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THE MULTIPLE HURWITZ ZETA FUNCTION AND THE MULTIPLE HURWITZ-EULER ETA FUNCTION

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Abstract. Almost eleven decades ago, Barnes introduced and made a systematic investigation on the multiple Gamma functions Γ_n. In about the middle of 1980s, these multiple Gamma functions were revived in the study of the determinants of Laplacians on the n-dimensional unit sphere $\mathbf{S}^n$ by using the multiple Hurwitz zeta functions $\zeta_n(s, a)$. In this paper, we first aim at presenting a generalized Hurwitz formula for $\zeta_n(s, a)$ together with its various special cases. Secondly, we give analytic continuations of multiple Hurwitz-Euler eta function $\eta_n(s, a)$ in two different ways. As a by-product of our second investigation, a relationship between $\eta_n(-\ell, a)$ $(\ell \in \mathbb{N}_0)$ and the generalized Euler polynomials $E_\ell^{(n)}(n - a)$ is also presented.

1. Introduction and Preliminaries

Barnes [4] (see also [1, 2, 3]) introduced and studied the generalized multiple Hurwitz zeta function $\zeta_n\,(s, a \,|\, w_1, \cdots, w_n)$ defined, for $\Re(s) > n$, by the following n-ple series:

$$\zeta_n\,(s, a \,|\, w_1, \cdots, w_n) := \sum_{k_1, \cdots, k_n = 0}^{\infty} \frac{1}{(a + \Omega)^s} \quad (\Re(s) > n;\; n \in \mathbb{N}), \tag{1.1}$$

where $\mathbb{N}$ denotes the set of positive integers,

$$\Omega = k_1\, w_1 + \cdots + k_n\, w_n$$

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and the general conditions for a and the parameters $w_1, \cdots, w_n$ are given in the work of Barnes [4] who used it in the study of the multiple Gamma functions Γ_n (see [12, Section 1.3]). In about the middle of 1980s, the multiple Gamma functions Γ_n were revived in the study of the determinants of Laplacians on the n-dimensional unit sphere $\mathbf{S}^n$ (see, e.g., [5, 8, 10, 11, 13, 14]).

Here we consider only the following simple case of (1.1) when

$$w_j = 1 \quad (j = 1, \cdots, n;\ j, n \in \mathbb{N}).$$

Thus, for

$$\mathbb{Z}_0^- := \mathbb{Z} \setminus \mathbb{N} \quad (\mathbb{Z} := \{0, \pm 1, \pm 2, \cdots\})$$

and $\mathbb{C}$ the set of complex numbers, we write this special case of (1.1) as follows:

$$\zeta_n(s, a) := \sum_{k_1, \cdots, k_n = 0}^{\infty} (a + k_1 + \cdots + k_n)^{-s} \tag{1.2}$$
$$\big(\Re(s) > n;\ n \in \mathbb{N};\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-\big),$$

which is also referred to as the *n-ple* (or, simply, the *multiple*) *Hurwitz zeta function*.

We first need to summarize some of the known (and useful) properties of the function $\zeta_n(s, a)$ in (1.2) (see, e.g., [12, Chapter 2]) with a view to making them readily and easily accessible for their application in our subsequent sections (especially in Section 2).

It is easy to see that

$$\zeta_1(s, a) = \zeta(s, a)$$

is the Hurwitz (or generalized) zeta function $\zeta(s, a)$ defined by

$$\zeta(s, a) := \sum_{k=0}^{\infty} (k + a)^{-s} \quad \big(\Re(s) > 1;\ a \in \mathbb{C} \setminus \mathbb{Z}_0^-\big), \tag{1.3}$$

which, upon setting $a = 1$, yields the Riemann zeta function $\zeta(s)$ defined by

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1 - 2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & \big(\Re(s) > 1\big) \\[4mm] \displaystyle\frac{1}{(1 - 2^{1-s})} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & \big(\Re(s) > 0;\ s \neq 1\big). \end{cases} \tag{1.4}$$

Both the Riemann zeta function $\zeta(s)$ and the Hurwitz zeta function $\zeta(s, a)$ can be continued meromorphically to the whole complex s-plane, except for a simple pole only at $s = 1$ with their respective residue 1, in many different ways (see also the recent investigations by Garg *et al.* [7] and Lin *et al.* [9]).

An integral representation of $\zeta_n(s,a)$ is given by

$$\Gamma(s)\,\zeta_n(s,a)=\int_0^\infty \frac{x^{s-1}e^{-ax}}{(1-e^{-x})^n}dx \quad \big(\Re(s)=\sigma>n;\ n\in\mathbb{N}\big). \tag{1.5}$$

As a way to extend $\zeta_n(s,a)$ to the half-plane on the left of the line $\sigma=\Re(s)=n$, we recall a contour integral as follows:

$$I_n(s,a)=-\frac{1}{2\pi i}\int_C \frac{(-z)^{s-1}e^{-az}}{(1-e^{-z})^n}\,dz \quad (a>0), \tag{1.6}$$

which defines an entire function of s. Moreover, we have

$$\zeta_n(s,a)=\Gamma(1-s)\,I_n(s,a) \quad \big(\Re(s)=\sigma>n;\ n\in\mathbb{N}\big). \tag{1.7}$$

The contour C is essentially a Hankel's loop (cf., e.g., Whittaker and Watson [15, p. 245]), which starts from ∞ along the upper side of the positive real axis, encircles the origin once in the positive (counter-clockwise) direction, and then returns to ∞ along the lower side of the positive real axis. In fact, the loop C can be decomposed into three parts C_1, C_2 and C_3, where C_2 is a positively-oriented circle of radius $c<2\pi$ centred at the origin, and C_1 and C_3 are the upper and lower edges of a cut in the complex z-plane along the positive real axis, traversed as given in Figure 1.

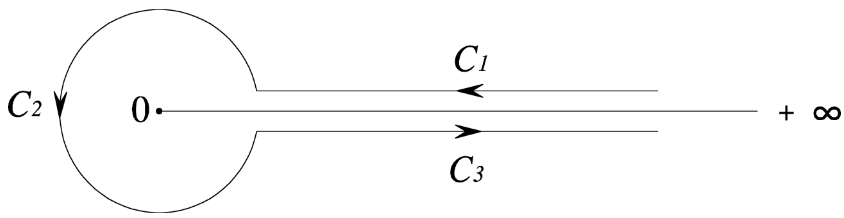

Fig. 1.

