$, *then the integral operator*

$$H(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t}\right)^{\frac{1}{\beta}} t^{n(\beta-1)} \mathrm{d}t,$$

is in $K(\delta)$, *where*

$$\delta = 1 - \left\{ \frac{1}{|\beta|} \sum_{i=1}^{n} \left[(2-\alpha_i)M_i^{\mu_i - 1} + 1\right] + n|\beta - 1| \right\}$$

and $\frac{1}{|\beta|} \sum_{i=1}^{n} \left[(2-\alpha_i)M_i^{\mu_i - 1} + 1\right] + n|\beta - 1| < 1,\ \beta \in \mathbb{C} \setminus \{0\}$.

Proof. Let $f_i(z)$ be in the class $B(\mu_i, \alpha_i)$, $\mu_i \geq 1$, $0 \leq \alpha_i < 1$, for all $i = 1, 2, \cdots, n$. It follows from (1.3) that

$$\frac{H''(z)}{H'(z)} = \frac{1}{\beta} \cdot \frac{f_1'(z) \cdot z - f_1(z)}{z f_1(z)} + \cdots + \frac{1}{\beta} \cdot \frac{f_n'(z) \cdot z - f_n(z)}{z f_n(z)} + \frac{n(\beta - 1)}{z}$$

and

$$\frac{zH''(z)}{H'(z)} = \frac{1}{\beta} \cdot \sum_{i=1}^{n} \left(\frac{f_i'(z) \cdot z}{f_i(z)} - 1 \right) + n(\beta - 1).$$

So that

$$\left| \frac{zH''(z)}{H'(z)} \right| \leq \frac{1}{|\beta|} \cdot \sum_{i=1}^{n} \left(\left| \frac{z f_i'(z)}{f_i(z)} \right| + 1 \right) + n|\beta - 1|$$

$$\leq \frac{1}{|\beta|} \cdot \sum_{i=1}^{n} \left(\left| f_i'(z) \left(\frac{z}{f_i(z)} \right)^{\mu_i} \right| \cdot \left| \left(\frac{f_i(z)}{z} \right)^{\mu_i - 1} \right| + 1 \right) + n|\beta - 1|. \tag{2.3}$$

Applying the General Schwarz lemma, we have $\left| \frac{f_i(z)}{z} \right| \leq M_i$, $(z \in U)$ for all $i = 1, 2, \cdots, n$. Therefore, from (2.3), we obtain

$$\left| \frac{zH''(z)}{H'(z)} \right| \leq \frac{1}{|\beta|} \cdot \sum_{i=1}^{n} \left(\left| f_i'(z) \left(\frac{z}{f_i(z)} \right)^{\mu_i} \right| \cdot M_i^{\mu_i - 1} + 1 \right) + n|\beta - 1|, \ (z \in U). \tag{2.4}$$

From (1.1) and (2.4), we see that

$$\left| \frac{zH''(z)}{H'(z)} \right| \leq \frac{1}{|\beta|} \sum_{i=1}^{n} \left[(2 - \alpha_i) M_i^{\mu_i - 1} + 1 \right] + n|\beta - 1| = 1 - \delta.$$

This completes the proof. □

Letting $M_i = M$ for all $i = 1, 2, \cdots, n$ in Theorem 2.6, we have

Corollary 2.7. *Let $f_i(z)$ be in the class $B(\mu_i, \alpha_i)$, $\mu_i \geq 1$, $0 \leq \alpha_i < 1$ for all $i = 1, 2, \cdots, n$. If $|f_i(z)| \leq M$ $(M \geq 1, \ z \in U)$ for all $i = 1, 2, \cdots, n$, then the integral operator*

$$H(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{f_i(t)}{t} \right)^{\frac{1}{\beta}} t^{n(\beta - 1)} \mathrm{d}t$$

is in $K(\delta)$, where

$$\delta = 1 - \left\{ \frac{1}{|\beta|} \sum_{i=1}^{n} \left[(2 - \alpha_i) M^{\mu_i - 1} + 1 \right] + n|\beta - 1| \right\}$$

and $\frac{1}{|\beta|} \sum_{i=1}^{n} [(2 - \alpha_i) M^{\mu_i - 1} + 1] + n|\beta - 1| < 1$, $\beta \in \mathbb{C} \setminus \{0\}$.

