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Lower and Upper Solutions Method for Nonlinear Second-order Differential Equations Involving a Φ-Laplacian Operator

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Abstract

In this paper, we consider the following nonlinear second-order differential equations: $-(\Phi(u'(t)))' = f(t,u(t),u'(t)) + \Xi(u(t))$ a.e on $\Omega = [0,T]$ with a discontinuous perturbation and multivalued boundary conditions. The nonlinear differential operator is not necessarily homogeneous and incorporates as a special case the one-dimensional p-Laplacian. By combining lower and upper solutions method, a fixed point theorem for multifunction and theory of monotone operators, we show the existence of solutions and existence of extremal solutions in the order interval $[\alpha,\beta]$ where α and β are assumed respectively an ordered pair of lower and upper solutions of the problem. Moreover, we show that our method of proof also applies to the periodic problem.

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Keywords:$\Phi-$laplacian, lower and upper solutions, Bernstein-Nagumo-Winter condition, Extremal solutions, Truncation and penalty functions, Multifunctions, Fixed point.

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1 Introduction

We consider the nonlinear second order problem:

$$\begin{cases} -(\Phi(u'(t)))' = f(t,u(t),u'(t)) + \Xi(u(t)) \text{ a.e on } \Omega = [0,T] \\ u'(0) \in B_1(u(0)), -u'(T) \in B_2(u(T)) \end{cases} \tag{1.1}$$

where B_1 and B_2 are maximal monotone graphs in $\mathbb{R}^2$ and are some multifunctions which describe the boundary conditions, $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is L^p−Caratheodory function, $p \geq 2$, $\Xi : \mathbb{R} \to \mathbb{R}$ is a not necessarily continuous map, $\Phi : \mathbb{R} \to \mathbb{R}$ is an increasing homeomorphism. The Φ−Laplacian operators in the problem (1.1) contain, for example, some version of Φ−Laplacian operators like the case when, for all $z \in \mathbb{R}$, $\Phi(z) = a(z)\Phi_p(z)$ with $a : \mathbb{R} \to]0,+\infty[$ is a continuous map and for all $z \in \mathbb{R}$, $\Phi_p(z) = |z|^{p-2}z$,.

Second order boundary value problems involving a Φ−Laplacian operators have received a lot of attention with respect to existence and multiplicity solutions. As examples, see [1,5,6,7,8,11,12,13,14,15] and references therein. In their book, Papageorgiou and Kyritsi [6] (see the problem (5.111) p.390) study the following single-valued version of the problem in Staicu-Papageorgiou [12] :

$$\begin{cases} -\Big(a(u'(t))\Phi_p(u'(t))\Big)' = f(t,u(t),u'(t)) + \Xi(u(t)) \text{ a.e on } \Omega = [0,T] \\ u'(0) \in B_1(u(0)), -u'(T) \in B_2(u(T)) \end{cases}$$

where B_1, B_2, Ξ and f are defined as in problem (1.1) and for all $z \in \mathbb{R}$, $a(z) = 1$. The results of the present paper hold for some $a : \mathbb{R} \to]0,+\infty[$. So, our work generalize the one of Kyritsi-Bader [6]. Furthermore, when the maps B_1 and B_2 of problem (1.1) are single- valued, continuous and monotone, by setting $B_1 = g_0$ and $B_2 = -g_T$, then problem (1.1) become Neumann-Steklov problem. So, the formulation of problem (1.1) incorporates Neumann-Steklov type problems considered by Goli and Adjé in [15] but, here, the kind of Φ−Laplacian differential operators are different. Also, the boundary conditions of problem (1.1) unify the classical problems of Dirichlet, Neumann, Sturm-Liouville but not the periodic case. However, as in [6], our method of proof stay true for the periodic case.

The goal of this paper is to establish existence of solution and extremal solutions concerning the problem (1.1). The method of proof, in this work, is the same of the one of Kyritsi-Papageorgiou[6]. It combines lower and upper solutions method, theory of monotone operators and a fixed point theorem for multifunctions in ordered Banach space, due to Heikilla-Hu [9].

2 Notations and preliminaries

In this section, we introduce our terminology and notations. We also recall some basic definitions and facts from multivalued analysis that we shall need in the sequel. Our main sources are the books of Hu-Papageorgiou [10] and Zeidler [2].
We denote:

- $L^1(\Omega)$: the Banach space of Lebesgue-integrable functions on Ω ;

- $||u||_1 = \int_0^T |u(t)|\, dt$: the norm on $L^1(\Omega)$;
- $L^1(\Omega)_+ = \left\{u \in L^1(\Omega) : u(t) \geq 0 \text{ a.e on } \Omega\right\}$;
- $L^p(\Omega) = \left\{u \text{ mesurable and } \int_\Omega |u(t)|^p\, dt < \infty\right\}$;
- $||u||_p = \left(\int_0^T |u(t)|^p\, dt\right)^{\frac{1}{p}}$: the norm on $L^p(\Omega)$;
- $L^\infty(\Omega) = \{u \text{ mesurable on } \Omega \text{ and there exists } C \text{ such that } |u(t)| \leq C \text{ a.e on } \Omega\}$;
- $W^{1,p}(\Omega) = \{u \in L^p(\Omega) : u' \in L^p(\Omega)\}$ with u' the weak derivative of u ;
- $||y|| = \left(\int_0^T |y(t)|^p + |y'(t)|^p\right)^{\frac{1}{p}}$: the norm on $W^{1,p}(\Omega)$;
- $W_0^{1,p}(\Omega) = \left\{u \in W^{1,p}(\Omega) : u = 0 \text{ on } \partial\Omega\right\}$ where $\partial\Omega$ denotes the boundary of Ω ;
- $W^{-1,p}(\Omega)$: the dual of $W_0^{1,p}(\Omega)$;
- $||.||_{\mathbb{R}^2}$: the euclidean norm in $\mathbb{R}^2$;
- $C(\Omega)$: the banach space of continuous functions on Ω ;
- $||u||_\infty = \max\{|u(t)| : t \in \Omega\}$: the norm on $C(\Omega)$;
- $C^1(\Omega) = \{u : \Omega \to \mathbb{R} : u' \in C(\Omega)\}$ with u' the derivative function of u ;
- $\rightharpoonup$: the weak convergence ;
- $\longrightarrow$: the strong convergence ;
- $s^+ = \max\{s, 0\}$;
- $|x|$: absolute value of number x ;
- $\mathrm{R}(A)$: image of operator A ;
- $\Phi_p(x) = |x|^{p-2}x$: the one-dimensional operator $p-$Laplacian ;
- $P(X^*)$: the family of subsets of space X^*.

Let X be a reflexive Banach space and X^* the topological dual of X. A map $A : D(A) \subseteq X \longrightarrow P(X^*)$ is said to be monotone, if for all $x, y \in D(A)$ and for all $x^* \in A(x), y^* \in A(y)$, we have $\langle x^* - y^*, x - y\rangle \geq 0$. By $\langle . \rangle$, we denote the duality brackets for the pair (X, X^*). If additionally, the fact that $\langle x^* - y^*, x - y\rangle = 0$ implies that $x = y$, then we say that A is strictly monotone. The map A is said to be maximal monotone, if it is monotone and for all $x \in D(A)$, $x^* \in A(x)$, the fact that $\langle x^* - y^*, x - y\rangle \geq 0$ implies that $y \in D(A)$ and $y^* \in A(y)$. It is clear from this definition that A is maximal monotone if and only if its graph $\mathrm{Gr}A = \{(x, x^*) \in X \times X^* : x^* \in A(x)\}$ is maximal with respect to inclusion among the graphs of monotone maps . If A is maximal monotone , for any $x \in D(A)$, the set $A(x)$ is nonempty, closed and convex. Moreover , $\mathrm{Gr}A$ is *demiclosed*, ie, if $(x_n, x_n^*) \in \mathrm{Gr}A, n \geq 1$, either $x_n \to x$ in X and $x_n^* \rightharpoonup x^*$ in X^*, or

$x_n \rightharpoonup x$ in X and $x_n^* \to x^*$ in X^*, then $(x,x^*) \in \mathrm{Gr}A$. If $A : X \longrightarrow X^*$ is everywhere defined and single-valued, we say that A is demicontinuous, if for every sequence $(x_n)_{n\geq 1}$ such that $x_n \longrightarrow x$ in X, we have that $A(x_n) \rightharpoonup A(x)$ in X^*. If map $A : X \supseteq D(A) \longrightarrow X^*$ is monotone and demicontinuous, then it is also maximal monotone. A map : $X \supseteq D(A) \longrightarrow P(X^*)$ is said to be coercive, if $D(A) \subseteq X$ is bounded or if $D(A)$ is unbounded and we have that

$$\frac{\inf\{\langle x^*, x\rangle_X : x^* \in A(x)\}}{\|x\|_X} \longrightarrow +\infty \text{ as } \|x\|_X \longrightarrow +\infty, x \in D(A).$$

A maximal monotone and coercive map is surjective. Let Y,Z be Banach spaces and $L : Y \longrightarrow Z$. We say:

(a) L is completely continuous, if $y_n \rightharpoonup y$ in Y implies $L(y_n) \longrightarrow L(y)$ in Z and

(b) L is compact, if it is continuous and maps bounded sets into relatively compact sets.

