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ON A NON-CLASSICAL BOUNDARY VALUE PROBLEM FOR THE HEAT EQUATION

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Abstract

In this paper, we are concerned in a non-classical boundary value problem for heat equation. More precisely, we study a linear heat equation without initial condition but with a homogeneous Dirichlet condition on the whole boundary and a nonhomogeneous Neumann condition on a part of the boundary. Under sufficient conditions on the data, we prove that the problem has a unique solution. The proof combines optimal control and controllability theories.

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1 Statement of problem

Let $N \in \mathbb{N}^*$ and Ω be a bounded open subset of $\mathbb{R}^N$ with boundary Γ of class $\mathbb{C}^2$. Let also Γ_0 be a nonempty open and connected subset of Γ. For a time $T > 0$, we set $Q = \Omega \times (0,T)$, $\Sigma = \Gamma \times (0,T)$ and $\Sigma_0 = \Gamma_0 \times (0,T)$. Then, we consider the following linear heat equation:

$$y' - \Delta y = f \quad \text{in } Q \tag{1.1}$$

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where

$$y' = \frac{\partial y}{\partial t}, \Delta = \frac{\partial^2}{\partial x_1^2} + ... + \frac{\partial^2}{\partial x_N^2},$$

and $f \in L^2(Q)$.

We are interested by the following problem: Given $h \in L^2(\Sigma_0)$, find y solution of (1.1) which satisfies

$$y = 0 \text{ on } \Sigma, \tag{1.2}$$

$$\frac{\partial y}{\partial \nu} = h \text{ on } \Sigma_0. \tag{1.3}$$

In (1.3), ν is the unit normal vector of Γ directed towards the exterior of Ω.

The usual conditions, that permit existence and uniqueness of solution of the heat equation, are an initial condition or Cauchy data and a condition on the whole lateral boundary Σ that can be of Dirichlet type, Neumann type, Newton type, etc., or combination of the different types ([7]). Actually, classical boundary problems associated to heat equation represent the evolution of the state y of a physical, chemical or economical system,etc. And if we want to act on that system in order to get a precise goal we talk about control theory, we look generally for the optimal control or the controllability of the system [7, 15, 19, 20, 18, 2].

But for problem (1.1)-(1.3) the boundary conditions (1.2), (1.3) are not classical because (1.2) is already a condition on Σ, consequently (1.3) is over-determined and there is no Cauchy data. Consequently, Problem (1.1), (1.2), (1.3) is nonclassical problem.

In [9] J.-L. Lions studied a similar problem in the case of the waves equation; he used the HUM method and an inverse observability inequality [4, 1]. As far as we know Problem (1.1)-(1.3) is an open problem because the case of heat equation is not yet treated. At last, in order to end these survey, we recall that these kind of problems is encountered in matter related to geophysical [9, 16].

In these paper we give necessary and sufficient condition to obtain a unique solution of Problem (1.1)-(1.3) . The method combines optimal control and null controllability. So, for $v \in L^2(\Sigma_0)$, let $\varphi = \varphi(v) \in L^2((0,T), H^2(\Omega)) \cap H^1((0,T), L^2(\Omega))$ be the unique solution of the backward problem:

$$\varphi' + \Delta\varphi = 0 \quad \text{in } Q, \tag{1.4a}$$

$$\varphi = v\chi_{\Sigma_0} \quad \text{on } \Sigma, \tag{1.4b}$$

$$\varphi(T) = 0 \quad \text{in } \Omega. \tag{1.4c}$$

Set

$$\mathcal{U} = \{v \in L^2(\Sigma_0), \varphi(v)(0) = 0\}, \tag{1.5}$$

the space of controls v which brings the solution $\varphi(v)$ of (1.4) to zero at initial time.

