TAIWANESE JOURNAL OF MATHEMATICS
Vol. 18, No. 6, pp. 1957-1979, December 2014
DOI: 10.11650/tjm.18.2014.3541
This paper is available online at http://journal.taiwanmathsoc.org.tw

NON-NEHARI MANIFOLD METHOD FOR SUPERLINEAR SCHRÖDINGER EQUATION

X. H. Tang

Abstract. We consider the boundary value problem

$$\begin{cases} -\triangle u + V(x)u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{0.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain, $\inf_\Omega V(x) > -\infty$, f is a superlinear, subcritical nonlinearity. Inspired by previous work of Szulkin and Weth (2009) [21] and (2010) [22], we develop a more direct and simpler approach on the basis of one used in [21], to deduce weaker conditions under which problem (0.1) has a ground state solution of Nehari-Pankov type or infinity many nontrivial solutions. Unlike the Nehari manifold method, the main idea of our approach lies on finding a minimizing Cerami sequence for the energy functional outside the Nehari-Pankov manifold by using the diagonal method.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and consider the boundary value problem

$$\begin{cases} -\triangle u + V(x)u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $V : \Omega \to \mathbb{R}$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$.

The problem (1.1) is one of the main nonlinear elliptic problems which has been studied extensively for many years where $\inf_\Omega V > -\lambda_1(\Omega)$ or $V \in L^{N/2}(\Omega)$, $\lambda_1(\Omega)$ denotes the first Dirichlet eigenvalue of $-\triangle$ in Ω. Since Ambrosetti and Rabinowitz

Received July 13, 2013, accepted February 17, 2014.
Communicated by Yingfei Yi.
2010 *Mathematics Subject Classification*: 35J20, 35J60.
Key words and phrases: Schrödinger equation, Strongly indefinite functional, Superlinear, Diagonal method, Boundary value problem, Ground state solutions of Nehari-Pankov type.
This work is partially supported by the NNSF (No. 11171351) and the SRFDP (No. 20120162110021) of China.

proposed the mountain-pass theorem in 1973 (see [16]), critical point theory has become one of the main tools for finding solutions to elliptic equations of variational type. Clearly, weak solutions to (1.1) correspond to critical points of the energy functional

$$\Phi(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x, \quad u \in H_0^1(\Omega), \tag{1.2}$$

where $F(x,t) = \int_0^t f(x,s)\mathrm{d}s$. For the case of $\inf_\Omega V > -\lambda_1(\Omega)$, 0 is a local minimum of Φ, it is very convenient to apply the mountain-pass theorem to construct non-trivial solutions of (1.1) with a variational method by a minimax procedure on Φ, see example, [1, 2, 13, 14, 16, 19, 25]. When $\inf_\Omega V \in (-\infty, -\lambda_1(\Omega))$, 0 is a saddle point rather than a local minimum of Φ, problem (1.1) is indefinite and it is not easy to obtain the boundedness of the Palais-Smale sequence for Φ. Under an additional assumption on potential: $V \in L^{N/2}(\Omega)$, applying Linking theorem (1978, Rabinowitz, see [16]), Willem [25] obtained the existence of one nontrivial solution to (1.1). In the aforementioned references, the following classical condition (AR) due to Ambrosetti and Rabinowitz [3] is commonly assumed:

(AR) there exist $\mu > 2$ and $R_0 > 0$ such that

$$0 < \mu F(x,t) \le t f(x,t), \quad \forall\, x \in \Omega, \quad |t| \ge R_0.$$

(AR) is a very convenient hypothesis since it readily achieves mountain pass geometry as well as satisfaction of Palais-Smale condition. However (AR) implies $F(x,t) \ge C|t|^\mu$ for large $|t|$ and some constant $C > 0$, one can not deal with (1.1) using the mountain-pass theorem directly if $f(x,t)$ is of asymptotically linear at ∞. During the past three decades, many results have been obtained for the existence of nontrivial solutions to (1.1) when $f(x,t)$ does not satisfy (AR) condition, see e.g. [6, 7, 8, 9, 10, 11, 18, 24] and the references therein.

Let $\mathcal{A}$ be the selfadjoint extension of the operator $-\triangle + V$ with domain $\mathfrak{D}(\mathcal{A})$ ($C_0^\infty(\Omega) \subset \mathfrak{D}(\mathcal{A}) \subset L^2(\Omega)$) and $|\mathcal{A}|$ be the absolute value of $\mathcal{A}$. Let $E = \mathfrak{D}(|\mathcal{A}|^{1/2})$ be the domain of $|\mathcal{A}|^{1/2}$. Then $E \subset H_0^1(\Omega)$ is a Hilbert space with the orthogonal decomposition $E = E^- \oplus E^0 \oplus E^+$, see section 2 in detail. Let

$$\mathcal{N}^- = \left\{u \in E \setminus (E^- \oplus E^0) : \langle \Phi'(u), u\rangle = \langle \Phi'(u), v\rangle = 0, \ \forall\, v \in E^- \oplus E^0\right\}. \tag{1.3}$$

The set $\mathcal{N}^-$ was first introduced by Pankov [15], which is a subset of the Nehari manifold

$$\mathcal{N} = \left\{u \in E \setminus \{0\} : \langle \Phi'(u), u\rangle = 0\right\}. \tag{1.4}$$

Suppose that $u \ne 0$ is a critical point of Φ, i.e. $\Phi'(u) = 0$. Then necessarily u is contained in set $\mathcal{N}^-$. Let

$$m = \inf_{u \in \mathcal{N}^-} \Phi(u). \tag{1.5}$$

Under appropriate conditions on Φ one hopes that m is attained at some $u_0 \in \mathcal{N}^-$ and that u_0 is a critical point. Since u_0 is a solution to the equation $\Phi'(u) = 0$ at which Φ has least "energy" in set $\mathcal{N}^-$, we shall call it a ground state solution of Nehari-Pankov type.

Clearly, there are more difficulties to overcome to find ground state solutions of Nehari-Pankov type for problem (1.1) than nontrivial weak solutions. The main difficulty is to find a Palais-Smale sequence or Cerami sequence $\{u_n\}$ with $\Phi(u_n) \to \inf_{u\in\mathcal{N}^-} \Phi(u)$.

In recent paper [22] (see also [21]), on the basis of the Nehari manifold method, Szukin and Weth developed a new approach to find ground state solutions of Nehari-Pankov type for the following special form of problem (1.1):

$$\begin{cases} -\triangle u - \lambda u = f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{1.6}$$

This approach transforms, by a direct and simple reduction, the indefinite variational problem to a definite one, in which $\lambda \geq \lambda_1(\Omega)$. To state the results obtained by Szukin and Weth in [21, 22], we first introduce the following assumptions:

(F1) $f \in C(\Omega \times \mathbb{R}, \mathbb{R})$, and there exist constants $p \in (2, 2^*)$ and $C_0 > 0$ such that

$$|f(x,t)| \leq C_0 \left(1 + |t|^{p-1}\right), \quad \forall\ (x,t) \in \Omega \times \mathbb{R};$$

(F2) $f(x,t) = o(|t|)$, as $|t| \to 0$, uniformly in $x \in \Omega$;

(SQ) $\lim_{|t|\to\infty} \frac{|F(x,t)|}{|t|^2} = \infty$ uniformly in $x \in \Omega$;

(Ne) $t \mapsto \frac{f(x,t)}{|t|}$ is strictly increasing on $(-\infty, 0) \cup (0, \infty)$.

Let $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots$ be the Dirichlet eigenvalues of $-\triangle$ in Ω and $e_1, e_2, e_3, \ldots$ the corresponding orthogonal eigenfunctions. Let

$$\Phi_\lambda(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 - \lambda u^2\right) \mathrm{d}x - \int_\Omega F(x,u)\mathrm{d}x, \quad u \in H_0^1(\Omega). \tag{1.7}$$

Now, we are able to state a main result in [21, Theorem 3.1, Theorem 3.2] or [22, Theorem 37].

Theorem 1.1. ([21, 22]). *Assume that* $\lambda < \lambda_{m+1}$ *and* f *satisfies* (F1), (F2), (SQ) *and* (Ne). *Then problem* (1.6) *has a solution* $u_0 \in H_0^1(\Omega)$ *such that* $\Phi_\lambda(u_0) = \inf_{\mathcal{N}_\lambda^-} \Phi_\lambda > 0$, *where*

$$\mathcal{N}_\lambda^- = \left\{u \in H_0^1(\Omega) \setminus E_m : \langle \Phi_\lambda'(u), u\rangle = \langle \Phi_\lambda'(u), v\rangle = 0,\ \forall\ v \in E_m\right\}, \tag{1.8}$$

$E_m := \mathrm{span}\{e_1, e_2, \ldots, e_m\}$. *Moreover, if* $f(x,t)$ *is odd in* t, *then* (1.6) *has infinitely many pairs of solutions.*

We point out that the Nehari type assumption (Ne) is very crucial in the argument of Szulkin and Weth [21, 22]. In fact, the starting point of their approach is to show that for each $u \in H_0^1(\Omega) \setminus E_m$, the Nehari-Pankov manifold $\mathcal{N}_\lambda^-$ intersects $E_m \oplus \mathbb{R}^+ u$ in exactly one point $\hat{m}(u)$. The uniqueness of $\hat{m}(u)$ enables one to define a map $u \mapsto \hat{m}(u)$, which is important in the remaining proof. If $t \mapsto f(x,t)/|t|$ is not strictly increasing, then $\hat{m}(u)$ may not be unique and their argument becomes invalid.