In general, these two notions are distinct. However, if Y is reflexive, then complete continuity implies compactness. Moreover, if Y is reflexive and L is linear, then the two notions are equivalent.

To establish the existence of a solution for problem (1.1), we will need the following fixed point theorem for multifunctions in ordered Banach spaces due to Heikkila-Hu [9].

Theorem 2.1. *Let X be a separable, reflexive and ordered Banach space, $U \subseteq X$ a nonempty and weakly closed set. Let $S : U \longrightarrow P(U)\setminus\{\emptyset\}$ be a multifunction with weakly closed values. We suppose that $S(U)$ is bounded and :*

(i) $V = \{u \in U : u \leq v$, for some $v \in S(u)\}$ is nonempty ;

(ii) if $u_1 \leq y_1, y_1 \in S(u_1)$ and $u_1 \leq u_2$, then we can find $y_2 \in S(u_2)$ such that $y_1 \leq y_2$.

Then S has a fixed point , that's mean there exists $u \in U$ such that $u \in S(u)$.

3 Auxiliary results

Let $p,q \in \mathbb{N}^*$ such that $\frac{1}{p}+\frac{1}{q} = 1$ and $p \geq 2$. First, let us define what we mean by solution of problem (1.1).

Definition 3.1. A function $u \in C^1(\Omega)$ such that $\Phi(u'(.)) \in W^{1,q}((0,T))$ is said to be a solution of the problem (1.1), if it verifies (1.1).

Next, we introduce the notions of upper and lower solutions for problem (1.1).

Definition 3.2. (a) A function $\beta \in C^1(\Omega)$ such that $\Phi(\beta'(.)) \in W^{1,q}((0,T))$ is said to be an upper solution of the problem (1.1), if:

$$\begin{cases} -(\Phi(\beta'(t)))' \geq f(t,\beta(t),\beta'(t)) + \Xi(\beta(t)) \text{ a.e on } \Omega = [0,T] \\ \beta'(0) \in B_1(\beta(0)) - \mathbb{R}_+, -\beta'(T) \in B_2(\beta(T)) - \mathbb{R}_+. \end{cases}$$

(b) A function $\alpha \in C^1(\Omega)$ such that $\Phi(\alpha'(.)) \in W^{1,q}((0,T))$ is said to be a lower solution of problem (1.1), if:

$$\begin{cases} -(\Phi(\alpha'(t)))' \leq f(t,\alpha(t),\alpha'(t)) + \Xi(\alpha(t)) \ \text{ a.e on } \ \Omega = [0,T] \\ \alpha'(0) \in B_1(\alpha(0)) + \mathbb{R}_+, -\alpha'(T) \in B_2(\alpha(T)) + \mathbb{R}_+. \end{cases}$$

Remark 3.3. In general, for a given problem, there is not methodology (single valued and multivalued alike) which allows to generate a lower and an upper solutions. But, one should try simple functions such as constants, linear, quadratic, exponentials, eigenfunctions of simple operator, etc.

Our hypotheses on the data of (1.1) are the following:

(H_0) : There exist a lower solution $\alpha \in C^1(\Omega)$ and an upper solution $\beta \in C^1(\Omega)$.

(H_Φ) $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$ is an increasing continuous map such that:

- (a) $\Phi(0) = 0$;
- (b) there exists $d_1 > 0$ such that: $\Phi(x)x \geq d_1 |x|^p$ for all $x \in \mathbb{R}$;
- (c) there exist $d_2, d_3 > 0$ such that for a.e $t \in \Omega$ and for all $x \in \mathbb{R}$:

$$|\Phi(x)| \leq d_2 + d_3 |x|^{p-1}.$$

Remark 3.4. Suppose that $\Phi(z) = \Phi_p(z) = |z|^{p-2} z, p \geq 2$. Then this function satisfies hypothesis (H_Φ). This function correspond to the one-dimensional operator $p-$Laplacian. Another interesting case which satisfies hypothesis (H_Φ) is when Φ is defined by $\Phi(z) = a(z)|z|^{p-2} z$ with $a : \mathbb{R} \to]0,+\infty[$ continuous, $a(y) \geq k > 0$ for all $y \geq 0$ and $y \mapsto a(y)|y|^{p-2} y$ is strictly increasing on $\mathbb{R}$ and $a(y)|y|^{p-1} \leq d_2 + d_3 |y|^{p-1}$. In fact, one can write $a(z) = \varphi(|z|)$ with $\varphi :]0,+\infty[\to]0,+\infty[$. For examples, we have:

$$\varphi(|y|) = \frac{\sqrt{1 + \left(1 + |y|^{p-1}\right)^2}}{1 + |y|^{p-1}} \qquad \text{and} \qquad \varphi(|y|) = 1 + \frac{1}{1 + |y|^{p-1}}.$$

It is well-know that under the monotonicity condition and hypotheses (a) and (b) , Φ is a homeomorphism from $\mathbb{R}$ onto $\mathbb{R}$. And Φ^{-1} is strictly monotone and $\left|\Phi^{-1}(y)\right| \to +\infty$ as $|y| \to +\infty$ (See Deimling [4] chap. 3).

(H_f) $f : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a function such that:

- (i) for all $x, y \in \mathbb{R}, t \longmapsto f(t,x,y)$ is measurable;
- (ii) for a.e $t \in \Omega, (x,y) \longmapsto f(t,x,y)$ is continuous;
- (iii) for a.e $t \in \Omega$, for all $(x,y) \in [\alpha(t),\beta(t)] \times \mathbb{R}$, we have :

$$|f(t,x,y)| < \eta(|\Phi(y)|)(\psi(t) + c|y|)$$

where $\psi \in L^1(\Omega)_+$, $c > 0$ and $\eta : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \setminus \{0\}$ a Borel measurable nondecreasing functions such that:

$$\int_{\Phi(\lambda)}^{+\infty} \frac{ds}{\eta(s)} > \|\psi\|_1 + c\left(\max_\Omega \beta - \min_\Omega \alpha\right) + \frac{T}{\eta(\lambda)} \sup\{|\Xi(z)| : |z| \leq \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}\}$$

with $\quad \lambda = \dfrac{\max\{|\alpha(T)-\beta(0)|, |\alpha(0)-\beta(T)|\}}{T}$;

(iv) for every $r > 0$, there exists $\gamma_r \in L^q(\Omega)$ such that for a.e $t \in \Omega$ and for all $x, y \in \mathbb{R}$ with
$|x|, |y| \leq r$, we have: $|f(t,x,y)| \leq \gamma_r(t)$.

Remark 3.5. Hypothesis $(H_f)(iii)$ is known as a BernsteinNagumo Wintner growth condition and produces an uniform a priori bound of the derivatives of the solutions of problem (1.1).
And the hypotheses $(H_f)(i), (ii)$ and (iv) are well known as L^p-Caratheodory conditions.

(H_B) : B_1 and $B_2 : \mathbb{R} \longrightarrow P(\mathbb{R})$ are maximal monotone maps such that $0 \in B_1(0) \cap B_2(0)$.