Letting $\delta = 0$ in Corollary 2.7 we have

Corollary 2.8. *Let $f_i(z)$ be in the class $B(\mu_i,\alpha_i),\ \mu_i\geq 1,\ 0\leq\alpha_i<1$ for all $i=1,2,\cdots,n$. If $|f_i(z)|\leq M\ (M\geq 1,\ z\in U)$ for all $i=1,2,\cdots,n$, then the integral operator defined in (1.3) is convex function in U, where*

$$\frac{1}{|\beta|}\sum_{i=1}^{n}\left[(2-\alpha_i)M^{\mu_i-1}+1\right]+n|\beta-1|=1,\ \beta\in\mathbb{C}\setminus\{0\}.$$

Letting $\mu_i=1$ for all $i=1,2,\cdots,n$ in Corollary 2.7, we have

Corollary 2.9. *Let $f_i(z)$ be in the class $S^*(\alpha_i),\ 0\leq\alpha_i<1$ for all $i=1,2,\cdots,n$. If $|f_i(z)|\leq M\ (M\geq 1,\ z\in U)$ for all $i=1,2,\cdots,n$, then the integral operator defined in (1.3) is in $K(\delta)$, where*

$$\delta=1-\left[\frac{1}{|\beta|}\sum_{i=1}^{n}(3-\alpha_i)+n|\beta-1|\right]$$

and $\frac{1}{|\beta|}\sum\limits_{i=1}^{n}(3-\alpha_i)+n|\beta-1|<1$ *for all* $\beta\in\mathbb{C}\setminus\{0\}$.

Letting $n=1$ and $\alpha_i=\delta=0$ for all $i=1,2,\cdots,n$ in Corollary 2.9, we have

Corollary 2.10. *Let $f(z)$ be a starlike function in U. If $|f(z)|\leq M\ (M\geq 1,\ z\in U)$ then the integral operator $\int_0^z\left(\frac{f(t)}{t}\right)^{\frac{1}{\beta}}t^{\beta-1}\mathrm{d}t$ is convex in U, where* $\frac{3}{|\beta|}+|\beta-1|=1,\ \beta\in\mathbb{C}\setminus\{0\}$.

Theorem 2.11. *Let $f_i(z)$ be in the class $B(\mu_i,\alpha_i),\ \mu_i\geq 1,\ 0\leq\alpha_i<1$ for all $i=1,2,\cdots,n$. If $|f_i(z)|\leq M_i\ (M_i\geq 1,\ z\in U)$, for all $i=1,2,\cdots,n$ then the integral operator*

$$G(z)=\int_0^z\prod_{i=1}^{n}(f_i(t))^{\beta-1}\mathrm{d}t$$

is in $K(\delta)$, where

$$\delta=1-|\beta-1|\cdot\sum_{i=1}^{n}(2-\alpha_i)M_i^{\mu_i-1}$$

and $|\beta-1|\cdot\sum\limits_{i=1}^{n}(2-\alpha_i)M_i^{\mu_i-1}<1,\ \beta\in\mathbb{C}$.

Proof. Let $f_i(z)$ be in the class $B(\mu_i,\alpha_i),\ \mu_i\geq 1,\ 0\leq\alpha_i<1$ for all $i=1,2,\cdots,n$. It follows from (1.4) that

$$\frac{G''(z)}{G'(z)}=(\beta-1)\sum_{i=1}^{n}\frac{f_i'(z)}{f_i(z)}$$

and, hence

$$\begin{aligned}\left|\frac{zG''(z)}{G'(z)}\right| &\leq |\beta-1|\left(\sum_{i=1}^{n}\left|\frac{zf_i'(z)}{f_i(z)}\right|\right)\\ &\leq |\beta-1|\left(\sum_{i=1}^{n}\left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu_i}\right|\cdot\left|\left(\frac{f_i(z)}{z}\right)^{\mu_i-1}\right|\right) \qquad (2.5)\end{aligned}$$