Motivated by papers [21, 22], in the present paper, we shall mainly study the existence of ground state solutions of Nehari-Pankov type and infinitely many solutions for problem (1.1). On the basis of the approach used in [21], we will develop a more direct and simpler one to generalize Theorem 1.1 by relaxing assumptions (F2), (SQ) and (Ne). Unlike the Nehari manifold method, our approach lies on finding a minimizing Cerami sequence for functional Φ outside the Nehari-Pankov manifold $\mathcal{N}^-$ by using the diagonal method, see Lemma 3.8.

To state our results, we make the following assumptions:

(V) $V \in C(\Omega, \mathbb{R})$, and $\inf_\Omega V(x) > -\infty$;

(F2′) $\lim_{t\to 0} \frac{f(x,t)}{t} = \bar{a} \in \mathbb{R}$ uniformly in $x \in \Omega$;

(F2″) $\limsup_{t\to 0} \frac{|f(x,t)|}{|t|} < \infty$ uniformly in $x \in \Omega$;

(F3) $\lim_{|t|\to\infty} \frac{|F(x,t)|}{|t|^2} = \infty, \quad a.e.\ x \in \Omega$;

(F4) there exist constants $c_0 > 0,\ R_0 > 0$ and $\kappa > \max\{1, N/2\}$ such that

$$0 \le [F(x,t)]^\kappa \le c_0 |t|^{2\kappa} \left[tf(x,t) - 2F(x,t)\right], \quad \forall\ x \in \Omega, \quad |t| \ge R_0;$$

(F5) $f(x,-t) = -f(x,t),\ \forall\ (x,t) \in \Omega \times \mathbb{R}$;

(WN) $t \mapsto \frac{f(x,t)}{|t|}$ is non-decreasing on $(-\infty, 0) \cup (0, \infty)$.

We are now in a position to state the main results of this paper.

Theorem 1.2. *Assume that V and f satisfy* (V), (F1), (F2), (F3) *and* (WN). *Then problem* (1.1) *has a solution $u_0 \in E$ such that $\Phi(u_0) = \inf_{\mathcal{N}^-} \Phi > 0$.*

Corollary 1.3. *Assume that V and f satisfy* (V), (F1), (F2′), (F3) *and* (WN). *Then problem* (1.1) *has a solution $u_0 \in E$ such that $\Phi(u_0) = \inf_{\mathcal{N}^-} \Phi > 0$.*

Theorem 1.4. *Assume that V and f satisfy* (V), (F1), (F2″), (F3), (F4) *and* (F5). *Then problem* (1.1) *possesses infinitely many pairs of solutions.*

Corollary 1.5. *In Theorem 1.4, if* (F4) *is replaced by the following assumption* (F4′) *there exist constants $c_1 > 0,\ R_0 > 0$ and $\nu > 0$ with $(p-\nu)/(p-2) > \max\{1, N/2\}$ such* *that*

$$0 \le \left(2 + \frac{1}{c_1|t|^\nu}\right) F(x,t) \le tf(x,t), \quad \forall\, x \in \Omega, \;\; |t| \ge R_0,$$

then the conclusion still holds.

Remark 1.6. After Szulkin and Weth [21, 22], Liu [12] also obtained the existence of "ground state solutions" for a periodic problem similar to (1.1). However, the "ground state solutions" for the problem in [12] is in fact a nontrivial solution u_0 which satisfies $\Phi(u_0) = \inf_{\mathcal{M}} \Phi$, where

$$\mathcal{M} = \left\{u \in E \setminus \{0\} : \Phi'(u) = 0\right\} \tag{1.9}$$

is a very small subset of $\mathcal{N}^-$. On the existence of this kind "ground state solutions", a weaker condition was obtained in very recent paper [24]. In general, it is much more difficult to find a solution u_0 for (1.1) which satisfies $\Phi(u_0) = \inf_{\mathcal{N}^-} \Phi$ than one satisfying $\Phi(u_0) = \inf_{\mathcal{M}} \Phi$.

Remark 1.7. It is easy to check that

$$f(x,t) = \vartheta(x) t \ln(2 + |t|) \tag{1.10}$$

satisfies (F2′) and (WN), and

$$F(x,t) = \vartheta(x)\left[|t|^\mu + (\mu-2)|t|^{\mu-\epsilon}\sin^2\left(\frac{|t|^\epsilon}{\epsilon}\right) + 1 - \cos t\right] \tag{1.11}$$

satisfies (F2″) and (F4), where $\vartheta \in C(\Omega, \mathbb{R})$, and $0 < \inf_\Omega \vartheta \le \sup_\Omega \vartheta < \infty$, and $\mu > 2$, $0 < \epsilon < \mu - 2$ if $N = 1, 2$ and $0 < \epsilon < \mu + N - \mu N/2$ if $N \ge 3$. Remark that these functions do not satisfy (F2) and (AR).

Throughout this paper, by $\|\cdot\|_s$ we denote the usual norm in $L^s(\Omega)$.

The remainder of this paper is organized as follows. In Section 2, we describe the space structure of the Hilbert space E in detail. The proofs of Theorem 1.2 and Corollary 1.3 are given in section3. In the last section, we show Theorem 1.4 and Corollary 1.5.

2. Variational Setting

In order to establish our existence results via the critical point theory, we first describe some properties of the space E.

In what follows V is assumed to satisfy assumption (V). Let $\{\mathcal{E}(\lambda) : -\infty < \lambda < +\infty\}$ be the spectral family of $\mathcal{A}$, and $|\mathcal{A}|^{1/2}$ be the square root of $|\mathcal{A}|$. Set $U = id - \mathcal{E}(0) - \mathcal{E}(0-)$. Then U commutes with $\mathcal{A}$, $|\mathcal{A}|$ and $|\mathcal{A}|^{1/2}$, and $\mathcal{A} = U|\mathcal{A}|$ is the polar decomposition of $\mathcal{A}$ (see [5, Theorem 4.3.3]). Define an inner product

$$(u,v)_0 = \left(|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v\right)_{L^2} + (u,v)_{L^2}, \quad \forall\, u, v \in E$$

and the corresponding norm

$$\|u\|_0 = \sqrt{(u,u)_0}, \quad \forall\, u \in E,$$

where, as usual, $(\cdot,\cdot)_{L^2}$ denotes the inner product of $L^2(\Omega,\mathbb{R})$. Then E is a Hilbert space with the above inner product. Clearly, $C_0^\infty(\Omega,\mathbb{R})$ is dense in E.

By (V), $V(x)$ is bounded from below and so there is an $a_0 > 0$ such that

$$V(x) + a_0 \geq 1, \quad \forall\, x \in \Omega. \tag{2.1}$$

Set

$$E_* = \left\{ u \in H_0^1(\Omega) : \int_\Omega \left[|\nabla u|^2 + (V(x)+a_0)u^2\right] \mathrm{d}x < +\infty \right\},$$

$$(u,v)_* = \int_\Omega \left[\nabla u \nabla v + (V(x)+a_0)uv\right] \mathrm{d}x, \quad \forall\, u, v \in E_*$$

and

$$\|u\|_* = \left\{ \int_\Omega \left[|\nabla u|^2 + (V(x)+a_0)u^2\right] \mathrm{d}x \right\}^{1/2}, \quad \forall\, u \in E_*.$$

Then E_* is also a Hilbert space with the above inner product $(\cdot,\cdot)_*$.

