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On the Hilali conjecture for configuration spaces of closed manifolds

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Abstract

The first author conjectured in 1990 (see [18]) that for any simply-connected elliptic space, the total dimension of the rational homotopy does not exceed that of its rational cohomology. Our main purpose in this paper is to investigate the following: does the Hilali conjecture holds for the configuration spaces of a rationally elliptic and simply connected topological space when it already holds for the space itself. We will prove that this statement is true for closed manifolds.

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1 Introduction

A topological space X is called rationally *elliptic* when both of $\pi_*(X)\otimes\mathbb{Q}$ and $H^*(X;\mathbb{Q})$ are of finite dimension, otherwise it is called *hyperbolic*. For that kind of spaces, the Hilali conjecture predicts that:

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Conjecture 1.1 (Topological version)**.** *If X is an elliptic and simply connected topological space, then*

$$\dim \pi_*(X)\otimes\mathbb{Q} \leq \dim H^*(X;\mathbb{Q}).$$

Until now, this conjecture holds in many interesting cases: for pure spaces ([18]), these are spaces whose Euler-Poincaré characteristic is nonzero, for $H-$spaces, for nilmanifolds, for symplectic and cosymplectic manifolds, for coformal spaces whose rational homotopy is concentrated in odd degrees, and for formal spaces (see [19]), [20])). Authors in [3] have extended the Hilali conjecture from pure spaces to the so called hyperelliptic spaces. Authors in [24] have checked the conjecture for elliptic spaces under some restrictive assumptions on the formal dimension. Our main result in this paper is to prove that:

Theorem 1.4. *If M is a closed and simply connected manifold satisfying the Hilali conjecture, then it is also for all its configurations spaces $F(M,k)$, provided that $F(M,k)$ is elliptic.*

Let us recall that

$$F(M,k) = \{(x_1,x_2,...,x_k) \in M^k \ , \ x_i \neq x_j \text{ for } i \neq j\}$$

denotes the space of ordered configurations of k distinct points in M.

The paper is organised as follows. In section 2 we will outline the main properties of the notion of Sullivan minimal models and summarize briefly the description of the rational cohomology and homotopy of configuration spaces as given in [4], [8], [15], [16]. In section 3, we prove our main result: Theorem 1.4, but also some other interesting results like:

Theorem 1.1. *If M is rationally elliptic, and $X = M-\{pt\}$ has a non-trivial rational homotopy group in dimension > 1, then $F(X,2)$ and $F(M,k)$ for $k > 2$, are rationally hyperbolic.*

Theorem 1.2. *If M is a simply connected manifold of dimension at least 3, and has at least two linearly independent elements in its rational cohomology, then $F(M,3)$ and in general $F(M,k), k \geq 3$ is rationally hyperbolic.*

Theorem 1.3. *If M is a closed and simply connected manifold, then $F(M,k)$ verifies the Hilali conjecture provided that $F(M,k)$ is elliptic.*

In section 4 we ask some open questions, answer some ones and propose some possible directions of research.

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2 Preliminaries

2.1 Sullivan minimal models

By the theory of Sullivan minimal models (see [13]), there exists a category equivalence between the homotopical category of rational and simply connected topological spaces X of finite type and that of 1-connected commutative differential graded $\mathbb{Q}$-algebras of finite type. Thus, the rational homotopy type of X is encoded in a differential algebra (A,d) called the *Sullivan minimal model* of X. This is a free graded algebra $A = \Lambda V$, generated by a graded vector space $V = \bigoplus_{k\geq 2} V^k$, and equipped with a decomposable differential $d : V^k \to (\Lambda^{\geq 2}V)^{k+1}$. It satisfies that:

$$\begin{array}{rcl} V^k & = & \mathrm{Hom}(\pi_k(X)\otimes\mathbb{Q},\mathbb{Q}); \\ H^k(\Lambda V,d) & = & H^k(X;\mathbb{Q}). \end{array} \tag{2.1}$$