Remark 1.1. $\mathcal{U}$ is not reduced to $\{0\}$. Indeed, let

$$v = \begin{cases} w & \in & L^2(\frac{T}{2}, T; L^2(\Gamma_0)), & w \neq 0, \\ \widehat{w} & \in & L^2(0, \frac{T}{2}; L^2(\Gamma_0)), & \end{cases}$$

where w is such that $\varphi(w)$ is solution of the backward system

$$\begin{cases} \varphi' + \Delta\varphi &= 0 & \text{in} & \Omega\times]\frac{T}{2}, T[, \\ \varphi &= w & \text{on} & \Gamma_0\times]\frac{T}{2}, T[, \\ \varphi &= 0 & \text{on} & (\Gamma\setminus\Gamma_0)\times]\frac{T}{2}, T[, \\ \varphi(T) &= 0 & \text{in} & \Omega, \end{cases}$$

and $\widehat{w}$ such that if $\varphi(\widehat{w})$ is solution of

$$\begin{cases} \varphi' + \Delta\varphi &= 0 & \text{in} & \Omega\times]0, \frac{T}{2}[\\ \varphi &= \widehat{w} & \text{on} & \Gamma_0\times]0, \frac{T}{2}[\\ \varphi &= 0 & \text{on} & (\Gamma\setminus\Gamma_0)\times]0, \frac{T}{2}[\\ \varphi(\frac{T}{2}) &= 0 & \text{in} & \Omega, \end{cases}$$

then

$$\varphi(\hat{w})(0) = 0.$$

Then, $v \neq 0$ and $v \in \mathcal{U}$.

Now, the main result of the paper is the following:

Theorem 1.2. *Let $f \in L^2(Q)$, $h \in L^2(\Sigma_0)$ and φ be the solution of (1.4). Then the following properties are equivalent :*

(i) Problem (1.1)-(1.3) admits unique solution y.

(ii) $\displaystyle\int_Q f\varphi(v)\,dxdt + \int_{\Sigma_0} hv\,d\sigma dt = 0 \quad \forall v \in \mathcal{U}.$

The paper is devoted to the proof of theorem 1.2. First, in Section 2 we deal with an optimal control problem . We then prove theorem 1.2 in Section 3.

2 Optimal control problem

From now on, we adopt, the following notation for heat operator:

$$L = \frac{\partial}{\partial t} - \Delta;$$

and the adjoint operator

$$L^* = -\frac{\partial}{\partial t} - \Delta.$$

We set

$$\mathcal{V} = \{\rho \in C^\infty(\overline{Q}), \rho = 0 \text{ on } \Sigma\} \tag{2.1}$$

and we recall the following Carleman inequality [5, 13]:

Proposition 2.1. *There exists a positive function θ with $\dfrac{1}{\theta}$ bounded, and a constant $C = C(\Omega,\Gamma_0) > 0$, such that for all $\rho \in \mathcal{V}$,*

$$\int_Q \frac{1}{\theta^2}|\rho|^2 dxdt \leq C\Big(\int_Q |L\rho|^2 dxdt + \int_{\Sigma_0}\Big|\frac{\partial\rho}{\partial\nu}\Big|^2 d\sigma dt\Big). \tag{2.2}$$

Consider the application $\rho \mapsto \Pi(\rho) = \Big(\int_Q |L\rho|^2 dxdt + \int_{\Sigma_0}\Big|\frac{\partial\rho}{\partial\nu}\Big|^2 d\sigma dt\Big)^{\frac{1}{2}}$. Then Π is a norm on $\mathcal{V}$. Let be V the completion of $\mathcal{V}$ for this norm. Then V is a Hilbert space.

For any $f \in L^2(Q)$, let $K(f)$ be defined by

$$K(f) = \Big\{\rho \in V\,, L\rho = f\Big\}.$$

Then the following results hold.

Lemma 2.2. *K is nonempty, closed and convex .*

Proof. First, it is easy to show that $K(f)$ is closed. Next, for $\rho^0 \in H_0^1(\Omega)$ we know that there exists a unique $\rho \in L^2((0,T),H^2(\Omega)\cap H_0^1(\Omega))\cap H^1((0,T),L^2(\Omega))$ solution of

$$\begin{cases} L\rho = f & \text{in} \quad Q,\\ \rho = 0 & \text{on} \quad \Sigma,\\ \rho(0) = \rho^0 & \text{in} \quad \Omega. \end{cases}$$

This means that $\rho \in K(f)$. Finally, $K(f)$ being an affine vector subspace of V, we have that $K(f)$ is convex.

