

Parity sheaves on the affine Grassmannian and the Mirković–Vilonen conjecture

by

PRAMOD N. ACHAR
*Louisiana State University
Baton Rouge, LA, U.S.A.*

LAURA RIDER
*Massachusetts Institute of Technology
Cambridge, MA, U.S.A.*

1. Introduction

1.1. Overview

Let G be a connected complex reductive group and let $\mathcal{G}r$ denote its affine Grassmannian. The space $\mathcal{G}r$ has the remarkable property that its topology encodes the representation theory of the split Langlands dual group G^\vee over any field \mathbb{k} (or even over a commutative ring). To be more precise, the *geometric Satake equivalence*, in the form due to Mirković–Vilonen [MV2] (see also [L], [G2]), asserts that there is an equivalence of tensor categories

$$\mathcal{S}: \text{Rep}(G^\vee) \xrightarrow{\sim} \text{Perv}_{G_{\mathbb{O}}}(\mathcal{G}r, \mathbb{k}), \quad (1.1)$$

where $\text{Perv}_{G_{\mathbb{O}}}(\mathcal{G}r, \mathbb{k})$ is the category of spherical perverse \mathbb{k} -sheaves on $\mathcal{G}r$. (A full explanation of the notation is given in §1.2 below.) This result raises the possibility of comparing representation theory over different fields via the universal coefficient theorem of topology.

For instance, let λ be a dominant coweight for G , and let $\mathcal{I}_1(\lambda, \mathbb{k})$ denote the “standard” perverse sheaf on the corresponding stratum of $\mathcal{G}r$. This perverse sheaf serves as a topological realization of a Weyl module for G^\vee . When $\mathbb{k}=\mathbb{C}$, it is simple, and its stalks are described by Kazhdan–Lusztig theory.

With a view to applications in modular representation theory, Mirković and Vilonen conjectured in the late 1990s [MV1] that the stalks of $\mathcal{I}_1(\lambda, \mathbb{Z})$ are torsion-free. This implies that the \mathbb{k} -stalks are “independent” of \mathbb{k} . Their conjecture was slightly too optimistic: counterexamples due to Juteau [Ju] reveal the presence of torsion, but

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Type	Bound	Type	Bound
A_n	any p	E_6, F_4, G_2	$p > 3$
B_n, D_n	$p > 2$	E_7	$p > 19$
C_n	$p > n$	E_8	$p > 31$

Table 1. Currently known bounds (see footnote ⁽¹⁾ on next page) for JMW primes.

only at bad primes. Juteau proposed a modified conjecture, asserting that there is no p -torsion as long as p is a good prime for G^\vee . In this paper, we prove the following result, confirming this conjecture in nearly all cases.

THEOREM 1.1. *If p is a JMW prime for G^\vee (see Table 1), then the stalks of $\mathcal{I}_1(\lambda, \mathbb{Z})$ have no p -torsion. Similarly, if \mathbb{k} is a field whose characteristic is a JMW prime, then the stalks of $\mathcal{I}_1(\lambda, \mathbb{k})$ have a parity-vanishing property.*

An outline of the proof will be explained below, after some preliminaries. In a subsequent paper [AR], the authors exploit this result to establish a modular analogue of the derived equivalence of [ABG, Theorem 9.4.3].

1.2. The constructible side

Recall that $\mathcal{G}r = G_{\mathbf{K}}/G_{\mathbf{O}}$, where $\mathbf{K} = \mathbb{C}((t))$ and $\mathbf{O} = \mathbb{C}[[t]]$. For the remainder of the introduction, \mathbb{k} will denote an algebraically closed field. Let $D_{(G_{\mathbf{O}})}^b(\mathcal{G}r, \mathbb{k})$ denote the bounded derived category of complexes of \mathbb{k} -sheaves on $\mathcal{G}r$ that are constructible with respect to the $G_{\mathbf{O}}$ -orbits, and let $\text{Perv}_{G_{\mathbf{O}}}(\mathcal{G}r, \mathbb{k}) \subset D_{(G_{\mathbf{O}})}^b(\mathcal{G}r, \mathbb{k})$ be the subcategory of perverse sheaves. Those $G_{\mathbf{O}}$ -orbits are naturally in bijection with the set \mathbf{X}^+ of dominant coweights for G . For $\lambda \in \mathbf{X}^+$, let $i_\lambda: \mathcal{G}r_\lambda \hookrightarrow \mathcal{G}r$ be the inclusion map of the corresponding orbit.

For $\lambda \in \mathbf{X}^+$, the irreducible (resp. Weyl, dual Weyl, indecomposable tilting) G^\vee -module of highest weight λ is denoted by $L(\lambda)$ (resp. $M(\lambda)$, $N(\lambda)$, $T(\lambda)$). The perverse sheaves corresponding to these objects under \mathcal{S} are denoted by $\text{IC}(\lambda, \mathbb{k})$, (resp. $\mathcal{I}_1(\lambda, \mathbb{k})$, $\mathcal{I}_*(\lambda, \mathbb{k})$, $\mathcal{T}(\lambda, \mathbb{k})$). Of course, $\text{IC}(\lambda, \mathbb{k})$ is a simple perverse sheaf. We saw $\mathcal{I}_1(\lambda, \mathbb{k})$ earlier; $\mathcal{I}_*(\lambda, \mathbb{k})$ is its Verdier dual, a costandard perverse sheaf.

What about the $\mathcal{T}(\lambda, \mathbb{k})$? It is a deep insight of Juteau–Mautner–Williamson that these perverse sheaves should be characterized by a topological property: specifically, they ought to be *parity sheaves* in the sense of [JMW1].

Definition 1.2. A prime number p is said to be a JMW prime for G^\vee if it is good for G^\vee and, whenever \mathbb{k} has characteristic p , each $\mathcal{T}(\lambda, \mathbb{k})$ is a parity sheaf on $\mathcal{G}r$.

Juteau, Mautner, and Williamson have shown in [JMW2, Theorem 1.8] that for quasi-simple G^\vee , the primes in Table 1 are JMW primes. They conjecture⁽¹⁾ that every good prime is JMW (see [JMW2, Conjecture 1.10]).

1.3. The coherent side

The main idea of the proof of Theorem 1.1 is to translate the problem into an algebraic question about coherent sheaves on the nilpotent cone \mathcal{N} for G^\vee . The motivation comes from an old result of Ginzburg [G2, Proposition 1.10.4]: when $\mathbb{k}=\mathbb{C}$, he showed that for all $V_1, V_2 \in \text{Rep}(G^\vee)$, there is an isomorphism of graded vector spaces

$$\text{Hom}_{\mathbf{D}_{(G^\vee)}^b(\mathcal{G}_{r,\mathbb{k}})}(\mathcal{S}(V_1), \mathcal{S}(V_2)) \cong \text{Hom}_{\text{Coh}^{G^\vee \times \mathbb{G}_m}(\mathcal{N})}^\bullet(V_1 \otimes \mathcal{O}_{\mathcal{N}}, V_2 \otimes \mathcal{O}_{\mathcal{N}}). \tag{1.2}$$

For details on the category $\text{Coh}^{G^\vee \times \mathbb{G}_m}(\mathcal{N})$, see §2.4.

To imitate this in positive characteristic, we need control over the algebraic geometry of \mathcal{N} . To this end, we impose the following condition on G^\vee :

$$\begin{aligned} &\text{The derived group of } G^\vee \text{ is simply connected and its Lie} \\ &\text{algebra admits a non-degenerate } G^\vee\text{-invariant bilinear form.} \end{aligned} \tag{1.3}$$

This condition holds for $\text{GL}(n)$ and for every simply connected quasi-simple group that is not of type A . See [J2, §2.9] for a discussion of other situations where condition (1.3) holds. Under this condition, it is feasible to adapt Ginzburg’s argument, provided that $\mathcal{S}(V_1)$ and $\mathcal{S}(V_2)$ are parity.

To push this result further, we need the following observation: coherent sheaves of the form $V \otimes \mathcal{O}_{\mathcal{N}}$ also lie in the category of *perverse-coherent sheaves*, denoted

$$\text{PCoh}^{G^\vee \times \mathbb{G}_m}(\mathcal{N}),$$

or simply $\text{PCoh}(\mathcal{N})$. This category, which is the heart of a certain t -structure on $\mathbf{D}^b\text{Coh}^{G^\vee \times \mathbb{G}_m}(\mathcal{N})$, looks very different from $\text{Coh}^{G^\vee \times \mathbb{G}_m}(\mathcal{N})$. For instance, every object of $\text{PCoh}(\mathcal{N})$ has finite length. We will not use the details of its definition in this paper, we just require a structural property discussed in §2.4.

Interpreting the right-hand side of (1.2) as a Hom-group in $\text{PCoh}(\mathcal{N})$ leads to new avenues for generalizing that result. For $\mu \in \mathbf{X}^+$, let $\text{PCoh}(\mathcal{N})_{\leq \mu} \subset \text{PCoh}(\mathcal{N})$ be the Serre subcategory generated by $\mathbf{N}(\nu) \otimes \mathcal{O}_{\mathcal{N}}\langle n \rangle$ with $\nu \leq \mu$. (Here, $\langle n \rangle$ indicates a twist of the \mathbb{G}_m -action.) In §5 we prove the following result, which seems to be new even for $\mathbb{k}=\mathbb{C}$.

⁽¹⁾ Since this paper appeared in preprint form, Mautner and Riche have proved that every good prime is a JMW prime [MR], confirming [JMW2, Conjecture 1.10].

THEOREM 1.3. *Let $\text{char } \mathbb{k}$ be a JMW prime for G^\vee , and assume that (1.3) holds for G^\vee . Let $j: \mathcal{G}r \setminus \overline{\mathcal{G}r}_\lambda \rightarrow \mathcal{G}r$ be the inclusion map and let*

$$\Pi: \text{PCoh}(\mathcal{N}) \longrightarrow \text{PCoh}(\mathcal{N})/\text{PCoh}(\mathcal{N})_{\leq \lambda}$$

be the Serre quotient functor. If $V_1 \in \text{Rep}(G^\vee)$ has a Weyl filtration and $V_2 \in \text{Rep}(G^\vee)$ has a good filtration, then there is a natural isomorphism

$$\text{Hom}^\bullet(j^* \mathcal{S}(V_1), j^* \mathcal{S}(V_2)) \cong \text{Hom}^\bullet(\Pi(V_1 \otimes \mathcal{O}_{\mathcal{N}}), \Pi(V_2 \otimes \mathcal{O}_{\mathcal{N}})).$$

Intuitively, this theorem gives us an algebraic counterpart in $\text{D}^b \text{Coh}^{G^\vee \times \mathbb{G}_m}(\mathcal{N})$ of the geometric notion of “restricting to an open subset” in $\mathcal{G}r$. Once we have that, it is not difficult to translate the problem of studying stalks of $\mathcal{I}_!(\lambda, \mathbb{k})$ into an algebraic question about certain objects in $\text{PCoh}(\mathcal{N})$ and its quotients. The latter question turns out to be quite easy (see Lemma 2.12).

1.4. Outline of the paper

In §2 we recall the necessary background on properly stratified categories and on the category $\text{PCoh}(\mathcal{N})$, largely following the work of Minn–Thu–Aye. We review the theory of parity sheaves in §3. In §4, which can be read independently of the rest of the paper, we study the cohomology of parity sheaves on flag varieties of Kac–Moody groups, generalizing earlier results of Soergel and Ginzburg. That result is a step on the way to Theorem 1.3, which is proved in §5. Finally, the main result, Theorem 1.1, is proved in §6.

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2. Properly stratified categories

2.1. Definition and background

Let \mathbb{k} be a field, and let \mathcal{C} be a \mathbb{k} -linear abelian category in which every object has finite length. Assume that \mathcal{C} is equipped with an automorphism $\langle 1 \rangle: \mathcal{C} \rightarrow \mathcal{C}$, which we will refer to as the *Tate twist*. For $X, Y \in \mathcal{C}$, let $\underline{\text{Hom}}(X, Y)$ be the graded vector space given by

$$\underline{\text{Hom}}(X, Y)_n = \text{Hom}(X, Y \langle n \rangle).$$

The Tate twist induces an action of \mathbb{Z} on the set $\text{Irr}(\mathcal{C})$ of isomorphism classes of simple objects in \mathcal{C} . Assume that this action is free, and let $\Omega = \text{Irr}(\mathcal{C})/\mathbb{Z}$. For each $\gamma \in \Omega$, choose a representative simple object $L_\gamma \in \mathcal{C}$ whose isomorphism class lies in the \mathbb{Z} -orbit $\gamma \subset \text{Irr}(\mathcal{C})$. Thus, every simple object in \mathcal{C} is isomorphic to some $L_\gamma\langle n \rangle$ with $\gamma \in \Omega$ and $n \in \mathbb{Z}$.

Assume that Ω is equipped with a partial order \leq , and that for any $\gamma \in \Omega$ the set $\{\xi \in \Omega \mid \xi \leq \gamma\}$ is finite. For any order ideal $\Gamma \subset \Omega$, let $\mathcal{C}_\Gamma \subset \mathcal{C}$ be the Serre subcategory generated by the simple objects $\{L_\gamma\langle n \rangle \mid \gamma \in \Gamma \text{ and } n \in \mathbb{Z}\}$. (Recall that an *order ideal* is a subset $\Gamma \subset \Omega$ such that if $\gamma \in \Gamma$ and $\xi \leq \gamma$, then $\xi \in \Gamma$.) As a special case, we write

$$\mathcal{C}_{\leq \gamma} = \mathcal{C}_{\{\xi \in \Omega \mid \xi \leq \gamma\}}. \quad (2.1)$$

The category $\mathcal{C}_{< \gamma}$ is defined similarly.

Definition 2.1. Suppose \mathcal{C} , Ω , and \leq are as above. We say that \mathcal{C} is a *graded properly stratified category* if for each $\gamma \in \Omega$ the following conditions hold:

- (1) $\text{End}(L_\gamma) \cong \mathbb{k}$.
- (2) There is an object $\bar{\Delta}_\gamma$ and a surjective morphism $\phi_\gamma: \bar{\Delta}_\gamma \rightarrow L_\gamma$ such that

$$\ker(\phi_\gamma) \in \mathcal{C}_{< \gamma} \quad \text{and} \quad \underline{\text{Hom}}(\bar{\Delta}_\gamma, L_\xi) = \underline{\text{Ext}}^1(\bar{\Delta}_\gamma, L_\xi) = 0 \quad \text{if } \xi \not\leq \gamma.$$

- (3) There is an object $\bar{\nabla}_\gamma$ and an injective morphism $\psi_\gamma: L_\gamma \rightarrow \bar{\nabla}_\gamma$ such that

$$\text{cok}(\psi_\gamma) \in \mathcal{C}_{< \gamma} \quad \text{and} \quad \underline{\text{Hom}}(L_\xi, \bar{\nabla}_\gamma) = \underline{\text{Ext}}^1(L_\xi, \bar{\nabla}_\gamma) = 0 \quad \text{if } \xi \not\leq \gamma.$$

- (4) In $\mathcal{C}_{\leq \gamma}$, L_γ admits a projective cover $\Delta_\gamma \rightarrow L_\gamma$. Moreover, Δ_γ admits a filtration whose subquotients are of the form $\bar{\Delta}_\gamma\langle n \rangle$ for various $n \in \mathbb{Z}$.

- (5) In $\mathcal{C}_{\leq \gamma}$, L_γ admits an injective envelope $L_\gamma \rightarrow \nabla_\gamma$. Moreover, ∇_γ admits a filtration whose subquotients are of the form $\bar{\nabla}_\gamma\langle n \rangle$ for various $n \in \mathbb{Z}$.

- (6) We have $\underline{\text{Ext}}^2(\Delta_\gamma, \bar{\nabla}_\xi) = \underline{\text{Ext}}^2(\bar{\Delta}_\gamma, \nabla_\xi) = 0$ for all $\gamma, \xi \in \Omega$.

An object in \mathcal{C} is said to be *standard* (resp. *costandard*, *proper standard*, *proper costandard*) if it is isomorphic to some $\Delta_\gamma\langle n \rangle$ (resp. $\nabla_\gamma\langle n \rangle$, $\bar{\Delta}_\gamma\langle n \rangle$, $\bar{\nabla}_\gamma\langle n \rangle$).

More generally, a *standard* (resp. *costandard*, *proper standard*, *proper costandard*) filtration of an object of \mathcal{C} is a filtration whose subquotients are all standard (resp. costandard, proper standard, proper costandard) objects.