Lemma 2.1. *Suppose that V satisfies* (V). *Then*

$$\frac{1}{\sqrt{2+a_0}}\|u\|_0 \leq \|u\|_* \leq \sqrt{1+a_0}\|u\|_0, \quad \forall\, u \in E_* = E. \tag{2.2}$$

Proof. For $u \in C_0^\infty(\Omega)$, one has

$$\begin{aligned} \|u\|_*^2 &= ((\mathcal{A}+a_0)u,u)_{L^2} = (\mathcal{A}u,u)_{L^2} + a_0\|u\|_2^2 \\ &= (|\mathcal{A}|Uu,u)_{L^2} + a_0\|u\|_2^2 \\ &= \left(U|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}u\right)_{L^2} + a_0\|u\|_2^2 \\ &\leq \left\|U|\mathcal{A}|^{1/2}u\right\|_2 \left\||\mathcal{A}|^{1/2}u\right\|_2 + a_0\|u\|_2^2 \\ &= \left\||\mathcal{A}|^{1/2}u\right\|_2^2 + a_0\|u\|_2^2 \leq (1+a_0)\|u\|_0^2 \end{aligned} \tag{2.3}$$

and

$$
\begin{aligned}
\|u\|_0^2 &= \left(|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}u\right)_{L^2} + \|u\|_2^2 \\
&= (|\mathcal{A}|u, u)_{L^2} + \|u\|_2^2 = (\mathcal{A}Uu, u)_{L^2} + \|u\|_2^2 \\
&= ((\mathcal{A}+a_0)Uu, u)_{L^2} - a_0(Uu, u)_{L^2} + \|u\|_2^2 \\
&= \left(U(\mathcal{A}+a_0)^{1/2}u, (\mathcal{A}+a_0)^{1/2}u\right)_{L^2} - a_0(Uu, u)_{L^2} + \|u\|_2^2 \\
&\le \left\|U(\mathcal{A}+a_0)^{1/2}u\right\|_2 \left\|(\mathcal{A}+a_0)^{1/2}u\right\|_2 + (1+a_0)\|u\|_2^2 \\
&= \left\|(\mathcal{A}+a_0)^{1/2}u\right\|_2^2 + (1+a_0)\|u\|_2^2 \\
&= ((\mathcal{A}+a_0)u, u)_{L^2} + (1+a_0)\|u\|_2^2 \\
&\le (2+a_0)((\mathcal{A}+a_0)u, u)_{L^2} = (2+a_0)\|u\|_*^2.
\end{aligned} \tag{2.4}
$$

Combining (2.3) with (2.4), we have

$$\frac{1}{\sqrt{2+a_0}}\|u\|_0 \le \|u\|_* \le \sqrt{1+a_0}\|u\|_0, \quad \forall\ u \in C_0^\infty(\Omega). \tag{2.5}$$

Since $C_0^\infty(\Omega)$ is dense in E and E_*, it follows from (2.5) that (2.2) holds. ∎

Lemma 2.2. ([25, Theorem 1.9]). *The embeddings $H_0^1(\Omega) \subset L^s(\Omega)$ are compact for $1 \le s < 2^*$.*

Note that (2.1) implies that $\|u\|_{H^1(\Omega)} \le \|u\|_*$ for all $u \in E_*$, we have the following corollary.

Corollary 2.3. *The embeddings $E_* \subset L^s(\Omega)$ are compact for $1 \le s < 2^*$.*

Lemma 2.4. *Suppose that V satisfies* (V). *Let*

$$E^- = \mathcal{E}(0-)E, \quad E^0 = [\mathcal{E}(0) - \mathcal{E}(0-)]E, \quad E^+ = [\mathcal{E}(+\infty) - \mathcal{E}(0)]E. \tag{2.6}$$

Then for the inner products $(\cdot,\cdot)_0$ and $(\cdot,\cdot)_{L^2}$ on E, we have

$$E^-\perp E^0, \quad E^-\perp E^+, \quad E^0\perp E^+, \quad E = E^- \oplus E^0 \oplus E^+. \tag{2.7}$$

Furthermore, there hold

$$\dim(\mathcal{E}(M)E) < +\infty, \quad \forall\ M \ge 0, \tag{2.8}$$

$$
\begin{aligned}
E^0 &= \mathit{Ker}(\mathcal{A}), \quad \mathcal{A}u = -|\mathcal{A}|u, \ \ \forall\ u \in E^- \cap \mathfrak{D}(\mathcal{A}), \\
\mathcal{A}u &= |\mathcal{A}|u, \ \ \forall\ u \in E^+ \cap \mathfrak{D}(\mathcal{A})
\end{aligned} \tag{2.9}
$$

and

$$u = u^- + u^0 + u^+, \quad \forall\, u \in E, \tag{2.10}$$

where

$$\begin{aligned} u^- &= \mathcal{E}(0-)u \in E^-, \quad u^0 = [\mathcal{E}(0) - \mathcal{E}(0-)]u \in E^0, \\ u^+ &= [\mathcal{E}(+\infty) - \mathcal{E}(0)]u \in E^+. \end{aligned} \tag{2.11}$$

Proof. For $u \in E$, it follows from $\mathcal{E}(+\infty) = id$ that

$$u = \mathcal{E}(0-)u + [\mathcal{E}(0) - \mathcal{E}(0-)]u + [\mathcal{E}(+\infty) - \mathcal{E}(0)]u. \tag{2.12}$$

Since $\mathcal{E}(\lambda)E \subset E$ for $\lambda \in \mathbb{R}$, the above equation shows that (2.10) holds and $E = E^- + E^0 + E^+$. On the other hand, for $u \in E^-, v \in E^0$ and $w \in E^+$, there are $\tilde{u}, \tilde{v}, \tilde{w} \in L^2(\Omega, \mathbb{R}^N)$ such that

$$u = \mathcal{E}(0-)\tilde{u}, \quad v = [\mathcal{E}(0) - \mathcal{E}(0-)]\tilde{v}, \quad w = [\mathcal{E}(+\infty) - \mathcal{E}(0)]\tilde{w}.$$

Hence

$$\begin{aligned} (u, v)_{L^2} &= (\mathcal{E}(0-)\tilde{u}, [\mathcal{E}(0) - \mathcal{E}(0-)]\tilde{v})_{L^2} \\ &= ([\mathcal{E}(0) - \mathcal{E}(0-)]\mathcal{E}(0-)\tilde{u}, \tilde{v})_{L^2} = 0, \end{aligned} \tag{2.13}$$

$$\begin{aligned} (u, v)_0 &= \left(|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v\right)_{L^2} + (u, v)_{L^2} \\ &= \left(\mathcal{E}(0-)|\mathcal{A}|^{1/2}\tilde{u}, [\mathcal{E}(0) - \mathcal{E}(0-)]|\mathcal{A}|^{1/2}\tilde{v}\right)_{L^2} \\ &= \left([\mathcal{E}(0) - \mathcal{E}(0-)]\mathcal{E}(0-)|\mathcal{A}|^{1/2}\tilde{u}, |\mathcal{A}|^{1/2}\tilde{v}\right)_{L^2} = 0. \end{aligned} \tag{2.14}$$

Similarly, one has $(v, w)_{L^2} = (u, w)_{L^2} = 0$ and $(v, w)_0 = (u, w)_0 = 0$, which, together with (2.13), (2.14) and $E = E^- + E^0 + E^+$, implies that (2.7) holds.

For $u \in E$, it follows from (2.11) that

$$Uu^- = [\mathcal{E}(+\infty) - \mathcal{E}(0) - \mathcal{E}(0-)]\mathcal{E}(0-)u = -\mathcal{E}(0-)u = -u^- \tag{2.15}$$

and

$$Uu^+ = [\mathcal{E}(+\infty) - \mathcal{E}(0) - \mathcal{E}(0-)][\mathcal{E}(+\infty) - \mathcal{E}(0)]u = [\mathcal{E}(+\infty) - \mathcal{E}(0)]u = u^+. \tag{2.16}$$

Both (2.15) and (2.16) imply that

$$\mathcal{A}u^- = |\mathcal{A}|Uu^- = -|\mathcal{A}|u^-, \quad \mathcal{A}u^+ = |\mathcal{A}|Uu^+ = |\mathcal{A}|u^+, \quad \forall\, u \in E \cap \mathfrak{D}(\mathcal{A}). \tag{2.17}$$

For $u \in E$, it follows from (2.11) that

$$\mathcal{A}u^0 = |\mathcal{A}|Uu^0 = |\mathcal{A}|[\mathcal{E}(+\infty) - \mathcal{E}(0) - \mathcal{E}(0-)][\mathcal{E}(0) - \mathcal{E}(0-)]u = 0.$$

Hence, $E^0 \subset \mathrm{Ker}(\mathcal{A})$. Conversely, For any $u \in \mathrm{Ker}(\mathcal{A})$, $\mathcal{A}u = 0$ and so $\mathcal{A}^2 u = 0$, it follows that

$$0 = (\mathcal{A}^2 u, u)_{L^2} = \int_{-\infty}^{+\infty} \lambda^2 \ \mathrm{d}(\mathcal{E}(\lambda)u, u)_{L^2}. \tag{2.18}$$