Now, for any $z \in K(f)$, consider the cost function J defined by:

$$J(z) = \frac{1}{2}\Big\|\frac{\partial z}{\partial\nu} - h\Big\|^2_{L^2(\Sigma_0)}. \tag{2.3}$$

Lemma 2.3. **J is coercive, i.e.:**

$$\lim_{\|z\|_V\to+\infty} J(z) = +\infty \quad \textit{when} \quad z \in K(f).$$

Proof. We define the norm in $K(f)$ by induction from the norm of V. According to the definition of the norm in V, we have

$$\|\rho\|_V^2 = \|L\rho\|^2_{L^2(Q)} + \Big\|\frac{\partial\rho}{\partial\nu}\Big\|^2_{L^2(\Sigma_0)}. \tag{2.4}$$

Therefore, for $\rho \in K(f)$ it comes

$$\|\rho\|_V^2 = \|f\|^2_{L^2(Q)} + \Big\|\frac{\partial\rho}{\partial\nu}\Big\|^2_{L^2(\Sigma_0)}. \tag{2.5}$$

Observing that the cost function defined by (2.3) can be rewritten as

$$2J(z) = \left\|\frac{\partial z}{\partial \nu} - h\right\|^2_{L^2(\Sigma_0)} = \left\|\frac{\partial z}{\partial \nu}\right\|^2_{L^2(\Sigma_0)} - 2\langle\frac{\partial z}{\partial \nu}, h\rangle_{L^2(\Sigma_0)} + \|h\|^2_{L^2(\Sigma_0)},$$

using on the one hand the Cauchy-Schwartz inequality , and on the second hand, the inequality of Young , we get

$$\int_{\Sigma_0} \frac{\partial z}{\partial \nu} h\, dxdt \leq \left\|\frac{\partial z}{\partial \nu}\right\| \|h\| \leq \frac{1}{2\varepsilon}\left\|\frac{\partial z}{\partial \nu}\right\|^2 + \frac{\varepsilon}{2}\|h\|^2 \ \ \forall \varepsilon > 0. \tag{2.6}$$

Thus

$$2J(z) \geq \left(1 - \frac{1}{\varepsilon}\right)\left\|\frac{\partial z}{\partial \nu}\right\|^2_{L^2(\Sigma_0)} + (1-\varepsilon)\|h\|^2_{L^2(\Sigma_0)}. \tag{2.7}$$

Now, let $\epsilon > 0$ be such that $\dfrac{1}{\varepsilon} < 1$ and $\alpha = \frac{1}{2}\left(1 - \frac{1}{\varepsilon}\right)$. Then $\alpha > 0$. Set

$$\beta = \frac{1}{2}(1-\varepsilon)\|h\|^2_{L^2(\Sigma_0)} - \frac{1}{2}\left(1 - \frac{1}{\varepsilon}\right)\|f\|^2_{L^2(Q)}.$$

Then, it follow from (2.7) that

$$J(z) \geq \alpha\|z\|^2_V + \beta, \quad \forall z \in K(f). \tag{2.8}$$

This shows J is coercive.

As J is strictly convex, continuous and coercive on $K(f)$, the minimization problem:

$$\inf_{z \in K(f)} J(z) \tag{2.9}$$

has a unique solution y characterized by the optimality condition:

$$\int_{\Sigma_0} \left(\frac{\partial y}{\partial \nu} - h\right)\frac{\partial \rho}{\partial \nu} d\sigma dt = 0, \ \ \forall \rho \in K(0), \tag{2.10}$$

where

$$K(0) = \left\{\rho \in V, L\rho = 0\right\}. \tag{2.11}$$

The relation (2.10) is not easy to interpret in terms of adjoint problem since it derives from the minimization problem with constraint (2.9). So, in order to obtain a suitable characterization of the optimal solution, we proceed by penalization [7]. More precisely, let $\varepsilon > 0$. Let us considere $\mathcal{A}$ the set of z satisfying

$$\begin{cases} Lz \in L^2(Q) \\ \dfrac{\partial z}{\partial \nu} \in L^2(\Sigma_0) \\ z = 0 \ \text{ on } \ \Sigma, \end{cases} \tag{2.12}$$

and define for any $z \in \mathcal{A}$, the cost function J_ε by:

$$J_\varepsilon(z) = \frac{1}{2}\left\|\frac{\partial z}{\partial \nu} - h\right\|^2_{L^2(\Sigma_0)} + \frac{1}{2\varepsilon}\left\|Lz - f\right\|^2_{L^2(Q)}. \tag{2.13}$$

Then the penalized problem:

$$\inf_{z\in\mathcal{A}} J_\varepsilon(z) \tag{2.14}$$

has a unique solution $y_\varepsilon \in \mathcal{A}$. Furthermore, we have the following results.