Remark 3.6. There exist functions $E_1, E_2 : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ proper, convex and lower semicontinuous which are not identically equal to $+\infty$ such that $B_1 = \partial E_1, B_2 = \partial E_2$. More exactly, there exists some increasing positives functions p_1 and p_2 such that $p_i(s) = \mathrm{Proj}(0; B_i(s))$ (the minimum absolute value element in the closed, convex set $B_i(s)$). Then $E_i(s) = \int_0^s p_i(t)dt$, $i = 1,2$. We have: for all $s \in \mathbb{R}$, $B_i(s) = [p_i(s-); p_i(s+)]$, where $p_i(s-) = \lim_{\varepsilon \to 0^+} p_i(s-\varepsilon)$ and $p_i(s+) = \lim_{\varepsilon \to 0^+} p_i(s+\varepsilon)$, $i = 1,2$.

(H_Ξ) : $\Xi : \mathbb{R} \longrightarrow \mathbb{R}$ is a function that maps bounded sets to bounded sets and there exists $M > 0$ such that $x \longrightarrow \Xi(x) + Mx$ is increasing.

Remark 3.7. We emphasize that Ξ need not be continuous.

Lemma 3.8. *If $u \in C^1(\Omega)$ and hypotheses (H_Φ) and $(H_f)(iii)$ hold,*

$$-(\Phi(u'(t)))' = f(t, u(t), u'(t)) + \Xi(u(t)) \text{ a.e on } \Omega = [0,T]$$

and if

$$\alpha(t) \leq u(t) \leq \beta(t) \text{ for all } t \in \Omega$$

then, there exists $M_1 > 0$ (depending only on $\alpha, \beta, \eta, \psi, \Xi, c$) such that: $\quad |u'(t)| \leq M_1 \quad$ for all $\quad t \in \Omega$.

Proof. Set $\mu = \left(\dfrac{1}{\eta(\lambda)}\right) \sup\{|\Xi(z)| : |z| \leq \max\{\|\alpha\|_\infty, \|\beta\|_\infty\}\}$(See hypothesis H_Ξ). By hypothesis (iii) of (H_f), we can find $M_1 > \lambda$ such that

$$\int_{\Phi(\lambda)}^{\Phi(M_1)} \frac{ds}{\eta(s)} > \|\psi\|_1 + c\left(\max_\Omega \beta - \min_\Omega \alpha\right) + \mu T.$$

We claim that $|u'(t)| \leq M_1$ for all $t \in \Omega$. Suppose that this is not the case. Then, we can find $t_1 \in \Omega$ such that

$$\left|u'(t_1)\right| > M_1.$$

By the mean value theorem, there exists $t_2 \in (0,T)$ such that $u(T) - u(0) = u'(t_2)T$. Without any loss of generality, we assume that $t_2 \leq t_1$. We obtain:

$$\left|u'(t_2)\right| = \frac{1}{T}\left|u(T)-u(0)\right| \leq \frac{1}{T}\max\left\{|\alpha(T)-\beta(0)|,|\beta(T)-\alpha(0)|\right\} \Longrightarrow \left|u'(t_2)\right| \leq \lambda < M_1.$$

Since $u \in C^1(\Omega)$, by the intermediate value theorem, there exists t_3 and $t_4 \in [t_2, t_1)$ with $t_3 < t_4$ such that $|u'(t_3)| = \lambda$ and $|u'(t_4)| = M_1$. Then $\lambda < |u'(t)| < M_1$ for all $t \in (t_3, t_4)$. So, we have two possibilities:

(a) $u'(t_3) = \lambda$, $u'(t_4) = M_1$ and $\lambda < u'(t) < M_1$ for $t_3 < t < t_4$,

(b) $u'(t_3) = -\lambda$, $u'(t_4) = -M_1$ and $-M_1 < u'(t) < -\lambda$ for $t_3 < t < t_4$.

We will treat case (a). In similarly fashion , case (b) can be analysed. We have:

$$\begin{aligned}
-(\Phi(u'(t)))' &= f(t,u(t),u'(t)) + \Xi(u(t)) \text{ a.e on } \Omega = [0,T] \\
&\Rightarrow \left|\Phi(u'(t))\right|' \leq \left|(\Phi(u'(t)))'\right| \leq \left|f(t,u(t),u'(t))\right| + |\Xi(u(t))| \\
&\leq \eta\left(\left|\Phi(u'(t))\right|\right)\left(\psi(t) + c\left|u'(t)\right|\right) + |\Xi(u(t))| \text{ ae on } \Omega.
\end{aligned}$$

Thus:

$$\frac{|\Phi(u'(t))|'}{\eta\left(|\Phi(u'(t))|\right)} \leq \psi(t) + c\left|u'(t)\right| + \frac{|\Xi(u(t))|}{\eta\left(|\Phi(u'(t))|\right)} \text{ a.e on } [t_3, t_4]$$

and then

$$\int_{t_3}^{t_4} \frac{|\Phi(u'(t))|'}{\eta\left(|\Phi(u'(t))|\right)} dt \leq \|\psi\|_1 + c\left(\max_{\Omega}\beta - \min_{\Omega}\alpha\right) + \mu T.$$

Setting $s = |\Phi(u'(t))|$, we have :

$$\int_{\Phi(\lambda)}^{\Phi(M_1)} \frac{ds}{\eta(s)} \leq \|\psi\|_1 + c\left(\max_{\Omega}\beta - \min_{\Omega}\alpha\right) + \mu T$$

which contradicts the choice of $M_1 > 0$. □

Now, we introduce the truncation map : $\varrho : \Omega \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^2$ defined by:

$$\varrho(t,x,y) = \begin{cases} (\alpha(t), \alpha'(t)) & \text{if } x < \alpha(t) \\ (\beta(t), \beta'(t)) & \text{if } x > \beta(t) \\ (x, M_0) & \text{if } \alpha(t) \leq x \leq \beta(t), y > M_0 \\ (x, -M_0) & \text{if } \alpha(t) \leq x \leq \beta(t), y < -M_0 \\ (x,y) & \text{if } \alpha(t) \leq x \leq \beta(t), |y| \leq M_0 \end{cases} \tag{3.1}$$

where $M_0 > \max\{M_1, \|\alpha'\|_\infty, \|\beta'\|_\infty\}$ and the penalty function $\Lambda : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ defined by:

$$\Lambda(t,x) = \begin{cases} \Phi_p(\alpha(t)) - \Phi_p(x) & \text{if } x < \alpha(t) \\ 0 & \text{if } \alpha(t) \leq x \leq \beta(t) \\ \Phi_p(\beta(t)) - \Phi_p(x) & \text{if } x > \beta(t). \end{cases} \tag{3.2}$$

We set $f_1(t,x,y) = f(t,\varrho(t,x,y))$. Note that for ae $x \in [\alpha(t),\beta(t)]$ and all $|y| < M_0$, we have $f_1(t,x,y) = f(t,x,y)$. Moreover, for almost all $t \in \Omega$ and all $x,y \in \mathbb{R}$, we have: $|f_1(t,x,y)| \leq \gamma_r(t)$ with $r = \max\{M_0, \|\alpha\|_\infty, \|\beta\|_\infty\}$. For every $u \in W^{1,p}((0,T))$, we set

$$N_1(u)(.) = f_1(.,u(.),u'(.))$$

and

$$\hat{\Lambda}(u)(.) = \Lambda(.,u(.))$$

the Nemitsky operators corresponding to f_1 and Λ respectively. We set $J(u) = N_1(u) + \hat{\Lambda}(u)$ for every $u \in W^{1,p}((0,T))$.

Proposition 3.9. *If hypothesis $(H_f)(ii)$ holds, then: $J : W^{1,p}((0,T)) \longrightarrow L^q(\Omega)$ is continuous* .