Since $I_n(s,a)$ is an entire function of s for $a>0$, the only possible singularities of $\zeta_n(s,a)$ are the poles of $\Gamma(1-s)$. It is known that $\Gamma(1-s)$ has simple poles at $s\in\mathbb{N}$. But the series definition (1.2) shows that $\zeta_n(s,a)$ is analytic in

$$\Re(s)=\sigma>n \quad (n\in\mathbb{N}),$$

and so $s=1,\cdots,n$ are the only poles of $\zeta_n(s,a)$. If, on the other hand,

$$\Re(s)=\sigma\leqq n \quad (n\in\mathbb{N}),$$

we define $\zeta_n(s,a)$ by

$$\zeta_n(s,a):=\Gamma(1-s)\,I_n(s,a), \tag{1.8}$$

where $I_n(s, a)$ is given in (1.6). This equation (1.8) provides the analytic continuation of $\zeta_n(s, a)$ to the whole complex s-plane. The function $\zeta_n(s, a)$ defined by (1.8) is analytic for all s except for simple poles at

$$s = k \quad (1 \leqq k \leqq n;\ n,\ k \in \mathbb{N}).$$

From (1.6) and (1.8) we can get the following relationship between $\zeta_n\,(s, a)$ and $B_n^{(\alpha)}(a)$:

$$\zeta_n(-\ell, a) = (-1)^n \frac{\ell!}{(n+\ell)!} B_{n+\ell}^{(n)}(a) \quad (\ell \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}), \tag{1.9}$$

where the generalized Bernoulli polynomials $B_n^{(\alpha)}(a)$ of degree n in a are defined by the generating function:

$$\left(\frac{z}{e^z - 1}\right)^{\alpha} e^{az} = \sum_{n=0}^{\infty} B_n^{(\alpha)}(a) \frac{z^n}{n!} \quad (|z| < 2\pi;\ 1^{\alpha} := 1) \tag{1.10}$$

for arbitrary (real or complex) parameter α. Clearly, we have

$$B_n^{(\alpha)}(a) = (-1)^n\, B_n^{(\alpha)}(\alpha - a), \tag{1.11}$$

so that

$$B_n^{(\alpha)}(\alpha) = (-1)^n\, B_n^{(\alpha)}(0) =: (-1)^n\, B_n^{(\alpha)} \tag{1.12}$$

in terms of the *generalized Bernoulli numbers* $B_n^{(\alpha)}$ defined by the generating function:

$$\left(\frac{z}{e^z - 1}\right)^{\alpha} = \sum_{n=0}^{\infty} B_n^{(\alpha)} \frac{z^n}{n!} \quad (|z| < 2\pi;\ 1^{\alpha} := 1). \tag{1.13}$$

It is easily observed that

$$B_n^{(1)}(a) = B_n(a) \quad \text{and} \quad B_n^{(1)} = B_n \quad (n \in \mathbb{N}_0), \tag{1.14}$$

where $B_n(a)$ and B_n are the familiar Bernoulli polynomials and Bernoulli numbers, respectively.

Setting $n = 1$ in (1.9), we have the following well-known result:

$$\zeta(-\ell, a) = -\frac{B_{\ell+1}(a)}{\ell + 1} \quad (\ell \in \mathbb{N}_0). \tag{1.15}$$

where $\zeta(s, a)$ is the Hurwitz (or generalized) zeta function given in (1.3).

Choi [6] (see also [12, Section 2.2]) presented another analytic continuation of $\zeta_n(s,a)$, which is different from the contour integral representation (1.8), by expressing it as a finite linear combination of the Hurwitz (or generalized) zeta functions $\zeta(s,a)$ as follows.

Proposition. *The number of solutions of*

$$k_1+\cdots+k_n=k \quad (k\in\mathbb{N}_0;\ (k_1,\cdots,k_n)\in{\mathbb{N}_0}^n)$$

is equal to the coefficient of a^k *in the following Taylor-Maclaurin series expansion of* $(1-a)^{-n}$:

$$(1-a)^{-n}=\sum_{k=0}^{\infty}\binom{-n}{k}(-a)^k=\sum_{k=0}^{\infty}\binom{k+n-1}{n-1}a^k. \tag{1.16}$$

Furthermore, *the multiple Hurwitz zeta function in* (1.2) *can be expressed as the following simple series*:

$$\zeta_n(s,a)=\sum_{k=0}^{\infty}\binom{k+n-1}{n-1}(a+k)^{-s}. \tag{1.17}$$

For further analysis of (1.17), we recall the Stirling numbers of the first kind $s(n,\ell)$ defined by the following generating functions:

$$a(a-1)\cdots(a-n+1)=\sum_{\ell=0}^{n}s(n,\ell)\,a^{\ell} \tag{1.18}$$

and

$$[\log(1+a)]^{\ell}=\ell!\sum_{n=\ell}^{\infty}s(n,\ell)\,\frac{a^n}{n!}\quad(|a|<1). \tag{1.19}$$

It is found from (1.18) that the Pochhammer symbol $(a)_n$ defined by

$$(a)_n:=\begin{cases}1 & (n=0)\\ a(a+1)\cdots(a+n-1) & (n\in\mathbb{N})\end{cases} \tag{1.20}$$

can be easily expanded in terms of $s(n,\ell)$:

$$(a)_n=a(a+1)\cdots(a+n-1)=\sum_{\ell=0}^{n}(-1)^{n+\ell}\,s(n,\ell)\,a^{\ell}, \tag{1.21}$$

where the coefficient

$$(-1)^{n+\ell}\, s(n,\ell) = |s(n,\ell)| := |s|(n,\ell)$$

is called the unsigned or absolute Stirling numbers of the first kind and represents the number of permutations of n symbols which have exactly ℓ cycles. It is known that the Stirling numbers $s(n,\ell)$ satisfy the following recurrence relations:

$$s(n+1,\ell) = s(n,\ell-1) - n\, s(n,\ell) \quad (n \geqq \ell \geqq 1) \tag{1.22}$$

and

$$\binom{k}{j} s(n,k) = \sum_{\ell=k-j}^{n-j} \binom{n}{\ell} s(n-\ell,j)\, s(\ell,k-j) \quad (n \geqq k \geqq j). \tag{1.23}$$