Therefore, Hilali conjecture can be rewritten in the equivalent algebraic version:

Conjecture 2.1 (Algebraic version)**.** *If $(\Lambda V,d)$ is a 1-connected and elliptic model of Sullivan, then*

$$\dim V \leq \dim H^*(\Lambda V,d).$$

2.2 Configuration spaces

Throughout this paper M denotes a m-dimensional closed and simply connected manifold and

$$F(M,k) = \{(x_1,x_2,...,x_k) \in M^k\ ,\ x_i \neq x_j \text{ for } i \neq j\}$$

its ordered configurations space of k distinct points in M, the importance of such spaces is well illustrated and detailed in [4] and [14].

F. Cohen and L. Taylor were the first ones who were interested in describing the cohomology of configuration spaces. They considered the 2-points configuration space of a closed and oriented manifold M, whose cohomological algebra $H^*(M)$ is a Poincaré duality algebra and showed that:

Theorem 2.2. *If M is a closed and oriented manifold of dimension m, whose cohomological algebra $H^*(M)$ is a Poincaré duality algebra, then*

$$H^*(F(M,2)) \cong \frac{H^*(M)\otimes H^*(M)}{(\Delta)}.$$

Where $\Delta := \sum_{i=1}^{n}(-1)^{|a_i|}a_i\otimes a_i^* \in (H^*(M)\otimes H^*(M))^m$ *is called the diagonal class,* $(a_i)_{1\leq i\leq N}$ *denotes a homogeneous basis of* $H^*(M)$ *and* $(a_i)^*_{1\leq i\leq N}$ *its dual.*

Inspired by this result, P. Lambrechts and D. Stanley studied in [23] the rational homotopy type of $F(M,2)$ when M is a closed manifold. They specially proved that the rational homotopy type of $F(M,2)$ is completely determined by that of M in the sense that:

Theorem 2.3. *If M is a connected closed and oriented manifold of dimension m such that $H^1(M;\mathbb{Q}) = H^2(M;\mathbb{Q}) = 0$. If (A,d) is minimal model of M such that A is a connected Poincaré duality algebra. Then there exists a model of $F(M,2)$ of the form*

$$\frac{A\otimes A}{(\Delta)},$$

where $\Delta := \sum_{i=1}^{n}(-1)^{|a_i|}a_i \otimes a_i^* \in (A\otimes A)^m$ *is a well defined diagonal class (unique up to a multiplicative unit).*

3 Results and proofs

Before proving our main theorem, we will check it in some informative special cases: when $M = \mathbb{C}\mathrm{P}^{\mathrm{n}}$, when M is a projective complex variety and in general when M is a monogenic closed manifold.

Let us recall this folkloric results from rational homotopy theory about the so called *formal dimension* of any simply connected and elliptic space X denoted $fd(X)$ and defined to be the greatest k such that $H^k(M;\mathbb{Q}) \neq 0$. It is well known that $fd(M) = \dim M$ when M is simply connected manifold and (see [11]) that $\dim V \leq fd(X)$.

Proposition 3.1. *The Hilali conjecture holds for $F(\mathbb{C}\mathrm{P}^{\mathrm{n}},2)$.*

Proof. We know from the proof of Theorem 1 in [28], that the only non null Betti numbers of $F(\mathbb{C}P^n,2)$ are $\beta_{2k} = k+1$ and $\beta_{2m+2k} = n-k$ where $0 \leq k \leq n-1$. Thus $\dim H^*(F(\mathbb{C}P^n,2);\mathbb{Q}) = n(n+1)$. But $F(\mathbb{C}P^n,2)$ has the homotopy type of a CW-complex of dimension $\leq 4n$. Thus, we have $\dim \pi_*(F(\mathbb{C}P^n,2))\otimes\mathbb{Q} \leq 4n \leq n(n+1) = \dim H^*(F(\mathbb{C}P^n,2);\mathbb{Q})$ if $n \geq 3$.