Proof. Since N_1 and $\hat{\Lambda}$ are Nemitsky operators, it is standard to show that they are continuous. It follows that J is continuous. □

We introduce the set

$$D = \left\{u \in C^1(\Omega) : \Phi(u') \in W^{1,q}(0,T), u'(0) \in B_1(u(0)) \text{ and } -u'(T) \in B_2(u(T))\right\}$$

and then we define the nonlinear operator: $\vartheta : D \subseteq L^p(\Omega) \longrightarrow L^q(\Omega)$ by

$$\vartheta(u)(.) = -(\Phi(u'(.)))' \quad \text{for all } u \in D.$$

Proposition 3.10. *If the hypotheses (H_Φ) and (H_B) hold, then ϑ is maximal monotone.*

Proof. Given $h \in L^q(\Omega)$, we consider the following nonlinear boundary value problem:

$$\begin{cases} -(\Phi(u'(t)))' + \Phi_p(u(t)) = h(t) \text{ a.e on } \Omega = [0,T] \\ u'(0) \in B_1(u(0)), -u'(T) \in B_2(u(T)). \end{cases} \tag{3.3}$$

We show that problem (3.3) has a unique solution $u \in C^1(\Omega)$. To this end, given $v,w \in \mathbb{R}$, we consider the following two-point boundary value problem:

$$\begin{cases} -(\Phi(u'(t)))' + \Phi_p(u(t)) = h(t) \text{ a.e on } \Omega = [0,T] \\ u(0) = v, u(T) = w. \end{cases} \tag{3.4}$$

We set $\gamma(t) = \left(1 - \frac{t}{T}\right)v + \frac{t}{T}w$. Then $\gamma(0) = v$ and $\gamma(T) = w$. We consider the function y defined by
$y(t) = u(t) - \gamma(t)$ and rewrite (3.4) in the terms of the function y :

$$\begin{cases} -(\Phi(y'(t) + \gamma'(t)))' + \Phi_p(y(t) + \gamma(t)) = h(t) \text{ a.e on } \Omega = [0,T] \\ y(0) = y(T) = 0. \end{cases} \tag{3.5}$$

This is a homogeneous Dirichlet problem for (3.4). To solve (3.5), we argue as follows: Let $V_1 : W_0^{1,p}(\Omega) \longrightarrow W^{-1,q}(\Omega)$ be nonlinear operator defined by :

$$\langle V_1(y), z\rangle_0 = \int_0^T \Phi(y'(t) + \gamma'(t))z'(t)dt + \int_0^T \Phi_p(y(t) + \gamma(t))z(t)dt, \forall y,z \in W_0^{1,p}(\Omega)$$

where $\langle\rangle_0$ denotes the duality brackets for the pair $\left(W^{-1,q}(\Omega), W_0^{1,p}(\Omega)\right)$.

• Let us show that V_1 is strictly monotone.
Let $y, z \in W_0^{1,p}(\Omega)$. We have:

$$\begin{aligned}\langle V_1(y)-V_1(z), y-z\rangle_0 &= \langle V_1(y), y-z\rangle_0 - \langle V_1(z), y-z\rangle_0\\ &= \int_0^T \Phi(y'(t)+\gamma'(t))(y'(t)-z'(t))dt + \int_0^T \Phi_p(y(t)+\gamma(t))(y(t)-z(t))dt\\ &\quad - \int_0^T \Phi(z'(t)+\gamma'(t))(y'(t)-z'(t))dt - \int_0^T \Phi_p(z(t)+\gamma(t))(y(t)-z(t))dt.\end{aligned}$$

Then

$$\begin{aligned}\langle V_1(y)-V_1(z), y-z\rangle_0 &= \int_0^T (\Phi(y'(t)+\gamma'(t)) - \Phi(z'(t)+\gamma'(t)))\left((y'(t)+\gamma'(t)) - (z'(t)+\gamma'(t))\right)dt\\ &\quad + \int_0^T \left(\Phi_p(y(t)+\gamma(t)) - \Phi_p(z(t)+\gamma(t))\right)((y(t)+\gamma(t)) - (z(t)+\gamma(t)))\,dt.\end{aligned}$$

Therefore, V_1 is *strictly monotone* because Φ is monotone and Φ_p is strictly monotone.

• Let us show that V_1 is demicontinuous.
Using the extended dominated convergence theorem (see for example Hu-Papageorgiou [10], Theorem A.2.54, p. 907 or Bader-Papageorgiou [13] page 75), it follows easily that V_1 is *demicontinuous*.

⊛ Recall that an operator monotone and demicontinuous is maximal monotone. So V_1 is *maximal monotone*.

• Let us show that V_1 is coercive.
For $y \in W_0^{1,p}(\Omega)$ we have:

$$\begin{aligned}\langle V_1(y), y\rangle_0 &= \int_0^T \Phi(y'(t)+\gamma'(t))y'(t)dt + \int_0^T \Phi_p(y(t)+\gamma(t))y(t)dt\\ &\geq \int_0^T \Phi(y'(t)+\gamma'(t))(y'(t)+\gamma'(t))dt - \int_0^T \left|\Phi(y'(t)+\gamma'(t))\right|\left|\gamma'(t)\right|dt\\ &+ \int_0^T \Phi_p(y(t)+\gamma(t))(y(t)+\gamma(t))dt - \int_0^T \left|\Phi_p(y(t)+\gamma(t))\right||\gamma(t)|\,dt.\end{aligned}$$

Using the hypotheses (b) and (c) on Φ, it follows

$$\langle V_1(y), y\rangle_0 \geq \eta_1 \|y+\gamma\|^p - \eta_2 \|y+\gamma\|^{p-1} - \eta_3, \text{ for some } \eta_1, \eta_2, \eta_3 > 0.$$

Therefore , V_1 is coercive.

⊛ Recall that an operator maximal monotone which is coercive is surjective. So, V_1 is surjective.

Moreover, since V_1 is strictly monotone, we infer that there exists an unique $y \in W_0^{1,p}((0,T))$ such that $V_1(y) = h$. For any test function ϕ , we have:

$$\begin{aligned}&\langle V_1(y), \phi\rangle_0 = \langle h, \phi\rangle_0\\ \Leftrightarrow &\int_0^T \Phi(y'(t)+\gamma'(t))\phi'(t)dt = \int_0^T \Big(h(t) - \Phi_p(y(t)+\gamma(t))\Big)\phi(t)dt.\end{aligned}$$

From the definition of the distributional derivative, it follows that:

$$-(\Phi(y'(t)+\gamma'(t)))' = h(t) - \Phi_p(y(t)+\gamma(t)) \text{ a.e on } \Omega.$$

Whence y is the unique solution of problem (3.5). Then $u = y+\gamma \in C^1(\Omega)$ is the unique solution of the problem (3.4). We can define the solution map $\sigma : \mathbb{R}\times\mathbb{R} \longrightarrow C^1(\Omega)$ which assigns to each pair (v,w) the unique solution of the problem (3.4). Let $Q : \mathbb{R}\times\mathbb{R} \longrightarrow \mathbb{R}\times\mathbb{R}$ be defined by:

$$Q(v,w) = (-\Phi(\sigma(v,w)'(0)), \Phi(\sigma(v,w)'((T)).$$

• We claim that Q is monotone.
Indeed, for $(v_1,w_1),(v_2,w_2) \in \mathbb{R}^2$, we have :

$$\begin{aligned}&\langle Q(v_1,w_1) - Q(v_2,w_2), (v_1,w_1) - (v_2,w_2)\rangle_2\\ &= \Big\langle -\Big(\Phi(u_1'(0)) - \Phi(u_2'(0)), \Phi(u_1'(T)) - \Phi(u_2'(T))\Big), (u_1(0) - u_2(0), u_1(T) - u_2(T))\Big\rangle_2\\ &= \Big(\Phi(u_2'(0)) - \Phi(u_1'(0))\Big)(u_1(0) - u_2(0)) + \Big(\Phi(u_1'(T)) - \Phi(u_2'(T))\Big)(u_1(T) - u_2(T))\\ &= \int_0^T \Big(\Phi(u_1'(t))\Big)' - \Big(\Phi(u_2'(t))\Big)'(u_1(t) - u_2(t))\,dt + \int_0^T \Big(\Phi(u_1'(t)) - \Phi(u_2'(t))\Big)\Big(u_1'(t) - u_2'(t)\Big)dt\end{aligned}$$

where $\langle\rangle_2$ is the scalar product in $\mathbb{R}^2$.
From (3.4), we have $\Big(\Phi(u_1'(t))\Big)' - \Big(\Phi(u_2'(t))\Big)' = \Phi_p(u_1(t)) - \Phi_p(u_2(t))$. Because of monotonicity of the operators Φ and Φ_p , we obtain the monotonicity of Q.