It is not difficult to see also that

$$\begin{gathered} s(n,0) = \begin{cases} 1 & (n=0) \\ 0 & (n \in \mathbb{N}), \end{cases} \qquad s(n,n) = 1, \\ s(n,1) = (-1)^{n+1}\,(n-1)! \quad \text{and} \quad s(n,n-1) = -\binom{n}{2}. \end{gathered} \tag{1.24}$$

Choi [6] observed that

$$\binom{k+n-1}{n-1} = \frac{1}{(n-1)!}\sum_{\ell=0}^{n-1} |s|(n,\ell+1)k^{\ell}. \tag{1.25}$$

Substituting from (1.25) into (1.17) and using the following double-series identity:

$$\sum_{\ell=0}^{n}\sum_{j=0}^{\ell} A_{\ell,j} = \sum_{j=0}^{n}\sum_{\ell=j}^{n} A_{\ell,j}, \tag{1.26}$$

Choi [6] found that $\zeta_n(s,a)$ is expressible as the following finite linear combination of the generalized zeta functions $\zeta(s,a)$ with polynomial coefficients in a:

$$\zeta_n(s,a) = \sum_{j=0}^{n-1} p_{n,j}(a)\zeta(s-j,a), \tag{1.27}$$

where

$$p_{n,j}(a) = \frac{1}{(n-1)!}\sum_{\ell=j}^{n-1}(-1)^{n+1-j}\binom{\ell}{j} s(n,\ell+1)a^{\ell-j}. \tag{1.28}$$

We find that $p_{n,j}(a)$ in (1.28) is a polynomial in a of degree $n-1-j$ with *rational* coefficients.

Since $\zeta(s,a)$ can be continued analytically to a meromorphic function having a simple pole at $s=1$ with its residue 1, the representation (1.27) shows that $\zeta_n(s,a)$ is analytic for all s except for simple poles only at $s=k$ $(k=1,\cdots,n;\ n\in\mathbb{N})$ with their respective residues given by

$$\operatorname*{Res}_{s=k}\zeta_n(s,a)=p_{n,k-1}(a)\quad (k=1,\cdots,n;\ n\in\mathbb{N}). \tag{1.29}$$

The series for $\zeta_n(s,a)$ can be evaluated explicitly for the first few values of n:

$$\begin{aligned}
\zeta_2(s,a)=&(1-a)\zeta(s,a)+\zeta(s-1,a),\\
\zeta_3(s,a)=&\frac{1}{2}\left(a^2-3a+2\right)\zeta(s,a)+\left(\frac{3}{2}-a\right)\zeta(s-1,a)+\frac{1}{2}\zeta(s-2,a),\\
\zeta_4(s,a)=&\frac{1}{6}\Big\{\left(-a^3+6a^2-11a+6\right)\zeta(s,a)+\left(3a^2-12a+11\right)\zeta(s-1,a)\\
&-(3a-6)\zeta(s-2,a)+\zeta(s-3,a)\Big\}.
\end{aligned} \tag{1.30}$$

On the other hand, it is found from (1.21) that

$$\binom{k+n-1}{n-1}=\frac{(k+1)_{n-1}}{(n-1)!}=\frac{1}{(n-1)!}\sum_{\ell=0}^{n-1}|s|(n-1,\ell)\,(k+1)^{\ell}. \tag{1.31}$$

Setting (1.31) in (1.17) and using the same technique as in [6] yields a seemingly different expression from (1.27):

$$\zeta_n(s,a)=\sum_{j=0}^{n-1}q_{n,j}(a)\,\zeta(s-j,a), \tag{1.32}$$

where

$$q_{n,j}(a)=\frac{1}{(n-1)!}\sum_{\ell=j}^{n-1}(-1)^{n+\ell-1}\binom{\ell}{j}s(n-1,\ell)(1-a)^{\ell-j}. \tag{1.33}$$

We show that (1.28) and (1.33) are equivalent. Since $s(n,0)=0$ $(n\in\mathbb{N})$, it follows from (1.21) that

$$a(a+1)\cdots(a+n-1)=\sum_{\ell=1}^{n}(-1)^{n+\ell}\,s(n,\ell)\,a^{\ell},$$

which, upon dividing by a, yields

$$(a+1)_{n-1} = (a+1)\cdots(a+n-1) = \sum_{\ell=1}^{n} (-1)^{n+\ell}\, s(n,\ell)\, a^{\ell-1}.$$

Now, using (1.21) for the left-hand side and letting $\ell - 1 = \ell'$ for the right-hand side, and then dropping the prime on ℓ', we are led to the following identity:

$$\begin{aligned} &\sum_{\ell=0}^{n-1} (-1)^{n+1+\ell}\, s(n-1,\ell)\, (a+1)^{\ell} \\ &\quad = \sum_{\ell=0}^{n-1} (-1)^{n+1+\ell}\, s(n,\ell+1)\, a^{\ell} \quad (n \in \mathbb{N}), \end{aligned} \tag{1.34}$$

which does indeed show that (1.28) and (1.33) are equivalent.

In this paper, we first extend the Hurwitz formula (2.9) for $\zeta(s,a)$ by presenting a generalized Hurwitz formula for $\zeta_n(s,a)$ and consider its various special cases. Secondly, we investigate and give the analytic continuation of the multiple Hurwitz-Euler eta function $\eta_n(s,a)$ in two different ways. As a by-product of the second investigation, a relationship between $\eta_n(-\ell,a)$ $(\ell \in \mathbb{N}_0)$ and the generalized Euler polynomials $E_{\ell}^{(n)}(n-a)$ of Definition 3 below is also provided.

2. A Generalized Hurwitz Formula for $\zeta_n(s,a)$

The series expression (1.2) for $\zeta_n(s,a)$ has a meaning whenever

$$\Re(s) = \sigma > n \quad (s = \sigma + it;\ \ n \in \mathbb{N}).$$

We present here another series representation for $\zeta_n(s,a)$, which is valid in the half-plane $\Re(s) = \sigma < 0$.

Theorem 1. *The following explicit representation holds true*:

$$\zeta_n(1-s,a) = \Gamma(s) \sum_{k=1}^{\infty} \{\mathcal{R}(n;k) + \mathcal{R}(n;-k)\} \tag{2.1}$$

$$\big(0 < a \leqq n;\ \ n \in \mathbb{N};\ \ \sigma = \Re(s) > 1\big),$$

where $\mathcal{R}(n;k)$ *denotes the residue of the function*

$$h(n,a,s;\,z) := \frac{(-z)^{-s}\, e^{(n-a)\,z}}{(e^{z}-1)^{n}} \tag{2.2}$$

at $z = 2k\pi i \ \ (k \in \mathbb{Z} \setminus \{0\})$.