Proposition 2.4. *$y_\varepsilon \in V$ is the optimal solution of the penalized problem (2.14) if and only if, there exists $p_\varepsilon \in L^2(0,T;H^{\frac{1}{2}}(\Omega))$ such that $(y_\varepsilon, p_\varepsilon)$ is solution of the approached optimality systems:*

$$\begin{cases} y_\varepsilon' - \Delta y_\varepsilon = f - \varepsilon p_\varepsilon \ \text{ in } \ Q, \\ y_\varepsilon = 0 \ \text{ on } \ \Sigma, \end{cases} \tag{2.15}$$

$$\begin{cases} -p_\varepsilon' - \Delta p_\varepsilon = 0 \ \text{ in } \ Q, \\ p_\varepsilon = \left(h - \dfrac{\partial y_\varepsilon}{\partial \nu}\right)\chi_{\Sigma_0} \ \text{ on } \ \Sigma, \\ p_\varepsilon(T) = p_\varepsilon(0) = 0. \end{cases} \tag{2.16}$$

Moreover there exists $C > 0$ such that

$$\|y_\varepsilon\|_{\mathrm{V}} \leq C, \tag{2.17}$$

$$\|p_\varepsilon\|_{\mathrm{L}^2\left(0,T;\mathrm{H}^{\frac{1}{2}}(\Omega)\right)} + \|p_\varepsilon^{'}\|_{\mathrm{L}^2\left(0,T;\mathrm{H}^{-\frac{1}{2}}(\Omega)\right)} \leq C. \tag{2.18}$$

Proof.

We express the Euler-Lagrange optimality condition which characterizes y_ε:

$$J_\varepsilon'(y_\varepsilon)(\rho) = 0 \ \ \forall \rho \in \mathcal{A}. \tag{2.19}$$

After calculations we have

$$\int_{\Sigma_0} \Big(\frac{\partial y_\varepsilon}{\partial \nu} - h\Big)\frac{\partial \rho}{\partial \nu} d\sigma dt + \frac{1}{\varepsilon}\int_Q (Ly_\varepsilon - f)L\rho dxdt = 0 \ \ \forall \rho \in \mathcal{A}. \tag{2.20}$$

Set

$$p_\varepsilon = -\frac{1}{\varepsilon}(Ly_\varepsilon - f). \tag{2.21}$$

Then $p_\varepsilon \in L^2(Q)$ and (2.20) can be rewritten as

$$\int_{\Sigma_0} \Big(\frac{\partial y_\varepsilon}{\partial \nu} - h\Big)\frac{\partial \rho}{\partial \nu} d\sigma dt - \int_Q p_\varepsilon L\rho dxdt = 0 \ \ \forall \rho \in \mathcal{A}. \tag{2.22}$$

Taking $\rho \in \mathcal{D}(Q)$ in (2.22) and integrating by parts over Q, we get

$$\langle L^* p_\varepsilon, \rho\rangle_{\mathcal{D}'(Q),\mathcal{D}(Q)} = 0, \tag{2.23}$$

from which we deduce that

$$L^* p_\varepsilon = -p_\varepsilon' - \Delta p_\varepsilon = 0 \text{ in } Q. \tag{2.24}$$

As $p_\varepsilon \in L^2(0,T;L^2(\Omega))$, we have $p_\varepsilon' \in H^{-1}(0,T;L^2(\Omega))$ which in view of (2.24) yields $\Delta p_\varepsilon \in H^{-1}(0,T;L^2(\Omega))$. Thus $p_\varepsilon' \in H^{-1}(0,T;L^2(\Omega))$ and $\Delta p_\varepsilon \in H^{-1}(0,T;L^2(\Omega))$. Consequently $p_\varepsilon|_\Sigma$ and $\frac{\partial p_\varepsilon}{\partial \nu}|_\Sigma$ exist and belong respectively to $H^{-1}(0,T;H^{-\frac{1}{2}}(\Gamma))$ and $H^{-1}(0,T;H^{-\frac{3}{2}}(\Gamma))$.