Since $(\mathcal{E}(\lambda)u, u)_{L^2}$ is non-decreasing on $\lambda \in (-\infty, +\infty)$, then for any $\varepsilon > 0$, it follows from (2.18) that

$$\begin{aligned} 0 &= \int_{-\infty}^{+\infty} \lambda^2 \ \mathrm{d}(\mathcal{E}(\lambda)u, u)_{L^2} \\ &\geq \int_{-\infty}^{-\varepsilon} \lambda^2 \ \mathrm{d}(\mathcal{E}(\lambda)u, u)_{L^2} \geq \varepsilon^2 \int_{-\infty}^{-\varepsilon} \mathrm{d}(\mathcal{E}(\lambda)u, u)_{L^2} \\ &= \varepsilon^2 (\mathcal{E}(-\varepsilon)u, u)_{L^2} = \varepsilon^2 \|\mathcal{E}(-\varepsilon)u\|_2^2 \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} 0 &= \int_{-\infty}^{+\infty} \lambda^2 \ \mathrm{d}(\mathcal{E}(\lambda)u, u)_{L^2} \geq \int_{\varepsilon}^{+\infty} \lambda^2 \ \mathrm{d}(\mathcal{E}(\lambda)u, u)_{L^2} \\ &\geq \varepsilon^2 \int_{\varepsilon}^{+\infty} \mathrm{d}(\mathcal{E}(\lambda)u, u)_{L^2} \\ &= \varepsilon^2 [(u, u)_{L^2} - (\mathcal{E}(\varepsilon)u, u)_{L^2}] = \varepsilon^2 \|u - \mathcal{E}(\varepsilon)u\|_2^2. \end{aligned} \tag{2.20}$$

From (2.19) and (2.20), we obtain $[\mathcal{E}(\varepsilon) - \mathcal{E}(-\varepsilon)]u = u, \ \forall\ \varepsilon > 0$. Let $\varepsilon \to 0+$, then we can conclude that

$$[\mathcal{E}(0+) - \mathcal{E}(0-)]u = [\mathcal{E}(0) - \mathcal{E}(0-)]u = u.$$

This shows that $u \in E^0$, and so $\mathrm{Ker}(\mathcal{A}) \subset E^0$. Therefore, $\mathrm{Ker}(\mathcal{A}) = E^0$, which, together with (2.17), implies that (2.9) holds.

Finally, we prove that $\dim(\mathcal{E}(M)E) < +\infty, \ \forall\ M \geq 0$. If $\dim[\mathcal{E}(M)E] = +\infty$ for some $M_0 \geq 0$, then there exists a $\lambda_0 \in \sigma_e(\mathcal{A}) \cap (-\infty, M_0]$. By virtue of [5, Theorem IX 1.3] or [20, Theorem 4.5.2], there exists a sequence $\{u_n\} \subset \mathfrak{D}(\mathcal{A})$ such that

$$u_n \rightharpoonup 0, \quad \|u_n\|_2 = 1, \quad \|(\mathcal{A} - \lambda_0)u_n\|_2 \to 0. \tag{2.21}$$

Let $v_n = (\mathcal{A} - \lambda_0)u_n$. Then by (2.1) and (2.21), we have

$$\begin{aligned} \|u_n\|_*^2 &= \int_{\mathbb{R}^N} \left[|\nabla u_n|^2 + (V(x) + a_0)u_n^2\right] \mathrm{d}x \\ &= ((\mathcal{A} + a_0)u_n, u_n)_{L^2} = (v_n + (a_0 + \lambda_0)u_n, u_n)_{L^2} \\ &\leq \|u_n\|_2 \|v_n\|_2 + (a_0 + \lambda_0)\|u_n\|_2^2 \\ &= a_0 + \lambda_0 + o(1). \end{aligned} \tag{2.22}$$

(2.22) shows that $\{\|u_n\|_*\}$ is bounded. Passing to a subsequence if necessary, it can be assumed that $u_n \rightharpoonup u_0$ in $H_0^1(\Omega)$. Since $u_n \rightharpoonup 0$ in $L^2(\mathbb{R}^N)$, then $u_0 = 0$. It follows from Lemma 2.2 that $u_n \to 0$ in $L^2(\mathbb{R}^N)$. This contradiction shows that (2.8) holds. ∎

In view of Lemma 2.4, we introduce on E the following inner product

$$(u, v) = \left(|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v\right)_{L^2} + \left(u^0, v^0\right)_{L^2}$$

and norm

$$\|u\|^2 = (u, u) = \left\||\mathcal{A}|^{1/2}u\right\|_2^2 + \left\|u^0\right\|_2^2,$$

where $u = u^- + u^0 + u^+,\ v = v^- + v^0 + v^+ \in E^- \oplus E^0 \oplus E^+ = E$. Then it is easy to check the following lemma.

Lemma 2.5. *Suppose that V satisfies* (V). *Then for the inner product $(\cdot, \cdot)$ on E, we have*

$$E^- \perp E^0, \quad E^- \perp E^+, \quad E^0 \perp E^+. \tag{2.23}$$

Lemma 2.6. *Suppose that V satisfies* (V). *Then there exists a constant $\beta > 0$ such that*

$$\|u\|_2 \le \beta \|u\|, \quad \forall\ u \in E. \tag{2.24}$$

Proof. Since $\dim[\mathcal{E}(1)E] < +\infty$, there exists a constant $\beta_1 > 0$ such that

$$\|u\|_2 \le \beta_1 \|u\|, \quad \forall\ u \in \mathcal{E}(1)E. \tag{2.25}$$

On the other hand, we have

$$\begin{aligned} \|u\|^2 &= \left(|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}u\right)_{L^2} + \left\|u^0\right\|_2^2 = (|\mathcal{A}|u, u)_{L^2} \\ &= \int_1^{+\infty} |\lambda| \mathrm{d}(\mathcal{E}(\lambda)u, u)_{L^2} \ge \|u\|_2^2, \quad \forall\ u \in [\mathcal{E}(+\infty) - \mathcal{E}(1)]E. \end{aligned} \tag{2.26}$$

The conclusion of Lemma 2.6 follows by the combination of (2.25) with (2.26).

The following lemma follows immediately from Lemma 2.6.

Lemma 2.7. *Suppose that V satisfies* (V). *Then*

$$\|u\| \le \|u\|_0 \le \sqrt{1 + \beta^2}\|u\|, \quad \forall\ u \in E. \tag{2.27}$$

Remark 2.8. *Under condition* (V), *Lemmas 2.1 and 2.7 show that three norms* $\|\cdot\|_0$, $\|\cdot\|_*$ *and* $\|\cdot\|$ *on* E *are equivalent.*

Set

$$b(u,v)=\int_\Omega (\nabla u\nabla v+V(x)uv)\,\mathrm{d}x, \quad \forall\ u,v\in E. \tag{2.28}$$

Then it is easy to check the following lemma.

Lemma 2.9. *Suppose that* V *satisfies* (V). *Then* $b(u,v)$ *is a bilinear function on* E*, and*

$$|b(u,v)|\le \left(1+a_0+\beta^2+2a_0\beta^2\right)\|u\|\|v\|, \quad \forall\ u,v\in E, \tag{2.29}$$

$$\begin{aligned} &b(u^+,u^+)=\|u^+\|^2, \quad b(u^-,u^-)=-\|u^-\|^2,\\ &b(u^+,u^-+u^0)=0, \quad \forall\ u\in E, \end{aligned} \tag{2.30}$$

$$b(u,u)=\|u^+\|^2-\|u^-\|^2, \quad \forall\ u\in E. \tag{2.31}$$

By (2.28) and (2.31), we have

$$\Phi(u)=\frac{1}{2}\left(\|u^+\|^2-\|u^-\|^2\right)-\int_\Omega F(x,u)\mathrm{d}x, \quad \forall\ u\in E. \tag{2.32}$$

Under assumptions (V) and (F1), Φ is of class $C^1(E,\mathbb{R})$, and

$$\langle\Phi'(u),v\rangle=\int_\Omega (\nabla u\nabla v+V(x)uv)\,\mathrm{d}x-\int_\Omega f(x,u)v\mathrm{d}x, \quad \forall\ u,v\in E \tag{2.33}$$

and

$$\langle\Phi'(u),u\rangle=\|u^+\|^2-\|u^-\|^2-\int_\Omega f(x,u)u\mathrm{d}x, \quad \forall\ u\in E. \tag{2.34}$$

3. Existence of Ground State Solutions of Nehari-pankov Type

In this section, we give the proofs of Theorem 1.2 and Corollary 1.3.