We first prove the following Lemma which will be used in the proof of the assertion (2.1) of Theorem 1.

Lemma. *Let $P(\delta)$ denote the region that remains when we remove from the complex z-plane all open circular disks of radius δ $(0 < \delta < \pi)$ with centers at $z = 2k\pi i$ $(k \in \mathbb{Z})$. Then the function*

$$f(z) = \frac{e^{-az}}{(1-e^{-z})^n} = \frac{e^{(n-a)z}}{(e^z-1)^n} \quad (0 < a \leqq n;\ n \in \mathbb{N})$$

is bounded in $P(\delta)$ with the bound depending upon δ and n).

Proof. Write $z = x + iy$ and consider the punctured rectangle $Q(\delta)$ given by

$$Q(\delta) = \{z : z \in \mathbb{C},\ |x| \leqq 1,\ |y| \leqq \pi \quad \text{and} \quad |z| \geqq \delta\}.$$

Since $f(z)$ is a continuous function on the compact punctured rectangle $Q(\delta)$, f is bounded on $Q(\delta)$. Moreover, since $|f(z+2\pi i)| = |f(z)|$, i.e., since $|f(z)|$ is a periodic function with a purely imaginary period $2\pi i$, f is bounded in the punctured infinite strip $S(\delta)$ given by

$$S(\delta) = \{z : z \in \mathbb{C},\ |x| \leqq 1 \quad \text{and} \quad |z - 2k\pi i| \geqq \delta \ \ (k \in \mathbb{Z})\}.$$

We next show that f is bounded outside the strip $S(\delta)$. Suppose that $|x| \geqq 1$ and consider

$$|f(z)| = \left|\frac{e^{(n-a)z}}{(e^z-1)^n}\right| = \frac{e^{(n-a)x}}{|(e^z-1)^n|} \leqq \frac{e^{(n-a)x}}{|e^x-1|^n}.$$

When $x \geqq 1$, we have

$$|e^x - 1| = e^x - 1 \quad \text{and} \quad e^{(n-a)x} \leqq e^{nx},$$

so that

$$|f(z)| \leqq \frac{e^{nx}}{(e^x-1)^n} = \frac{1}{(1-e^{-x})^n} \leqq \frac{1}{(1-e^{-1})^n} = \frac{e^n}{(e-1)^n}.$$

On the other hand, when $x \leqq -1$, we have

$$|e^x - 1| = 1 - e^x \quad \text{and} \quad e^{(n-a)x} \leqq 1,$$

so that

$$|f(z)| \leqq \frac{1}{(1-e^x)^n} \leqq \frac{1}{(1-e^{-1})^n} = \frac{e^n}{(e-1)^n}.$$

Hence

$$|f(z)| \leqq \frac{e^n}{(e-1)^n} \quad \bigl(|\Re(z)| \geqq 1\bigr),$$

which completes the proof of the Lemma.

Proof of Theorem 1. Consider the function

$$F_N(s,a) = -\frac{1}{2\pi i} \int_{C(N)} \frac{(-z)^{s-1}\, e^{-az}}{(1-e^{-z})^n}\, dz \quad (N \in \mathbb{N}), \tag{2.3}$$

where

$$C(N) := C_1 \cup C_2 \cup C_3 \cup C_R \quad \bigl(R := (2N+1)\pi;\ \ N \in \mathbb{N}\bigr)$$

is a simple closed contour as shown in Figure 2. In particular,

$$C_R :\ z = (2N+1)\pi\, e^{i\theta} \quad (\theta \ \text{ varies from } \ 2\pi \ \text{ to } \ 0;\ \ N \in \mathbb{N}).$$

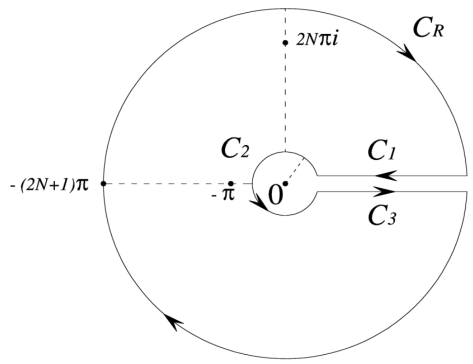

Fig. 2.

We first prove that

$$\lim_{N\to\infty} F_N(s,a) = I_n(s,a) \quad \bigl(\Re(s) = \sigma < 0\bigr),$$

where $I_n(s,a)$ is given by (1.6). To do this, in view of (1.6), it suffices to show that the integral along the outer circle C_R tends to 0 as $N \to \infty$. On C_R, we have

$$\bigl|(-z)^{s-1}\bigr| = \Bigl|(-R)^{s-1}\, e^{i\theta(s-1)}\Bigr| = R^{\sigma-1}\, e^{-t\,\theta} \leqq R^{\sigma-1}\, e^{2\pi|t|} \quad (s = \sigma + it).$$

Let M be the bound for the function

$$\left|\frac{e^{(n-a)z}}{(e^z-1)^n}\right|,$$

which is implied by the above Lemma. Then it is seen that

$$\left|\int_{C_R} \frac{(-z)^{s-1}\, e^{-az}}{(1-e^{-z})^n}\, dz\right| \leqq 2\pi\, M\, e^{2\pi|t|}\, R^{\sigma} \to 0 \quad \big(R \to \infty;\ \sigma = \Re(s) < 0\big).$$

Hence, upon replacing s by $1-s$ in (2.3), we find that

$$\lim_{N\to\infty} F_N(1-s,a) = I_n(1-s,a) \quad \big(\Re(s) = \sigma > 1\big). \tag{2.4}$$