• We claim that Q is continuous.
Indeed, let $(b_n)_{n\geq 1}$ and $(e_n)_{n\geq 1}$ be real sequences converging respectively to b and e.
Let us set :

$$\begin{aligned}&u_n = \sigma(b_n,e_n),\ u = \sigma(b,e),\ \ \gamma_n(t) = (1-\frac{t}{T})b_n + \frac{t}{T}e_n,\ \ \gamma(t) = (1-\frac{t}{T})b + \frac{t}{T}e\\ &\text{and}\quad y_n = u_n - \gamma_n,\quad \text{for all } n \geq 1.\end{aligned}$$

Now, we consider the following sequence of problems:

$$\begin{cases} -(\Phi(y_n'(t)+\gamma_n'(t)))' + \Phi_p(y_n(t)+\gamma_n(t)) = h(t) \ \text{ a.e on } \ \Omega = [0,T]\\ y_n(0) = y_n(T) = 0.\end{cases} \tag{3.6}$$

• We claim that $\{u_n = y_n+\gamma_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded.
Let us multiply (3.6) by $y_n(t)$ and then integrate on Ω. We obtain:

$$\int_0^T -(\Phi(y_n'(t)+\gamma_n'(t)))'y_n(t)dt + \int_0^T (\Phi_p(y_n(t)+\gamma_n(t)))y_n(t)dt = \int_0^T h(t)y_n(t)dt.$$

By using green's formula, we obtain:

$$\int_0^T (\Phi(y_n'(t)+\gamma_n'(t)))'y_n'(t)dt + \int_0^T (\Phi_p(y_n(t)+\gamma_n(t)))y_n(t)dt = \int_0^T h(t)y_n(t)dt$$
$$\geq \int_0^T (\Phi(y_n'(t)+\gamma_n'(t))' \left(y_n'(t)+\gamma_n'(t)\right)dt + \int_0^T (\Phi_p(y_n(t)+\gamma_n(t))\left(y_n(t)+\gamma_n(t)\right)dt$$
$$-\int_0^T \left|(\Phi(y_n'(t)+\gamma_n'(t))'\right|\left|\gamma_n'(t)\right|dt - \int_0^T \left|(\Phi_p(y_n(t)+\gamma_n(t))\right||\gamma_n(t)|dt$$

Whence:

$$\int_0^T h(t)y_n(t)dt \geq \eta_1' \|y_n+\gamma_n\|^P - \eta_2' \|y_n+\gamma_n\|^{p-1} - \eta_3' \quad \text{for some } \eta_1', \eta_2', \eta_3' > 0.$$

Furthermore, using the Cauchy-Schwartz inequality and then the triangular inequality, we obtain the following inequalities:

$$\int_0^T h(t)y_n(t)dt \leq \|h\|_q \|y_n\|_p \leq \|h\|_q \left(\|y_n+\gamma_n\|_p + \|\gamma_n\|_p\right) \leq \|h\|_q \left(\|y_n+\gamma_n\| + \|\gamma_n\|_p\right).$$

Then:

$$\eta_1' \|y_n+\gamma_n\|^p \leq \eta_2' \|y_n+\gamma_n\|^{p-1} + \|h\|_q \|y_n+\gamma_n\| + \|h\|_q \|\gamma_n\|_p + \eta_3'.$$

So

$$\eta_4 \|y_n+\gamma_n\|^p \leq \eta_5 \|y_n+\gamma_n\|^{p-1} + \eta_6 \|y_n+\gamma_n\| + \eta_7 \quad \text{for some } \eta_4, \eta_5, \eta_6, \eta_7 > 0.$$

Therefore, the sequence $\{u_n = y_n + \gamma_n\}_{n\geq 1} \subseteq W^{1,p}(\Omega)$ is bounded. It follows immediately that the sequences $\left\{\Phi_p(u_n)\right\}_{n\geq 1} \subseteq L^q(\Omega)$ is bounded. Then, directly from the problem (3.4), we get that the sequence $\{(\Phi(u_n'))'\}_{n\geq 1} \subseteq L^q(\Omega)$ is bounded . By integration, we obtain $\{\Phi(u_n')\}_{n\geq 1} \subseteq L^q(\Omega)$ is bounded. So the sequence $\{\Phi(u_n')\}_{n\geq 1} \subseteq W^{1,q}((0,T))$ is bounded. Then we have respectively

$$u_n \rightharpoonup u \text{ in } W^{1,p}(\Omega), \Phi_p(u_n) \rightharpoonup v \text{ in } L^q(0,T) \text{ and } \Phi(u_n') \rightharpoonup w \text{ in } W^{1,q}((0,T))$$

Due to the compact embedding of $W^{1,p}((0,T))$ in $C(\Omega)$, we have:

$$u_n \longrightarrow u \text{ in } C(\Omega) \text{ and } \Phi(u_n') \longrightarrow w \text{ in } C(\Omega).$$

Since Φ is an homeomorphism, Φ^{-1} exists and is continuous. So, we have: $u_n' \longrightarrow \Phi^{-1}(w)$ in $C(\Omega)$. Whence $u' = \Phi^{-1}(w)$ (ie $\Phi(u') = w$). Therefore passing to the limit as $n \longrightarrow +\infty$, we have:

$$\begin{cases} -(\Phi(u'(t))' + \Phi_p(u(t)) = h(t) \text{ a.e on } \Omega = [0,T] \\ u(0) = b, u(T) = e \end{cases}$$

$$\Longrightarrow u = \sigma(b,e)\Big(\text{ ie } \sigma : \mathbb{R}\times\mathbb{R} \longrightarrow C^1(\Omega) \text{ is continuous }\Big). \text{ So, } Q \text{ is continuous.}$$

• We claim that Q is coercive.
For $(v,w)\in\mathbb{R}^2$, we have:

$$\frac{\langle Q(v,w),(v,w)\rangle_2}{\|(v,w)\|_{\mathbb{R}^2}}=\frac{\Phi(u'(T))u(T)-\Phi(u'(0))u(0)}{\|(v,w)\|_{\mathbb{R}^2}}=\frac{\int_0^T(\Phi(u'(t)))'\,u(t)dt+\int_0^T\Phi(u'(t))u'(t)dt}{\|(v,w)\|_{\mathbb{R}^2}}$$

$$\frac{\int_0^T(\Phi(u'(t)))'\,u(t)dt+\int_0^T\Phi(u'(t))u'(t)dt}{\|(v,w)\|_{\mathbb{R}^2}}=\frac{\int_0^T\Phi_p(u(t))u(t)dt-\int_0^T h(t)u(t)dt+\int_0^T\Phi(u'(t))u'(t)dt}{\|(v,w)\|_{\mathbb{R}^2}}.$$

$$\geq\frac{\int_0^T(|u(t)|^p+d_1|u'(t)|^p)\,dt-\left(\int_0^T|h(t)|^q\right)^{\frac{1}{q}}\left(\int_0^T|u(t)|^p\right)^{\frac{1}{p}}}{\|(v,w)\|_{\mathbb{R}^2}}\geq\frac{d_4\left(\|u\|_p^p+\|u'\|_p^p\right)-\|h\|_q\|u\|_p}{\|(v,w)\|_{\mathbb{R}^2}}.$$

Then

$$\frac{\langle Q(v,w),(v,w)\rangle_2}{\|(v,w)\|_{\mathbb{R}^2}}\geq\frac{d_4\|u\|^p-\|h\|_q\|u\|_p}{\|(v,w)\|_{\mathbb{R}^2}}\quad\text{for some}\quad d_4>0$$

Since $u\in W^{1,p}(\Omega)$, by mean value theorem, there is some $t_0\in(0,T)$ such that $|u(t_0)|T=\int_0^T|u(t)|dt$.
As $u(t)-u(t_0)=\int_{t_0}^t u'(s)ds$, we have :
$|u(t)|\leq|u(t_0)|+\int_0^T|u'(s)|ds\leq\frac{1}{T}\|u\|_1+T^{\frac{1}{q}}\|u'\|_P\leq\frac{T^{\frac{1}{q}}}{T}\|u\|_P+T^{\frac{1}{q}}\|u'\|_p$ for all $t\in\Omega$. In particular, we have:

$$\|(v,w)\|_{\mathbb{R}^2}\leq\sqrt{2}\left(\frac{T^{\frac{1}{q}}}{T}\|u\|_p+T^{\frac{1}{q}}\left\|u'\right\|_P\right)\leq\sqrt{2}\max\left\{\frac{T^{\frac{1}{q}}}{T},T^{\frac{1}{q}}\right\}\left(\|u\|_P+\left\|u'\right\|_P\right).$$

So

$$\|(v,w)\|_{\mathbb{R}^2}\leq\zeta\|u\|\text{ for some }\zeta>0.$$

Then

$$\frac{\langle Q(v,w),(v,w)\rangle_2}{\|(v,w)\|_{\mathbb{R}^2}}\geq\frac{d_4\|u\|^p-\|h\|_q\|u\|_p}{\zeta\|u\|}.$$

Therefore Q is coercive.