For $n = 1$, the complex projective space $\mathbb{C}P^1$ is nothing but the Riemann sphere $\mathbb{S}^2$ which obviously verifies the Hilali conjecture. Moreover $F(\mathbb{S}^2,2) \simeq \mathbb{S}^2$ verifies also the Hilali conjecture.

For $n = 2$, $\mathbb{C}P^2$ is the complex projective plane whose cohomological dimension is given in Theorem 1, [28]. Indeed, given an elliptic topological space X, we define its Poincaré polynomial to be

$$P_X(t) := \sum_k \dim H^k(X;\mathbb{Q})t^k.$$

It is proven in Theorem 1, [28] that

$$P_{F(\mathbb{C}P^m,2)}(t) = \prod_{\substack{d|m(m+1)\\ d\neq 1}} \varphi_d(t^2),$$

where φ_d denote the cyclotomic polynomials. Thus

$$\dim H^*(F(\mathbb{C}P^m,2)) = \varphi_2(1).\varphi_3(1) = 6.$$

On other hands, in the rational homotopy of $F(\mathbb{C}P^n,2)$ is easy to work out. Loop the space, and fibre: the base is $\mathbb{S}^1\times\Omega(\mathbb{S}^{2n+1})$ while the fibre is $\mathbb{S}^1\times\Omega(S^{2n-1})$. So the rank of the rational homotopy is 4 if $n > 1$. □

Second proof. (suggested by S. Kallel to prove that $\dim \pi_*(F(\mathbb{C}P^2,2))\otimes\mathbb{Q} = 4$). Consider the fibration

$$\mathbb{C}P^{n-1} \longrightarrow F(\mathbb{C}P^n,2) \longrightarrow \mathbb{C}P^n$$

which admits a section. Thus, it can be splitting on a long and right rational exact sequence, so

$$\begin{aligned}\pi_k(F(\mathbb{C}P^n,2)) &= \pi_k(\mathbb{C}P^n)+\pi_k(\mathbb{C}P^{n-1})\\ &= \pi_k(\mathbb{S}^{2n+1})+\pi_k(\mathbb{S}^{2n-1}) \text{ for } k>2.\end{aligned}$$

For $\mathbb{C}P^2$, we have $\pi_5(F(\mathbb{C}P^n,2)) = \mathbb{Q}, \pi_3(F(\mathbb{C}P^n,2)) = \mathbb{Q}$ and $\pi_2(F(\mathbb{C}P^n,2)) = \mathbb{Q}\otimes\mathbb{Q}$, hence the total rank of the homotopy of $F(\mathbb{C}P^n,2)$ is indeed 4. □

Proposition 3.2. *If M is a smooth projective complex variety, then $F(M,2)$ verify the Hilali conjecture, provided that $F(M,2)$ is elliptic.*

Proof. We know from [9] that smooth projective closed varieties are formal. Corollary 5.6 of [23] states that $F(M,2)$ is formal when M is a closed connected formal manifold such that $H^1(M;\mathbb{Q}) = H^2(M;\mathbb{Q}) = 0$, and finally (see [19]) all formal and simply connected elliptic spaces verify the Hilali conjecture. Thus the 2-points ordered configuration spaces of smooth projective closed varieties verify the Hilali conjecture. □

Proposition 3.3. *If M is a closed and simply connected manifold whose rational cohomology is generated by one element, then $F(M,2)$ verify the Hilali conjecture.*

Proof. Let M be a closed, simply connected and monogenic manifold, then its cohomological algebra is one of the the two following forms:

$$H^*(M;\mathbb{Q}) = \mathbb{Q}[x]/(x^k) \text{ with } |x| = 2\ell$$

or

$$H^*(M;\mathbb{Q}) = \Lambda x \text{ with } |x| = 2\ell+1.$$

- **First case**: if $H^*(M;\mathbb{Q}) = \Lambda x$, then M and $\mathbb{S}^{2\ell+1}$ are of the same rational homotopy type, i.e.,

  $$M \simeq_{\mathbb{Q}} \mathbb{S}^{2\ell+1}.$$

  From [23], we conclude that

  $$F(M,2) \simeq_{\mathbb{Q}} F(\mathbb{S}^{2\ell+1},2) \simeq_{\mathbb{Q}} \mathbb{S}^{2\ell+1}.$$

  Thus $\mathbb{S}^{2\ell+1}$ and $F(\mathbb{S}^{2\ell+1},2)$ satisfy the Hilali conjecture.