The integrand in $F_N(1-s,a)$ has poles of order n at $z = 2k\pi i \ \ (k \in \mathbb{Z} \setminus \{0\})$ and so we have

$$\begin{aligned}\mathcal{R}(n;k) &:= \operatorname*{Res}_{z=2k\pi i}\ h(n,a,s\,;\,z)\\ &= \frac{1}{(n-1)!} \lim_{z\to 2k\pi i} \frac{d^{n-1}}{dz^{n-1}} \left\{(z-2k\pi i)^n\, h(n,a,s;\,z)\right\},\end{aligned} \tag{2.5}$$

where, in particular,

$$\mathcal{R}(1;k) = \frac{e^{-2k\pi i a}}{(-2k\pi i)^s}, \tag{2.6}$$

$$\mathcal{R}(2;k) = \frac{s\, e^{-2k\pi i a}}{(-2k\pi i)^{s+1}} + \frac{(1-a)\, e^{-2k\pi i a}}{(-2k\pi i)^s}, \tag{2.7}$$

and

$$\begin{aligned}\mathcal{R}(3;k) &= \frac{1}{2} s(s+1)\, \frac{e^{-2k\pi i a}}{(-2k\pi i)^{s+2}} + \left(\frac{3}{2} - a\right) \frac{s\, e^{-2k\pi i a}}{(-2k\pi i)^{s+1}}\\ &\quad + \frac{1}{2}\left(a^2 - 3a + 2\right) \frac{e^{-2k\pi i a}}{(-2k\pi i)^s}.\end{aligned} \tag{2.8}$$

The proof of Theorem 1 is thus completed.

By applying (2.6), (2.7) and (2.8) to (2.1), we obtain Corollary 1 below.

Corollary 1. *The following relationships hold true*:

$$\begin{aligned}&\zeta(1-s,a) = \frac{\Gamma(s)}{(2\pi)^s} \left\{e^{-\frac{1}{2}\pi i s}\, L(a,s) + e^{\frac{1}{2}\pi i s}\, L(-a,s)\right\}\\ &\big(0 < a \leqq 1 \ \textit{and}\ \ \sigma = \Re(s) > 1;\ 0 < a < 1 \ \textit{and}\ \ \sigma = \Re(s) > 0\big),\end{aligned} \tag{2.9}$$

$$\begin{aligned}
&\zeta_2(1-s,a)\\
&\quad=\frac{\Gamma(s+1)}{(2\pi)^{s+1}}\left\{e^{\frac{1}{2}\pi i(s+1)}\,L(-a,s+1)+e^{-\frac{1}{2}\pi i(s+1)}\,L(a,s+1)\right\}\\
&\quad\quad+(1-a)\,\frac{\Gamma(s)}{(2\pi)^s}\,\left\{e^{\frac{1}{2}\pi is}\,L(-a,s)+e^{-\frac{1}{2}\pi is}\,L(a,s)\right\}\\
&\quad\quad\big(0<a\leqq 1 \text{ and } \sigma=\Re(s)>1;\; 0<a<1 \text{ and } \sigma=\Re(s)>0\big)
\end{aligned}\tag{2.10}$$

and

$$\begin{aligned}
&\zeta_3(1-s,a)\\
&\quad=\frac{\Gamma(s+2)}{2\,(2\pi)^{s+2}}\left\{e^{\frac{1}{2}\pi i(s+2)}\,L(-a,s+2)+e^{-\frac{1}{2}\pi i(s+2)}\,L(a,s+2)\right\}\\
&\quad\quad+\left(\frac{3}{2}-a\right)\,\frac{\Gamma(s+1)}{(2\pi)^{s+1}}\left\{e^{\frac{1}{2}\pi i(s+1)}\,L(-a,s+1)+e^{-\frac{1}{2}\pi i(s+1)}\,L(a,s+1)\right\}\\
&\quad\quad+\frac{1}{2}\,\left(a^2-3a+2\right)\,\frac{\Gamma(s)}{(2\pi)^s}\,\left\{e^{\frac{1}{2}\pi is}\,L(-a,s)+e^{-\frac{1}{2}\pi is}\,L(a,s)\right\}\\
&\quad\quad\big(0<a\leqq 1 \text{ and } \sigma=\Re(s)>1;\; 0<a<1 \text{ and } \sigma=\Re(s)>0),
\end{aligned}\tag{2.11}$$

where $L(a,s)$ denotes the Dirichlet series defined by

$$L(a,s):=\sum_{n=1}^{\infty}\frac{e^{2n\pi ia}}{n^s}\quad\Big(a\in\mathbb{R};\;\sigma=\Re(s)>1\Big).\tag{2.12}$$

The relationship (2.9) is, in fact, Hurwitz's formula for $\zeta(s,a)$ (see, e.g., [12, p. 89, Eq. 2.2(6)]). Furthermore, the Dirichlet series $L(a,s)$ defined by (2.12) is often referred to as the *periodic* (or Lerch) zeta function. Indeed the Dirichlet series in (2.12) is a periodic function of a with period 1 and

$$L(1,s)=\zeta(s),$$

in terms of the Riemann zeta function $\zeta(s)$. The series in (2.12) converges absolutely for

$$\sigma=\Re(s)>1.$$

Moreover, if $a\notin\mathbb{Z}$, the series in (2.12) can also be seen to converge conditionally for

$$\sigma=\Re(s)>0.$$

Consequently, the formulas (2.9) to (2.11) are all valid for

$$\sigma=\Re(s)>0\quad(a\neq 1).$$

We observe that the function $L(a,s)$ in (2.12) is a linear combination of the Hurwitz zeta functions when a is a rational number. Since the set $\mathbb{N}$ of positive integers is a disjoint union of the residue classes $\mod q$ $(q \in \mathbb{N})$ as follows:

$$\mathbb{N} = \bigcup_{r=1}^{q} \{kq + r \mid k \in \mathbb{N}_0\},$$

we are led easily to the familiar series identity:

$$\sum_{n=1}^{\infty} \Theta(n) = \sum_{r=1}^{q} \sum_{k=0}^{\infty} \Theta(k\,q + r) \quad (q \in \mathbb{N}), \tag{2.13}$$

if the involved series is absolutely convergent.

Setting

$$a = \frac{p}{q} \quad (1 \leqq p \leqq q;\ p,\, q \in \mathbb{N})$$

in (2.12) and using the series identity (2.13), we see, for $\Re(s) = \sigma > 1$, that

$$L\left(\frac{p}{q}, s\right) = \frac{1}{q^s} \sum_{r=1}^{q} \exp\left(\frac{2\pi i r p}{q}\right) \zeta\left(s, \frac{r}{q}\right).$$

Therefore, if we take $a = \frac{p}{q}$ in the formulas (2.9) to (2.11), we obtain Corollary 2 below.