On the other hand, in view of (2.24), we have that $p_\varepsilon' \in L^2(0,T;H^{-2}(\Omega))$ since $p_\varepsilon \in L^2(0,T;L^2(\Omega))$ and then $\Delta p_\varepsilon \in L^2(0,T;H^{-2}(\Omega))$. Therefore, $p_\varepsilon \in C([0,T],H^{-1}(\Omega))$, so that $p_\varepsilon(0)$ and $p_\varepsilon(T)$ have a sense in $H^{-1}(\Omega)$.

Multiplying (2.24) by $\rho \in C^\infty(\overline{Q})$ such that $\rho = 0$ on Σ and integrating by parts over Q, we obtain that

$$\begin{aligned} 0 &= -\langle p_\varepsilon(T), \rho(T)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \langle p_\varepsilon(0), \rho(0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &+ \int_Q L\rho\, p_\varepsilon\, dxdt + \langle p_\varepsilon, \frac{\partial \rho}{\partial \nu}\rangle_{H^{-1}((0,T),H^{-\frac{1}{2}}(\Gamma)),H_0^1((0,T),H^{\frac{1}{2}}(\Gamma))}, \end{aligned} \tag{2.25}$$

where $\langle ., .\rangle_{X,X'}$ is the duality bracket between X and its dual X'.

Combining (2.22) and (2.25), we have

$$\begin{aligned} 0 &= -\langle p_\varepsilon(T), \rho(T)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + \langle p_\varepsilon(0), \rho(0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &+ \int_{\Sigma_0} \Big(\frac{\partial y_\varepsilon}{\partial \nu} - h\Big)\frac{\partial \rho}{\partial \nu} d\sigma dt \\ &+ \langle p_\varepsilon, \frac{\partial \rho}{\partial \nu}\rangle_{H^{-1}(0,T;H^{\frac{1}{2}}(\Gamma)),H_0^1(0,T;H^{\frac{1}{2}}(\Gamma))}, \\ & \forall \rho \in C^\infty(\overline{Q}) \text{ such that } \rho = 0 \text{ on } \Sigma. \end{aligned} \tag{2.26}$$

Choosing in $\rho(T) = \rho(0) = 0$ in Ω in (2.26), we deduce that

$$0 = \Big\langle \Big[p_\varepsilon + \Big(\frac{\partial y_\varepsilon}{\partial \nu} - h\Big)\chi_{\Sigma_0}\Big], \frac{\partial \rho}{\partial \nu}\Big\rangle_{H^{-1}(0,T;H^{\frac{1}{2}}(\Gamma)),H_0^1(0,T;H^{\frac{1}{2}}(\Gamma))}.$$

Thus

$$p_\varepsilon = \Big(h - \frac{\partial y_\varepsilon}{\partial \nu}\Big)\chi_{\Sigma_0} \text{ on } \Sigma. \tag{2.27}$$

Choosing again in (2.26) $\rho(T) = 0$ in Ω, then $\rho(0) = 0$ in Ω, we deduce successively,

$$p_\varepsilon(0) = 0, \tag{2.28}$$

and

$$p_\varepsilon(T) = 0. \tag{2.29}$$

Equations (2.24), (2.27) (2.28) and (2.29) gives (2.16). It remains to show (2.17) and (2.18) to complete the proof of Proposition 2.4.

Observing that $K(f) \subset \mathcal{A}$, we have that the optimal solution y of problem (2.9) belongs to $\mathcal{A}$. Thus, we can write

$$J_\varepsilon(y_\varepsilon) \leq J_\varepsilon(y) = J(y), \tag{2.30}$$

Therefore, from the structure of J_ε we have

$$\left\|\frac{\partial y_\varepsilon}{\partial \nu} - h\right\|_{L^2(\Sigma_0)} \leq \sqrt{2J(y)}. \tag{2.31}$$

and

$$\left\|Ly_\varepsilon - f\right\|_{L^2(Q)} \leq C\sqrt{2\varepsilon J(y)}. \tag{2.32}$$

Hence we deduce that there exists $C > 0$ such that

$$\|y_\varepsilon\|_V^2 = \left\|\frac{\partial y_\varepsilon}{\partial \nu}\right\|_{L^2(\Sigma_0)}^2 + \|Ly_\varepsilon\|_{L^2(Q)}^2 \leq C \tag{2.33}$$

and we have (2.17).
Finally the relations (2.24), (2.27), (2.29) in the one hand and, the estimation (2.31) in the other hand allow us to conclude that p_ε is bounded in $L^2(0,T;H^{\frac{1}{2}}(\Omega))$ and consequently to obtain that (2.18) holds.