- **Second case**: if $H^*(M;\mathbb{Q}) = \mathbb{Q}[a]/(a^k)$ with $k \geq 3$ (the case when $k = 2$ was already considered here above). Hence, the Sullivan minimal model of M is of the form

  $$(\Lambda(a,b),d) \text{ with } da = 0, db = a^k.$$

  The model of $F(M,2)$, as described in Corollary 3.1, [27], is of the form

  $$(\Lambda(x,y,z,t),d)$$

where $dx = dz = 0, dy = x^k, dt = \sum_{i=0}^{k-1} x^i z^{k-i-1}$. To finish the proof, it suffices to remark that $\dim H^*(F(M,2);\mathbb{Q}) \geq 4$, since that in general for any closed and orientable simply-connected manifold, the cohomology with field coefficients of $F(M,2)$ is additively that of $M \times \overset{\circ}{M}$.

□

Let us now prove our main result by announcing some intermediate one. We will use the following notations

$$\overset{\circ}{M} := M - \{\text{point}\}\,,\ \overset{\circ\circ}{M} := M - \{2 \text{ points}\}.$$

Proof of Theorem 1.1. Consider the fibration:

$$F(M,3) \longrightarrow M$$

with fibre $F(X,2)$ where $X = \overset{\circ}{M}$. It suffices to give conditions which imply that $F(X,2)$ is hyperbolic. Notice that there is a fibration

$$F(X,2) \longrightarrow X$$

with fibre $\overset{\circ\circ}{M}$. Furthermore, this fibration has a cross-section. It suffices to see that $\overset{\circ\circ}{M}$ is rationally hyperbolic. Notice that $\overset{\circ\circ}{M}$ has the homotopy type of $S^{m-1} \vee \overset{\circ}{M}$. where $m = \dim(M)$. The homotopy fibre of

$$S^{m-1} \vee \overset{\circ}{M} \longrightarrow S^{m-1} \times \overset{\circ}{M}$$

is $\Sigma(\Omega(\overset{\circ}{M}) \wedge \Omega(S^{m-1}))$, since that in general the homotopy fibre of $X \vee Y \longrightarrow X \times Y$ is $\Sigma[\Omega(X) \wedge \Omega(Y)]$. If $\overset{\circ}{M}$ has a non-trivial rational homotopy group, then $\Sigma(\Omega(\overset{\circ}{M}) \wedge \Omega(S^{m-1}))$ is rationally hyperbolic. That suffices. □

Remark 3.4. There is one exception in the case of '3 configurations' where $M = \mathbb{S}^n$. In fact, '3 configurations' is still elliptic. Indeed, $F(\mathbb{S}^n, 3)$ is homotopy equivalent to the Stiefel manifold of orthonormal two frames in $\mathbb{R}^{n+1}$ which is elliptic,

Proof of Theorem 1.2. In this case,

$$\overset{\circ}{X} = \overset{\circ\circ}{M} = \overset{\circ}{M} \bigvee \mathbb{S}^{m-1}.$$

Now by the hypothesis that M is simply-connected of dimension at least 3, the rational homotopy of $\overset{\circ}{M}$ is non-zero as the rational homology is non-zero. Then the argument below gives that the fibre of

$$\overset{\circ}{M} \bigvee \mathbb{S}^{m-1} \longrightarrow \overset{\circ}{M} \times \mathbb{S}^{m-1}.$$

is hyperbolic. Thus $\overset{\circ\circ}{M}$ is hyperbolic. This applies to 3 or more configurations in $\mathbb{C}\mathrm{P}^n$ also. □