Corollary 2. *Each of the following relationships holds true*:

$$\zeta\left(1 - s, \tfrac{p}{q}\right) = \frac{2\,\Gamma(s)}{(2\pi q)^s} \sum_{r=1}^{q} \cos\left(\frac{\pi s}{2} - \frac{2\pi r p}{q}\right) \zeta\left(s, \frac{r}{q}\right) \tag{2.14}$$
$$\big(1 \leqq p \leqq q;\ p,\, q \in \mathbb{N};\ \sigma = \Re(s) > 1\big),$$

$$\begin{aligned} \zeta_2&\left(1 - s, \frac{p}{q}\right) \\ &= \left(1 - \frac{p}{q}\right) \frac{2\,\Gamma(s)}{(2\pi q)^s} \sum_{r=1}^{q} \cos\left(\frac{\pi s}{2} - \frac{2\pi r p}{q}\right) \zeta\left(s, \frac{r}{q}\right) \\ &\quad + \frac{2\,\Gamma(s+1)}{(2\pi q)^{s+1}} \sum_{r=1}^{q} \cos\left(\frac{\pi (s+1)}{2} - \frac{2\pi r p}{q}\right) \zeta\left(s + 1, \frac{r}{q}\right) \\ &\quad \big(1 \leqq p \leqq q;\ p,\, q \in \mathbb{N};\ \sigma = \Re(s) > 1\big) \end{aligned} \tag{2.15}$$

and

$$
\begin{aligned}
&\zeta_3\left(1-s,\frac{p}{q}\right)\\
&\quad=\left[\left(\frac{p}{q}\right)^2-\frac{3p}{q}+2\right]\frac{\Gamma(s)}{(2\pi q)^s}\sum_{r=1}^{q}\cos\left(\frac{\pi s}{2}-\frac{2\pi rp}{q}\right)\zeta\left(s,\frac{r}{q}\right)\\
&\qquad+\left(3-\frac{2p}{q}\right)\frac{\Gamma(s+1)}{(2\pi q)^{s+1}}\sum_{r=1}^{q}\cos\left(\frac{\pi(s+1)}{2}-\frac{2\pi rp}{q}\right)\zeta\left(s+1,\frac{r}{q}\right)\\
&\qquad+\frac{\Gamma(s+2)}{(2\pi q)^{s+2}}\sum_{r=1}^{q}\cos\left(\frac{\pi(s+2)}{2}-\frac{2\pi rp}{q}\right)\zeta\left(s+2,\frac{r}{q}\right)\\
&\qquad\big(1\leqq p\leqq q;\ p,\,q\in\mathbb{N};\ \sigma=\Re(s)>1\big).
\end{aligned}
\tag{2.16}
$$

The relationship (2.14) is the familiar Rademacher's formula (see, e.g., [12, p. 90, Eq. 2.2(8)]). Moreover, each of the formulas (2.14) to (2.16) holds true, by the principle of analytic continuation, for all admissible values of $s\in\mathbb{C}$.

3. Multiple Hurwitz-Euler Eta Function

The *alternating* Hurwitz zeta function (or, equivalently, Hurwitz-Euler eta function) $\eta(s,a)$ is defined by

$$
\eta(s,a)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+a)^s}\qquad\big(\Re(s)>0;\ a>0\big).
\tag{3.1}
$$

The special case of (3.1) when $a=1$, denoted by

$$
\eta(s,1):=\eta(s)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^s}\qquad\big(\Re(s)>0\big),
\tag{3.2}
$$

is called the Dirichlet eta function (or the alternating Riemann zeta function).

It is known that $\eta(s,a)$ $\big($and so also $\eta(s)\big)$ can be continued analytically to the whole complex s-plane. Clearly, therefore, each of the functions $\eta(s,a)$ and $\eta(s)$ is an entire function of $s\in\mathbb{C}$.

The multiple Hurwitz-Euler eta function $\eta_n(s,a)$ is defined by

$$
\eta_n(s,a):=\sum_{k_1,\cdots,k_n=0}^{\infty}\frac{(-1)^{k_1+\cdots+k_n}}{(a+k_1+\cdots+k_n)^s}\qquad\big(\Re(s)>0;\ a>0;\ n\in\mathbb{N}\big).
\tag{3.3}
$$

By using the same process as in getting (1.32), we find that $\eta_n(s,a)$ is expressible as a finite linear combination of the alternating Hurwitz zeta function $\eta(s,a)$ with polynomials in a as the coefficients.

Theorem 2. *The following relationship holds true*:

$$\eta_n(s,a) = \sum_{j=0}^{n-1} q_{n,j}(a)\, \eta(s-j,a), \tag{3.4}$$

where the polynomials $q_{n,j}(a)$ *are given in* (1.33).

It follows from Theorem 2 that, since $\eta(s,a)$ can be continued analytically as an entire function of $s \in \mathbb{C}$, $\eta_n(s,a)$ can be continued analytically as an entire function on $\mathbb{C}$.

By using the familiar relation:

$$\frac{\Gamma(s)}{p^s} = \int_0^\infty e^{-pt}\, t^{s-1}\, dt \quad \big(\Re(s) > 0;\ \Re(p) > 0\big), \tag{3.5}$$

we obtain an integral representation of $\eta_n(s,a)$ given by Theorem 3 below.

Theorem 3. *The following relationship holds true*:

$$\Gamma(s)\, \eta_n(s,a) = \int_0^\infty \frac{x^{s-1} e^{-ax}}{(1+e^{-x})^n} dx \quad \big(\Re(s) > 0;\ n \in \mathbb{N}\big). \tag{3.6}$$

By slightly modifying the method described in [12, p. 81], we can extend the integral representation (3.6) to all complex $s = \sigma + it$ with $\sigma > 0$. Indeed, we note that both functions on the left-hand side of (3.6) are analytic for $\sigma > 0$. To show that the right–hand dise of (3.6) is also analytic, we first assume that

$$\delta \leqq \sigma \leqq c \quad (c > 0;\ \ \delta > 0)$$

and write

$$\begin{aligned}\int_0^\infty \left| \frac{e^{-at} t^{s-1}}{(1+e^{-t})^n} \right| dt &= \int_0^\infty \frac{e^{-at} t^{\sigma-1}}{(1+e^{-t})^n} dt \\ &= \left(\int_0^1 + \int_1^\infty \right) \frac{e^{-at} t^{\sigma-1}}{(1+e^{-t})^n} dt.\end{aligned}$$