Now we give the singular optimality system which characterizes the optimal control problem (2.9).

Proposition 2.5. *y is an optimal solution of (2.9) if and only if there exists $p \in L^2(0,T;H^{\frac{1}{2}}(\Omega))$ such that (y, p) satisfy the following singular optimality system (SOS)*

$$\begin{cases} y' - \Delta y = f \text{ in } Q, \\ y = 0 \text{ on } \Sigma, \end{cases} \tag{2.34}$$

$$\begin{cases} -p' - \Delta p = 0 \text{ in } Q, \\ p = \left(h - \dfrac{\partial y}{\partial \nu}\right)\chi_{\Sigma_0} \text{ on } \Sigma, \\ p(T) = p(0) = 0. \end{cases} \tag{2.35}$$

Proof. In view of (2.17), there exists $\widehat{y} \in V$ such that

$$y_\varepsilon \rightharpoonup \widehat{y} \quad \text{in} \quad V \quad \text{weakly}. \tag{2.36}$$

This means that

$$Ly_\varepsilon \rightharpoonup L\widehat{y} \quad \text{in} \quad L^2(Q) \quad \text{weakly}, \tag{2.37}$$

$$\frac{\partial y_\varepsilon}{\partial \nu} \rightharpoonup \frac{\partial \widehat{y}}{\partial \nu} \quad \text{in} \quad L^2(\Sigma_0) \quad \text{weakly} \tag{2.38}$$

because of the definition of the norm on the Hilbert space V. Therefore combining (2.32) and (2.37), we deduce that

$$L\widehat{y} = f.$$

In short we have prove that $\widehat{y} \in V$ and $L\widehat{y} = f$. This means that

$$\widehat{y} \in K(f).$$

From the definition of J_ε, we have

$$\left\| \frac{\partial y_\varepsilon}{\partial \nu} - h \right\|_{L^2(\Sigma_0)} \leq J_\varepsilon(y_\varepsilon),$$

which in view of (2.38) implies that

$$J(\widehat{y}) = \left\| \frac{\partial \widehat{y}}{\partial \nu} - h \right\|_{L^2(\Sigma_0)} \leq \liminf_{\varepsilon \to 0} J_\varepsilon(y_\varepsilon). \tag{2.39}$$

On the other hand, as from (2.30) one obtains that

$$\liminf J_\varepsilon(y_\varepsilon) \leq J(y), \tag{2.40}$$

combining this latter inequality with (2.39), we get $J(\widehat{y}) \leq J(y)$. Hence, we deduce that

$$\widehat{y} = y. \tag{2.41}$$

It remains to establish (2.35).

Using (2.18) we have that there exists $p \in L^2(0,T;H^{\frac{1}{2}}(\Omega))$ such that

$$p_\varepsilon \rightharpoonup p \quad \text{weakly in } L^2(0,T;H^{\frac{1}{2}}(\Omega)); \tag{2.42}$$

in particular,

$$p_\varepsilon \rightharpoonup p \quad \text{weakly } \mathcal{D}'(Q). \tag{2.43}$$

Consequently,

$$L^* p_\varepsilon \rightharpoonup L^* p \text{ in } \mathcal{D}'(Q). \tag{2.44}$$