Lemma 3.1. ([16, 17]). *Let* $X=Y\oplus Z$ *be a Banach space with* $\dim Y<\infty$. *Let* $r>\rho>0$, $\kappa>0$ *and* $e\in Z$ *with* $\|e\|=1$. *If* $\varphi\in C^1(X,\mathbb{R})$ *satisfies*

$$\inf\varphi(S_\rho)\ge\kappa>\sup\varphi(\partial Q),$$

where

$$S_\rho = \{u \in Z : \|u\| = \rho\}, \qquad Q = \{v + se : \|v + se\| \le r,\ v \in Y,\ s \ge 0\},$$

then there exist $c \in [\kappa, \sup \varphi(Q)]$ *and a sequence* $\{u_n\} \subset X$ *satisfying*

$$\varphi(u_n) \to c, \qquad \|\varphi'(u_n)\|(1 + \|u_n\|) \to 0. \tag{3.1}$$

Lemma 3.2. *Suppose that* (V), (F1) *and* (WN) *are satisfied. Then*

$$\begin{aligned}\Phi(u) \ge \Phi(tu + w) + \frac{1}{2}\|w^-\|^2 + \frac{1-t^2}{2}\langle \Phi'(u), u\rangle - t\langle \Phi'(u), w\rangle,\\ \forall\ u \in E,\quad t \ge 0,\quad w \in E^- \oplus E^0.\end{aligned} \tag{3.2}$$

Proof. For any $x \in \Omega$ and $\tau \ne 0$, (WN) yields

$$f(x,s) \le \frac{f(x,\tau)}{|\tau|}|s|, \qquad s \le \tau; \qquad f(x,s) \ge \frac{f(x,\tau)}{|\tau|}|s|, \qquad s \ge \tau. \tag{3.3}$$

It follows that

$$\left(\frac{1-t^2}{2}\tau^2 - t\tau\sigma\right)\frac{f(x,\tau)}{\tau} \ge \int_{t\tau+\sigma}^{\tau} f(x,s)\mathrm{d}s, \qquad t \ge 0, \quad \sigma \in \mathbb{R}. \tag{3.4}$$

To show (3.4), we consider four possible cases. Since $sf(x,s) \ge 0$, it follows from (3.3) that

(Case 1). $0 \le t\tau + \sigma \le \tau$ or $t\tau + \sigma \le \tau < 0$,

$$\int_{t\tau+\sigma}^{\tau} f(x,s)\mathrm{d}s \le \frac{f(x,\tau)}{|\tau|}\int_{t\tau+\sigma}^{\tau}|s|\mathrm{d}s \le \left(\frac{1-t^2}{2}\tau^2 - t\tau\sigma\right)\frac{f(x,\tau)}{\tau};$$

(Case 2). $t\tau + \sigma \le 0 < \tau$,

$$\int_{t\tau+\sigma}^{\tau} f(x,s)\mathrm{d}s \le \int_{0}^{\tau} f(x,s)\mathrm{d}s \le \frac{f(x,\tau)}{|\tau|}\int_{0}^{\tau}|s|\mathrm{d}s \le \left(\frac{1-t^2}{2}\tau^2 - t\tau\sigma\right)\frac{f(x,\tau)}{\tau};$$

(Case 3). $0 < \tau \le t\tau + \sigma$ or $\tau \le t\tau + \sigma \le 0$,

$$\int_{\tau}^{t\tau+\sigma} f(x,s)\mathrm{d}s \ge \frac{f(x,\tau)}{|\tau|}\int_{\tau}^{t\tau+\sigma}|s|\mathrm{d}s \ge -\left(\frac{1-t^2}{2}\tau^2 - t\tau\sigma\right)\frac{f(x,\tau)}{\tau};$$

(Case 4). $\tau < 0 < t\tau + \sigma$,

$$\int_{\tau}^{t\tau+\sigma} f(x,s)\mathrm{d}s \ge \int_{\tau}^{0} f(x,s)\mathrm{d}s \ge \frac{f(x,\tau)}{|\tau|}\int_{\tau}^{0}|s|\mathrm{d}s \ge -\left(\frac{1-t^2}{2}\tau^2 - t\tau\sigma\right)\frac{f(x,\tau)}{\tau}.$$

The above four cases show that (3.4) holds. By (1.2), (2.28) and (2.33), one has

$$\Phi(u) = \frac{1}{2}b(u,u) - \int_\Omega F(x,u)\mathrm{d}x, \quad \forall\, u \in E \tag{3.5}$$

and

$$\langle \Phi'(u), v\rangle = b(u,v) - \int_\Omega f(x,u)v\mathrm{d}x, \quad \forall\, u,v \in E. \tag{3.6}$$

Thus, by (2.30), (3.4), (3.5) and (3.6), one has

$$\begin{aligned}
&\Phi(u) - \Phi(tu+w)\\
=\ &\frac{1}{2}[b(u,u) - b(tu+w,tu+w)] + \int_\Omega [F(x,tu+w) - F(x,u)]\mathrm{d}x\\
=\ &\frac{1-t^2}{2}b(u,u) - tb(u,w) - \frac{1}{2}b(w,w) + \int_\Omega [F(x,tu+w) - F(x,u)]\mathrm{d}x\\
=\ &\frac{1}{2}\|w^-\|^2 + \frac{1-t^2}{2}\langle \Phi'(u),u\rangle - t\langle \Phi'(u),w\rangle\\
&+ \int_\Omega \left[\frac{1-t^2}{2}f(x,u)u - tf(x,u)w - \int_{tu+w}^{u} f(x,s)\mathrm{d}s\right]\mathrm{d}x\\
\geq\ &\frac{1}{2}\|w^-\|^2 + \frac{1-t^2}{2}\langle \Phi'(u),u\rangle - t\langle \Phi'(u),w\rangle, \quad \forall\, t \geq 0, \ \ w \in E^- \oplus E^0.
\end{aligned}$$

This shows that (3.2) holds. ■

From Lemma 3.2, we have the following two corollaries.

Corollary 3.3. *Suppose that* (V), (F1) *and* (WN) *are satisfied. Then for* $u \in \mathcal{N}^-$

$$\Phi(u) \geq \Phi(tu+w), \quad \forall\, t \geq 0, \ \ w \in E^- \oplus E^0. \tag{3.7}$$

Corollary 3.4. *Suppose that* (V), (F1) *and* (WN) *are satisfied. Then*

$$\begin{aligned}
\Phi(u) \geq\ &\frac{t^2}{2}\left(\|u^+\|^2 + \|u^-\|^2\right) - \int_\Omega F(x,tu^+)\mathrm{d}x + \frac{1-t^2}{2}\langle \Phi'(u),u\rangle\\
&+t^2\langle \Phi'(u), u^0 + u^-\rangle, \quad \forall\, u \in E, \ \ t \geq 0.
\end{aligned} \tag{3.8}$$

Applying Corollary 3.3, we can demonstrate the following lemma in the same way as [21, Lemma 2.4].

Lemma 3.5. *Suppose that* (V), (F1), (F2) *and* (WN) *are satisfied. Then*

(i) *there exists* $\rho > 0$ *such that*

$$m = \inf_{\mathcal{N}^-} \Phi \geq \zeta := \inf \left\{ \Phi(u) : u \in E^+, \|u\| = \rho \right\} > 0. \tag{3.9}$$

(ii) $\|u^+\| \geq \max\left\{ \|u^-\|, \sqrt{2m} \right\}$ *for all* $u \in \mathcal{N}^-$.

For $u \in E \setminus (E^- \oplus E^0)$, we define

$$\hat{E}(u) := E^- \oplus E^0 \oplus \mathbb{R}^+ u = E^- \oplus E^0 \oplus \mathbb{R}^+ u^+, \tag{3.10}$$

where as usual, $\mathbb{R}^+ = [0, \infty)$.

Lemma 3.6. *Suppose that* (V), (F1) *and* (F3) *are satisfied. Let* $e \in E^+$ *with* $\|e\| = 1$. *Then there is a* $r_e > \rho$ *such that* $\sup \Phi(\partial Q(r)) \leq 0$ *for* $r \geq r_e$, *where*

$$Q(r) = \left\{ w + se : \|w + se\| \leq r,\ w \in E^- \oplus E^0,\ s \geq 0 \right\}. \tag{3.11}$$

The proof of Lemma 3.6 is standard, so we omit it.

Lemma 3.7. *Suppose that* (V), (F1), (F2), (F3) *and* (WN) *are satisfied. Then for* $r \geq r_e$, *there exist* $c \in [\zeta, \sup \Phi(Q(r))]$ *and a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c, \qquad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0. \tag{3.12}$$

Proof. Let $X = E,\ Y = E^- \oplus E^0$ and $Z = E^+$. Then Lemma 3.7 is a direct consequence of Lemmas 2.4, 3.1, 3.5 (i) and 3.6. ∎

The following lemma is crucial to demonstrate the existence of ground state solutions for problem (1.1).