If $0 \leqq t \leqq 1$, we have

$$t^{\sigma-1} \leqq t^{\delta-1}.$$

And, if $t \geqq 1$, we have

$$t^{\sigma-1} \leqq t^{c-1}.$$

Moreover, since

$$e^t \geqq 1 \quad (t \geqq 0),$$

we have

$$\begin{aligned}\int_0^1 \frac{e^{-at}t^{\sigma-1}}{(1+e^{-t})^n}dt &\leqq \int_0^1 \frac{e^{(n-a)t}t^{\delta-1}}{(e^t+1)^n}dt \leqq \frac{1}{2^n}\int_0^1 e^{(n-a)t}t^{\delta-1}\,dt \\ &\leqq \begin{cases} e^{n-a}\displaystyle\int_0^1 t^{\delta-1}\,dt = \frac{e^{n-a}}{\delta} & (0 < a \leqq n) \\ \displaystyle\int_0^1 t^{\delta-1}dt = \frac{1}{\delta} & (a > n) \end{cases}\end{aligned}$$

and

$$\int_1^\infty \frac{e^{-at}t^{\sigma-1}}{(1+e^{-t})^n}dt \leqq \int_0^\infty \frac{e^{-at}t^{c-1}}{(1+e^{-t})^n}dt = \Gamma(c)\,\eta_n(c,a).$$

This shows that the integral in (3.6) converges uniformly in every strip given by

$$\delta \leqq \sigma \leqq c \quad (\delta > 0)$$

and, therefore, it represents an analytic function in every such strip; hence also in the half-plane $\sigma > 0$. Consequently, by the principle of analytic continuation, (3.6) holds true for all s with $\Re(s) = \sigma > 0$.

In order to extend $\eta_n(s,a)$ to the half-plane on the left of the line $\sigma = 0$, we derive another representation in terms of a contour integral. The contour C is essentially the same as in (1.6) or in Figure 1. Here C_1 and C_2 are the same as in Figure 1. But the radius c of the positively-oriented circle C_2 should be constrained by $c < \pi$. Thus we can use the following parameterizations:

$$-z = re^{-\pi i} \quad \text{on} \quad C_1$$

and

$$-z = re^{\pi i} \quad \text{on} \quad C_3,$$

where r varies from c to ∞.

Theorem 4. *If* $a > 0$, *then the function defined by the following contour integral*:

$$J_n(s,a) = -\frac{1}{2\pi i}\int_C \frac{(-z)^{s-1}e^{-az}}{(1+e^{-z})^n}\,dz \tag{3.7}$$

is an entire function of s. *Furthermore*

$$\eta_n(s,a) = \Gamma(1-s)\,J_n(s,a) \quad \big(\Re(s) = \sigma > 0\big). \tag{3.8}$$

Proof. Here $(-z)^s$ means $r^s e^{-\pi i s}$ on C_1 and $r^s e^{\pi i s}$ on C_3. We consider an arbitrary compact disk $|s| \leqq M$ and prove that the integrals along C_1 and C_3

converge uniformly on every such disk. Since the integrand in (3.7) is an entire function of s, this will prove that $J_n(s,a)$ is also an entire function of s for $a>0$. Along C_1 we have, for $r \geqq 1$,

$$|(-z)^{s-1}| = r^{\sigma-1}\left|e^{-\pi i(\sigma-1+it)}\right| = r^{\sigma-1}\,e^{\pi t} \leqq r^{M-1}e^{\pi M},$$

since $|s| \leqq M$. Similarly, along C_3 we have, for $r \geqq 1$,

$$\left|(-z)^{s-1}\right| = r^{\sigma-1}|e^{\pi i\,(\sigma-1+it)}| = r^{\sigma-1}e^{-\pi t} \leqq r^{M-1}e^{\pi M}.$$

Hence, on either C_1 or C_3, we have, for $r \geqq 1$,

$$\left|\frac{(-z)^{s-1}e^{-az}}{(1+e^{-z})^n}\right| \leqq \frac{r^{M-1}e^{\pi\,M}e^{-ar}}{(1-e^{-r})^n} \leqq \frac{e^{\pi\,M}}{(1-e^{-1})^n}\cdot r^{M-1}\cdot e^{-ar}.$$

But the following integral:

$$\int_c^\infty r^{M-1}\,e^{-ar}\,dr$$

converges if $c>0$; this shows that the integrals along C_1 and C_3 converge uniformly on every compact disk $|s| \leqq M$, and hence $J_n(s,a)$ is an entire function of s. To prove (3.8), we write

$$-2\pi i\,J_n(s,a) = \left(\int_{C_1} + \int_{C_2} + \int_{C_3}\right)(-z)^{s-1}g(-z)\,dz,$$

where

$$g(-z) = \frac{e^{-az}}{(1+e^{-z})^n}.$$

On C_1 and C_3 we have $g(-z)=g(-r)$, and on C_2 we write $-z=ce^{i\theta}$, where θ varies from 2π to 0. This yields

$$\begin{aligned}
-2\pi i\,J_n(s,a) &= \int_\infty^c r^{s-1}\,e^{-\pi i\,(s-1)}\,g(-r)\,dr \\
&\quad -i\int_{2\pi}^0 c^{s-1}e^{(s-1)i\theta}ce^{i\theta}g(ce^{i\theta})\,d\theta \\
&\quad +\int_c^\infty r^{s-1}e^{\pi i\,(s-1)}g(-r)\,dr \\
&= -2i\sin(\pi s)\int_c^\infty r^{s-1}g(-r)\,dr - ic^s\int_{2\pi}^0 e^{is\theta}\,g(ce^{i\theta})\,d\theta.
\end{aligned} \tag{3.9}$$

Dividing both sides of (3.9) by $-2i$, we obtain

$$\pi\,J_n(s,a) = \sin(\pi s)I_1(s,c) + I_2(s,c),$$

where

$$I_1(s,c) = \int_c^\infty r^{s-1} g(-r)dr \quad \text{and} \quad I_2(s,c) = \frac{c^s}{2}\int_{2\pi}^0 e^{is\theta} g(ce^{i\theta})d\theta.$$

We now let $c \to 0$. In view of (3.6), we thus find that

$$\lim_{c\to 0} I_1(s,c) = \int_0^\infty \frac{r^{s-1}e^{-ar}}{(1-e^{-r})^n}dr = \Gamma(s)\,\eta_n(s,a) \quad \bigl(\sigma = \Re(s) > 0\bigr).$$