Routine arguments (see [B, Lemma 1]) show that when objects $\bar{\Delta}_\gamma$, $\bar{\nabla}_\gamma$, Δ_γ , and ∇_γ with the above properties exist, they are unique up to isomorphism. It may happen that $\bar{\Delta}_\gamma \cong \Delta_\gamma$ and $\bar{\nabla}_\gamma \cong \nabla_\gamma$; in that case, \mathcal{C} is usually called a *highest weight* or *quasi-hereditary category*. The class of objects in \mathcal{C} admitting a standard (resp. costandard, proper standard, proper costandard) filtration is denoted

$$\mathcal{F}(\Delta) \quad (\text{resp. } \mathcal{F}(\nabla), \mathcal{F}(\bar{\Delta}), \mathcal{F}(\bar{\nabla})).$$

The relationship between the notions above and the notion of a *properly stratified algebra* [D], [FM] is explained in [Mi]. In particular, results in [Mi] explain how to transfer results from the literature on properly stratified algebras to our setting. For instance, the following result is a restatement of [D, Definition 4 and Theorem 5].

PROPOSITION 2.2. *Let $\Gamma \subset \Omega$ be a finite-order ideal. Then the Serre quotient $\mathcal{C}/\mathcal{C}_\Gamma$ is again a graded properly stratified category, and $\text{Irr}(\mathcal{C}/\mathcal{C}_\Gamma)/\mathbb{Z}$ is naturally identified with $\Omega \setminus \Gamma$. Indeed, we have a recollement diagram*

$$\begin{array}{ccccc}
 & \overset{\iota^L}{\curvearrowright} & & \overset{\Pi^L}{\curvearrowright} & \\
 D^b\mathcal{C}_\Gamma & \xrightarrow{\iota} & D^b\mathcal{C} & \xrightarrow{\Pi} & D^b(\mathcal{C}/\mathcal{C}_\Gamma) \\
 & \underset{\iota^R}{\curvearrowleft} & & \underset{\Pi^R}{\curvearrowleft} &
 \end{array}$$

Here, the superscripts L and R indicate the left and right adjoints, respectively, of ι and Π . An important property implied by the preceding proposition is that

$$\underline{\text{Ext}}^k(\Delta_\gamma, \bar{\nabla}_\xi) = \underline{\text{Ext}}^k(\bar{\Delta}_\gamma, \nabla_\xi) = 0 \quad \text{for all } k > 0.$$

Also implicit in Proposition 2.2 (or explicit in its proof) are the next two lemmas, which express the compatibility of the various functors with the properly stratified structure. For analogues in the quasi-hereditary case, see [CPS].

LEMMA 2.3. *The functors ι and Π are t -exact and preserve the property of having a standard (resp. costandard, proper standard, proper costandard) filtration.*

The remaining functors in the recollement diagram are not t -exact in general, but they do send certain classes of objects to the heart of the t -structure.

LEMMA 2.4. *The functors ι^L and Π^L preserve the property of having a standard or proper standard filtration. The functors ι^R and Π^R preserve the property of having a costandard or proper costandard filtration.*

2.2. Tilting objects

In contrast with the quasi-hereditary case, there are, in general, two inequivalent notions of “tilting” in a properly stratified category.

Definition 2.5. A *tilting object* is an object in $\mathcal{F}(\Delta) \cap \mathcal{F}(\bar{\nabla})$. A *cotilting object* is an object in $\mathcal{F}(\bar{\Delta}) \cap \mathcal{F}(\nabla)$.

The next proposition gives the classification of indecomposable tilting and cotilting objects. (See [ÁHLU] for a similar statement for properly stratified algebras.)

PROPOSITION 2.6. *For each $\gamma \in \Omega$, there is an indecomposable tilting object T_γ , unique up to isomorphism, that fits into the short exact sequences*

$$0 \longrightarrow \Delta_\gamma \longrightarrow T_\gamma \longrightarrow X \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow Y \longrightarrow T_\gamma \longrightarrow \bar{\nabla}_\gamma \longrightarrow 0$$

with $X \in \mathcal{F}(\Delta)_{<\gamma}$ and $Y \in \mathcal{F}(\bar{\nabla})_{\leq\gamma}$. Dually, there is an indecomposable cotilting object T'_γ , unique up to isomorphism, with short exact sequences

$$0 \longrightarrow \bar{\Delta}_\gamma \longrightarrow T'_\gamma \longrightarrow X' \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow Y' \longrightarrow T'_\gamma \longrightarrow \nabla_\gamma \longrightarrow 0$$

with $X' \in \mathcal{F}(\bar{\Delta})_{\leq\gamma}$ and $Y' \in \mathcal{F}(\nabla)_{<\gamma}$. Also, every indecomposable tilting (resp. cotilting) object is isomorphic to some $T_\gamma\langle n \rangle$ (resp. $T'_\gamma\langle n \rangle$).

LEMMA 2.7. *Assume that the tilting and cotilting objects in \mathcal{C} coincide, i.e., that for each $\gamma \in \Omega$, there is an integer m_γ such that $T_\gamma \cong T'_\gamma\langle m_\gamma \rangle$. Then*

- (1) if $\gamma \in \Omega$ is minimal, then $\Delta_\gamma \cong T_\gamma \cong T'_\gamma\langle m_\gamma \rangle \cong \nabla_\gamma\langle m_\gamma \rangle$;
- (2) for any $\gamma \in \Omega$, we have $\underline{\text{Ext}}^1(\nabla_\gamma, \bar{\nabla}_\gamma) = 0$;
- (3) for any $\gamma \in \Omega$, there are natural isomorphisms

$$\underline{\text{Hom}}(\Delta_\gamma, \Delta_\gamma) \cong \underline{\text{Hom}}(\Delta_\gamma, \nabla_\gamma\langle m_\gamma \rangle) \cong \underline{\text{Hom}}(\nabla_\gamma, \nabla_\gamma).$$

Proof. (1) This is immediate from the short exact sequences in Proposition 2.6.

(2) Consider the long exact sequence

$$\dots \longrightarrow \underline{\text{Hom}}(Y', \bar{\nabla}_\gamma) \longrightarrow \underline{\text{Ext}}^1(\nabla_\gamma, \bar{\nabla}_\gamma) \longrightarrow \underline{\text{Ext}}^1(T'_\gamma, \bar{\nabla}_\gamma) \longrightarrow \dots$$

The first term vanishes because $Y' \in \mathcal{C}_{<\gamma}$, and the last term vanishes because

$$T'_\gamma \cong T_\gamma\langle -m_\gamma \rangle \in \mathcal{F}(\Delta).$$

The result follows.

(3) It is easy to see that the natural maps

$$\underline{\text{Hom}}(\Delta_\gamma, \Delta_\gamma) \longrightarrow \underline{\text{Hom}}(\Delta_\gamma, T_\gamma) \quad \text{and} \quad \underline{\text{Hom}}(\Delta_\gamma, T'_\gamma\langle m_\gamma \rangle) \longrightarrow \underline{\text{Hom}}(\Delta_\gamma, \nabla_\gamma\langle m_\gamma \rangle)$$

are both isomorphisms. The proof of the second isomorphism in the statement is similar. \square

PROPOSITION 2.8. ([Mi]; cf. [BBM, Proposition 1.5]) *Assume that the tilting and cotilting objects in \mathcal{C} coincide. Let $\mathcal{T} \subset \mathcal{C}$ be the full subcategory of tilting objects, and consider its homotopy category $K^b\mathcal{T}$. The obvious functor*

$$K^b\mathcal{T} \longrightarrow D^b\mathcal{C} \tag{2.2}$$

is fully faithful. In case \mathcal{C} is quasi-hereditary, it is an equivalence.

PROPOSITION 2.9. *Assume that the tilting and cotilting objects in \mathcal{C} coincide. The following conditions are equivalent:*

- (1) $X \in \mathcal{F}(\Delta)$;
- (2) *there is an exact sequence $0 \rightarrow X \rightarrow T^0 \rightarrow T^1 \rightarrow \dots \rightarrow T^k \rightarrow 0$, where all the T^i 's are tilting.*

Before proving this, we record one immediate consequence.

Definition 2.10. For $X \in \mathcal{F}(\Delta)$, we define the *tilting dimension* of X , denoted by $\text{tdim } X$, to be the smallest integer k such that there exists a resolution of X of length k by tilting objects, as in Proposition 2.9.

COROLLARY 2.11. *If $X \in \mathcal{F}(\Delta)$, then there is a short exact sequence*

$$0 \longrightarrow X \longrightarrow T \longrightarrow X' \longrightarrow 0, \tag{2.3}$$

where T is tilting, $X' \in \mathcal{F}(\Delta)$, and $\text{tdim } X' = \text{tdim } X - 1$.

Proof of Proposition 2.9. Let $\mathcal{F}(\Delta)'$ be the class of objects X satisfying condition (2) above. The notion of tilting dimension makes sense for objects of $\mathcal{F}(\Delta)'$. Moreover, if we replace every occurrence of $\mathcal{F}(\Delta)$ by $\mathcal{F}(\Delta)'$ in the statement of Corollary 2.11, then the resulting statement is true. An argument by induction on tilting dimension, using the short exact sequence (2.3), shows that $\mathcal{F}(\Delta)' \subset \mathcal{F}(\Delta)$.

Next, let $K^0 \subset K^b\mathcal{T}$ be the full subcategory consisting of objects isomorphic to a bounded complex of tilting modules (X^\bullet, d) satisfying the following two conditions:

- (1) the complex is concentrated in non-negative degrees;
- (2) the cohomology of the complex vanishes, except possibly in degree 0.

It is easy to see that $\mathcal{F}(\Delta)'$ consists precisely of the objects that lie in the image of K^0 under the functor (2.2). In particular, we see that $\mathcal{F}(\Delta)'$ is stable under extensions, because K^0 is. Thus, to prove that $\mathcal{F}(\Delta) \subset \mathcal{F}(\Delta)'$, it suffices to show that each Δ_γ lies in $\mathcal{F}(\Delta)'$. This follows from the first short exact sequence in Proposition 2.6, by induction on γ . \square

The next lemma is ultimately the source of the torsion-freeness in Theorem 1.1.

LEMMA 2.12. *Assume that the tilting and cotilting objects in \mathcal{C} coincide. If $X \in \mathcal{F}(\Delta)$, then $\underline{\mathrm{Hom}}(X, \nabla_\gamma)$ is a free module over the graded ring $\underline{\mathrm{End}}(\nabla_\gamma)$.*

Proof. We proceed by induction on the number of steps in a standard filtration of X . If $0 \rightarrow X' \rightarrow X \rightarrow X'' \rightarrow 0$ is an exact sequence with $X', X'' \in \mathcal{F}(\Delta)$, then we obtain a short exact sequence

$$0 \longrightarrow \underline{\mathrm{Hom}}(X'', \nabla_\gamma) \longrightarrow \underline{\mathrm{Hom}}(X, \nabla_\gamma) \longrightarrow \underline{\mathrm{Hom}}(X', \nabla_\gamma) \longrightarrow 0$$

of $\underline{\mathrm{End}}(\nabla_\gamma)$ -modules. If the first and last terms are free, the middle term must be as well. Thus, we are reduced to considering the case where X is a standard object, say $X = \Delta_\xi \langle n \rangle$. If $\xi \neq \gamma$, then $\underline{\mathrm{Hom}}(X, \nabla_\gamma) = 0$. If $\xi = \gamma$, then $\underline{\mathrm{Hom}}(X, \nabla_\gamma)$ is a free $\underline{\mathrm{End}}(\nabla_\gamma)$ -module by Lemma 2.7 (3). \square

2.3. Quotients of the category of tilting objects

The next result compares the Serre quotient $\mathcal{C}/\mathcal{C}_\Gamma$ to a “naive” quotient category. If \mathcal{A} is an additive category and $\mathcal{B} \subset \mathcal{A}$ is a full subcategory, we write $\mathcal{A} // \mathcal{B}$ for the category with the same objects as \mathcal{A} , but with morphisms given by

$$\mathrm{Hom}_{\mathcal{A} // \mathcal{B}}(X, Y) = \mathrm{Hom}_{\mathcal{A}}(X, Y) / \{f \mid f \text{ factors through an object of } \mathcal{B}\}. \quad (2.4)$$

PROPOSITION 2.13. *Assume that the tilting and cotilting objects in \mathcal{C} coincide, and let $\Gamma \subset \Omega$ be a finite-order ideal. Then the quotient functor $\Pi: \mathcal{C} \rightarrow \mathcal{C}/\mathcal{C}_\Gamma$ induces an equivalence of categories*

$$\bar{\Pi}: \mathrm{Tilt}(\mathcal{C}) // \mathrm{Tilt}(\mathcal{C}_\Gamma) \xrightarrow{\sim} \mathrm{Tilt}(\mathcal{C}/\mathcal{C}_\Gamma). \quad (2.5)$$

Proof. Let $Q: \mathrm{Tilt}(\mathcal{C}) \rightarrow \mathrm{Tilt}(\mathcal{C}) // \mathrm{Tilt}(\mathcal{C}_\Gamma)$ be the quotient functor. It is clear that $\Pi(\mathrm{Tilt}(\mathcal{C}_\Gamma)) = 0$, so there is a unique functor $\bar{\Pi}$ such that $\bar{\Pi} \circ Q \cong \Pi$. From the classification of tilting objects in Proposition 2.6, it is clear that every indecomposable object in $\mathrm{Tilt}(\mathcal{C}/\mathcal{C}_\Gamma)$ occurs as a direct summand of some object in the image of $\bar{\Pi}$. If $\bar{\Pi}$ were already known to be fully faithful, it would send indecomposable objects to indecomposable objects, and would therefore be essentially surjective.

It suffices, then, to prove that $\bar{\Pi}$ is fully faithful. We proceed by induction on the size of Γ . Suppose first that Γ is a singleton. Let $T, T' \in \mathrm{Tilt}(\mathcal{C})$ and consider the diagram

$$\underline{\mathrm{Hom}}_{\mathcal{C}}(T, T') \xrightarrow{Q} \underline{\mathrm{Hom}}_{\mathrm{Tilt}(\mathcal{C}) // \mathrm{Tilt}(\mathcal{C}_\Gamma)}(T, T') \xrightarrow{\bar{\Pi}} \underline{\mathrm{Hom}}_{\mathcal{C}/\mathcal{C}_\Gamma}(\Pi(T), \Pi(T')). \quad (2.6)$$

$\xrightarrow{\quad \Pi \quad}$

By Lemma 2.4, all three terms of the functorial distinguished triangle $\Pi^L\Pi(T) \rightarrow T \rightarrow i^L(T) \rightarrow$ lie in \mathcal{C} , so that the distinguished triangle is actually a short exact sequence. Apply $\underline{\mathrm{Hom}}(-, T')$ to get the long exact sequence

$$\begin{aligned} 0 \longrightarrow \underline{\mathrm{Hom}}(i^L(T), i^R(T')) \longrightarrow \underline{\mathrm{Hom}}(T, T') \longrightarrow \underline{\mathrm{Hom}}(\Pi(T), \Pi(T')) \\ \longrightarrow \underline{\mathrm{Ext}}^1(i^L(T), i^R(T')) \longrightarrow \dots \end{aligned} \quad (2.7)$$

The last term vanishes because (by Lemma 2.4 again) $i^L(T)$ has a standard filtration and $i^R(T')$ has a costandard filtration. It follows that the map labeled Π in (2.6) is surjective, and its kernel can be identified with the space

$$K = \{f: T \rightarrow T' \mid f \text{ factors as } T \rightarrow i^L(T) \rightarrow i^R(T') \rightarrow T'\}.$$

We deduce that $\bar{\Pi}$ is surjective as well. Now, the kernel of Q in (2.6) is the space

$$K' = \{f: T \rightarrow T' \mid f \text{ factors through an object of } \mathrm{Tilt}(\mathcal{C}_\Gamma)\}.$$

We already know that $K' \subset K$. But since Γ is a singleton $\{\gamma\}$ with γ necessarily minimal in Ω , we see from Lemma 2.7 (1) that $i^L(T)$ is actually tilting (and not merely in $\mathcal{F}(\Delta)$), and likewise for $i^R(T')$. So $K = K'$, and we conclude that $\bar{\Pi}$ in (2.6) is an isomorphism.