Lemma 3.8. *Suppose that* (V), (F1), (F2), (F3) *and* (WN) *are satisfied. Then there exist a constant* $c_* \in [\zeta, m]$ *and a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c_*, \qquad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0. \tag{3.13}$$

Proof. Choose $v_k \in \mathcal{N}^-$ such that

$$m \leq \Phi(v_k) < m + \frac{1}{k}, \qquad k \in \mathbb{N}. \tag{3.14}$$

By Lemma 3.5, $\|v_k^+\| \geq \sqrt{2m} > 0$. Set $e_k = v_k^+/\|v_k^+\|$. Then $e_k \in E^+$ and $\|e_k\| = 1$. In view of Lemma 3.7, there exists $r_k > \max\{\rho, \|v_k\|\}$ such that $\sup \Phi(\partial Q_k) \leq 0$, where

$$Q_k = \left\{ w + se_k : \|w + se_k\| \leq r_k,\ w \in E^- \oplus E^0, s \geq 0 \right\}, \qquad k \in \mathbb{N}. \tag{3.15}$$

Hence, applying Lemma 3.7 to the above set Q_k, there exist a sequence $\{u_{k,n}\}_{n\in\mathbb{N}} \subset E$ satisfying

$$\Phi(u_{k,n}) \to c_k, \quad \|\Phi'(u_{k,n})\|(1+\|u_{k,n}\|) \to 0, \quad k \in \mathbb{N}, \tag{3.16}$$

where $c_k \in [\zeta, \sup \Phi(Q_k)]$. By virtue Corollary 3.3, one has

$$\Phi(v_k) \geq \Phi(tv_k + w), \quad \forall\ t \geq 0, \ \ w \in E^- \oplus E^0. \tag{3.17}$$

Since $v_k \in Q_k$, it follows from (3.15) and (3.17) that $\Phi(v_k) = \sup \Phi(Q_k)$. Hence, by (3.14) and (3.16), one has

$$\Phi(u_{k,n}) \to c_k < m + \frac{1}{k}, \quad \|\Phi'(u_{k,n})\|(1+\|u_{k,n}\|) \to 0, \quad k \in \mathbb{N}. \tag{3.18}$$

Now, we can choose a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$\Phi(u_{k,n_k}) < m + \frac{1}{k}, \quad \|\Phi'(u_{k,n_k})\|(1+\|u_{k,n_k}\|) < \frac{1}{k}, \quad k \in \mathbb{N}. \tag{3.19}$$

Let $u_k = u_{k,n_k}, k \in \mathbb{N}$. Then, going if necessary to a subsequence, we have

$$\Phi(u_n) \to c_* \in [\zeta, m], \quad \|\Phi'(u_n)\|(1+\|u_n\|) \to 0. \qquad \blacksquare$$

Lemma 3.9. *Suppose that* (V), (F1), (F2), (F3) *and* (WN) *are satisfied. Then any sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c \geq 0, \quad \langle \Phi'(u_n), u_n^{\pm} \rangle \to 0, \quad \langle \Phi'(u_n), u_n^0 \rangle \to 0 \tag{3.20}$$

is bounded in E.

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$, then $\|v_n\| = 1$. By Corollary 2.3 and Remark 2.8, passing to a subsequence, we may assume that $v_n \to v$ in $L^s(\Omega)$, $2 \leq s < 2^*$, $v_n \to v$ a.e. on Ω. If $v^+ + v^0 = 0$ then $v_n^+ \to 0$ in $L^s(\Omega)$ for $2 \leq s < 2^*$ and $v_n^0 \to 0$ in E. Fix $R > [2(1+c)]^{1/2}$. By (F1) and (F2), there exists $C_1 > 0$ such that

$$\limsup_{n\to\infty} \int_\Omega F(x, Rv_n^+)\mathrm{d}x \leq R^2 \lim_{n\to\infty} \|v_n^+\|_2^2 + R^p C_1 \lim_{n\to\infty} \|v_n^+\|_p^p = 0. \tag{3.21}$$

Let $t_n = R/\|u_n\|$. Hence, by using (3.20), (3.21) and Corollary 3.4, one has

$$\begin{aligned} c+o(1) &= \Phi(u_n) \\ &\geq \frac{t_n^2}{2}\left(\|u_n^+\|^2+\|u_n^-\|^2\right)-\int_\Omega F(x,t_nu_n^+)\mathrm{d}x+\frac{1-t_n^2}{2}\langle\Phi'(u_n),u_n\rangle \\ &\quad +t_n^2\langle\Phi'(u_n),u_n^-+u_n^0\rangle \\ &= \frac{R^2}{2}\left(\|v_n^+\|^2+\|v_n^-\|^2\right)-\int_\Omega F(x,Rv_n^+)\mathrm{d}x+\left(\frac{1}{2}-\frac{R^2}{2\|u_n\|^2}\right)\langle\Phi'(u_n),u_n\rangle \\ &\quad +\frac{R^2}{\|u_n\|^2}\langle\Phi'(u_n),u_n^-+u_n^0\rangle \\ &= \frac{R^2}{2}+o(1)>c+1+o(1), \end{aligned}$$

which is a contradiction. Thus $v^+ + v^0 \neq 0$ and so $v \neq 0$.

For $x \in \{z \in \mathbb{R}^N : v(z) \neq 0\}$, we have $\lim_{n\to\infty}|u_n(x)| = \infty$. Hence, it follows from (3.20), (F3), (WN) and Fatou's lemma that

$$\begin{aligned} 0 &= \lim_{n\to\infty}\frac{c+o(1)}{\|u_n\|^2}=\lim_{n\to\infty}\frac{\Phi(u_n)}{\|u_n\|^2} \\ &= \lim_{n\to\infty}\left[\frac{1}{2}\left(\|v_n^+\|^2-\|v_n^-\|^2\right)-\int_\Omega\frac{F(x,u_n)}{u_n^2}v_n^2\mathrm{d}x\right] \\ &\leq \frac{1}{2}-\liminf_{n\to\infty}\int_\Omega\frac{F(x,u_n)}{u_n^2}v_n^2\mathrm{d}x\leq\frac{1}{2}-\int_\Omega\liminf_{n\to\infty}\frac{F(x,u_n)}{u_n^2}v_n^2\mathrm{d}x \\ &= -\infty. \end{aligned}$$

This contradiction shows that $\{u_n\}$ is bounded. ■

Proof of Theorem 1.2. Applying Lemmas 3.8 and 3.9. we deduce that there exists a bounded sequence $\{u_n\} \subset E$ satisfying (3.13). Going if necessary to a subsequence, we can assume that $u_n \rightharpoonup u_0$ in E. Base on Corollary 2.3 and Remark 2.8, $u_n \to u_0$ in $L^s(\Omega)$ for $2 \leq s < 2^*$ and $u_n \to u_0$ a.e. on Ω. Employing [24, Lemma 2.3], one can get that

$$\int_\Omega |f(x,u_n)-f(x,u_0)||u_n-u_0|\mathrm{d}x \to 0. \tag{3.22}$$

Observe that

$$\begin{aligned} &b(u_n-u_0,u_n-u_0) \\ &= \langle\Phi'(u_n)-\Phi'(u_0),u_n-u_0\rangle+\int_\Omega[f(x,u_n)-f(x,u_0)](u_n-u_0)\mathrm{d}x. \end{aligned} \tag{3.23}$$

It is clear that

$$\langle\Phi'(u_n)-\Phi'(u_0),u_n-u_0\rangle \to 0. \tag{3.24}$$

From (3.22)-(3.24), we have $b(u_n - u_0, u_n - u_0) \to 0$, it follows that

$$\|u_n^+ - u_0^+\|^2 - \|u_n^- - u_0^-\|^2 \to 0. \tag{3.25}$$

Since $u_n \rightharpoonup u_0$ in E, it follows that

$$u_n^- \rightharpoonup u_0^- \ \text{ in } \ E^-, \quad u_n^0 \rightharpoonup u_0^0 \ \text{ in } \ E^0, \quad u_n^+ \rightharpoonup u_0^+ \ \text{ in } \ E^+.$$

Note that $\dim(E^- \oplus E^0) < +\infty$, it follows that

$$\left\|u_n^0 - u_0^0\right\|^2 + \left\|u_n^- - u_0^-\right\|^2 \to 0. \tag{3.26}$$

Combining (3.25) with (3.26), we have

$$\|u_n - u_0\|^2 = \left\|u_n^+ - u_0^+\right\|^2 + \left\|u_n^0 - u_0^0\right\|^2 + \left\|u_n^- - u_0^-\right\|^2 \to 0. \tag{3.27}$$

Hence, it follows from (3.13) and (3.27) that $\Phi(u_0) = c_* \leq m$ and $\Phi'(u_0) = 0$. This shows that $u_0 \in \mathcal{N}^-$ and so $\Phi(u_0) \geq m$. Therefore $\Phi(u_0) = m = \inf_{\mathcal{N}^-} \Phi > 0$. ■

Proof of Corollary 1.3. Let $\bar{V}(x) := V(x) - \bar{a}$ and $\bar{f}(x,t) = f(x,t) - \bar{a}t$. Then problem (1.1) is equivalent to the following problem

$$\begin{cases} -\triangle u + \bar{V}(x)u = \bar{f}(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{3.28}$$

It is easy to see that $\bar{V}$ satisfies (V), moreover, (F1), (F2′), (F3) and (WN) imply that $\bar{f}$ satisfies (F1), (F2), (F3) and (WN). Furthermore,

$$\Phi(u) = \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \bar{V}(x)u^2\right) \mathrm{d}x - \int_\Omega \bar{F}(x,u)\mathrm{d}x, \quad \forall\ u \in E$$

and

$$\langle \Phi'(u), u\rangle = \int_\Omega \left(|\nabla u|^2 + \bar{V}(x)u^2\right) \mathrm{d}x - \int_\Omega \bar{f}(x,u)u\mathrm{d}x, \quad \forall\ u \in E,$$

where $\bar{F}(x,t) = \int_0^t \bar{f}(x,s)\mathrm{d}s$. Hence, Theorem 1.2 yields Corollary 1.3. ■

4. Existence of Infinitely Many Solutions

In this section, we are concerned with the existence of infinitely many solutions for (1.1).