We next show that

$$\lim_{c\to 0} I_2(s,c) = 0.$$

To do this, we note that $g(-z)$ is analytic in $|z| < \pi$. Hence $g(-z)$ is bounded there, that is,

$$|g(-z)| \leqq A \quad (|z| = c < \pi; \;\; A > 0)$$

for some positive constant A. This leads us to the following two-sided inequalities:

$$|I_2(s,c)| \leqq \frac{c^\sigma}{2}\int_0^{2\pi} e^{-t\theta}\cdot A\,d\theta \leqq \pi A\,e^{2\pi|t|}\,c^\sigma \quad (A > 0).$$

If $\sigma > 0$ and $c \to 0$, we find that $I_2(s,c) \to 0$. Hence

$$\pi J_n(s,a) = \sin(\pi s)\,\Gamma(s)\,\eta_n(s,a), \tag{3.10}$$

which, by means of the following well-known relation:

$$\Gamma(s)\,\Gamma(1-s) = \frac{\pi}{\sin(\pi s)} \quad (s \in \mathbb{C}\setminus\mathbb{Z}), \tag{3.11}$$

is seen to be equivalent to (3.8).

In the equation (3.8), valid for $\sigma > 0$, the function $J_n(s,a)$ is an entire function of s, and $\Gamma(1-s)$ is analytic for every complex s $(s \in \mathbb{C}\setminus\mathbb{N})$. We, therefore, can use this equation to define $\eta_n(s,a)$ for

$$\sigma = \Re(s) \leqq 0,$$

that is, outside

$$\sigma = \Re(s) > 0$$

as desired.

Definition 1. For

$$\Re(s) = \sigma \leqq 0,$$

we define the function $\eta_n(s,a)$ by

$$\eta_n(s,a) := \Gamma(1-s)\, J_n(s,a), \tag{3.12}$$

where $J_n(s,a)$ is given in (3.7).

Clearly, the equation (3.12) provides an analytic continuation of $\eta_n(s,a)$ to the whole complex s-plane.

Just as in (1.9), in order to get a relationship between $\eta_n(s,a)$ and the generalized Euler polynomials $E_n^{(\alpha)}(a)$, we recall the following definitions.

Definition 2. The classical *Euler polynomials* $E_k(a)$ and the classical *Euler numbers* E_k are defined by the following generating functions (see, e.g., [12, pp. 63–66]):

$$\frac{2e^{az}}{e^z+1} = \sum_{k=0}^{\infty} E_k(a)\,\frac{z^k}{k!} \quad (|z|<\pi) \tag{3.13}$$

and

$$\frac{2e^{z}}{e^{2z}+1} = \operatorname{sech} z = \sum_{k=0}^{\infty} E_k\,\frac{z^k}{k!} \quad \left(|z|<\frac{\pi}{2}\right), \tag{3.14}$$

respectively.

Definition 3. For an arbitrary (real or complex) parameter α, the *generalized Euler polynomials* $E_k^{(\alpha)}(a)$ and the *generalized Euler numbers* $E_k^{(\alpha)}$ are defined by the generating functions:

$$\left(\frac{2}{e^z+1}\right)^{\alpha} e^{az} = \sum_{k=0}^{\infty} E_k^{(\alpha)}(a)\,\frac{z^k}{k!} \quad (|z|<\pi;\ 1^{\alpha}:=1) \tag{3.15}$$

and

$$\left(\frac{2\,e^{z}}{e^{2z}+1}\right)^{\alpha} = \sum_{k=0}^{\infty} E_k^{(\alpha)}\,\frac{z^k}{k!} \quad \left(|z|<\frac{\pi}{2};\ 1^{\alpha}:=1\right), \tag{3.16}$$

respectively. Clearly, we find from Definitions 2 and 3 that

$$E_k^{(1)}(a) = E_k(a) \quad \text{and} \quad E_k^{(1)} = E_k \quad (k\in\mathbb{N}_0). \tag{3.17}$$

Setting

$$s=-\ell \quad (\ell\in\mathbb{N}_0)$$

in (3.12), we obtain

$$\eta_n(-\ell, a) = \ell!\, J_n(-\ell, a),$$

where

$$J_n(-\ell, a) = -\frac{1}{2\pi i} \int_C \frac{(-z)^{-\ell-1} e^{(n-a)z}}{(e^z+1)^n}\, dz \quad (\ell \in \mathbb{N}_0)\,. \tag{3.18}$$

The integrand in the contour integral (3.18) for $J_n(-\ell, a)$ takes on the same values on both C_1 and C_3, but with opposite signs. Hence the integrals along C_1 and C_3 cancel. Thus, In view of (3.15), we find that

$$\begin{aligned} J_n(-\ell, a) &= -\frac{1}{2\pi i} \int_{C_2} \frac{(-z)^{-\ell-1} e^{(n-a)z}}{(e^z+1)^n}\, dz \\ &= \frac{(-1)^\ell}{2^n} \operatorname*{Res}_{z=0} \left\{ z^{-\ell-1} \left(\frac{2}{e^z+1} \right)^n e^{(n-a)z} \right\} \\ &= \frac{(-1)^\ell}{2^n\, \ell!}\, E_\ell^{(n)}(n-a). \end{aligned}$$

We finally get a relationship between $\eta_n\,(s, a)$ and $E_n^{(\alpha)}(a)$.

Corollary 3. *The following relationship holds true*:

$$\eta_n(-\ell, a) = \frac{(-1)^\ell}{2^n}\, E_\ell^{(n)}(n-a) \quad (\ell \in \mathbb{N}_0;\ n \in \mathbb{N})\,. \tag{3.19}$$

In particular, *for the Hurwitz-Euler eta function* $\eta(s, a)$,

$$\eta(-\ell, a) = \frac{(-1)^\ell}{2}\, E_\ell(1-a) \quad (\ell \in \mathbb{N}_0) \tag{3.20}$$

in terms of the classical Euler polynomials $E_k(a)$.

We conclude this paper by remarking that, as in Section 2, it is not difficult to get a formula for $\eta_n(s, a)$ analogous to the assertion (2.1) of Theorem 1. However, we do not find it be worthwhile to try to get such an analogous formula for $\eta_n(s, a)$, since $\eta_n\,(s, a)$ is already continued analytically as an entire function of $s \in \mathbb{C}$.

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