For the general case, choose a non-empty proper ideal $\Upsilon \subset \Gamma$. Then Υ and $\Gamma \setminus \Upsilon$ are both smaller than Γ , and by induction, we have natural equivalences

$$\begin{aligned} \mathrm{Tilt}(\mathcal{C}) // \mathrm{Tilt}(\mathcal{C}_\Upsilon) &\cong \mathrm{Tilt}(\mathcal{C}/\mathcal{C}_\Upsilon), \\ \mathrm{Tilt}(\mathcal{C}_\Gamma) // \mathrm{Tilt}(\mathcal{C}_\Upsilon) &\cong \mathrm{Tilt}(\mathcal{C}_\Gamma/\mathcal{C}_\Upsilon), \\ \mathrm{Tilt}(\mathcal{C}/\mathcal{C}_\Upsilon) // \mathrm{Tilt}(\mathcal{C}_\Gamma/\mathcal{C}_\Upsilon) &\cong \mathrm{Tilt}((\mathcal{C}/\mathcal{C}_\Upsilon)/(\mathcal{C}_\Gamma/\mathcal{C}_\Upsilon)) \cong \mathrm{Tilt}(\mathcal{C}/\mathcal{C}_\Gamma). \end{aligned}$$

It is also easy to see that there is a canonical equivalence

$$\mathrm{Tilt}(\mathcal{C}) // \mathrm{Tilt}(\mathcal{C}_\Gamma) \cong (\mathrm{Tilt}(\mathcal{C}) // \mathrm{Tilt}(\mathcal{C}_\Upsilon)) // (\mathrm{Tilt}(\mathcal{C}_\Gamma) // \mathrm{Tilt}(\mathcal{C}_\Upsilon)).$$

Combining all these yields the desired equivalence (2.5). \square

The next corollary is immediate from (2.7) and the discussion following it.

COROLLARY 2.14. *Assume that the tilting and cotilting objects in \mathcal{C} coincide, and let $\Gamma \subset \Omega$ be a finite-order ideal. If $X \in \mathcal{F}(\Delta)$ and $Y \in \mathcal{F}(\nabla)$, then the map $\underline{\mathrm{Hom}}_{\mathcal{C}}(X, Y) \rightarrow \underline{\mathrm{Hom}}_{\mathcal{C}/\mathcal{C}_\Gamma}(\Pi(X), \Pi(Y))$ is surjective.*

2.4. Perverse-coherent sheaves on the nilpotent cone

In this subsection, we assume that \mathbb{k} is an algebraically closed field whose characteristic is good for G^\vee , and that the derived group of G^\vee is simply connected. Recall that \mathcal{N} denotes the nilpotent cone of G^\vee . Let $G^\vee \times \mathbb{G}_m$ act on \mathcal{N} by $(g, z) \cdot x = z^{-2} \text{Ad}(g)(x)$. We write $\text{Coh}^{G^\vee \times \mathbb{G}_m}(\mathcal{N})$, or simply $\text{Coh}(\mathcal{N})$, for the category of $(G^\vee \times \mathbb{G}_m)$ -equivariant coherent sheaves on \mathcal{N} .

Let $\text{PCoh}(\mathcal{N})$ be the category of $(G^\vee \times \mathbb{G}_m)$ -equivariant *perverse-coherent sheaves* on \mathcal{N} . This is the heart of a certain remarkable t -structure on $\text{D}^b \text{Coh}(\mathcal{N})$. We refer the reader to [AB], [B], and [A] for details on the definition and properties of this category. Here are some basic facts about $\text{PCoh}(\mathcal{N})$:

- every object in $\text{PCoh}(\mathcal{N})$ has finite length;
- it is stable under $\mathcal{F} \mapsto \mathcal{F}\langle 1 \rangle$, where $\langle 1 \rangle: \text{D}^b \text{Coh}(\mathcal{N}) \rightarrow \text{D}^b \text{Coh}(\mathcal{N})$ is given by a twist of the \mathbb{G}_m -action;
- the set $\text{Irr}(\text{PCoh}(\mathcal{N}))/\mathbb{Z}$ is naturally in bijection with \mathbf{X}^+ .

Remark 2.15. For any $V \in \text{Rep}(G^\vee)$, the coherent sheaf $V \otimes \mathcal{O}_{\mathcal{N}}$ is perverse-coherent. The proof of [A, Lemma 5.4] can be generalized to work for any $V \in \text{Rep}(G^\vee)$. Alternatively, one can give a more direct argument using the definition of the perverse-coherent t -structure from [B] and the fact that \mathcal{N} is Cohen–Macaulay.

For $\lambda \in \mathbf{X}^+$, let δ_λ be the length of the shortest Weyl group element w such that $w\lambda$ is antidominant. We define a subcategory $\text{PCoh}(\mathcal{N})_{\leq \lambda} \subset \text{PCoh}(\mathcal{N})$ as in (2.1). (The theorem below implies that this agrees with the definition of $\text{PCoh}(\mathcal{N})_{\leq \lambda}$ given in §1.3.) For our purposes, the most important fact about $\text{PCoh}(\mathcal{N})$ is the following result of Minn–Thu–Aye, which refines the description given in [A] and [B].

THEOREM 2.16. (Minn–Thu–Aye [Mi]) *Assume that $\text{char } \mathbb{k}$ is good for G^\vee , and that the derived group of G^\vee is simply connected. Then the category $\text{PCoh}(\mathcal{N})$ is a graded properly stratified category. Moreover,*

- (1) *the tilting and cotilting objects in $\text{PCoh}(\mathcal{N})$ coincide and are given by*

$$T_\lambda = (\text{T}(\lambda) \otimes \mathcal{O}_{\mathcal{N}})\langle -\delta_\lambda \rangle \quad \text{and} \quad T'_\lambda = (\text{T}(\lambda) \otimes \mathcal{O}_{\mathcal{N}})\langle \delta_\lambda \rangle;$$

- (2) *the object $\text{M}(\lambda) \otimes \mathcal{O}_{\mathcal{N}}$ lies in $\text{PCoh}(\mathcal{N})_{\leq \lambda}$ and has a standard filtration;*
 (3) *the object $\text{N}(\lambda) \otimes \mathcal{O}_{\mathcal{N}}$ lies in $\text{PCoh}(\mathcal{N})_{\leq \lambda}$ and has a costandard filtration.*

For completeness, we include a proof of this theorem. The following argument is adapted from [Mi, Chapter 4].

Proof. Throughout this proof, we will freely make use of the main result of [A], which states that $\text{D}^b \text{PCoh}(\mathcal{N})$ and $\text{D}^b \text{Coh}(\mathcal{N})$ are equivalent. In particular, we will compute Ext-groups for $\text{PCoh}(\mathcal{N})$ by computing Hom-groups in $\text{D}^b \text{Coh}(\mathcal{N})$.

Let $\tilde{\mathcal{N}}$ denote the cotangent bundle of the flag variety for G^\vee , and let $\pi: \tilde{\mathcal{N}} \rightarrow \mathcal{N}$ be the Springer resolution. Any weight $\lambda \in \mathbf{X}$ determines a line bundle $\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)$ on $\tilde{\mathcal{N}}$, obtained by pulling back from the flag variety. For $\lambda \in \mathbf{X}^+$, let

$$\bar{\Delta}_\lambda = \pi_* \mathcal{O}_{\tilde{\mathcal{N}}}(w_0 \lambda) \langle \delta_\lambda \rangle \quad \text{and} \quad \bar{\nabla}_\lambda = \pi_* \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda) \langle -\delta_\lambda \rangle,$$

where w_0 is the longest element of the Weyl group. (Here, π_* is the derived functor $\mathrm{D}^b\mathrm{Coh}(\tilde{\mathcal{N}}) \rightarrow \mathrm{D}^b\mathrm{Coh}(\mathcal{N})$.) According to [A, Proposition 6.1], parts (2) and (3) of Definition 2.1 hold⁽²⁾ for $\mathrm{PCoh}(\mathcal{N})$. For $\lambda \in \mathbf{X}^+$, let L_λ denote the unique simple subobject of $\bar{\nabla}_\lambda$, or equivalently the unique simple quotient of $\bar{\Delta}_\lambda$.

That result also says that the $\{\bar{\nabla}_\lambda\}_{\lambda \in \mathbf{X}^+}$ form a “graded quasi-exceptional sequence” (see [A, Definition 2.4]). This implies that part (1) of Definition 2.1 also holds. Furthermore, by [B, Lemma 4], the recollement formalism is available, and hence so are the parts of Lemmas 2.3 and 2.4 involving proper standard or proper costandard objects.

Fix $\lambda \in \mathbf{X}^+$ and let

$$i: \mathrm{D}^b\mathrm{PCoh}(\mathcal{N})_{<\lambda} \longrightarrow \mathrm{D}^b\mathrm{PCoh}(\mathcal{N}) \quad \text{and} \quad \Pi: \mathrm{D}^b\mathrm{PCoh}(\mathcal{N}) \longrightarrow \mathrm{D}^b(\mathrm{PCoh}(\mathcal{N})/\mathrm{PCoh}(\mathcal{N})_{<\lambda})$$

be the inclusion and quotient functors, respectively. We will denote their adjoints as in Proposition 2.2. Let

$$\Delta_\lambda = \Pi^L \Pi(\mathrm{M}(\lambda) \otimes \mathcal{O}_{\mathcal{N}}) \langle -\delta_\lambda \rangle.$$

By [A, Lemma 5.4], $\mathrm{M}(\lambda) \otimes \mathcal{O}_{\mathcal{N}}$ lies in $\mathrm{PCoh}(\mathcal{N})_{\leq \lambda}$, so Δ_λ also lies in $\mathrm{PCoh}(\mathcal{N})_{\leq \lambda}$ and, by Lemma 2.4, it has a filtration by various $\bar{\Delta}_\lambda \langle n \rangle$. We claim that

$$\mathrm{Hom}(\Delta_\lambda, \bar{\nabla}_\mu \langle n \rangle) \cong \begin{cases} 0, & \text{if } \mu < \lambda, \text{ or if } \mu = \lambda \text{ and } n \neq 0, \\ \mathbb{k}, & \text{if } \mu = \lambda \text{ and } n = 0. \end{cases} \quad (2.8)$$

By adjunction, we have $\mathrm{Hom}(\Delta_\lambda, \bar{\nabla}_\mu \langle n \rangle) \cong \mathrm{Hom}(\mathrm{M}(\lambda) \otimes \mathcal{O}_{\mathcal{N}}, \Pi^R \Pi(\bar{\nabla}_\mu) \langle n + \delta_\lambda \rangle)$. If $\mu < \lambda$ then clearly $\Pi(\bar{\nabla}_\mu) = 0$, and if $\mu = \lambda$ then $\Pi^R \Pi(\bar{\nabla}_\lambda) \cong \bar{\nabla}_\lambda$. Due to these observations, (2.8) follows from [A, Lemma 5.5].

We will next show that, for all $\mu \in \mathbf{X}^+$, we have

$$\underline{\mathrm{Ext}}^k(\mathrm{M}(\lambda) \otimes \mathcal{O}_{\mathcal{N}}, \bar{\nabla}_\mu) = 0 \quad \text{for all } k > 0. \quad (2.9)$$

By [KLT, Theorem 2], the object $\bar{\nabla}_\mu$ is actually a coherent sheaf. Let $\Gamma(\bar{\nabla}_\mu)$ be its space of global sections. As in the proof of [A, Lemma 5.5], we have

$$\underline{\mathrm{Ext}}^k(\mathrm{M}(\lambda) \otimes \mathcal{O}_{\mathcal{N}}, \bar{\nabla}_\mu) \cong \underline{\mathrm{Ext}}^k_{G^\vee}(\mathrm{M}(\lambda), \Gamma(\bar{\nabla}_\mu)),$$

⁽²⁾ That proposition, like [B], actually asserts that $\mathrm{PCoh}(\mathcal{N})$ is quasi-hereditary, but the papers [A] and [B] use that term in a non-standard way, imposing weaker Ext-vanishing conditions on standard objects. Of course, $\mathrm{PCoh}(\mathcal{N})$ is not quasi-hereditary in the sense of this paper.

and the latter vanishes because, by [KLT, Theorem 7], as a G^\vee -representation, $\Gamma(\bar{\nabla}_\mu)$ has a good filtration.

We also claim that, for all $\mu \in \mathbf{X}^+$, we have

$$\underline{\text{Ext}}^k(\Delta_\lambda, \bar{\nabla}_\mu) = 0 \quad \text{for all } k > 0. \tag{2.10}$$

If $\mu \neq \lambda$, the claim follows from the recollement formalism. If $\mu = \lambda$, it follows from (2.9) by a calculation like that used to prove (2.8) (again invoking [A, Lemma 5.5]).

Since every simple object in $\text{PCoh}(\mathcal{N})_{\leq \lambda}$ occurs as the socle of some $\bar{\nabla}_\mu \langle n \rangle$ with $\mu \leq \lambda$, we see from (2.8) that Δ_λ has a unique simple quotient, namely L_λ .

Next, let K_μ denote the cokernel of $L_\mu \hookrightarrow \bar{\nabla}_\mu$, and consider the exact sequence

$$\dots \longrightarrow \text{Hom}(\Delta_\lambda, K_\mu \langle n \rangle) \longrightarrow \text{Ext}^1(\Delta_\lambda, L_\mu \langle n \rangle) \longrightarrow \text{Ext}^1(\Delta_\lambda, \bar{\nabla}_\mu \langle n \rangle) \longrightarrow \dots$$

If $\mu \leq \lambda$, then K_μ must lie in $\text{PCoh}(\mathcal{N})_{< \lambda}$, and the preceding paragraph implies that the first term vanishes. The last term vanishes by (2.10), so the middle term vanishes for all $\mu \leq \lambda$. Thus, Δ_λ is a projective object in $\text{PCoh}(\mathcal{N})_{\leq \lambda}$, and hence the projective cover of L_λ . Since L_λ is the unique simple quotient of $\bar{\Delta}_\lambda$, Δ_λ is also the projective cover of $\bar{\Delta}_\lambda$. We have now established part (4) of Definition 2.1. The first half of part (6) holds by (2.10).

Let \mathcal{S} be the Serre–Grothendieck duality functor on $\text{D}^b\text{Coh}(\mathcal{N})$. This is an antiautoequivalence that preserves $\text{PCoh}(\mathcal{N})$ and swaps $\bar{\Delta}_\lambda$ with $\bar{\nabla}_{-w_0\lambda}$. Define

$$\nabla_\lambda = \mathcal{S}(\Delta_{-w_0\lambda}).$$

(In fact, one can check that $\nabla_\lambda \cong \Pi^R \Pi(N(\lambda) \otimes \mathcal{O}_{\mathcal{N}}) \langle \delta_\lambda \rangle$.) It follows from the previously established properties of Δ_λ that part (5) and the second half of part (6) of Definition 2.1 hold for $\text{PCoh}(\mathcal{N})$. We have completed the proof of the fact that $\text{PCoh}(\mathcal{N})$ is a graded properly stratified category.

We saw earlier that $M(\lambda) \otimes \mathcal{O}_{\mathcal{N}}$ lies in $\text{PCoh}(\mathcal{N})_{\leq \lambda}$. By (2.9) and the criterion in [ÁHLU, Theorem 1.6 (iii)], we see that $M(\lambda) \otimes \mathcal{O}_{\mathcal{N}}$ has a standard filtration. By Serre–Grothendieck duality, $N(\lambda) \otimes \mathcal{O}_{\mathcal{N}}$ lies in $\text{PCoh}(\mathcal{N})_{\leq \lambda}$ and has a costandard filtration. It follows immediately that $T(\lambda) \otimes \mathcal{O}_{\mathcal{N}}$ lies in $\text{PCoh}(\mathcal{N})_{\leq \lambda}$ and is both tilting and cotilting. There are non-zero maps $\Delta_\lambda \rightarrow M(\lambda) \otimes \mathcal{O}_{\mathcal{N}} \langle -\delta_\lambda \rangle \rightarrow T(\lambda) \otimes \mathcal{O}_{\mathcal{N}} \langle -\delta_\lambda \rangle$ and $T(\lambda) \otimes \mathcal{O}_{\mathcal{N}} \langle \delta_\lambda \rangle \rightarrow N(\lambda) \otimes \mathcal{O}_{\mathcal{N}} \langle \delta_\lambda \rangle \rightarrow \nabla_\lambda$. We thus obtain the desired formulas for T_λ and T'_λ . \square

Note that this theorem does *not* say that $M(\lambda) \otimes \mathcal{O}_{\mathcal{N}}$ is itself a standard object. Indeed, the standard objects in $\text{PCoh}(\mathcal{N})$ do not, in general, belong to $\text{Coh}(\mathcal{N})$. The costandard objects of $\text{PCoh}(\mathcal{N})$ do happen to lie in $\text{Coh}(\mathcal{N})$, but they are not generally of the form $N(\lambda) \otimes \mathcal{O}_{\mathcal{N}}$.