Let $\tilde{V}(x) := V(x) + a_0$, $\tilde{f}(x,t) = f(x,t) + a_0 t$ and $\tilde{F}(x,t) = \int_0^t \tilde{f}(x,s)\mathrm{d}s$. Then

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\int_\Omega \left(|\nabla u|^2 + \tilde{V}(x)u^2\right) \mathrm{d}x - \int_\Omega \tilde{F}(x,u)\mathrm{d}x \\ &= \frac{1}{2}\|u\|_*^2 - \int_\Omega \tilde{F}(x,u)\mathrm{d}x, \quad \forall\ u \in E_* \end{aligned} \tag{4.1}$$

and

$$(4.2)\qquad \begin{aligned}\langle \Phi'(u), v\rangle &= \int_\Omega \Big(\nabla u \nabla u + \tilde{V}(x)uv\Big)\,\mathrm{d}x - \int_\Omega \tilde{f}(x,u)v\mathrm{d}x\\ &= (u,v)_* - \int_\Omega \tilde{f}(x,u)v\mathrm{d}x, \qquad \forall\ u,v\in E_*.\end{aligned}$$

Lemma 4.1. *Suppose that* (V), (F1), (F2″), (F3) *and* (F4) *are satisfied. Then any sequence* $\{u_n\}\subset E_*$ *satisfying*

$$(4.3)\qquad \Phi(u_n)\to c\geq 0, \qquad \langle \Phi'(u_n), u_n\rangle \to 0$$

is bounded in E_*.

Proof. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\|_*\to\infty$. Let $v_n = u_n/\|u_n\|_*$. Then $\|v_n\|_* = 1$ and $\|v_n\|_s \leq \gamma_s\|v_n\|_* = \gamma_s$ for $2\leq s<2^*$, where γ_s is the embedding constant. Observe that for n large

$$(4.4)\qquad c+1\geq \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle = \int_\Omega \left[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right]\mathrm{d}x.$$

It follows from (4.1) and (4.3) that

$$(4.5)\qquad \frac{1}{2}\leq \limsup_{n\to\infty}\int_\Omega \frac{|\tilde{F}(x,u_n)|}{\|u_n\|_*^2}\mathrm{d}x.$$

For $0\leq \xi<\eta$, let

$$(4.6)\qquad \Omega_n(\xi,\eta) = \{x\in\Omega : \xi\leq |u_n(x)|<\eta\}.$$

Passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E_*, then by Corollary 2.3, $v_n\to v$ in $L^s(\Omega)$, $2\leq s<2^*$, and $v_n\to v$ a.e. on Ω.

If $v=0$, then $v_n\to 0$ in $L^s(\Omega)$, $2\leq s<2^*$, $v_n\to 0$ a.e. on Ω. Hence, it follows from (F2″) that there exists a constant $C_2>0$ such that

$$(4.7)\qquad \int_{\Omega_n(0,R_0)} \frac{|\tilde{F}(x,u_n)|}{|u_n|^2}|v_n|^2\mathrm{d}x \leq C_2\int_{\Omega_n(0,R_0)} |v_n|^2\mathrm{d}x \leq C_2\int_\Omega |v_n|^2\mathrm{d}x\to 0.$$

Set $\kappa' = \kappa/(\kappa-1)$. Since $\kappa>\max\{1, N/2\}$, one sees that $2\kappa'\in(2,2^*)$. Hence, from (F1), (F2″), (F4) and (4.4), one has

$$
\begin{aligned}
&\int_{\Omega_n(R_0,\infty)} \frac{|\tilde{F}(x,u_n)|}{|u_n|^2}|v_n|^2\mathrm{d}x \\
&\leq \left[\int_{\Omega_n(R_0,\infty)} \left(\frac{\tilde{F}(x,u_n)}{|u_n|^2}\right)^{\kappa}\mathrm{d}x\right]^{1/\kappa}\left[\int_{\Omega_n(R_0,\infty)}|v_n|^{2\kappa'}\mathrm{d}x\right]^{1/\kappa'} \\
&\leq 2\left\{\int_{\Omega_n(R_0,\infty)}\left[a_0^{\kappa}+\left(\frac{F(x,u_n)}{|u_n|^2}\right)^{\kappa}\right]\mathrm{d}x\right\}^{1/\kappa}\left(\int_{\Omega}|v_n|^{2\kappa'}\mathrm{d}x\right)^{1/\kappa'} \\
&\leq 2\left\{\int_{\Omega_n(R_0,\infty)}[a_0^{\kappa}+c_0(f(x,u_n)u_n-2F(x,u_n)]\,\mathrm{d}x\right\}^{1/\kappa}\left(\int_{\Omega}|v_n|^{2\kappa'}\mathrm{d}x\right)^{1/\kappa'} \\
&\leq 2\left[a_0^{\kappa}|\Omega|+2c_0(c+1)+c_0\int_{\Omega_n(0,R_0)}|f(x,u_n)u_n-2F(x,u_n)|\mathrm{d}x\right]^{1/\kappa} \\
&\qquad\left(\int_{\Omega}|v_n|^{2\kappa'}\mathrm{d}x\right)^{1/\kappa'} \\
&\leq C_3\left(\int_{\Omega}|v_n|^{2\kappa'}\mathrm{d}x\right)^{1/\kappa'}\to 0.
\end{aligned}
\tag{4.8}
$$

Combining (4.7) with (4.8), we have

$$
\int_{\Omega}\frac{|\tilde{F}(x,u_n)|}{\|u_n\|_*^2}\mathrm{d}x=\int_{\Omega_n(0,R_0)}\frac{|\tilde{F}(x,u_n)|}{|u_n|^2}|v_n|^2\mathrm{d}x+\int_{\Omega_n(R_0,\infty)}\frac{|\tilde{F}(x,u_n)|}{|u_n|^2}|v_n|^2\mathrm{d}x\to 0,
$$

which contradicts (4.5). Hence, $v\neq 0$.

Set $A:=\{x\in\Omega: v(x)\neq 0\}$. If $v\neq 0$, then meas$(A)>0$. For a.e. $x\in A$, we have $\lim_{n\to\infty}|u_n(x)|=\infty$. Hence $A\subset\Omega_n(R_0,\infty)$ for large $n\in\mathbb{N}$, it follows from (F1), (F2$''$), (F4), (4.1) and Fatou's lemma that

$$
\begin{aligned}
0&=\lim_{n\to\infty}\frac{c+o(1)}{\|u_n\|_*^2}=\lim_{n\to\infty}\frac{\Phi(u_n)}{\|u_n\|_*^2} \\
&=\lim_{n\to\infty}\left[\frac{1}{2}-\int_{\Omega}\frac{\tilde{F}(x,u_n)}{u_n^2}v_n^2\mathrm{d}x\right] \\
&=\lim_{n\to\infty}\left[\frac{1}{2}-\int_{\Omega_n(0,R_0)}\frac{\tilde{F}(x,u_n)}{u_n^2}v_n^2\mathrm{d}x-\int_{\Omega_n(R_0,\infty)}\frac{\tilde{F}(x,u_n)}{u_n^2}v_n^2\mathrm{d}x\right] \\
&\leq\limsup_{n\to\infty}\left[\frac{1}{2}+C_2\int_{\Omega}|v_n|^2\mathrm{d}x-\int_{\Omega_n(R_0,\infty)}\frac{\tilde{F}(x,u_n)}{u_n^2}v_n^2\mathrm{d}x\right] \\
&\leq\frac{1}{2}+C_4-\liminf_{n\to\infty}\int_{\Omega_n(R_0,\infty)}\frac{\tilde{F}(x,u_n)}{u_n^2}v_n^2\mathrm{d}x \\
&=\frac{1}{2}+C_4-\liminf_{n\to\infty}\int_{\Omega}\frac{|\tilde{F}(x,u_n)|}{u_n^2}[\chi_{\Omega_n(R_0,\infty)}(x)]v_n^2\mathrm{d}x \\
&\leq\frac{1}{2}+C_4-\int_{\Omega}\liminf_{n\to\infty}\frac{\tilde{F}(x,u_n)}{u_n^2}[\chi_{\Omega_n(R_0,\infty)}(x)]v_n^2\mathrm{d}x \\
&=-\infty,
\end{aligned}
\tag{4.9}
$$

which is a contradiction. Thus $\{u_n\}$ is bounded in E. ∎

Lemma 4.2. *Suppose that* (V), (F1), (F2″), (F3) *and* (F4) *are satisfied. Then any sequence* $\{u_n\} \subset E_*$ *satisfying* (4.3) *has a convergent subsequence in* E_*.

The proof is similar to that of [24, Lemma 2.6], so we omit it.

Lemma 4.3. *Suppose that* (V), (F1), (F2″), (F3) *and* (F4) *are satisfied. Then for any finite dimensional subspace* $\tilde{E} \subset E_*$, *there is* $R = R(\tilde{E}) > 0$ *such that*

$$\Phi(u) \leq 0, \quad \forall\ u \in \tilde{E}, \ \ \|u\| \geq R. \tag{4.10}$$

The proof is similar to that of [24, Corollary 2.9], so we omit it.