Since $L^* p_\varepsilon = 0$ in Q, we deduce that

$$L^* p = -p' - \triangle p = 0 \text{ in } Q. \tag{2.45}$$

As $p \in L^2(0,T;L^2(\Omega))$, we have $p' \in H^{-1}(0,T;L^2(\Omega))$ which in view of (2.45) yields $\Delta p \in H^{-1}(0,T;L^2(\Omega))$. Thus $p \in H^{-1}(0,T;L^2(\Omega))$ and $\Delta p \in H^{-1}(0,T;L^2(\Omega))$. Consequently $p|_\Sigma$ and $\frac{\partial p}{\partial \nu}|_\Sigma$ exist and belong respectively to $H^{-1}(0,T;H^{-\frac{1}{2}}(\Gamma))$ and $H^{-1}(0,T;H^{-\frac{3}{2}}(\Gamma))$. On the other hand, in view (2.45), we have that $p' \in L^2(0,T;H^{-2}(\Omega))$ since $p \in L^2(0,T;L^2(\Omega))$ and $\Delta p \in L^2(0,T;H^{-2}(\Omega))$. Therefore, $p \in C([0,T],H^{-1}(\Omega))$, so that $p(0)$ and $p(T)$ have a sense in $H^{-1}(\Omega)$.

Multiplying (2.45) by $\rho \in C^\infty(\overline{Q})$ such that $\rho = 0$ on Σ we get

$$\begin{aligned} 0 \;&=\; \int_Q L\rho\, p_\varepsilon\, dxdt \\ &+\; \left\langle \left(h - \frac{\partial y_\varepsilon}{\partial \nu} \right), \frac{\partial \rho}{\partial \nu} \right\rangle_{H^{-1}(0,T;H^{\frac{1}{2}}(\Gamma_0)),\, H_0^1(0,T;H^{\frac{1}{2}}(\Gamma_0))} . \end{aligned} \tag{2.46}$$

Passing (2.46) to the limit while using (2.42), (2.38) and (2.41), we obtain that

$$\begin{aligned} 0 &= \int_Q L\rho\, p\, dxdt + \left\langle \left(h - \frac{\partial y}{\partial \nu}\right), \frac{\partial \rho}{\partial \nu} \right\rangle_{H^{-1}(0,T;H^{\frac{1}{2}}(\Gamma_0)),H_0^1(0,T;H^{\frac{1}{2}}(\Gamma_0))} \\ &\quad \forall \rho \in C^\infty(\overline{Q}) \text{ such that } \rho = 0 \text{ on } \Sigma. \end{aligned}$$

Integrating this latter identity by parts over Q, we have

$$\begin{aligned} 0 &= \langle p(T), \rho(T)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} - \langle p(0), \rho(0)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ &- \left\langle \frac{\partial \rho}{\partial \nu}, p \right\rangle_{H^{-1}(0,T;H^{\frac{1}{2}}(\Gamma)),H_0^1(0,T;H^{\frac{1}{2}}(\Gamma))} \\ &+ \left\langle \left(h - \frac{\partial y_\varepsilon}{\partial \nu}\right), \frac{\partial \rho}{\partial \nu} \right\rangle_{H^{-1}(0,T;H^{\frac{1}{2}}(\Gamma_0)),H_0^1(0,T;H^{\frac{1}{2}}(\Gamma_0))} \\ &\quad \forall \rho \in C^\infty(\overline{Q}) \text{ such that } \rho = 0 \text{ on } \Sigma. \end{aligned} \tag{2.47}$$

Choosing $\rho(0) = \rho(T) = 0$ in (2.47) it comes

$$\left\langle \left(p + \frac{\partial y}{\partial \nu} - h\right)\chi_{\Sigma_0}, \frac{\partial \rho}{\partial \nu} \right\rangle_{H^{-1}(0,T;H^{-\frac{1}{2}}(\Gamma)),H_0^1(0,T;H^{\frac{1}{2}}(\Gamma))} = 0,$$

from which we deduce that

$$p = \left(h - \frac{\partial y}{\partial \nu}\right)\chi_{\Sigma_0} \text{ on } \Sigma. \tag{2.48}$$

Now, choosing successively in $\rho(T) = 0$ and $\rho(0) = 0$, we get successively

$$p(0) = 0 \text{ in } \Omega \tag{2.49}$$

and

$$p(T) = 0 \text{ in } \Omega. \tag{2.50}$$

the relations (2.45), (2.48), (2.49) and (2.50) show that p is solution of system (2.35).