COROLLARY 2.17. *Let $\Gamma \subset \mathbf{X}^+$ be a finite-order ideal. Suppose $V_1 \in \text{Rep}(G^\vee)$ has a Weyl filtration, and that $V_2 \in \text{Rep}(G^\vee)$ has a good filtration. Then the graded vector space $\underline{\text{Hom}}(\Pi(V_1 \otimes \mathcal{O}_{\mathcal{N}}), \Pi(V_2 \otimes \mathcal{O}_{\mathcal{N}}))$ is concentrated in even degrees.*

Proof. When $\Gamma = \emptyset$, it is clear that the space

$$\underline{\text{Hom}}_{\text{Coh}(\mathcal{N})}(V_1 \otimes \mathcal{O}_{\mathcal{N}}, V_2 \otimes \mathcal{O}_{\mathcal{N}}) \cong \underline{\text{Hom}}_{\text{Rep}(G^\vee)}(V_1, V_2 \otimes \mathbb{k}[\mathcal{N}])$$

is concentrated in even degrees, since the coordinate ring $\mathbb{k}[\mathcal{N}]$ is concentrated in even degrees. For general Γ , the result then follows from Corollary 2.14. \square

3. Background on parity sheaves

Let X be a complex algebraic variety or ind-variety equipped with a fixed algebraic stratification (as in [CG, Definition 3.2.23]) $X = \bigsqcup_{\gamma \in \Omega} X_\gamma$, where Ω is some indexing set. In the ind-variety case, we assume that the closure of each X_γ is an ordinary finite-dimensional variety; in particular, the closure of each stratum should contain only finitely many other strata. Let \mathbb{k} be a field. Assume that the following conditions hold:

- each stratum X_γ is simply connected;
- the cohomology groups $H^k(X_\gamma; \mathbb{k})$ vanish when k is odd.

Let $D^b(X, \mathbb{k})$, or simply $D^b(X)$, denote the bounded derived category of \mathbb{k} -sheaves on X (in the analytic topology). Let $D_\Omega^b(X, \mathbb{k})$, or simply $D_\Omega^b(X)$, denote the full triangulated subcategory consisting of complexes that are supported on the union of finitely many strata and are constructible with respect to the given stratification. For each stratum X_γ , let $j_\gamma: X_\gamma \rightarrow X$ be the inclusion map. For a locally closed subspace $Y \subset X$, we denote the constant sheaf on Y by $\underline{\mathbb{k}}_Y$.

Definition 3.1. An object $\mathcal{F} \in D_\Omega^b(X)$ is said to be **-even* (resp. *!-even*) if for each γ , the cohomology sheaves $\mathcal{H}^k(j_\gamma^* \mathcal{F})$ (resp. $\mathcal{H}^k(j_\gamma^! \mathcal{F})$) vanish for k odd. It is *even* if it is both **-even* and *!-even*.

The terms **-odd*, *!-odd*, and *odd* are defined similarly. An object is *parity* if it is a direct sum of an even object and an odd object.

The assumptions above are significantly more restrictive than those in [JMW1], but we will not require the full generality of loc. cit. The following statement classifies the indecomposable parity objects.

THEOREM 3.2. ([JMW1, Theorem 2.12]) *Let \mathcal{E} be an indecomposable parity object. Then there is a stratum X_γ such that \mathcal{E} is supported on \overline{X}_γ , and $\mathcal{E}|_{X_\gamma}$ is a shift of the constant sheaf $\underline{\mathbb{k}}_{X_\gamma}$. Moreover, if \mathcal{E}' is another indecomposable parity object with the same support as \mathcal{E} , then \mathcal{E}' is (up to shift) isomorphic to \mathcal{E} .*

Definition 3.3. The variety X is said to *have enough parity objects* if for every stratum X_γ , there is an indecomposable parity object \mathcal{E}_γ that is supported on the closure \overline{X}_γ , and such that $\mathcal{E}_\gamma|_{X_\gamma} \cong \mathbb{k}_{X_\gamma}[\dim X_\gamma]$.

For X as above, let $\text{Parity}(X) \subset D_\Omega^b(X)$ denote the full additive subcategory consisting of parity objects. The main result of this section is the following geometric analogue of Proposition 2.13, comparing a Verdier quotient of $D_\Omega^b(X)$ to a “naive” quotient (cf. (2.4)). The statement makes use of the following observation: for any closed inclusion of a union of strata $i: Y \rightarrow X$, we can identify $\text{Parity}(Y)$ with a full subcategory of $\text{Parity}(X)$ via i_* .

PROPOSITION 3.4. *Assume that X has enough parity objects, and let $Y \subset X$ be a closed union of finitely many strata. Then the open inclusion $j: X \setminus Y \rightarrow X$ induces an equivalence of categories*

$$\bar{j}^*: \text{Parity}(X) // \text{Parity}(Y) \xrightarrow{\sim} \text{Parity}(X \setminus Y). \tag{3.1}$$

Proof. Let $Q: \text{Parity}(X) \rightarrow \text{Parity}(X) // \text{Parity}(Y)$ be the quotient functor, and let $i: Y \rightarrow X$ be the inclusion map. It is clear that $j^*(\text{Parity}(Y)) = 0$, so there is unique functor \bar{j}^* such that $\bar{j}^* \circ Q \cong j^*$. Because X has enough parity objects, every indecomposable parity object occurs as a direct summand of some object in the image of \bar{j}^* . By [JMW1, Proposition 2.11], \bar{j}^* sends indecomposable objects to indecomposable objects, so it is essentially surjective.

It remains to prove that \bar{j}^* is fully faithful. We proceed by induction on the number of strata in Y . Suppose first that Y consists of a single closed stratum X_0 . Let $\mathcal{E}, \mathcal{F} \in D_\Omega^b(X)$ be parity objects, and consider the diagram

$$\text{Hom}(\mathcal{E}, \mathcal{F}) \xrightarrow{Q} \text{Hom}_{\text{Parity}(X) // \text{Parity}(Y)}(\mathcal{E}, \mathcal{F}) \xrightarrow{\bar{j}^*} \text{Hom}(j^*\mathcal{E}, j^*\mathcal{F}). \tag{3.2}$$

$\xrightarrow{j^*}$

It suffices to consider the case where \mathcal{E} and \mathcal{F} are both indecomposable. If \mathcal{E} is even and \mathcal{F} is odd, or vice versa, then both $\text{Hom}(\mathcal{E}, \mathcal{F})$ and $\text{Hom}(j^*\mathcal{E}, j^*\mathcal{F})$ vanish by [JMW1, Corollary 2.8], so \bar{j}^* is trivially an isomorphism. We henceforth assume that \mathcal{E} and \mathcal{F} are both even. (The case where they are both odd is identical.) Apply $\text{Hom}(-, \mathcal{F})$ to the distinguished triangle $j_!j^*\mathcal{E} \rightarrow \mathcal{E} \rightarrow i_*i^*\mathcal{E} \rightarrow$ to get the long exact sequence

$$\dots \rightarrow \text{Hom}(i^*\mathcal{E}, i^!\mathcal{F}) \rightarrow \text{Hom}(\mathcal{E}, \mathcal{F}) \xrightarrow{j^*} \text{Hom}(j^*\mathcal{E}, j^*\mathcal{F}) \rightarrow \text{Hom}^1(i^*\mathcal{E}, i^!\mathcal{F}) \rightarrow \dots$$

Since $i^*\mathcal{E}$ is $*$ -even and $i^!\mathcal{F}$ is $!$ -even, we see from [JMW1, Corollary 2.8] that the last term above vanishes. It follows that the map labeled j^* in (3.2) is surjective, and its kernel can be identified with the space

$$K = \{f: \mathcal{E} \rightarrow \mathcal{F} \mid f \text{ factors as } \mathcal{E} \rightarrow i_*i^*\mathcal{E} \rightarrow i_*i^!\mathcal{F} \rightarrow \mathcal{F}\}.$$

We deduce that \bar{j}^* is surjective as well. Now, the kernel of Q in (3.2) is the space

$$K' = \{f: \mathcal{E} \rightarrow \mathcal{F} \mid f \text{ factors through an object of } \text{Parity}(Y)\}.$$

We already know that $K' \subset K$. But since Y consists of a single closed stratum, the object $i^*\mathcal{E}$ is actually even (not just $*$ -even), and likewise for $i^!\mathcal{F}$. So $K=K'$, and we conclude that \bar{j}^* in (3.2) is an isomorphism.

For the general case, let S be an open stratum in Y , and let $B=Y \setminus S$ and $X'=X \setminus B$. Then S is closed in X' , and by induction, we have natural equivalences

$$\begin{aligned} \text{Parity}(X') // \text{Parity}(S) &\xrightarrow{\sim} \text{Parity}(X' \setminus S) \cong \text{Parity}(X \setminus Y), \\ \text{Parity}(X) // \text{Parity}(B) &\xrightarrow{\sim} \text{Parity}(X \setminus B) \cong \text{Parity}(X'), \\ \text{Parity}(Y) // \text{Parity}(B) &\xrightarrow{\sim} \text{Parity}(Y \setminus B) \cong \text{Parity}(S). \end{aligned}$$

The desired equivalence (3.1) follows from these and the general observation that

$$\text{Parity}(X) // \text{Parity}(Y) \cong (\text{Parity}(X) // \text{Parity}(B)) // (\text{Parity}(Y) // \text{Parity}(B)). \quad \square$$

4. Parity sheaves on Kac–Moody flag varieties

In this section, we study Ext-groups of parity sheaves on flag varieties for Kac–Moody groups. The result below will be applied elsewhere in the paper only to affine Grassmannians, but it is no more effort to prove it in this generality. As in the previous section, \mathbb{k} denotes an arbitrary field.

THEOREM 4.1. *Let X be a generalized flag variety for a Kac–Moody group, equipped with the Bruhat stratification, and let \mathcal{E} and \mathcal{F} be two parity objects with respect to that stratification. The natural map*

$$\text{Hom}_{\mathbf{D}^b(X)}(\mathcal{E}, \mathcal{F}) \longrightarrow \text{Hom}_{H^\bullet(X; \mathbb{k})}(H^\bullet(\mathcal{E}), H^\bullet(\mathcal{F}))$$

is an isomorphism.

For finite flag varieties, this result (with some minor restrictions on $\text{char } \mathbb{k}$) is due to Soergel [S1], [S2]. In [G1], Ginzburg proved a very similar result for simple perverse \mathbb{C} -sheaves on smooth projective varieties equipped with a suitable \mathbb{C}^\times -action. The proof below follows the outline of Ginzburg’s argument quite closely. One exception occurs at a step (see [G1, Proposition 3.2]), where Ginzburg invokes the theory of mixed Hodge modules: here, we substitute an argument of Fiebig–Williamson that relies on the geometry of Schubert varieties.

Remark 4.2. When $\mathbb{k}=\mathbb{C}$, Ginzburg had already observed in a remark at the end of [G1] that his result could be generalized to the Kac–Moody case. Thus, in that case, this section can be regarded as an exposition of Ginzburg’s remark.

Remark 4.3. Although only field coefficients are used in the present paper, the arguments in this section go through unchanged if we instead take \mathbb{k} to be a complete discrete valuation ring, so Theorem 4.1 holds in that setting.

We begin with some notation. Let \mathcal{G} be a Kac–Moody group (over \mathbb{C}), with maximal torus $\mathcal{T}\subset\mathcal{G}$ and standard Borel subgroup $\mathcal{B}\subset\mathcal{G}$. Let $\mathcal{B}^-\subset\mathcal{G}$ denote the opposite Borel subgroup to \mathcal{B} (with respect to \mathcal{T}). Let $\mathcal{P}\subset\mathcal{G}$ be a standard parabolic subgroup of finite type (in the sense of [K, §1.2.2]), with Levi factor $L_{\mathcal{P}}$.

For the remainder of the section, X will denote the generalized flag variety $X=\mathcal{G}/\mathcal{P}$. Let W (resp. $W_{\mathcal{P}}$) be the Weyl group of \mathcal{G} (resp. $L_{\mathcal{P}}$), and let $W^{\mathcal{P}}$ be the set of minimal-length representatives of the set of cosets $W/W_{\mathcal{P}}$. The length of an element $w\in W^{\mathcal{P}}$ will be denoted by $\ell(w)$. It is well known that the \mathcal{T} -fixed points and the \mathcal{B} -orbits on X are both naturally in bijection with $W^{\mathcal{P}}$. For $w\in W^{\mathcal{P}}$, let $e_w\in X$ be the corresponding \mathcal{T} -fixed point, and let $X_w=\mathcal{B}\cdot e_w$ be the corresponding Bruhat cell. We will also need the “opposite Bruhat cell” $X_w^-=\mathcal{B}^-\cdot e_w$. Recall that $X_w\cap X_w^-=\{e_w\}$, and that the intersection is transverse. Moreover, X_w is isomorphic to an affine space of dimension $\ell(w)$. In general, X_w^- may have infinite dimension, but it has codimension $\ell(w)$ (see [K, Lemma 7.3.10]).

Recall that $D^b(X)$ is the bounded derived category of all \mathbb{k} -sheaves on X . Let $D_{(\mathcal{B})}^b(X)$ be the full subcategory of $D^b(X)$ consisting of complexes \mathcal{F} such that

- (1) \mathcal{F} is constructible with respect to the stratification by \mathcal{B} -orbits;
- (2) the support of \mathcal{F} is contained in the union of finitely many \overline{X}_w .

Let $\text{Parity}_{(\mathcal{B})}(X)\subset D_{(\mathcal{B})}^b(X)$ be the additive category of parity objects. Let

$$j_w: X_w \longrightarrow X \quad \text{and} \quad \bar{j}_w: X_w^- \longrightarrow X$$

denote the inclusion maps. For any closed subset $Z\subset X$, we let $i_Z: Z\rightarrow X$ be the inclusion map, and for an object $\mathcal{E}\in D_{(\mathcal{B})}^b(X)$, we put

$$\mathcal{E}_Z := i_{Z*}i_Z^*\mathcal{E} \quad \text{and} \quad \mathcal{E}^Z := i_{Z*}i_Z^!\mathcal{E}.$$

For simplicity, the inclusions of closures of \mathcal{B} - and \mathcal{B}^- -orbits are denoted

$$i_w: \overline{X}_w \longrightarrow X \quad \text{and} \quad \bar{i}_w: \overline{X}_w^- \longrightarrow X,$$

rather than $i_{\overline{X}_w}$ and $i_{\overline{X}_w^-}$. Recall that \overline{X}_w is known as a *Schubert variety*, and \overline{X}_w^- as a *Birkhoff variety*.

Certain complexes $\mathcal{F} \in D^b(X)$ that do *not* belong to $D^b_{(\mathcal{B})}(X)$ will also appear in our arguments, including objects whose support may be infinite-dimensional and hence not locally compact. Some caution is required when working with such objects, especially when applying functors of the form $f_!$ and $f^!$. In this section, whenever infinite-dimensional supports are involved, the functors $f_!$ and $f^!$ will only be used when f is an inclusion of a locally closed subset. See Appendix A for the definition and basic properties of these functors on non-locally compact spaces.