Let $\{e_j\}$ is an orthonomormal basis of E_* and define $X_j = \mathbb{R}e_j$,

$$Y_k = \oplus_{j=1}^{k} X_j, \quad Z_k = \oplus_{j=k+1}^{\infty} X_j, \quad k \in \mathbb{Z}. \tag{4.11}$$

Lemma 4.4. *Suppose that* (V) *is satisfied. Then for* $2 \leq s < 2^*$,

$$\beta_k(s) := \sup_{u \in Z_k, \|u\|_* = 1} \|u\|_s \to 0, \quad k \to \infty. \tag{4.12}$$

The proof is similar to that of [25, Lemma 3.8], so we omit it.

By (F1) and (F2″), there exists a constant $C_5 > 0$ such that

$$\tilde{F}(x,t) \leq C_5 \left(|t^2| + |t|^p\right), \quad \forall\ (x,t) \in \Omega \times \mathbb{R}.$$

In view of Lemma 4.4, we can choose an integer $m \geq 1$ such that

$$\|u\|_2^2 \leq \frac{1}{4C_5}\|u\|_*^2, \quad \|u\|_p^p \leq \frac{1}{4C_5}\|u\|_*^p, \quad \forall\ u \in Z_m. \tag{4.13}$$

Lemma 4.5. *Suppose that* (V), (F1) *and* (F2″) *are satisfied. Then there exist constants* $\rho, \alpha > 0$ *such that* $\Phi|_{\partial B_\rho \cap Z_m} \geq \alpha$.

The proof is similar to that of [24, Lemma 2.11], so we omit it.

We say that $I \in C^1(X, \mathbb{R})$ satisfies (C)$_c$-condition if any sequence $\{u_n\}$ such that

$$I(u_n) \to c, \quad \|I'(u_n)\|(1 + \|u_n\|) \to 0 \tag{4.14}$$

has a convergent subsequence.

Lemma 4.6. [4, 16] *Let X be an infinite dimensional Banach space, $X = Y \oplus Z$, where Y is finite dimensional. If $I \in C^1(X, \mathbb{R})$ satisfies (C)$_c$-condition for all $c > 0$, and*

(I1) *$I(0) = 0$, $I(-u) = I(u)$ for all $u \in X$;*

(I2) *there exist constants $\rho, \alpha > 0$ such that $\Phi|_{\partial B_\rho \cap Z} \geq \alpha$;*

(I3) *for any finite dimensional subspace $\tilde{X} \subset X$, there is $R = R(\tilde{X}) > 0$ such that $I(u) \leq 0$on $\tilde{X} \setminus B_R$;*

then I possesses an unbounded sequence of critical values.

Proof of Theorem 1.4. Let $X = E_*, Y = Y_m$ and $Z = Z_m$. By (F1), (F2$''$) and (F5), Lemmas 4.1, 4.2, 4.3 and 4.5, all conditions of Lemma 4.6 are satisfied. Thus, the following problem

$$\begin{cases} -\triangle u + \tilde{V}(x)u = \tilde{f}(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases} \tag{4.15}$$

possesses infinitely many nontrivial solutions, and so problem (1.1) also possesses infinitely many nontrivial solutions. Note that $\Phi(-u) = \Phi(u)$ for all $u \in E_*$, then problem (1.1) possesses infinitely many pairs of solutions. ■

Proof of Corollary 1.5. By (F1), there exists a constant $C_6 > 0$ such that $|F(x,t)| \leq C_6|t|^p$ for all $(x,|t|) \in \Omega \times [R_0, \infty)$. Let $\kappa = (p-\nu)/(p-2)$. Then $\kappa > \max\{1, N/2\}$, and

$$\frac{|F(x,t)|^{\kappa-1}}{c_1 C_6^{\kappa-1}|t|^{2\kappa}} \leq \frac{1}{c_1|t|^{2\kappa-p(\kappa-1)}} = \frac{1}{c_1|t|^\nu}, \qquad (x,t) \in \Omega \times \mathbb{R}, \quad |t| \geq R_0,$$

which, together with (F4$'$), implies that (F4) holds with $c_0 = c_1 C_6^{\kappa-1}$. ■

ACKNOWLEDGEMENTS

The authors would like to thank the referees for valuable comments and suggestions.

REFERENCES

1. A. Ambrosetti, H. Brezis and G. Cerami, Combined effects of concave and convex nonlinearities in some elliptic problems, *J. Funct. Anal.*, **122** (1994), 519-543.

2. A. Ambrosetti and A. Malchiodi, *Nonlinear Analysis and Semilinear Elliptic Problems*, Cambridge Studies in Advanced Mathematics 104, Cambridge University Press, Cambridge, 2007.

3. A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Anal.*, **14** (1973), 349-381.

4. T. Bartolo, V. Benci and D. Fortunato, Abstract critical point theorems and applications to some nonlinear problems with strong resonance at infinity, *Nonlinear Anal.*, **7** (1983), 241-273.

5. D. E. Edmunds and W. D. Evans, *Spectral Theory and Differential Operators*, Clarendon Press, Oxford, 1987.

6. L. Jeanjean, On the existence of bounded Palais-Smale sequences and application to a Landesman-Lazer type problem set on $\mathbb{R}^N$, *Proc. Roy. Soc. Edinburgh Sect. A*, **129** (1999), 787-809.

7. G. B. Li and A. Szulkin, An asymptotically periodic Schrödinger equation with indefinite linear part, *Commun. Contemp. Math.*, **4** (2002), 763-776.

8. G. B. Li and C. Wang, The existence of a nontrivial solution to a nonlinear elliptic problem of linking type without the Ambrosetti-Rabinowitz condition, *Annales Academiæ Scientiarum Fennicæ Mathematica*, **36** (2011), 461-480.

9. G. B. Li and H. S. Zhou, Asympotically linear Dirichlet problem for p-Laplacian, *Nonlinear Anal.*, **43** (2001), 1043-1055.

10. G. B. Li and H. S. Zhou, Multiple solutions to p-Laplacian problems with asympotic nonlinearity as u^{p-1} at infinity, *J. London Math. Soc.*, **65(2)** (2002), 123-138.

11. G. B. Li and H. S. Zhou, Dirichlet problem of p-Laplacian with nonlinear term $f(x;t) \sim t^{p-1}$ at infinity, In: Morse theory, Minimax theory and their applications to nonlinear partial differential equations, edited by H. Brezis, S. J. Li, J. Q. Liu and P. H. Rabinowitz, *New Stud. Adv. Math.*, **1** (2003), 77-89.

12. S. Liu, On superlinear Schrödinger equations with periodic potential, *Calc. Var. Partial Differential Equations*, **45** (2012), 1-9.

13. T.-C. Ouyang and J.-P. Shi, Exact multiplicity of positive solutions for a class of semilinear problem, *J. Differential Equations*, **146** (1998), 121-156.

14. T.-C. Ouyang and J.-P. Shi, Exact multiplicity of positive solutions for a class of semilinear problem: II, *J. Differential Equations*, **158** (1999), 94-151.

15. A. Pankov, Periodic nonlinear Schrödinger equation with application to photonic crystals, *Milan J. Math.*, **73** (2005), 259-287.

16. P. H. Rabinowitz, *Minimax Methods in Critical Point Theory with Applications to Differential Equations*, CBMS Reg. Conf. Series Math. 65, Amer. Math. Soc., Providence, R.I., 1986.

17. M. Schechter, *Minimax Systems and Critical Point Theory*, Birkhäuser, Boston, 2009.

18. M. Schechter and W. Zou, Superlinear problems, *Pacific J. Math.*, **214** (2004), 145-160.

19. M. Struwe, *Variational Methods*, Springer-Verlag, Berlin, 1990.

20. J. Sun and Z. Wang, *Spectral Analysis for Linear Operators*, Science Press, Beijing, 2005 (Chinese).

21. A. Szulkin and T. Weth, Ground state solutions for some indefinite variational problems, *J. Funct. Anal.*, **257(12)** (2009), 3802-3822.

22. A. Szulkin and T. Weth, The method of Nehari manifold, in: *Handbook of Nonconvex Analysis and Applications*, D. Y. Gao and D. Motreanu eds., International Press, Boston, 2010, pp. 597-632.

23. X. H. Tang, Infinitely many solutions for semilinear Schrödinger equations with sign-changing potential and nonlinearity, *J. Math. Anal. Appl.*, **401** (2013), 407-415.

24. X. H. Tang, New super-quadratic conditions on ground state solutions for superlinear Schrödinger equation, *Advance Nonlinear Studies*, **14** (2014), 349-361.

25. M. Willem, *Minimax Theorems*, Birkhäuser, Boston, 1996.

X. H. Tang
School of Mathematics and Statistics
Central South University
Changsha, Hunan 410083
P. R. China
E-mail: tangxh@mail.csu.edu.cn