Example 3.5. To well illustrate Theorems 1.2 and 1.1, we propose here below some informative examples

- If $M = \mathbb{S}^n$ or $\mathbb{C}\mathrm{P}^n$, then $F(M,2)$ is elliptic;
- If M is a product of two spheres $\mathbb{S}^p \times \mathbb{S}^q$ for $p,q > 0$, then $F(M,2)$ is hyperbolic;
- In the case of 2-configurations of a monogenic simply connected and closed manifold M, observe that there is a fibration

$$\overset{\circ}{M} \longrightarrow F(M,2) \longrightarrow M$$

  Thus $F(M,2)$ is hyperbolic if and only if $\overset{\circ}{M}$ is hyperbolic.

Remark 3.6. In Theorem 1.1, the case of manifolds of dimension 1 or 2 are classical:

- **In dimension 1**: manifolds without boundary are either $\mathbb{S}^1$ or disjoint unions of intervals;
- **In dimension 2**:
  - If M is not $\mathbb{S}^2$ or $\mathbb{R}\mathrm{P}^2$, then the configuration space is a $K(\pi,1)$.
  - If $M = \mathbb{R}\mathrm{P}^2$, this has been considered in [31] and [32]. These arise from a construction which F. Cohen considered in [6] and given by the $SO(3)$-Borel construction for configurations in $\mathbb{S}^2$ which is a $K(\pi,1)$ where π is a certain choice of mapping class group. However, we read from [12]-page 13, that

$$\pi_*(F(\mathbb{R}^n,2)) \otimes \mathbb{Q} \cong \pi_*(\mathbb{R}^n - pt) \otimes \mathbb{Q}$$

    and that $\mathbb{R}^n - pt$ is homotopy equivalent to $\mathbb{S}^n$. Thus $\dim \pi_*(F(\mathbb{R}^n,2)) \otimes \mathbb{Q} = 1$ or 2.
    On other hands, the integral cohomology of $F(\mathbb{R}^n,2)$ is well described in [12]-page 95: It is a graded-commutative algebra over $\mathbb{Z}$ on generators $(e_{ij})_{1\le i<j\le n} \in H^{n-1}(F(\mathbb{R}^m,2))$, subject to the relations

$$\begin{aligned} e_{ij} - e_{ji} &= 0 \\ e_{ij}^2 &= 0 \\ e_{ij}e_{jk} + e_{jk}e_{ki} + e_{ki}e_{ij} &= 0, \end{aligned}$$

    where $1 \le i < j < k \le n$. In particular, $H^*(F(\mathbb{R}^n,2);\mathbb{Q})$ is nonzero only in degrees $p(n-1)$ for $p = 0,1$. Thus $F(\mathbb{R}^n,2)$ verifies the Hilali conjecture.
  - The case of $M = \mathbb{S}^2$ is clear: $F(\mathbb{S}^2,k)$ is hyperbolic if and only if $k > 3$.

Proof of Theorem 1.3. From theorems 1.1 and 1.2 and from Remarks 3.4 and 3.6, we know that $F(M,k)$ is elliptic if and only if ($k \le 2$ and M is monogenic) or ($k = 3$ and $M = \mathbb{S}^n$).

- If $k = 2$ and M is monogenic. From Theorem 1.1, $F(M,2)$ verifies the Hilali conjecture.

- If $k=3$ and $M=\mathbb{S}^n$. Then $F(\mathbb{S}^n,3)$ is homotopy equivalent to the Stiefel manifold of orthonormal two frames in $\mathbb{R}^{n+1}$. By a result of Fadell (Theorem 2.4, [10]) there is a fiber homotopy equivalence between $F(\mathbb{S}^m,3)$ and $V_{m+1,2}$, the Stiefel manifold. Stiefel manifolds are homogeneous spaces, and the Hilali conjecture was already proved for such spaces since there are rationally H-spaces ([20]). Since the finite rational dimension of the homotopy and cohomology of two spaces joined by a fiber homotopy equivalence are the same, we conclude that the Hilali conjecture holds for $F(\mathbb{S}^m,3)$.