We infer that Q is maximal monotone (being continuous, monotone) and coercive. Thus Q is surjective. Now, let $B:\mathbb{R}\times\mathbb{R}\longrightarrow P(\mathbb{R}\times\mathbb{R})$ be defined by:

$$B(v,w)=(\Phi\circ B_1(v),\Phi\circ B_2(w))\text{ for all }(v,w)\in\mathbb{R}\times\mathbb{R}.$$

We have B is maximal monotone (see Claim 4 in the proof of Proposition 3.8 in BaderPapageorgiou [7]). Next, let $\theta:\mathbb{R}\times\mathbb{R}\longrightarrow P(\mathbb{R}\times\mathbb{R})$ be defined by:

$$\theta(v,w)=Q(v,w)+B(v,w)\text{ for all }(v,w)\in\mathbb{R}\times\mathbb{R}.$$

Then θ is maximal monotone (see Brezis [3, Corollary 2.7, p. 36] or Zeidler [2, Theorem 32.I, p. 897]). Moreover, since Q is coercive, B is maximal monotone and $(0,0)\in B(0,0)$, it follows that θ is coercive. Thus θ is surjective. We infer that we can find $(b,e)\in\mathbb{R}\times\mathbb{R}$ such that $(0,0)\in\theta(b,e)$. So $\Phi(u(0))\in\Phi\circ B_1(b)$ and $-\Phi(u(T))\in\Phi\circ B_2(e)$. Whence, by acting with Φ^{-1}, we obtain $(u'(0), -u'(T)) \in (B_1(b), B_2(e))$. Therefore $x_0 = \sigma(b, e)$ is the unique solution of the problem (3.3).

Let $H : L^p(\Omega) \longrightarrow L^q(\Omega)$ be the operator defined by:

$$H(u)(.) = \Phi_p(u(.)).$$

Since Φ_p is continuous and monotone, then H is continuous and monotone. Therefore H is maximal monotone. Also, since Φ_p is strictly monotone, H is strictly monotone.

Since in (3.3), the choice of h is arbitrary, by the previous arguments, we have:

$$R(\vartheta + H) = L^q(\Omega) \qquad (\text{ ie } \vartheta + H \text{ is surjective }). \tag{3.7}$$

We denote by $< ., . >_p$ the duality brackets between the pair $(L^p(\Omega), L^q(\Omega))$.

• Let us show that $\vartheta + H$ surjective implies ϑ is maximal monotone

For this purpose, we suppose that, for some $y \in L^p(\Omega)$ and some $v \in L^q(\Omega)$:

$$< \vartheta(u) - v, u - y >_p \geq 0 \text{ for all } u \in D. \tag{3.8}$$

Because of (3.7), we can find $u_1 \in D$ such that:

$$\vartheta(u_1) + H(u_1) = v + H(y).$$

We use this in (3.8) with $u = u_1$, to obtain :

$$< \vartheta(u_1) - \vartheta(u_1) - H(u_1) + H(y), u_1 - y >_p \geq 0.$$

$$\Rightarrow < H(y) - H(u_1), u_1 - y >_p \geq 0 \tag{3.9}$$

Because H is strictly monotone, from (3.9), we conclude that $y = u_1 \in D$ and $v = \vartheta(u_1)$. So ϑ is maximal monotone. In addition, since ϑ is monotone, we have $< \vartheta(u) + H(u), u >_p \geq < H(u), u >_2 = |u|^p$. Whence the operator $\vartheta + H : D \subseteq L^p(\Omega) \longrightarrow L^q(\Omega)$ is maximal monotone , strictly monotone and coercive. Therefore $\Psi = (\vartheta + H)^{-1} : L^q(\Omega) \longrightarrow D \subseteq W^{1,p}((0, T))$ is well defined, single valued, and maximal monotone (From $L^q(\Omega)$ into $L^p(\Omega)$). □

Proposition 3.11. *If hypotheses (H_Φ) and (H_B) hold, then $\Psi : L^q(\Omega) \longrightarrow D \subseteq W^{1,p}((0, T))$ is completly continuous.*

Proof. Suppose that $v_n \rightharpoonup v$ in $L^q(\Omega)$. We have to show that $\Psi(v_n) \longrightarrow \Psi(v)$ in $W^{1,p}(\Omega)$. let us set $u_n = \Psi(v_n)$ for all $n \geq 1$. We have

$$u_n \in D \text{ and } \vartheta(u_n) + H(u_n) = v_n.$$

$$\Longrightarrow \langle \vartheta(u_n), u_n \rangle_p + \langle H(u_n), u_n \rangle_p = \langle v_n, u_n \rangle_p . \tag{3.10}$$

By integration by part, we obtain:

$$-\Phi(u_n'(T))u_n(T)+\Phi(u_n'(0))u_n(0)+\int_0^T \Phi(u_n'(t))u_n'(t)dt+\int_0^T \Phi_p(u_n(t))u_n(t)dt=\langle v_n,u_n\rangle_p. \tag{3.11}$$

Since $u_n \in D$, we have $u_n'(0) \in B_1(u_n(0))$ and $-u_n'(T) \in B_1(u_n(T))$ for all $n \geq 1$. We recall that
$(0,0) \in Gr(B_i), i = 1,2$, then :

$$u_n'(0)u_n(0) \geq 0 \quad \text{and} \quad u_n'(T)u_n(T) \leq 0. \tag{3.12}$$

Moreover, the map Φ being increasing , we have :

$$\Phi(u_n'(0))u_n'(0) \geq 0 \quad \text{and} \quad \Phi(u_n'(T))u_n'(T) \geq 0. \tag{3.13}$$

From (3.12) and (3.13), we obtain :

$$\Phi(u_n'(0))u_n(0) \geq 0 \quad \text{and} \quad \Phi(u_n'(T))u_n(T) \leq 0. \tag{3.14}$$

From (3.11) and (3.14), we infer that :

$$\langle v_n,u_n\rangle_2 \geq \int_0^T \Phi(u_n'(t))u_n'(t)dt+\int_0^T \Phi_p(u_n(t))u_n(t)dt. \tag{3.15}$$

By hypothesis $b)$ on Φ , we have :

$$\int_0^T \Phi(u_n'(t))u_n'(t)dt+\int_0^T \Phi_p(u_n(t))u_n(t)dt \geq \int_0^T \left(d_1\left|u_n'(t)\right|^p+|u_n(t)|^p\right)dt. \tag{3.16}$$

It follows from (3.15) and (3.16) that :

$$\langle v_n,u_n\rangle_p \geq \int_0^T \left(d_1\left|u_n'(t)\right|^p+|u_n(t)|^p\right)dt.$$

Whence:

$$||u_n||^{p-1} \leq \eta_7 \quad \text{for some} \quad \eta_7>0.$$

Therefore the sequence $\{u_n\}_{n\geq 1} \subseteq W^{1,p}((0,T))$ is bounded. Then we can find a convergent subsequence of $\{u_n\}_{n\geq 1}$. So $u_n \rightharpoonup u$ in $W^{1,p}((0,T))$. Due to the compact embedding of $W^{1,p}((0,T))$ in $C(\Omega)$, we have $u_n \longrightarrow u$ in $C(\Omega)$. Since $\{u_n\}_{n\geq 1} \subseteq W^{1,p}((0,T))$ is bounded, we have $\{u_n'\}_{n\geq 1} \subseteq L^p(\Omega)$ and $\{u_n\}_{n\geq 1} \subseteq L^p(\Omega)$ are bounded. It follows that: $\left\{\Phi_p(u_n)\right\}_{n\geq 1} \subseteq L^q(\Omega)$ is bounded. Then $\vartheta(u_n)+H(u_n)=v_n$ implies that $\left\{(\Phi(u_n'))'\right\}_{n\geq 1} \subseteq L^q(\Omega)$ is bounded. Whence, by integration, $\{\Phi(u_n')\}_{n\geq 1} \subseteq W^{1,q}((0,T))$ is bounded. So we can suppose that $\Phi(u_n') \rightharpoonup h$ in $W^{1,q}((0,T))$. Due to the compact embedding of $W^{1,q}(0,T)$ in $C(\Omega)$, we obtain $\Phi(u_n') \to h$ in $C(\Omega)$. Since Φ is an homeomorphism , Φ^{-1} exists and is continuous. So, we have $\Phi^{-1}(\Phi(u_n'(t))) \longrightarrow \Phi^{-1}(h(t))$ for all $t \in \Omega$. Then $u_n'(t) \longrightarrow \Phi^{-1}(h(t))$ for all $t \in \Omega$. It follows that $u_n'(.) \longrightarrow \Phi^{-1}(h(.))$ in $L^P(\Omega)$ (By Lebesgue dominated convergence theorem). We infer that : $u' = \Phi^{-1}(h(.))$. So, $h = \Phi(u')$. We have:

$$\Phi(u_n') \longrightarrow \Phi(u') \text{ in } C(\Omega)$$
$$\Longrightarrow u_n' \longrightarrow u' \text{ in } L^p(\Omega).$$