LEMMA 4.4. *Let $Z \subset X$ be a finite union of Schubert varieties, and let $X_w \subset Z$ be a Bruhat cell that is open in Z . If \mathcal{E} is $*$ -even, then for each k , there is a natural short exact sequence*

$$0 \longrightarrow H^k(j_{w!}j_w^!\mathcal{E}_Z) \longrightarrow H^k(\mathcal{E}_Z) \longrightarrow H^k(\mathcal{E}_{Z \setminus X_w}) \longrightarrow 0.$$

Proof. The constant map $a: Z \rightarrow \{\text{pt}\}$ is a proper, even morphism in the sense of [JMW1, Definition 2.33] so, by [JMW1, Proposition 2.34], if $\mathcal{E}' \in D^b_{(\mathcal{B})}(Z)$ is $*$ -even, then $H^k(\mathcal{E}')$ vanishes when k is odd. All three terms in the distinguished triangle $j_{w!}j_w^!\mathcal{E}_Z \rightarrow \mathcal{E}_Z \rightarrow \mathcal{E}_{Z \setminus X_w} \rightarrow$ are $*$ -even, so in the long exact sequence in cohomology, all odd terms vanish, and the even terms give short exact sequences as above. \square

LEMMA 4.5. *Let $Z \subset X$ be a finite union of Schubert varieties, and let $X_w \subset Z$ be a Bruhat cell that is open in Z . If \mathcal{E} is a parity object, then the natural map $H^k(\mathcal{E}_Z) \rightarrow H^k(j_w^*\mathcal{E}_Z)$ is surjective.*

Proof. Let $k_w: \{e_w\} \rightarrow X$ be the inclusion map. Since k_w factors through j_w , and j_w factors through i_Z , there is a natural sequence of maps

$$\mathcal{E} \longrightarrow i_{Z*}i_Z^*\mathcal{E} \longrightarrow j_{w*}j_w^*\mathcal{E} \longrightarrow k_{w*}k_w^*\mathcal{E}.$$

Taking cohomology, we obtain a natural sequence of maps

$$H^k(\mathcal{E}) \longrightarrow H^k(\mathcal{E}_Z) \longrightarrow H^k(j_w^*\mathcal{E}) \longrightarrow H^k(k_w^*\mathcal{E}). \quad (4.1)$$

We claim first that the composition $H^k(\mathcal{E}) \rightarrow H^k(k_w^*\mathcal{E})$ is surjective. This is essentially the content of [FW, Theorem 5.7 (2)]. That result is stated in an abstract axiomatic setting, but [FW, Proposition 7.1] tells us that it applies to Schubert varieties. Another concern is that [FW, Theorem 5.7 (2)] deals with \mathcal{T} -equivariant rather than ordinary cohomology. The reader may check that the proof goes through with ordinary cohomology as well. Alternatively, note that both $H^k_{\mathcal{T}}(\mathcal{E})$ and $H^k_{\mathcal{T}}(k_w^*\mathcal{E})$ are free modules over the equivariant cohomology ring of a point $H^k_{\mathcal{T}}(\text{pt})$ by [FW, Proposition 5.6]. In that situation, the

ordinary cohomology is obtained from the equivariant cohomology by applying the right-exact functor $\mathbb{k} \otimes_{H_{\mathcal{T}}^*(\text{pt})} -$. In particular, the surjectivity of $H_{\mathcal{T}}^*(\mathcal{E}) \rightarrow H_{\mathcal{T}}^*(k_w^* \mathcal{E})$ implies the surjectivity of the corresponding map in ordinary cohomology.

Next, we claim that the third map in the sequence (4.1) is an isomorphism. Since $j_w^* \mathcal{E}$ lies in the triangulated subcategory of $D^b(X_w)$ generated by the constant sheaf \mathbb{k}_{X_w} , it suffices to check that $H^k(\mathbb{k}_{X_w}) \rightarrow H^k(\mathbb{k}_{\{e_w\}})$ is an isomorphism. That last claim is obvious.

From these observations it follows that $H^k(\mathcal{E}_Z) \rightarrow H^k(j_w^* \mathcal{E}) \cong H^k(j_w^* \mathcal{E}_Z)$ is surjective as well. \square

LEMMA 4.6. *There is a canonical isomorphism $\bar{j}_w^! \mathbb{k}_X \cong \mathbb{k}_{X_w^-}[-2\ell(w)]$.*

Proof. Let $\mathcal{U} \subset \mathcal{B}$ be the pro-unipotent radical of the Borel subgroup. For $w \in W^{\mathcal{P}}$, let $\mathcal{U}_w \subset \mathcal{U}$ be the subgroup generated by the root subgroups \mathcal{U}_α where α is a positive root but $w^{-1}\alpha$ is negative. Then \mathcal{U}_w is a finite-dimensional unipotent algebraic group. Let $O_w = \mathcal{U}_w \cdot X_w^- \subset X$. According to [K, Lemma 7.3.10], the multiplication map

$$\mathcal{U}_w \times X_w^- \longrightarrow O_w$$

is an isomorphism of ind-varieties. Let $i: X_w^- \rightarrow \mathcal{U}_w \times X_w^-$ be the inclusion map $x \mapsto (e, x)$, where $e \in \mathcal{U}_w$ is the identity element. Since O_w is open in X , $\bar{j}_w^! \mathbb{k}_X$ is naturally isomorphic to $i^! \mathbb{k}_{\mathcal{U}_w \times X_w^-}$.

To compute $i^! \mathbb{k}_{\mathcal{U}_w \times X_w^-}$, let us consider the Cartesian square

$$\begin{array}{ccc} X_w^- & \xrightarrow{i} & \mathcal{U}_w \times X_w^- \\ a \downarrow & & \downarrow q \\ \{e\} & \xrightarrow{i_0} & \mathcal{U}_w, \end{array}$$

where i_0 is the inclusion map, and a and q are the obvious projection maps. The space X_w^- is contractible, and each cohomology sheaf $\mathcal{H}^k(i^! \mathbb{k}_{\mathcal{U}_w \times X_w^-})$ is \mathcal{B}^- -equivariant, so in fact each such cohomology sheaf must be a constant sheaf. It follows from, say, [KS, Corollary 2.7.7] that the adjunction map

$$a^* a_* i^! \mathbb{k}_{\mathcal{U}_w \times X_w^-} \longrightarrow i^! \mathbb{k}_{\mathcal{U}_w \times X_w^-}$$

is an isomorphism. Thus, to finish the proof, we must show that

$$a_* i^! \mathbb{k}_{\mathcal{U}_w \times X_w^-} \cong \mathbb{k}_{\{e\}}[-2\ell(w)].$$

Using Lemma A.4 and [KS, Corollary 2.7.7] again, we find that

$$a_* i^! \mathbb{k}_{\mathcal{U}_w \times X_w^-} \cong i_0^! q_* \mathbb{k}_{\mathcal{U}_w \times X_w^-} \cong i_0^! q_* q^* \mathbb{k}_{\mathcal{U}_w} \cong i_0^! \mathbb{k}_{\mathcal{U}_w}.$$

Since \mathcal{U}_w is isomorphic as a variety to an affine space $\mathbb{A}^{\ell(w)}$, we have a well-known canonical isomorphism $i_0^! \mathbb{k}_{\mathcal{U}_w} \cong \mathbb{k}_{\{e\}}[-2\ell(w)]$, and the result follows. \square

Now, let $Y \subset X$ be a finite union of Birkhoff varieties, and let

$$\Lambda_Y = i_Y^! \mathbb{k}_X.$$

LEMMA 4.7. *Let $d_Y = \min\{\ell(w) \mid X_w^- \subset Y\}$. Then the cohomology sheaves $\mathcal{H}^i(\Lambda_Y)$ vanish for $i < 2d_Y$.*

Proof. Let $X_w^- \subset Y$ be such that $\ell(w) = d_Y$. Then X_w^- is necessarily open in Y . Let $A_1 = X_w^-$, and then inductively define A_k , for $k \geq 2$, by setting A_k to be some opposite Bruhat cell X_y^- that is open in $Y \setminus (A_1 \cup \dots \cup A_{k-1})$. We also let $B_k = A_1 \cup \dots \cup A_k$. The B_k 's form an increasing sequence of open subsets of Y whose union is all of Y . To show that $\mathcal{H}^i(\Lambda_Y)$ vanishes for $i < 2d_Y$, it suffices to show that for all k ,

$$\mathcal{H}^i(\Lambda_Y|_{B_k}) = 0 \quad \text{for } i < 2d_Y. \tag{4.2}$$

We proceed by induction on k . For $k=1$, we have $\Lambda_Y|_{B_1} \cong j_w^! \mathbb{k}_X$, so (4.2) follows from Lemma 4.6. For $k > 1$, let $a: A_k \rightarrow B_k$ and $b: B_{k-1} \rightarrow B_k$ be the inclusion maps. We have a distinguished triangle

$$a_* a^! (\Lambda_Y|_{B_k}) \longrightarrow \Lambda_Y|_{B_k} \longrightarrow b_* b^* (\Lambda|_{B_k}) \longrightarrow.$$

If $A_k = X_y^-$, the first term can be identified with $a_* j_y^! \mathbb{k}_X$. Since $\ell(y) \geq d_Y$, it follows from Lemma 4.6 again that $\mathcal{H}^i(a_* a^! (\Lambda_Y|_{B_k})) = 0$ for $i < 2d_Y$. On the other hand, $b^* (\Lambda|_{B_k}) \cong \Lambda|_{B_{k-1}}$, so $\mathcal{H}^i(b_* b^* (\Lambda_Y|_{B_k}))$ vanishes for $i < 2d_Y$ by induction. Thus, (4.2) holds, as desired. \square

LEMMA 4.8. *For any $w \in W^P$, there is a canonical morphism $q_w: \mathbb{k}_{\bar{X}_w^-} \rightarrow \Lambda_{\bar{X}_w^-}[2\ell(w)]$ whose restriction to $X_w^- \subset \bar{X}_w^-$ is an isomorphism.*

Proof. Let $Y = \bar{X}_w^- \setminus X_w^-$, and let $y: Y \rightarrow \bar{X}_w^-$ and $u: X_w^- \rightarrow \bar{X}_w^-$ be the inclusion maps. The space Y is a finite union of Birkhoff varieties, and, moreover, the integer d_Y defined in Lemma 4.7 satisfies $d_Y > \ell(w)$. Consider the distinguished triangle

$$y_! \Lambda_Y \longrightarrow \Lambda_{\bar{X}_w^-} \longrightarrow u_* u^* \Lambda_{\bar{X}_w^-} \longrightarrow.$$

Note that $u^* \Lambda_{\bar{X}_w^-} \cong j_w^! \mathbb{k}_X$. It follows from Lemma 4.6 by adjunction that we have a canonical morphism

$$\mathbb{k}_{\bar{X}_w^-} \longrightarrow u_* u^* \Lambda_{\bar{X}_w^-}[2\ell(w)]. \tag{4.3}$$

By construction, this map restricts to an isomorphism over X_w^- . Next, we deduce from Lemma 4.7 that

$$\text{Hom}(\mathbb{k}_{\bar{X}_w^-}, y_! \Lambda_Y[2\ell(w)]) = \text{Hom}(\mathbb{k}_{\bar{X}_w^-}, y_! \Lambda_Y[2\ell(w) + 1]) = 0.$$

These facts imply that the map in (4.3) factors in a unique way through $\Lambda_{\bar{X}_w^-}$. The resulting map $\mathbb{k}_{\bar{X}_w^-} \rightarrow \Lambda_{\bar{X}_w^-}[2\ell(w)]$ is the one we seek. \square

Remark 4.9. The map $q_w: \mathbb{k}_{\bar{X}_w^-} \rightarrow \Lambda_{\bar{X}_w^-}[2\ell(w)]$ plays a role similar to that of the fundamental class in Borel–Moore homology.

Define $c_w: \mathbb{k}_X \rightarrow \mathbb{k}_X[2\ell(w)]$ to be the composition

$$\mathbb{k}_X \longrightarrow \bar{i}_w^* \mathbb{k}_{\bar{X}_w^-} \xrightarrow{\bar{i}_w^* q_w} \bar{i}_w^* \Lambda_{\bar{X}_w^-}[2\ell(w)] \longrightarrow \mathbb{k}_X[2\ell(w)]. \quad (4.4)$$

We now study the map $\bar{c}_w: \mathrm{Hom}^k(\mathbb{k}_X, \mathcal{E}_Z) \rightarrow \mathrm{Hom}^{k+2\ell(w)}(\mathbb{k}_X, \mathcal{E}_Z)$ induced by c_w .

LEMMA 4.10. *Let Z be a finite union of Schubert varieties, let $X_w \subset Z$ be a Bruhat cell that is open in Z , and let $\mathcal{E} \in \mathrm{Parity}_{(\mathcal{B})}(X)$. Then \bar{c}_w induces an isomorphism $H^k(j_w^* \mathcal{E}_Z) \xrightarrow{\sim} H^{k+2\ell(w)}(j_w! j_w^! \mathcal{E}_Z)$ that makes the following diagram commute:*

$$\begin{array}{ccc} H^k(\mathcal{E}_Z) & \longrightarrow & H^k(j_w^* \mathcal{E}_Z) \\ \downarrow \bar{c}_w & & \wr \downarrow \bar{c}_w \\ 0 \leftarrow H^{k+2\ell(w)}(\mathcal{E}_Z \setminus X_w) & \leftarrow & H^{k+2\ell(w)}(\mathcal{E}_Z) \leftarrow H^{k+2\ell(w)}(j_w! j_w^! \mathcal{E}_Z) \leftarrow 0. \end{array}$$

Proof. Let $Y = Z \setminus X_w$, and let $y: Y \rightarrow X$ be the inclusion map. Since c_w factors through an object supported on \bar{X}_w^- , and since $\bar{X}_w^- \cap Y = \emptyset$, we see that the composition $H^k(y^! \mathcal{E}_Z) \rightarrow H^k(\mathcal{E}_Z) \xrightarrow{\bar{c}_w} H^{k+2\ell(w)}(\mathcal{E}_Z)$ vanishes. In other words, \bar{c}_w must factor through $H^k(\mathcal{E}_Z) \rightarrow H^k(j_w^* \mathcal{E}_Z)$. The resulting map $H^k(j_w^* \mathcal{E}_Z) \rightarrow H^{k+2\ell(w)}(\mathcal{E}_Z)$ must factor through $H^{k+2\ell(w)}(j_w! j_w^! \mathcal{E}_Z) \rightarrow H^{k+2\ell(w)}(\mathcal{E}_Z)$ for the same reason, so we at least have a commutative diagram as shown above.

It remains to show that the right vertical map is an isomorphism. Let $p: \{e_w\} \rightarrow X_w$ be the inclusion map. Applying i_w^* to (4.4) yields the composition

$$\mathbb{k}_{\bar{X}_w} \longrightarrow j_w^* p_* \mathbb{k}_{e_w} \longrightarrow \mathbb{k}_{\bar{X}_w}[2\ell(w)],$$

where the second map comes from adjunction and the identification $\mathbb{k}_{e_w} \cong p^! \mathbb{k}_{X_w}[2\ell(w)]$. Note that $j_w! p_* \mathbb{k}_{e_w} \cong j_w^* p_* \mathbb{k}_{e_w}$. Thus, \bar{c}_w is given by the following composition:

$$\begin{aligned} H^k(j_w^* \mathcal{E}_Z) &= \mathrm{Hom}^k(\mathbb{k}_{\bar{X}_w}, j_w^* j_w^* \mathcal{E}_Z) \longrightarrow \mathrm{Hom}^k(j_w^* p_* \mathbb{k}_{e_w}[-2\ell(w)], j_w^* j_w^* \mathcal{E}_Z) \\ &\cong \mathrm{Hom}^{k+2\ell(w)}(p_* \mathbb{k}_{e_w}, j_w^! \mathcal{E}_Z) \cong \mathrm{Hom}^{k+2\ell(w)}(j_w! p_* \mathbb{k}_{e_w}, j_w! j_w^! \mathcal{E}_Z) \\ &\longrightarrow \mathrm{Hom}^{k+2\ell(w)}(\mathbb{k}_{\bar{X}_w}, j_w! j_w^! \mathcal{E}_Z) = H^{k+2\ell(w)}(j_w! j_w^! \mathcal{E}_Z). \end{aligned}$$

We claim that each step of this is an isomorphism. If it happens that $j_w^* \mathcal{E}_Z \cong \mathbb{k}_{X_w}[m]$, this can be checked explicitly. But because \mathcal{E}_Z is $*$ -parity, $j_w^* \mathcal{E}_Z$ is always isomorphic to a direct sum of various $\mathbb{k}_{X_w}[m]$. \square

A very similar argument establishes the following result, whose proof we omit.

LEMMA 4.11. *Let Z be a finite union of Schubert varieties, let $X_w \subset Z$ be a Bruhat cell that is open in Z , and let $\mathcal{F} \in \text{Parity}_{(\mathcal{B})}(X)$. Then \bar{c}_w induces an isomorphism $H^k(j_w^* \mathcal{F}_Z) \xrightarrow{\sim} H^{k+2\ell(w)}(j_{w!} j_w^! \mathcal{F}^Z)$ that makes the following diagram commute:*

$$\begin{array}{ccccccc}
 0 & \longrightarrow & H^k(\mathcal{F}^Z \setminus X_w) & \hookrightarrow & H^k(\mathcal{F}^Z) & \twoheadrightarrow & H^k(j_w^* \mathcal{F}^Z) \longrightarrow 0 \\
 & & & & \downarrow \bar{c}_w & & \downarrow \wr \bar{c}_w \\
 & & & & H^{k+2\ell(w)}(\mathcal{F}^Z) & \xleftrightarrow{\quad} & H^{k+2\ell(w)}(j_{w!} j_w^! \mathcal{F}^Z).
 \end{array}$$

With the following proposition, we complete the proof of Theorem 4.1.