□

Proof of Theorem 1.4. If $k=1$, then $F(M,k)=M$ verifies the Hilali conjecture. If $k\neq 2$ then $F(M,k)$ verifies the Hilali conjecture from 1.3 since $F(M,k)$ is supposed to be elliptic. □

4 Open questions

To enrich this work, we suggest many other directions of research that can be explored. For example we ask if:

4.1 On the Hilali conjecture for unordered configurations spaces

It is legitimate to try looking after theorem 1.3 for $C(M,k)$ where $C(M,k)$ denotes the unordered configurations of k distinct points in M defined by

$$C(M,k) := F(M,k)/\Sigma_k.$$

Where Σ_n denotes the symmetric group whose right action on $F(M,k)$ is given by

$$\sigma.(x_1,\dots,x_k)=(x_{\sigma(1)},\dots,x_{\sigma(k)}).$$

It is well known that the computing of the homology of unordered configuration spaces is well studied, that of their homotopy is less. For example, it was proved in [25] that Betti numbers of $C(M,n)$ can be determined by that of M with in $\mathbb{F}_2$. This result has been extended in [2] to $\mathbb{F}$-Betti numbers for odd-dimensional closed manifolds, where $\mathbb{F}=\mathbb{F}_p$ or $\mathbb{Q}$. J.-C. Thomas and Y. Félix in [17] were interested in computing rational Betti numbers of $C(M,k)$ for an even-dimensional orientable closed manifold M.

For example, a cohomological basis for $C(\mathbb{C}P^3,k)$ when $k\in\{1,2,3\}$ is explicitly described in [17] from what we know that $\dim H^*(C(\mathbb{C}P^3,2);\mathbb{Q})=6$.

Note that in general there is a strong relation between the rational cohomology of $C(M,2)$ and that of $F(M,2)$:

$$H^*(C(M,2);\mathbb{Q})\cong H^*(F(M,2);\mathbb{Q})^{\Sigma_2},$$

and that if M is closed, orientable, and simply-connected, then the cohomological dimension of $F(M,2)$ is equal to $\dim(M)+dim(\overset{\circ}{M})$.

On the other hand, the map from ordered to unordered configurations is a covering space projection with covering group given by the symmetric group. So the map is an isomorphism on homotopy groups above dimension 1 (if M is simply-connected), so now

$$\pi_*(C(M,2)) = \pi_*(F(M,2)).$$

Thus $F(C,k)$ is rationally elliptic $\Rightarrow (n = 1,2)$ or ($n = 3$ and $M = \mathbb{S}^n$. In particular $\dim \pi_*(C(\mathbb{C}\mathrm{P}^3,2)) \otimes \mathbb{Q} = \dim \pi_*(\mathrm{F}(\mathbb{C}\mathrm{P}^3,2)) \otimes \mathbb{Q} = 4$ (i.e., Hilali conjecture holds for $C(\mathbb{C}\mathrm{P}^3,2)$

4.2 On the Hilali conjecture for configuration spaces (ordered or not) of manifolds (compact or not)

One may ask what about this precedent results if we omits the condition that M is closed or that when M is compact. From Remark 3.4, we have a first positive answer for $F(\mathbb{R}^n,2)$. To cover the case of elliptic manifolds M which are not closed, observe that $\overset{\circ\circ}{M}$ is homotopy equivalent to

$$M \vee S^{m-1} \vee S^{m-1}$$

which is hyperbolic as its rational homotopy contains a free graded Lie algebra with at least 2 generator. [7] is also a well recommended reference that one have to over look.

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