Applying the General Schwarz lemma, we have $\left|\frac{f_i(z)}{z}\right| \leq M_i,\ (z \in U)$ for all $i = 1, 2, \cdots, n$. Therefore, from (2.5), we obtain

$$\left|\frac{zG''(z)}{G'(z)}\right| \leq |\beta - 1| \left(\sum_{i=1}^{n} \left|f_i'(z)\left(\frac{z}{f_i(z)}\right)^{\mu_i}\right| \cdot M_i^{\mu_i - 1}\right),\ z \in U. \qquad (2.6)$$

From (1.1) and (2.6), we see that

$$\left|\frac{zG''(z)}{G'(z)}\right| \leq |\beta - 1| \cdot \sum_{i=1}^{n}(2 - \alpha_i)M_i^{\mu_i - 1} = 1 - \delta.$$

This completes the proof. □

Letting $M_i = M$ for all $i = 1, 2, \cdots, n$ in Theorem 2.11, we have

Corollary 2.12. *Let $f_i(z)$ be in the class $B(\mu_i, \alpha_i)$, $\mu_i \geq 1$, $0 \leq \alpha_i < 1$ for all $i = 1, 2, \cdots, n$. If $|f_i(z)| \leq M$ $(M \geq 1,\ z \in U)$ then the integral operator*

$$G(z) = \int_0^z \prod_{i=1}^{n}(f_i(t))^{\beta - 1}\mathrm{d}t$$

is in $K(\delta)$, where

$$\delta = 1 - |\beta - 1| \sum_{i=1}^{n}(2 - \alpha_i)M^{\mu_i - 1}$$

and $|\beta - 1| \sum_{i=1}^{n}(2 - \alpha_i)M^{\mu_i - 1} < 1,\ \beta \in \mathbb{C}$.

Letting $\delta = 0$ in Corollary 2.12 we have

Corollary 2.13. *Let $f_i(z)$ be in the class $B(\mu_i, \alpha_i)$, $\mu_i \geq 1$, $0 \leq \alpha_i < 1$ for all $i = 1, 2, \cdots, n$. If $|f_i(z)| \leq M$ $(M \geq 1,\ z \in U)$ then the integral operator defined in (1.4) is convex function in U, where*

$$|\beta - 1| = \frac{1}{\sum_{i=1}^{n}(2 - \alpha_i)M^{\mu_i - 1}},\ \beta \in \mathbb{C}.$$

Letting $\mu_i = 1$ for all $i = 1, 2, \cdots, n$ in Corollary 2.12, we have

Corollary 2.14. *Let $f_i(z)$ be in the class $S^*(\alpha_i)$, $0 \leq \alpha_i < 1$ for all $i = 1, 2, \cdots, n$. If $|f_i(z)| \leq M$ $(M \geq 1,\ z \in U)$ then the integral operator defined in (1.4) is in $K(\delta)$, where*

$$\delta = 1 - |\beta - 1| \sum_{i=1}^{n}(2 - \alpha_i)$$

and $|\beta - 1| \sum_{i=1}^{n}(2 - \alpha_i) < 1,\ \beta \in \mathbb{C}$.

Letting $n = 1$ and $\alpha_i = \delta = 0$ for all $i = 1, 2, \cdots, n$ in Corollary 2.14, we have

Corollary 2.15. *Let $f(z)$ be a starlike function in U. If $|f(z)| \leq M$ $(M \geq 1,\ z \in U)$ then the integral operator $\int_0^z f(t)^{\beta-1}\mathrm{d}t$ is convex in U, where $|\beta - 1| = \frac{1}{2},\ \beta \in \mathbb{C}$.*

References

1. B.A. Frasin and J. Jahangiri, *A new and comprehensive class of analytic functions*, Anal. Univ. Oradea, Fasc. Math. **15** (2008), 59–62.
2. Z. Nehari, *Conformal Mapping*, Dover, New York, NY, USA, 1975.

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