PROPOSITION 4.12. *Let $Z \subset X$ be a finite union of Schubert varieties, and let $\mathcal{E}, \mathcal{F} \in \text{Parity}_{(\mathcal{B})}(X)$. The natural map*

$$\text{Hom}_{D_{(\mathcal{B})}^b(X)}(\mathcal{E}_Z, \mathcal{F}^Z) \longrightarrow \text{Hom}_{H^*(X; \mathbb{k})}(\mathcal{H}^*(\mathcal{E}_Z), \mathcal{H}^*(\mathcal{F}^Z))$$

is an isomorphism.

Sketch of proof. This is proved by induction on the number of Bruhat cells in Z , via a diagram chase relying on formal consequences of the commutative diagrams in Lemmas 4.10 and 4.11. The argument is essentially identical to the proof of [G1, Proposition 3.10]; see also [G1, equation (3.8a–b)]. We omit further details. \square

5. Ext-groups of parity sheaves on the affine Grassmannian

In this section, \mathbb{k} will denote an algebraically closed field whose characteristic is good for G^\vee . We also assume that G^\vee satisfies (1.3). Recall that the $G_{\mathbf{O}}$ -orbits are parameterized by \mathbf{X}^+ . If $\Gamma \subset \mathbf{X}^+$ is a finite-order ideal, we can form the closed subset $\mathcal{G}r_\Gamma = \bigcup_{\gamma \in \Gamma} \mathcal{G}r_\gamma$. Let

$$j_\Gamma: U_\Gamma = \mathcal{G}r \setminus \mathcal{G}r_\Gamma \hookrightarrow \mathcal{G}r \tag{5.1}$$

be the complementary open inclusion. For the remainder of the paper, all constructible complexes on $\mathcal{G}r$ or on any subset of $\mathcal{G}r$ should be understood to be constructible with respect to the $G_{\mathbf{O}}$ -orbits. In particular, $\text{Parity}_{(G_{\mathbf{O}})}(\mathcal{G}r)$ will denote the category of $G_{\mathbf{O}}$ -constructible parity objects.

Our goal is compute certain Ext-groups in $D_{(G_{\mathbf{O}})}^b(\mathcal{G}r)$ or in some $D_{(G_{\mathbf{O}})}^b(U_\Gamma)$ in terms of $\text{PCoh}(\mathcal{N})$. The main result, a modular generalization of [G2, Proposition 1.10.4], depends on a result of Yun–Zhu [YZ] describing the cohomology of $\mathcal{G}r$. We begin by recalling that result.

Let $\mathfrak{e} \in \text{Lie}(G^\vee)$ be the principal nilpotent element described in [YZ, Proposition 5.6]. Let $B^\vee \subset G^\vee$ be the unique Borel subgroup such that $\mathfrak{e} \in \text{Lie}(B^\vee)$, and let $U^\vee \subset B^\vee$ be its

unipotent radical. Recall that $G^\vee \times \mathbb{G}_m$ acts on \mathcal{N} by $(g, z) \cdot x = z^{-2} \text{Ad}(g)(x)$. Below, for any subgroup $H \subset G^\vee \times \mathbb{G}_m$, we write H_e for the stabilizer of e in H .

Let $\mathcal{G}r^\circ$ be the identity component of $\mathcal{G}r$, and let $\text{Dist}(U_e^\vee)$ denote the algebra of distributions on U_e^\vee with support at the identity (see [J1, §I.7.1 and §I.7.7]).

THEOREM 5.1. (Yun–Zhu [YZ]) *There is a natural isomorphism*

$$H^\bullet(\mathcal{G}r^\circ; \mathbb{k}) \cong \text{Dist}(U_e^\vee). \tag{5.2}$$

The “naturality” in this proposition refers to a certain compatibility with \mathcal{S} . To be more precise, given $M \in \text{Perv}_{G_O}(\mathcal{G}r, \mathbb{k})$, the isomorphism (5.2) endows $H^\bullet(M)$ with the structure of a $\text{Dist}(U_e^\vee)$ -module. On the other hand, if we forget the grading on $H^\bullet(M)$, we obtain the underlying vector space of $\mathcal{S}^{-1}(M) \in \text{Rep}(G^\vee)$. Thus, we can regard $H^\bullet(M)$ as a representation of $U_e^\vee \subset G^\vee$, and hence as a $\text{Dist}(U_e^\vee)$ -module. In fact, these two $\text{Dist}(U_e^\vee)$ -module structures on $H^\bullet(M)$ coincide.

Remark 5.2. Theorem 5.1 is stated in [YZ] only when G is quasi-simple and simply connected (in which case $\mathcal{G}r = \mathcal{G}r^\circ$), but it is easily extended to general G by routine arguments. One caveat is that the element e may depend on a choice in general (it is uniquely determined in the quasi-simple case). Once e is fixed, however, the isomorphism (5.2) is still natural in the sense described above.

Let \mathbb{G}_m act on G^\vee by conjugation via the cocharacter $2\rho: \mathbb{G}_m \rightarrow T^\vee$, where 2ρ is the sum of the positive roots for G . The resulting semidirect product will be denoted $\mathbb{G}_m \ltimes_{2\rho} G^\vee$. This action preserves the subgroups G_e^\vee , B_e^\vee , and U_e^\vee , so groups such as $\mathbb{G}_m \ltimes_{2\rho} U_e^\vee$ also make sense.

LEMMA 5.3. *There are isomorphisms*

$$(G^\vee \times \mathbb{G}_m)_e \cong \mathbb{G}_m \ltimes_{2\rho} B_e^\vee \cong \mathbb{G}_m \ltimes_{2\rho} U_e^\vee \times Z(G^\vee),$$

where $Z(G^\vee)$ denotes the center of G^\vee .

Proof. Let $\phi: G^\vee \times \mathbb{G}_m \rightarrow \mathbb{G}_m \ltimes_{2\rho} G^\vee$ be the map $\phi(g, z) = (z, 2\rho(z^{-1})g)$. This map is an isomorphism, and it is easily checked that it takes $(G^\vee \times \mathbb{G}_m)_e$ to $\mathbb{G}_m \ltimes_{2\rho} G_e^\vee$. By [Sp, Theorem 5.9 (b)], $G_e^\vee = B_e^\vee \cong U_e^\vee \times Z(G^\vee)$. \square

LEMMA 5.4. *Let $M_1, M_2 \in \text{Perv}_{G_O}(\mathcal{G}r^\circ, \mathbb{k})$. Then there is a natural isomorphism*

$$\underline{\text{Hom}}_{\text{Coh}(\mathcal{N})}(\mathcal{S}^{-1}(M_1) \otimes \mathcal{O}_{\mathcal{N}}, \mathcal{S}^{-1}(M_2) \otimes \mathcal{O}_{\mathcal{N}}) \xrightarrow{\sim} \underline{\text{Hom}}_{\text{Dist}(U_e^\vee)}(H^\bullet(M_1), H^\bullet(M_2)).$$

In the course of the proof of this lemma, we will encounter an analogous statement (see (5.4) below) that is entirely in terms of G^\vee -modules, and that does not involve the geometric Satake equivalence.

Proof. The assumptions on \mathbb{k} imply that \mathcal{N} is a normal variety (see, e.g., [J2, Proposition 8.5]). Let $\mathcal{N}_{\text{reg}} \subset \mathcal{N}$ be the subvariety consisting of regular nilpotent elements, and let \mathcal{O}_{reg} denote its structure sheaf. Let $h: \mathcal{N}_{\text{reg}} \hookrightarrow \mathcal{N}$ be the inclusion map, and let $V \in \text{Rep}(G^\vee)$. Since the complement of \mathcal{N}_{reg} has codimension at least 2, the restriction map $h^*: \Gamma(V \otimes \mathcal{O}_{\mathcal{N}}) \rightarrow \Gamma(V \otimes \mathcal{O}_{\text{reg}})$ is an isomorphism of $(G^\vee \times \mathbb{G}_m)$ -modules. Equivalently, the adjunction map $V \otimes \mathcal{O}_{\mathcal{N}} \rightarrow R^0 h_* h^*(V \otimes \mathcal{O}_{\mathcal{N}})$ is an isomorphism. It follows that h^* is fully faithful on the category of free sheaves in $\text{Coh}(\mathcal{N})$. In particular, if we set

$$V_1 = \mathcal{S}^{-1}(M_1) = H^\bullet(M_1) \quad \text{and} \quad V_2 = \mathcal{S}^{-1}(M_2) = H^\bullet(M_2), \quad (5.3)$$

then h^* gives us a natural isomorphism

$$\underline{\text{Hom}}_{\text{Coh}(\mathcal{N})}(V_1 \otimes \mathcal{O}_{\mathcal{N}}, V_2 \otimes \mathcal{O}_{\mathcal{N}}) \xrightarrow{\sim} \underline{\text{Hom}}_{\text{Coh}^{G^\vee \times \mathbb{G}_m}(\mathcal{N}_{\text{reg}})}(V_1 \otimes \mathcal{O}_{\text{reg}}, V_2 \otimes \mathcal{O}_{\text{reg}}).$$

Now, \mathcal{N}_{reg} is the orbit of the point \mathfrak{e} under G^\vee or $G^\vee \times \mathbb{G}_m$. Due to condition (1.3), the natural map $G^\vee/G_{\mathfrak{e}}^\vee \rightarrow \mathcal{N}_{\text{reg}}$ is an isomorphism of varieties. (See [J2, §2.9], for an example.) Factoring this map as $G^\vee/G_{\mathfrak{e}}^\vee \rightarrow (G^\vee \times \mathbb{G}_m)/(G^\vee \times \mathbb{G}_m)_{\mathfrak{e}} \rightarrow \mathcal{N}_{\text{reg}}$, we see that $(G^\vee \times \mathbb{G}_m)/(G^\vee \times \mathbb{G}_m)_{\mathfrak{e}} \rightarrow \mathcal{N}_{\text{reg}}$ is also an isomorphism. Therefore, restriction to \mathfrak{e} induces an equivalence of categories $\text{Coh}^{G^\vee \times \mathbb{G}_m}(\mathcal{N}_{\text{reg}}) \xrightarrow{\sim} \text{Rep}((G^\vee \times \mathbb{G}_m)_{\mathfrak{e}})$. In view of Lemma 5.3, we have a natural isomorphism

$$\underline{\text{Hom}}_{\text{Coh}(\mathcal{N})}(V_1 \otimes \mathcal{O}_{\mathcal{N}}, V_2 \otimes \mathcal{O}_{\mathcal{N}}) \xrightarrow{\sim} \underline{\text{Hom}}_{\mathbb{G}_m \times_{2_{\mathfrak{e}}} U_{\mathfrak{e}}^\vee \times Z(G^\vee)}(V_1, V_2). \quad (5.4)$$

Since the M_i 's are supported on $\mathcal{G}r^\circ$, $Z(G^\vee)$ acts trivially on the V_i 's, so we may simply omit mentioning it and consider $\text{Hom}_{\mathbb{G}_m \times_{2_{\mathfrak{e}}} U_{\mathfrak{e}}^\vee}(V_1, V_2)$.

Now, the category of finite-dimensional $U_{\mathfrak{e}}^\vee$ -representations can be identified with a full subcategory of the finite-dimensional $\text{Dist}(U_{\mathfrak{e}}^\vee)$ -modules [J1, Lemma I.7.16]. Similarly, the category of finite-dimensional $(\mathbb{G}_m \times_{2_{\mathfrak{e}}} U_{\mathfrak{e}}^\vee)$ -modules can be identified with a full subcategory of *graded* finite-dimensional $\text{Dist}(U_{\mathfrak{e}}^\vee)$ -modules, where $\text{Dist}(U_{\mathfrak{e}}^\vee)$ itself is graded by the action of \mathbb{G}_m via the cocharacter $2_{\mathfrak{e}}$. This is precisely the grading appearing in (5.2), according to the remarks following [YZ, Theorem 1.1]. On the other hand, the grading on the right-hand side of each equation in (5.3) is also given by $2_{\mathfrak{e}}$, as seen in [MV2, Theorem 3.6]. Thus, we have

$$\underline{\text{Hom}}_{\mathbb{G}_m \times_{2_{\mathfrak{e}}} U_{\mathfrak{e}}^\vee}(V_1, V_2) \cong \underline{\text{Hom}}_{\text{Dist}(U_{\mathfrak{e}}^\vee)}(H^\bullet(M_1), H^\bullet(M_2)),$$

and the result follows. \square

PROPOSITION 5.5. *For all $V_1, V_2 \in \text{Rep}(G^\vee)$, there is a natural map*

$$\mathcal{S}: \text{Hom}_{\text{D}_{(\mathcal{G}_O)}^i(\mathcal{G}r, \mathbb{k})}(\mathcal{S}(V_1), \mathcal{S}(V_2)) \longrightarrow \text{Hom}_{\text{Coh}(\mathcal{N})}(V_1 \otimes \mathcal{O}_{\mathcal{N}}, V_2 \otimes \mathcal{O}_{\mathcal{N}}\langle i \rangle). \quad (5.5)$$

When $\mathcal{S}(V_1)$ and $\mathcal{S}(V_2)$ are parity sheaves, this is an isomorphism. For any V_1 and V_2 , this map is compatible with composition; i.e., the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Hom}^\bullet(\mathcal{S}(V_2), \mathcal{S}(V_3)) \otimes \mathrm{Hom}^\bullet(\mathcal{S}(V_1), \mathcal{S}(V_2)) & \longrightarrow & \mathrm{Hom}^\bullet(\mathcal{S}(V_1), \mathcal{S}(V_3)) \\ \downarrow & & \downarrow \\ \underline{\mathrm{Hom}}(V_2 \otimes \mathcal{O}_{\mathcal{N}}, V_3 \otimes \mathcal{O}_{\mathcal{N}}) \otimes \underline{\mathrm{Hom}}(V_1 \otimes \mathcal{O}_{\mathcal{N}}, V_2 \otimes \mathcal{O}_{\mathcal{N}}) & \longrightarrow & \underline{\mathrm{Hom}}(V_1 \otimes \mathcal{O}_{\mathcal{N}}, V_3 \otimes \mathcal{O}_{\mathcal{N}}). \end{array} \quad (5.6)$$

Note that the naturality of (5.5) just means that it is compatible with composition of morphisms in $\mathrm{Perv}_{G_{\mathcal{O}}}(\mathcal{G}r)$. The diagram (5.6) expresses a stronger property, allowing arbitrary morphisms in $\mathrm{D}_{(G_{\mathcal{O}})}^b(\mathcal{G}r)$.

Proof. We construct the map (5.5) as the following composition:

$$\begin{aligned} \mathrm{Hom}^\bullet(\mathcal{S}(V_1), \mathcal{S}(V_2)) &\xrightarrow{H^\bullet} \underline{\mathrm{Hom}}_{H^\bullet(\mathcal{G}r)}(H^\bullet(\mathcal{S}(V_1)), H^\bullet(\mathcal{S}(V_2))) \\ &\xrightarrow[\sim]{\text{Lemma 5.4}} \underline{\mathrm{Hom}}(V_1 \otimes \mathcal{O}_{\mathcal{N}}, V_2 \otimes \mathcal{O}_{\mathcal{N}}). \end{aligned} \quad (5.7)$$

When $\mathcal{S}(V_1)$ and $\mathcal{S}(V_2)$ are parity sheaves, Theorem 4.1 tells us that the first map in (5.7) is an isomorphism. Note that the first map in (5.7) is induced by a functor defined on all of $\mathrm{D}_{(G_{\mathcal{O}})}^b(\mathcal{G}r)$, while the second map is essentially the inverse of (5.4), which is induced by a functor defined on all of $\mathrm{Coh}(\mathcal{N})$. These two observations imply the commutativity of (5.6). \square

For the remainder of this section, we will assume that $\mathrm{char} \mathbb{k}$ is also a JMW prime (Definition 1.2) for G^\vee . Recall that most good primes are known to be JMW.