3 Proof of Theorem 1.1

Assume that (i) hold and let $v \in \mathcal{U}$. If we multiply (1.1) by $\varphi(v)$, solution of (1.4), it follows from integration by parts over Q that

$$\begin{aligned} &\int_\Omega y(T)\varphi(v)(T)dx - \int_\Omega y(0)\varphi(v)(0)dx + \int_Q yL^*\varphi(v)dxdt - \int_\Sigma \frac{\partial y}{\partial \nu}\varphi(v)d\sigma dt + \\ &\qquad + \int_\Sigma y\frac{\partial \varphi(v)}{\partial \nu}d\sigma dt = \int_Q f\varphi(v)dxdt. \end{aligned} \tag{3.1}$$

That is,

$$-\int_{\Sigma_0} hv d\sigma dt = \int_Q f\varphi(v)dxdt. \tag{3.2}$$

Conversely, assume that (ii) hold and let y be the optimal solution of minimization problem (2.9). Since $y \in K(f)$ we have $y \in V$ and $Ly = f$. In other words,

$$\begin{aligned} y' - \Delta y &= f \quad \text{in} \quad Q, & (3.3)\\ y &= 0 \quad \text{on} \quad \Sigma, & (3.4)\\ \frac{\partial y}{\partial \nu} &\in L^2(\Sigma_0). & (3.5) \end{aligned}$$

Now, let $v \in \mathcal{U}$ and $\varphi(v)$ be the solution of (1.4). Multiplying (3.3) by $\varphi(v)$ and integrating by parts over Q, we obtain

$$-\int_{\Sigma_0} \frac{\partial y}{\partial \nu} v d\sigma dt = \int_Q f \varphi(v) dx dt.$$

Combining this latter identity with Theorem 1.2-(ii), it follows

$$\int_{\Sigma_0} \frac{\partial y}{\partial \nu} v d\sigma dt = \int_{\Sigma_0} h v d\sigma dt \quad \forall v \in \mathcal{U},$$

which can be rewritten as

$$\int_{\Sigma_0} \left(\frac{\partial y}{\partial \nu} - h\right) v d\sigma dt = 0 \quad \forall v \in \mathcal{U}. \tag{3.6}$$

since $v \in \mathcal{U}$ which is a subset of $L^2(\Sigma_0)$, choosing $v = \left(\frac{\partial y}{\partial \nu} - h\right)\chi_{\Sigma_0}$, it comes

$$\int_{\Sigma_0} \left(\frac{\partial y}{\partial \nu} - h\right)^2 d\sigma dt = 0,$$

from which we deduce that

$$\frac{\partial y}{\partial \nu} = h \text{ on } \Sigma_0.$$

This complete the proof of Theorem 1.2.∎

We end this paper by the stability result. So let $\mathcal{W}$ be the vector space of data defined by:

$$\mathcal{W} = \left\{(f,h) \in L^2(Q) \times L^2(\Sigma_0) \;\Big|\; \int_Q f\varphi(v) dx dt + \int_{\Sigma_0} h v d\sigma dt = 0 \quad \forall v \in \mathcal{U}\right\};$$

Then $\mathcal{W}$ is a vectorial subspace of $L^2(Q) \times L^2(\Sigma_0)$. Define the norm on $\mathcal{W}$ by:

$$\|(f,h)\|_{\mathcal{W}} = (\|f\|^2_{L^2(Q)} + \|h\|^2_{L^2(\Sigma_0)})^{\frac{1}{2}}.$$

Theorem 3.1. *Let y be the solution of (1.1)-(1.3). The application* $(f,h) \mapsto y = y(f,h)$ *is an isometry from* $\mathcal{W}$ *to* V.

Proof. It's clear that $(f,h) \mapsto y = y(f,h)$ is linear, and we have

$$\|y\|_V = \|(f,h)\|_{\mathcal{W}}. \tag{3.7}$$

In fact by the definition of the norm on V we have

$$\begin{aligned} \|y\|_V^2 &= \|Ly\|^2_{L^2(Q)} + \left\|\frac{\partial y}{\partial \nu}\right\|^2_{L^2(\Sigma_0)} \\ &= \|f\|^2_{L^2(Q)} + \|h\|^2_{L^2(\Sigma_0)} \\ &= \|(f,h)\|^2_{\mathcal{W}}. \end{aligned}$$

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