But recall that $u_n \longrightarrow u$ in $L^p(\Omega)$. Thus $u_n \longrightarrow u$ in $W^{1,p}((0,T))$. This prove that the operator Ψ is completely continuous. □

4 Existence results

We introduce the order interval :

$$U=[\alpha,\beta]=\left\{u\in W^{1,p}((0,T)):\ \alpha(t)\leq u(t)\leq\beta(t)\text{ for all }\ t\in\Omega\right\}$$

We consider the operator $\tau: W^{1,p}((0,T))\longrightarrow W^{1,p}((0,T))$ defined by :

$$\tau(u)(t)=\max\{\alpha(t),\min\{u(t),\beta(t)\}\}=\begin{cases}\alpha(t) & \text{if } u(t)<\alpha(t)\\ u(t) & \text{if } \alpha(t)\leq u(t)\leq\beta(t)\\ \beta(t) & \text{if } u(t)>\beta(t).\end{cases}$$

We see that τ is bounded and is continuous.
Let $w\in U$. we consider the following auxiliary boundary problem :

$$\begin{cases}-(\Phi(u'(t)))'=f_1(t,u(t),u'(t))+\Lambda(t,u(t))+\Xi(w(t))-M\tau(u(t))+Mw(t)\ \text{ a.e on }\ \Omega=[0,T]\\ u'(0)\in B_1(u(0)),-u'(T)\in B_2(u(T)).\end{cases}\tag{4.1}$$

Proposition 4.1. *If the hypotheses* $(H_0),(H_f),(H_\Phi)$ *and* (H_Ξ) *hold, then the problem* (4.1) *has a solution* $u\in C^1(\Omega)\cap U$.

Proof. Let $J_1: W^{1,p}((0,T))\longrightarrow L^q(\Omega)$ be the nonlinear operator defined by:

$$J_1(u)=J(u)+H(u)-M\tau(u)+\hat{\Xi}(w)+Mw,\ \ \forall u\in W^{1,p}((0,T)).$$

From the proposition 3.1 and the continuity of the operators H and τ, we infer that J_1 is continuous. For all $u\in W^{1,p}((0,T))$, we have:

$$\|J_1(u)\|_q\leq\|\gamma_r\|_q+T^{\frac{1}{q}}\max\left\{\|\alpha\|_\infty^{p-1},\|\beta\|_\infty^{p-1})\right\}+T^{\frac{1}{q}}M\max\{\|\alpha\|_\infty,\|\beta\|_\infty\}+\left\|\hat{\Xi}(w)\right\|_q+M\|w\|_q=M_2$$

We recall that :

$$r=\max\{M_0,\|\alpha\|_\infty,\|\beta\|_\infty\}.$$

We set

$$\Gamma=\left\{v\in L^q(\Omega):\|v\|_q\leq M_2\right\}.$$

We see that J_1 maps bounded sets to bounded ones.
And $\Psi\left(J_1\left(W^{1,p}(0,T)\right)\right)\subseteq\Psi(\Gamma)$ which is relatively compact in $W^{1,p}(0,T)$
(see proposition 3.3). Therefore, we can find $u\in D\subseteq W^{1,p}((0,T))$ such that :

$$\begin{aligned}&u=\Psi(J_1(u))\\&\Longrightarrow\vartheta(u)+H(u)=J(u)+H(u)-M\tau(u)+\hat{\Xi}(w)+Mw\\&\Longrightarrow\vartheta(u)=J(u)-M\tau(u)+\hat{\Xi}(w)+Mw.\end{aligned}$$

Then $u\in D\subseteq C^1(\Omega)$ solves the problem (4.1).
It remains to show that $u\in U$. Since $\alpha\in C^1(\Omega)$ is a lower solution of the problem (1.1), we have :

$$\begin{cases}-(\Phi(\alpha'(t)))'\leq f(t,\alpha(t),\alpha'(t))+\Xi(\alpha(t))\text{ a.e on }\Omega=[0,T]\\ \alpha'(0)\in B_1(\alpha(0))+\mathbb{R}_+,-\alpha'(T)\in B_2(\alpha(T))+\mathbb{R}_+.\end{cases}\tag{4.2}$$

Soustraying (4.2) from (4.1), we obtain:

$$\begin{aligned}&(\Phi(\alpha'(t)))' - (\Phi(u'(t))' \geq f_1(t,u(t),u'(t)) + \Lambda(t,u(t)) + \Xi(w(t)) - M\tau(u(t)) + Mw(t)\\&- f(t,\alpha(t),\alpha'(t)) - \Xi(\alpha(t))\end{aligned} \tag{4.3}$$

We multiply (4.3) by $(\alpha-u)^+ \in W^{1,p}((0,T))$ and then integrate on Ω. We obtain:

$$\begin{aligned}&\int_0^T [(\Phi(\alpha'(t)))' - (\Phi(u'(t)))'](\alpha-u)^+(t)dt\\&\geq \int_0^T [f_1(t,u(t),u'(t)) - f(t,\alpha(t),\alpha'(t))](\alpha-u)^+(t)dt + \int_0^T \Lambda(t,u(t))(\alpha-u)^+(t)dt +\\&\int_0^T [\Xi(w(t)) - \Xi(\alpha(t)) - M\tau(u(t)) + Mw(t)](\alpha-u)^+(t)dt.\end{aligned} \tag{4.4}$$

The integration by parts of the left-hand side in inequality, yields :

$$\begin{aligned}&\int_0^T [(\Phi(\alpha'(t))' - (\Phi(u'(t))'](\alpha-u)^+(t)dt\\&= (\Phi(\alpha'(T)) - \Phi(u'(T)))(\alpha-u)^+(T) - (\Phi(\alpha'(0)) - \Phi(u'(0)))(\alpha-u)^+(0)\\&- \int_0^T [\Phi(\alpha'(t)) - \Phi(u'(t))](\alpha-u)'^+(t)dt\\&\geq \int_0^T [f_1(t,u(t),u'(t)) - f(t,\alpha(t),\alpha'(t))](\alpha-u)^+(t)dt + \int_0^T \Lambda(t,u(t))(\alpha-u)^+(t)dt +\\&\int_0^T [\Xi(w(t)) - \Xi(\alpha(t)) - M\tau(u(t)) + Mw(t)](\alpha-u)^+(t)dt.\end{aligned} \tag{4.5}$$

We set

$$[(\alpha-u)^+]'(t) = \begin{cases} (\alpha(t)-u(t))' & \text{if } \alpha(t) > u(t)\\ 0 & \text{if } \alpha(t) \leq u(t).\end{cases} \tag{4.6}$$

Also, from the boundary conditions in (4.1) and (4.2), we have:

$$-u'(T) \in B_1(u(T)) \text{ and } -\alpha'(T) \in B_1(\alpha(T)) + e_T \quad \text{with} \quad e_T \geq 0.$$

If $\alpha(T) \geq u(T)$, then from the monotony of B_2 (See hypothesis (H_B)), we have:

$$\alpha'(T) \leq u'(T). \text{ Since } \Phi \text{ is increasing, we have : } \Phi(\alpha'(T)) \leq \Phi(u'(T)).$$

So, it follows that

$$(\Phi(\alpha'(T)) - \Phi(u'(T)))(\alpha(T) - u(T)) \leq 0. \tag{4.7}$$

In a similar fashion, using the boundary conditions $u'(0) \in B_1(u(0))$ and $\alpha'(0) \in B_1(\alpha(0)) + e_0$ with $e_0 \geq 0$, if $\alpha(0) \geq u(0)$, we have:

$$\alpha'(0) \geq u'(0). \text{ We infer that } \Phi(\alpha'(0)) \geq \Phi(u'(0)).$$