THEOREM 5.6. ([JMW2, Theorem 1.8]) *Assume that G^\vee is quasi-simple. If $\mathrm{char} \mathbb{k}$ satisfies the bounds in Table 1, then \mathcal{S} sends every tilting module to a parity sheaf.*

PROPOSITION 5.7. *There is an equivalence of additive categories*

$$\mathbb{S}: \mathrm{Parity}_{(G_{\mathcal{O}})}(\mathcal{G}r) \longrightarrow \mathrm{Tilt}(\mathrm{PCoh}(\mathcal{N}))$$

such that for any tilting G^\vee -module V , we have $\mathbb{S}(\mathcal{S}(V)[n]) \cong V \otimes \mathcal{O}_{\mathcal{N}}\langle n \rangle$.

Proof. Every indecomposable object in $\mathrm{Parity}_{(G_{\mathcal{O}})}(\mathcal{G}r)$ is isomorphic to an object of the form $\mathcal{S}(V)[n]$, where $V \in \mathrm{Rep}(G^\vee)$ is a tilting module. Similarly, every indecomposable tilting object in $\mathrm{PCoh}(\mathcal{N})$ is of the form $V \otimes \mathcal{O}_{\mathcal{N}}\langle n \rangle$ for such a V . Thus, Proposition 5.5 implies that the full subcategory of indecomposable objects in $\mathrm{Parity}_{(G_{\mathcal{O}})}(\mathcal{G}r)$ is equivalent to the full subcategory of indecomposable objects in $\mathrm{Tilt}(\mathrm{PCoh}(\mathcal{N}))$. Such an equivalence extends in a unique way (up to isomorphism) to an equivalence

$$\mathrm{Parity}_{(G_{\mathcal{O}})}(\mathcal{G}r) \xrightarrow{\sim} \mathrm{Tilt}(\mathrm{PCoh}(\mathcal{N})). \quad \square$$

COROLLARY 5.8. *Let $\Gamma \subset \mathbf{X}^+$ be a finite-order ideal. There is an equivalence of categories \mathbb{S}_Γ , unique up to isomorphism, that makes the following diagram commute up to isomorphism:*

$$\begin{array}{ccc} \text{Parity}_{(G_{\mathcal{O}})}(\mathcal{G}r) & \xrightarrow{\mathbb{S}} & \text{Tilt}(\text{PCoh}(\mathcal{N})) \\ j_\Gamma^* \downarrow & & \downarrow \Pi_\Gamma \\ \text{Parity}_{(G_{\mathcal{O}})}(U_\Gamma) & \xrightarrow{\mathbb{S}_\Gamma} & \text{Tilt}(\text{PCoh}(\mathcal{N})/\text{PCoh}(\mathcal{N})_\Gamma). \end{array}$$

Proof. The functor \mathbb{S} of Proposition 5.7 restricts to an equivalence of categories $\text{Parity}_{(G_{\mathcal{O}})}(\mathcal{G}r_\Gamma) \xrightarrow{\sim} \text{Tilt}(\text{PCoh}(\mathcal{N})_\Gamma)$, and so it induces an equivalence

$$\text{Parity}_{(G_{\mathcal{O}})}(\mathcal{G}r) // \text{Parity}_{(G_{\mathcal{O}})}(\mathcal{G}r_\Gamma) \xrightarrow{\sim} \text{Tilt}(\text{PCoh}(\mathcal{N})) // \text{Tilt}(\text{PCoh}(\mathcal{N})_\Gamma).$$

Propositions 2.13 and 3.4 then give us the result. \square

THEOREM 5.9. *Let $\Gamma \subset \mathbf{X}^+$ be a finite-order ideal. If V_1 has a Weyl filtration and V_2 a good filtration, then there is a natural isomorphism of graded vector spaces*

$$\begin{aligned} \widehat{\mathbb{S}}_\Gamma: \text{Hom}_{\mathbf{D}_{(G_{\mathcal{O}})}^b(U_\Gamma)}^\bullet(\mathcal{S}(V_1)|_{U_\Gamma}, \mathcal{S}(V_2)|_{U_\Gamma}) \\ \xrightarrow{\sim} \underline{\text{Hom}}_{\text{PCoh}(\mathcal{N})/\text{PCoh}(\mathcal{N})_\Gamma}(\Pi_\Gamma(V_1 \otimes \mathcal{O}_\mathcal{N}), \Pi_\Gamma(V_2 \otimes \mathcal{O}_\mathcal{N})). \end{aligned} \quad (5.8)$$

This map is compatible with (5.5), in the sense that the diagram

$$\begin{array}{ccc} \text{Hom}_{\mathbf{D}_{(G_{\mathcal{O}})}^b(\mathcal{G}r)}^\bullet(\mathcal{S}(V_1), \mathcal{S}(V_2)) & \xrightarrow[\sim]{\mathbb{S}} & \underline{\text{Hom}}_{\text{PCoh}(\mathcal{N})}(V_1 \otimes \mathcal{O}_\mathcal{N}, V_2 \otimes \mathcal{O}_\mathcal{N}) \\ j_\Gamma^* \downarrow & & \downarrow \Pi_\Gamma \\ \text{Hom}_{\mathbf{D}_{(G_{\mathcal{O}})}^b(U_\Gamma)}^\bullet(\mathcal{S}(V_1)|_{U_\Gamma}, \mathcal{S}(V_2)|_{U_\Gamma}) & \xrightarrow[\sim]{\widehat{\mathbb{S}}_\Gamma} & \underline{\text{Hom}}_{\text{PCoh}(\mathcal{N})/\text{PCoh}(\mathcal{N})_\Gamma}(\Pi_\Gamma(V_1 \otimes \mathcal{O}_\mathcal{N}), \Pi_\Gamma(V_2 \otimes \mathcal{O}_\mathcal{N})) \end{array} \quad (5.9)$$

commutes. Moreover, (5.8) is compatible with composition: if V_1 has a Weyl filtration, V_2 is tilting, and V_3 has a good filtration, then the following diagram commutes:

$$\begin{array}{ccc} \text{Hom}^\bullet(\mathcal{S}(V_2)|_{U_\Gamma}, \mathcal{S}(V_3)|_{U_\Gamma}) \\ \otimes \text{Hom}^\bullet(\mathcal{S}(V_1)|_{U_\Gamma}, \mathcal{S}(V_2)|_{U_\Gamma}) & \longrightarrow & \text{Hom}^\bullet(\mathcal{S}(V_1)|_{U_\Gamma}, \mathcal{S}(V_3)|_{U_\Gamma}) \\ \downarrow & & \downarrow \\ \underline{\text{Hom}}(\Pi_\Gamma(V_2 \otimes \mathcal{O}_\mathcal{N}), \Pi_\Gamma(V_3 \otimes \mathcal{O}_\mathcal{N})) & \longrightarrow & \underline{\text{Hom}}(\Pi_\Gamma(V_1 \otimes \mathcal{O}_\mathcal{N}), \Pi_\Gamma(V_3 \otimes \mathcal{O}_\mathcal{N})). \\ \otimes \underline{\text{Hom}}(\Pi_\Gamma(V_1 \otimes \mathcal{O}_\mathcal{N}), \Pi_\Gamma(V_2 \otimes \mathcal{O}_\mathcal{N})) & & \end{array} \quad (5.10)$$

Note that, in contrast to (5.5), the map $\widehat{\mathbb{S}}_\Gamma$ is only defined when V_1 has a Weyl filtration and V_2 has a good filtration, and not for general objects of $\text{Rep}(G^\vee)$.

$$\begin{array}{ccc}
 \mathrm{Hom}^{i-1}(\mathcal{S}(T)|_U, \mathcal{S}(V_2)|_U) = 0 & & \\
 \downarrow & & \\
 \mathrm{Hom}^{i-1}(\mathcal{S}(V_1)|_U, \mathcal{S}(V_2)|_U) & & 0 \\
 \downarrow & & \downarrow \\
 \mathrm{Hom}^i(\mathcal{S}(V_1')|_U, \mathcal{S}(V_2)|_U) \xrightarrow[\sim]{\widehat{\mathbb{S}}} \mathrm{Hom}(\Pi(V_1' \otimes \mathcal{O}_{\mathcal{N}}), \Pi(V_2 \otimes \mathcal{O}_{\mathcal{N}})\langle i \rangle) & & \\
 \downarrow u & (*) & \downarrow u' \\
 \mathrm{Hom}^i(\mathcal{S}(T)|_U, \mathcal{S}(V_2)|_U) \xrightarrow[\sim]{\widehat{\mathbb{S}}} \mathrm{Hom}(\Pi(T \otimes \mathcal{O}_{\mathcal{N}}), \Pi(V_2 \otimes \mathcal{O}_{\mathcal{N}})\langle i \rangle) & & \\
 \downarrow v & & \downarrow v' \\
 \mathrm{Hom}^i(\mathcal{S}(V_1)|_U, \mathcal{S}(V_2)|_U) \dashrightarrow \mathrm{Hom}(\Pi(V_1 \otimes \mathcal{O}_{\mathcal{N}}), \Pi(V_2 \otimes \mathcal{O}_{\mathcal{N}})\langle i \rangle) & & \\
 \downarrow & & \downarrow \\
 \mathrm{Hom}^{i+1}(\mathcal{S}(V_1')|_U, \mathcal{S}(V_2)|_U) \xrightarrow[\sim]{\widehat{\mathbb{S}}} 0 & &
 \end{array}$$

Figure 1. Commutative diagram for the proof of Theorem 5.9

Proof. For brevity, we will write $\widehat{\mathbb{S}}$ for $\widehat{\mathbb{S}}_{\Gamma}$, U for U_{Γ} , and Π for Π_{Γ} throughout the proof. We proceed by induction on the tilting dimensions of V_1 and V_2 .

If V_1 and V_2 both have tilting dimension 0, i.e., if they are both tilting, then we simply take $\widehat{\mathbb{S}}$ to be the map induced by the functor \mathbb{S}_{Γ} from Corollary 5.8. The commutativity of both (5.9) and (5.10) is immediate from that proposition.

Suppose now that the result is known when V_1 has tilting dimension $\leq n_1$ and V_2 has tilting dimension $\leq n_2$. Now, let $V_1 \in \mathrm{Rep}(G^{\vee})$ have a Weyl filtration and tilting dimension $n_1 + 1$. By Corollary 2.11, we can find a short exact sequence

$$0 \longrightarrow V_1 \longrightarrow T \longrightarrow V_1' \longrightarrow 0, \quad (5.11)$$

where T is tilting and V_1' has a Weyl filtration and tilting dimension n_1 .

Let V_2 have a good filtration and tilting dimension $\leq n_2$. Let i be an even integer, and consider the commutative diagram in Figure 1. The left-hand column is the long exact sequence induced by (5.11). The right-hand column is also induced by (5.11). It is a short exact sequence because, by Theorem 2.16 and Lemma 2.3, $\Pi(V_1' \otimes \mathcal{O}_{\mathcal{N}})$ and $\Pi(V_2 \otimes \mathcal{O}_{\mathcal{N}})$ have standard and costandard filtrations, respectively, in $\mathrm{PCoh}(\mathcal{N})/\mathrm{PCoh}(\mathcal{N})_{\Gamma}$, and so $\underline{\mathrm{Ext}}^1(\Pi(V_1' \otimes \mathcal{O}_{\mathcal{N}}), \Pi(V_2 \otimes \mathcal{O}_{\mathcal{N}})) = 0$.

By induction, we have $\mathrm{Hom}^\bullet(\mathcal{S}(T)|_U, \mathcal{S}(V_2)|_U) \cong \underline{\mathrm{Hom}}(\Pi(T \otimes \mathcal{O}_{\mathcal{N}}), \Pi(V_2 \otimes \mathcal{O}_{\mathcal{N}}))$. In particular, since i is even, we have $\mathrm{Hom}^{i-1}(\mathcal{S}(T)|_U, \mathcal{S}(V_2)|_U) = 0$ by Corollary 2.17. Next, because the square (*) involving u and u' in Figure 1 commutes, the map u must be injective. It follows that

$$\mathrm{Hom}^{i-1}(\mathcal{S}(V_1)|_U, \mathcal{S}(V_2)|_U) = 0 \quad \text{for } i-1 \text{ odd.}$$

The left-hand column in Figure 1 has now been reduced to a short exact sequence. It is clear that there is a unique isomorphism

$$\widehat{\mathbb{S}}: \mathrm{Hom}^i(\mathcal{S}(V_1)|_U, \mathcal{S}(V_2)|_U) \xrightarrow{\sim} \mathrm{Hom}(\Pi(V_1 \otimes \mathcal{O}_{\mathcal{N}}), \Pi(V_2 \otimes \mathcal{O}_{\mathcal{N}})\langle i \rangle) \quad (5.12)$$

that makes the diagram in Figure 1 commute. For now, the map we have constructed appears to depend on the choice of (5.11). We will address this issue later.

First, let us consider the special case where $\Gamma = \emptyset$, so that $U = \mathcal{G}r$, and Π is the identity functor. In this case, the solid horizontal arrows in Figure 1 are given by (5.5), by induction. Since the dotted arrow is uniquely determined, it too must be given by (5.5). In particular, we have now shown that the top horizontal arrow in (5.9) is an isomorphism for the pair (V_1, V_2) .

Now, compare the special case ($\Gamma = \emptyset$) of Figure 1 with the general case. Since (5.9) holds for the pairs (V'_1, V_2) and (T, V_2) by induction, one can see by an easy diagram chase that it also holds for the pair (V_1, V_2) .

Recall from Corollary 2.14 that the right-hand vertical map in (5.9) is surjective. Since the horizontal maps are isomorphisms, the left-hand vertical map must be surjective as well. Once we know that both vertical maps are surjective, we can see that the bottom horizontal map is uniquely determined. Thus, the map (5.12) is independent of (5.11).

It remains to show that (5.12) is natural in both variables, and that (5.10) commutes. The former is essentially a special case of the latter, so we focus on the latter. In the special case $\Gamma = \emptyset$, the commutativity of (5.10) is contained in Proposition 5.5. For general Γ , we deduce the result by a diagram chase using the special case $\Gamma = \emptyset$ together with several instances of (5.9).

An entirely similar argument establishes the required induction step involving the tilting dimension of V_2 . \square

6. Proof of the Mirković–Vilonen conjecture

In this section \mathbb{k} may be any field. We begin with a lemma about sheaves on a single $G_{\mathcal{O}}$ -orbit $\mathcal{G}r_\lambda$ in $\mathcal{G}r$. Note that $D_{(G_{\mathcal{O}})}^b(\mathcal{G}r_\lambda)$ is the category of complexes of sheaves whose cohomology sheaves are (locally) constant.

LEMMA 6.1. *The following conditions on an object $\mathcal{F} \in D_{(G_{\mathbf{O}})}^b(\mathcal{G}r_\lambda)$ are equivalent:*

- (1) \mathcal{F} is even.
- (2) $\text{Hom}^\bullet(\mathcal{F}, \mathbb{k}_{\mathcal{G}r_\lambda})$ is a free $H^\bullet(\mathcal{G}r_\lambda)$ -module generated in even degrees.