It follows that

$$(\Phi(\alpha'(0)) - \Phi(u'(0)))(\alpha(0) - u(0)) \geq 0. \tag{4.8}$$

Also, since Φ is an increasing homeomorphism, we have:

$$\int_0^T (\Phi(\alpha'(t)) - \Phi(u'(t)))(\alpha - u)'^{+}(t)dt = \int_{\{\alpha>u\}} (\Phi(\alpha'(t)) - \Phi(u'(t)))(\alpha - u)'(t)dt \geq 0 \quad (4.9)$$

where $\{\alpha > u\} = \{t \in [0,T] : \alpha(t) > u(t)\}$.
Using the inequalities (4.7), (4.8) and (4.9) in the first member of (4.5), we obtain:

$$\int_0^T [(\Phi(u'(t))' - (\Phi(\alpha'(t))'](\alpha - u)^+(t)dt \leq 0. \quad (4.10)$$

Furthermore:

$$\begin{aligned} &f_1(t,u(t),u'(t)) - f(t,\alpha(t),\alpha'(t)) = f(t,\alpha(t),\alpha'(t)) - f(t,\alpha(t),\alpha'(t)) = 0 \text{ a.e on } \{\alpha > u\} \\ &\Rightarrow \int_0^T (f_1(t,u(t),u'(t)) - f(t,\alpha(t),\alpha'(t)))(\alpha - u)^+(t)dt = 0. \end{aligned} \quad (4.11)$$

Also from the definiton of the penalty map Λ , if $|\{\alpha > u\}| > 0$ (By $|.|$, we denote the Lebesgue mesure in $\mathbb{R}$), then:

$$\int_0^T \Lambda(t,u(t))(\alpha - u)^+(t)dt = \int_{\{\alpha>u\}} (\Phi(\alpha(t)) - \Phi(u(t)))(\alpha - u)^+(t)dt > 0. \quad (4.12)$$

Finally, by virtue of hypothesis (H_Ξ) and, since $w \in U$, we see that:

$$\begin{aligned} &\int_0^T (\Xi(w(t)) - \Xi(\alpha(t)) - M\tau(u(t)) - Mw(t))(\alpha - u)^+(t)dt \\ &= \int_{\{\alpha>u\}} (\Xi(w(t)) - \Xi(\alpha(t)) + Mw(t) - M\alpha(t))(\alpha - u)^+(t)dt \geq 0. \end{aligned} \quad (4.13)$$

Using the inequalities (4.11), (4.12) and (4.13) in the second member of (4.5), we infer that :

$$\begin{aligned} &\int_0^T [f_1(t,u(t),u'(t)) - f(t,\alpha(t),\alpha'(t))](\alpha - u)^+dt + \int_0^T \Lambda(t,u(t))(\alpha - u)^+dt + \\ &\int_0^T (\Xi(w(t)) - \Xi(\alpha(t)) + Mw(t) - M\alpha(t))(\alpha - u)^+(t)dt > 0. \end{aligned} \quad (4.14)$$

We consider (4.5) and using (4.10) and (4.14), we have a contradiction when $|\{\alpha > u\}| > 0$. Therefore , for all $t \in \Omega, \alpha(t) \leq u(t)$. In a similar fashion we show that $u(t) \leq \beta(t)$ for all $t \in \Omega$; thus $u \in U$. □

We use the solvability of the auxiliary problem (4.1) in order to produce a solution for the original problem (1.1). For this purpose, we need the fixed point theorem 2.1 for multifunctions in an ordered Banach space. To apply this theorem, we use the following data: $X = W^{1,p}((0,T))$ is the separable, reflexive, ordered Banach space, $U = [\alpha,\beta] \subseteq W^{1,p}((0,T))$ and $S : U \longrightarrow P(U) \setminus \{\emptyset\}$ is the solution multifunction for the auxiliary problem (4.1) . Then for every $w \in U$, $S(w)$ is subset of U solutions of the problem (4.1) . From Proposition 4.1 , we know that: $S(w) \neq \emptyset$ and $S(w) \subseteq U$.

Theorem 4.2. *If the hypotheses* $(H_f),(H_B)$*, and* (H_Ξ) *hold, then problem (1.1) has a solution* $u \in C^1(\Omega)$.

Proof. Some simple modifications in the proof of theorem 5.2.21 p.400 of Kyritsi-Papageorgiou [6] concerning the differential operator allows easily to obtain the proof . □

4.1 Existence of extremal solutions

We establish the existence of a greatest and of a smallest solution in the order interval U. So let

$$C_1 = \left\{u \in W^{1,p}(\Omega) : u \text{ be a solution of (1.1) and } u \in U\right\}.$$

On $L^\infty(\Omega)$, we consider the partial order structure induced by the order cone

$$L^\infty(\Omega)_+ = \{u \in L^\infty(\Omega) : u(t) \geq 0 \text{ a.e on } \Omega\}.$$

So $u \leq y$ in $L^\infty(\Omega)_+$ if and only if $u(t) \leq y(t)$ almost everywhere on Ω. From the theorem 2.1, we know that under hypotheses $(H_0),(H_f),(H_B)$ and (H_Ξ), the set C_1 is nonempty. Recall that the espace C in a partially ordered set is a chain (or totally ordered subset) , if for every $x,y \in C$ either $x \leq y$ or $y \leq x$.

Proposition 4.3. *If hypotheses* $(H_0),(H_f),(H_B),(H_g)$ *and* (H_Ξ) *hold, then every chain* C *in* C_1 *has an upper bound.*

Proof. Simple modifications in the proof of proposition 5.2.22 p.401 of Kyritsi-Papageorgiou [6] concerning the differential operator allows easily to obtain the proof. □

Recall if $(C_0,\leq)$ is a partially ordered set, we say that C_0 is directed, if every pair of u_1 and $u_2 \in C_0$ such that $u_1 \leq u_3$ and $u_2 \leq u_3$.

Proposition 4.4. *If the hypotheses* $(H_0),(H_f),(H_g)$ *and* $H(\Xi)$ *hold, then the partially ordered set* $C_1 \subseteq W^{1,p}((0,T))$ *is directed.*

Proof. Simple modifications in the proof of proposition 5.2.23 p.402 of Kyritsi-Papageorgiou [6] concerning the differential operator allows easily to obtain the proof. □

Theorem 4.5. *If the hypotheses* $(H_0),(H_f),(H_B)$*, and* (H_Ξ) *hold, then the problem (1.1) have some extremal solutions in the order interval* $U = [\alpha,\beta]$.

Proof. Using the same arguments of the proof of theorem 5.2.24 p. 403 of Kyritsi-Papageorgiou [6], we obtain the proof. □

5 Example and periodic problem

5.1 Example

Let us consider the following problem:

$$\begin{cases} -\left(\dfrac{\sqrt{1+\left(1+|u'(t)|^{p-1}\right)^2}}{1+|u'(t)|^{p-1}}\left|u'(t)\right|^{p-2}u'(t)\right)' = f(t,u(t),u'(t)) + \Xi(u(t)) \text{ a.e on } \Omega = [0,T] \\ u'(0) \in B_1(u(0)) \quad , \quad -u'(T) \in B_2(u(T)) \end{cases} \tag{5.1}$$

where f, Ξ, B_1 and B_2 are defined as in problem (1.1). Here, $\Phi(z) = \dfrac{\sqrt{1+\left(1+|z|^{p-1}\right)^2}}{1+|z|^{p-1}}\,|z|^{p-2}\,z$, for all $z \in \mathbb{R}$ and by the remark 3.2, it satisfies hypothesis (H_Φ) . Therefore, theorem 4.1 and theorem 4.2 are true for the problem (5.1). Moreover, by [6] (see example 5.2.25 page 404), this problem unifies classical problems of Dirichlet, Neumann and Sturm-Liouville and go beyong them.

5.2 Periodic problem

Let us consider the following periodic problem:

$$\begin{cases} -(\Phi(u'(t)))' = f(t,u(t),u'(t)) + \Xi(u(t)) \text{ a.e on } \Omega = [0,T] \\ u(0) = u(T) \ , \ u'(0) = u'(T). \end{cases} \tag{5.2}$$

Remark 5.1. The theorems 4.1 and 4.2 stay true for this problem (see [6] remark 5.2.26 page 404 and also [7] section 6 page 23).

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