Proof. Every even object is a direct sum of objects of the form $\mathbb{k}_{\mathcal{G}r_\lambda}[2n]$, so it is clear that the first condition implies the second. Suppose now that the second condition holds. Choose a basis e_1, \dots, e_k for $\text{Hom}^\bullet(\mathcal{F}, \mathbb{k}_{\mathcal{G}r_\lambda})$ as a $H^\bullet(\mathcal{G}r_\lambda)$ -module, and suppose each e_i is homogeneous of degree $2n_i$. That is, each e_i is a morphism $\mathcal{F} \rightarrow \mathbb{k}_{\mathcal{G}r_\lambda}[2n_i]$. Let

$$\mathcal{F}' = \bigoplus_{i=1}^k \mathbb{k}_{\mathcal{G}r_\lambda}[2n_i],$$

and consider the map $f = (e_1, \dots, e_k): \mathcal{F} \rightarrow \mathcal{F}'$. The map $\text{Hom}^\bullet(\mathcal{F}', \mathbb{k}_{\mathcal{G}r_\lambda}) \rightarrow \text{Hom}^\bullet(\mathcal{F}, \mathbb{k}_{\mathcal{G}r_\lambda})$ induced by f is surjective (all the e_i lie in its image), and since these are finite-dimensional graded vector spaces, it is an isomorphism. Therefore, letting \mathcal{G} denote the cone of $f: \mathcal{F} \rightarrow \mathcal{F}'$, we have

$$\text{Hom}^\bullet(\mathcal{G}, \mathbb{k}_{\mathcal{G}r_\lambda}) = 0. \tag{6.1}$$

We claim that $\mathcal{G} = 0$. If not, let n be the top degree in which $\mathcal{H}^n(\mathcal{G}) \neq 0$. Then, there is a non-zero truncation morphism $\mathcal{G} \rightarrow \tau_{\geq n} \mathcal{G} \cong \mathcal{H}^n(\mathcal{G})[-n]$. The constant sheaf $\mathcal{H}^n(\mathcal{G})$ is a direct sum of copies of $\mathbb{k}_{\mathcal{G}r_\lambda}$, so there is a non-zero map $\mathcal{G} \rightarrow \mathbb{k}_{\mathcal{G}r_\lambda}[-n]$, contradicting (6.1). Thus, $\mathcal{G} = 0$, and f is an isomorphism. In particular, $\mathcal{F} \cong \bigoplus_{i=1}^k \mathbb{k}_{\mathcal{G}r_\lambda}[2n_i]$ is even. \square

THEOREM 6.2. (Cf. [JMW2, Conjecture 1.10]) *Assume that $\text{char } \mathbb{k}$ is a JMW prime for G^\vee . Then the perverse sheaf $\mathcal{I}_!(\lambda, \mathbb{k})$ is \ast -parity.*

More precisely, $\mathcal{I}_!(\lambda, \mathbb{k})$ is \ast -even (resp. \ast -odd) if $\dim \mathcal{G}r_\lambda$ is even (resp. odd).

Proof. Let $\bar{\mathbb{k}}$ be an algebraic closure of \mathbb{k} . For any $x \in \mathcal{G}r$, we have that

$$\mathcal{I}_!(\lambda, \bar{\mathbb{k}})_x \cong \mathcal{I}_!(\lambda, \mathbb{k})_x \otimes \bar{\mathbb{k}}.$$

Thus, if $\mathcal{I}_!(\lambda, \bar{\mathbb{k}})$ were known to be \ast -parity, the result would follow for $\mathcal{I}_!(\lambda, \mathbb{k})$. In other words, it suffices to prove the theorem for algebraically closed fields. For the remainder of the proof, we assume that \mathbb{k} is algebraically closed.

It is well known that every component of $\mathcal{G}r$ is isomorphic to a component of the affine Grassmannian for the group $G/Z(G)$, via an isomorphism compatible with the stratification by $G_{\mathbf{O}}$ -orbits. For groups of type A , there is a similar statement in the opposite direction: every component of the affine Grassmannian of $\text{PGL}(n)$ is isomorphic to some component of the affine Grassmannian of $\text{GL}(n)$. These two observations imply that to prove the theorem in general, it suffices to consider groups of the form

$$\text{GL}(n_1) \times \dots \times \text{GL}(n_k) \times \left(\begin{array}{l} \text{a semisimple group of adjoint type} \\ \text{containing no factors of type } A \end{array} \right). \tag{6.2}$$

Assume henceforth that G has this form. Then G^\vee is a product of $\mathrm{GL}(n_i)$'s with a semisimple, simply connected group containing no factors of type A . According to [J2, §2.9], such a group satisfies (1.3), so we can invoke the results of §5.

For simplicity, let us assume that $\dim \mathcal{G}r_\lambda$ is even; the argument in the odd case is the same. Let $\mu \in \mathbf{X}^+$ be a weight such that $\mathcal{G}r_\mu \subset \overline{\mathcal{G}r}_\lambda$. Recall that this implies that $\dim \mathcal{G}r_\mu$ is also even.

Let $\Gamma \subset \mathbf{X}^+$ be the set of weights that are strictly smaller than μ . Let $U = U_\Gamma = \mathcal{G}r \setminus \mathcal{G}r_\Gamma$, and let $j = j_\Gamma$ as in (5.1). By Theorem 5.9, we have a natural isomorphism

$$\mathrm{Hom}^\bullet(j^* \mathcal{I}_l(\lambda, \mathbb{k}), j^* \mathcal{T}(\mu, \mathbb{k})) \cong \underline{\mathrm{Hom}}(\Pi(\mathcal{M}(\lambda) \otimes \mathcal{O}_\mathcal{N}), \Pi(\mathcal{T}(\mu) \otimes \mathcal{O}_\mathcal{N})).$$

In particular, by (5.10), this is an isomorphism of graded modules over the ring

$$\mathrm{Hom}^\bullet(j^* \mathcal{T}(\mu, \mathbb{k}), j^* \mathcal{T}(\mu, \mathbb{k})) \cong \underline{\mathrm{End}}(\Pi(\mathcal{T}(\mu) \otimes \mathcal{O}_\mathcal{N})).$$

Finally, in the quotient category $\mathrm{PCoh}(\mathcal{N})/\mathrm{PCoh}(\mathcal{N})_\Gamma$, the tilting object $\Pi(\mathcal{T}(\mu) \otimes \mathcal{O}_\mathcal{N})$ coincides with the costandard object $\Pi(\mathcal{N}(\mu) \otimes \mathcal{O}_\mathcal{N})$, by Lemma 2.7 (1). By Lemma 2.12, the space $\underline{\mathrm{Hom}}(\Pi(\mathcal{M}(\lambda) \otimes \mathcal{O}_\mathcal{N}), \Pi(\mathcal{N}(\mu) \otimes \mathcal{O}_\mathcal{N}))$ is a free $\underline{\mathrm{End}}(\Pi(\mathcal{N}(\mu) \otimes \mathcal{O}_\mathcal{N}))$ -module, and by Corollary 2.17, it is generated in even degrees.

Now, let $i: \mathcal{G}r_\mu \rightarrow U$ be the inclusion map. This is a closed inclusion, and we clearly have $j^* \mathcal{T}(\mu, \mathbb{k}) \cong i_* \mathbb{k}_{\mathcal{G}r_\mu}[\dim \mathcal{G}r_\mu]$. Rephrasing the conclusion of the previous paragraph, we have that

$$\mathrm{Hom}^\bullet(j^* \mathcal{I}_l(\lambda, \mathbb{k}), i_* \mathbb{k}_{\mathcal{G}r_\mu}) \cong \mathrm{Hom}^\bullet(i^* j^* \mathcal{I}_l(\lambda, \mathbb{k}), \mathbb{k}_{\mathcal{G}r_\mu})$$

is a free module generated in even degrees over the ring

$$\mathrm{Hom}^\bullet(i_* \mathbb{k}_{\mathcal{G}r_\mu}, i_* \mathbb{k}_{\mathcal{G}r_\mu}) \cong H^\bullet(\mathcal{G}r_\mu).$$

By Lemma 6.1, $i^* j^* \mathcal{I}_l(\lambda, \mathbb{k})$ is even, as desired. □

THEOREM 6.3. (Cf. [MV1, Conjecture 6.3] or [MV2, Conjecture 13.3]) *If p is a JMW prime for G^\vee , then the stalks of $\mathcal{I}_l(\lambda, \mathbb{Z})$ have no p -torsion.*

Proof. Let M be a \mathbb{Z} -module. It is a routine exercise to show that if M has p -torsion, then $H^i(M \otimes^L \mathbb{F}_p) \neq 0$ for both $i=0$ and $i=-1$. Now, let $x \in \mathcal{G}r$, and consider the stalk $\mathcal{I}_l(\lambda, \mathbb{Z})_x$, which is an object in the derived category of finitely generated \mathbb{Z} -modules. Since \mathbb{Z} has global dimension 1, $\mathcal{I}_l(\lambda, \mathbb{Z})_x$ is isomorphic to the direct sum of its cohomology modules, and if any cohomology module had p -torsion, the object $\mathcal{I}_l(\lambda, \mathbb{Z})_x \otimes^L \mathbb{F}_p$ would have non-zero cohomology in both even and odd degrees. But by [MV2, Proposition 8.1(a)], we have

$$\mathcal{I}_l(\lambda, \mathbb{Z})_x \otimes^L \mathbb{F}_p \cong \mathcal{I}_l(\lambda, \mathbb{F}_p)_x,$$

and Theorem 6.2 tells us that the latter cannot have cohomology in both even and odd degrees. Thus, $\mathcal{I}_l(\lambda, \mathbb{Z})_x$ has no p -torsion. □

Appendix A. Sheaves on non-locally compact spaces

Let X and Y be Hausdorff topological spaces, and let $h: Y \rightarrow X$ be a continuous map. If X is not locally compact, then $h^!$ may not be defined, and $h_!$ may fail to have some familiar properties. However, if h is an inclusion of a locally closed subset, these difficulties can be largely circumvented. In this appendix, we briefly explain how to define the functors $h_!$ and $h^!$, and we discuss some of their properties.

Let $\text{Sh}(X)$ denote the category of sheaves of k -modules on X . The discussion below will make heavy use of the two functors $\text{Sh}(X) \rightarrow \text{Sh}(X)$ discussed in [KS, Proposition 2.3.6 and Definition 2.3.8], denoted by

$$\mathcal{F} \mapsto \mathcal{F}_U \quad \text{and} \quad \mathcal{F} \mapsto \Gamma_U(\mathcal{F}).$$

As in the main body of this paper, $h_!$ and h_* will always denote derived functors. Their non-derived analogues will be denoted by ${}^\circ h_!$ and ${}^\circ h_*$, respectively.

As explained in [KS, equation (2.5.1)], the functor ${}^\circ h_!: \text{Sh}(Y) \rightarrow \text{Sh}(X)$ (and hence its derived functor $h_!$) can be defined without any local compactness assumption. According to [KS, Proposition 2.5.4], when h is a locally closed inclusion, we have a natural isomorphism

$${}^\circ h_! h^* \mathcal{F} \cong \mathcal{F}_Y. \tag{A.1}$$

On the other hand, inspired by [KS, Proposition 3.1.12], we define a new left-exact functor ${}^\circ h^!: \text{Sh}(X) \rightarrow \text{Sh}(Y)$ by

$${}^\circ h^!(\mathcal{F}) = h^* \Gamma_Y(\mathcal{F}). \tag{A.2}$$

Let $h^!: D^b(X) \rightarrow D^b(Y)$ be its right derived functor.

LEMMA A.1. *Let $h: Y \rightarrow X$ be a locally closed inclusion.*

- (1) *The functor ${}^\circ h_!$ is exact.*
- (2) *If h is an open inclusion, then $h^! \cong h^*$.*
- (3) *There are natural isomorphisms $h_! h^* \mathcal{F} \cong \mathcal{F}_Y$ and $h^! \mathcal{F} \cong h^* R\Gamma_Y(\mathcal{F})$.*
- (4) *Let $k: Z \rightarrow Y$ be another locally closed inclusion. Then there are natural isomorphisms $(h \circ k)_! \cong h_! \circ k_!$ and $(h \circ k)^! \cong k^! \circ h^!$.*
- (5) *There are natural isomorphisms $h_! \mathcal{F} \cong (h_* \mathcal{F})_Y$ and $h_* h^! \mathcal{F} \cong R\Gamma_Y(\mathcal{F})$.*

Comparing part (3) with [KS, Proposition 3.1.12], we see that our definition of $h^!$ agrees with the usual one in case X is locally compact. For part (4), the usual proofs (see [KS, equation (2.6.6) and Proposition 3.1.8]) require the use of c -soft sheaves, which may not be available on X .

Proof. (1) This is [KS, Proposition 2.5.4 (i)].

(2) We see from [KS, Proposition 2.3.9 (iii)] that ${}^\circ h^! \cong h^* \circ {}^\circ h_* \circ h^* \cong h^*$, and so $h^! \cong h^*$ as well.

(3) According to [KS, Propositions 2.3.6] and part (1), every functor in (A.1) is exact, so its derived version follows. In (A.2), since h^* is exact, the derived functor of the right-hand side is just $h^* R\Gamma_Y$.

(4) Since ${}^\circ h_!$ and ${}^\circ k_!$ are exact, the first assertion follows immediately from its non-derived analogue, found in [KS, equation (2.5.3)]. Next, a routine argument (cf. [KS, Proposition 2.4.6]) shows that ${}^\circ h^!$ takes flabby sheaves to flabby sheaves, and that its derived functor $h^!$ can be computed using flabby resolutions. Therefore, the desired result follows from its non-derived analogue, which says that

$$k^* \Gamma_Z h^* \Gamma_Y \cong (h \circ k)^* \Gamma_Z.$$

(Note that on the left-hand side, Γ_Z is a functor on $\text{Sh}(Y)$, while on the right-hand side, it is a functor on $\text{Sh}(X)$.) This can be checked directly from the definition, in the spirit of [KS, Proposition 2.3.9].

(5) Since $h^* h_* \mathcal{F} \cong \mathcal{F}$, it follows immediately from part (3) that $h_! \mathcal{F} \cong (h_* \mathcal{F})_Y$. For the second assertion, due to part (4), it suffices to consider the cases where Y is open or closed. If Y is open, the non-derived analogue ${}^\circ h_* \circ {}^\circ h^! \cong \Gamma_Y$ is [KS, Proposition 2.3.9 (iii)], and the derived version follows because ${}^\circ h^! \cong h^*$ takes flabby sheaves to flabby sheaves. If Y is closed, one can check from the definitions that ${}^\circ h_* \circ {}^\circ h^! = {}^\circ h_* h^* \Gamma_Y \cong \Gamma_Y$. Since ${}^\circ h_*$ is exact, it follows that $h_* h^! \cong R\Gamma_Y$. \square

LEMMA A.2. *The functor $h^!$ is right adjoint to $h_!$.*

Proof. It follows from [KS, equation (2.6.9)] that $\text{Hom}(\mathcal{F}_Y, \mathcal{G}) \cong \text{Hom}(\mathcal{F}, R\Gamma_Y(\mathcal{G}))$. We therefore have the following sequence of natural isomorphisms:

$$\begin{aligned} \text{Hom}(h_!(\mathcal{F}), \mathcal{G}) &\cong \text{Hom}((h_* \mathcal{F})_Y, \mathcal{G}) \cong \text{Hom}(h_* \mathcal{F}, R\Gamma_Y(\mathcal{G})) \\ &\cong \text{Hom}(h_* \mathcal{F}, h_* h^! \mathcal{G}) \cong \text{Hom}(h^* h_* \mathcal{F}, h^! \mathcal{G}) \cong \text{Hom}(\mathcal{F}, h^! \mathcal{G}). \end{aligned} \quad \square$$

LEMMA A.3. *Let $i: Z \rightarrow X$ be the inclusion of a closed subset, and let $j: U \rightarrow X$ be the inclusion of the complementary open subset. For any $\mathcal{F} \in \text{D}^b(X)$, there are functorial distinguished triangles*

$$i_* i^! \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow j_* j^* \mathcal{F} \longrightarrow \quad \text{and} \quad j_! j^* \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow i_* i^* \mathcal{F} \longrightarrow.$$

Proof. According to [KS, equation (2.6.32)], there is a functorial distinguished triangle $R\Gamma_Z(\mathcal{F}) \rightarrow \mathcal{F} \rightarrow R\Gamma_U(\mathcal{F}) \rightarrow$. Using Lemma A.1 (5), we obtain the first distinguished triangle above. The second follows similarly from [KS, equation (2.6.33)]. \square

LEMMA A.4. *Suppose we have a Cartesian square*

$$\begin{array}{ccc} Y' & \xrightarrow{h'} & X' \\ f' \downarrow & & \downarrow f \\ Y & \xrightarrow{h} & X, \end{array}$$

where h and h' are locally closed inclusions. Then we have $h^! f_* \cong f'_* h'^!$.

Proof. According to [KS, equation (2.3.20)], $\Gamma_{Y^\circ} f_* \cong f_* \Gamma_{Y'}$. Since all these functors take flabby sheaves to flabby sheaves, we also have the derived version $R\Gamma_Y f_* \cong f_* R\Gamma_{Y'}$. In other words, $h_* h^! f_* \cong f_* h'_* h'^! \cong h_* f'_* h'^!$. Compose with h^* to obtain $h_! f^* \cong f'_* h'^!$. \square

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PRAMOD N. ACHAR
 Department of Mathematics
 Louisiana State University
 Baton Rouge, LA 70803
 U.S.A.
pramod@math.lsu.edu

LAURA RIDER
 Department of Mathematics
 Massachusetts Institute of Technology
 Cambridge, MA 02139
 U.S.A.
laurajoy@mit.edu

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