TSUKUBA J. MATH.
Vol. 16 No. 2 (1992), 321–334

3-DIMENSIONAL SUBMANIFOLDS OF SPHERES WITH PARALLEL MEAN CURVATURE VECTOR*

By

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Abstract. In this paper, for a 3-dimensional complete submanifold M with parallel mean curvature vector in $S^{3+p}(c)$, we give a pinching condition of the Ricci curvature under which M is a 3-dimensional small sphere.

1. Introduction

Let M be an n-dimensional complete submanifold immersed in a sphere $S^{n+p}(c)$. It is well-known that properties of M can be described by a pinching condition of some curvatures. When M is a minimal submanifold or a submanifold with parallel mean curvature vector, many authors studied the pinching problem with respect to the sectional curvature or the scalar curvature of M and a lot of beautiful results were obtained. It is natural to consider whether we can describe the properties of M by a pinching condition of the Ricci curvature. When M is minimal, Ejiri [2] and Shen [5] studied the pinching problem. Shun [6] researched compact submanifolds of a sphere with parallel mean curvature vector for $n>3$. He gave a pinching condition of the Ricci curvature under which M is totally umbilic.

In this paper, for $n=3$, we consider same problem. That is, we prove the following:

THEOREM 1. *Let M be a 3-dimensional complete submanifold of $S^{3+p}(c)(p\leqq 2)$ with parallel mean curvature vector $\boldsymbol{h}$. If*

$$Ric(M)\geqq\frac{3}{4}c+\frac{39}{64}H^2+\frac{1}{8}\sqrt{\frac{1521}{64}H^4+\frac{45}{2}H^2c},$$

then M is totally umbilic. Hence M is a 3-dimensional small sphere, where $Ric(M)$ and $H=|\boldsymbol{h}|$ denote the Ricci curvature and the norm of the mean cur-

* The Project Supported by National Natural Science Fundation of China.
Received January 7, 1991, Revised September 18, 1991.

vature vector $\boldsymbol{h}$ *respectively.*

THEOREM 2. *Let M be a 3-dimensional complete submanifold with parallel mean curvature vector of $S^{3+p}(c)(p>2)$. If*

$$Ric(M) \geqq \delta,$$

then M is totally umbilic. Hence M is a 3-dimensional small sphere, where

$$\delta = Max\left\{\frac{3}{4}c+\frac{39}{64}H^2+\frac{1}{8}\sqrt{\frac{1521}{64}H^4+\frac{45}{2}H^2c},\ \frac{5p-p}{2(2p-3)}(c+H^2)\right\}.$$

2. Preliminaries.

Throughout this paper all manifolds are assumed to be smooth, connected without boundary. We discuss in smooth category.

Let M be a 3-dimensional submanifold of a $(3+p)$-dimensional sphere $S^{3+p}(c)$. We choose a local field of orthonormal frame $e_1, \cdots, e_{3+p}$ in $S^{3+p}(c)$ and the dual coframe $\omega_1, \cdots, \omega_{3+p}$ in such a way that e_1, e_2 and e_3 are tangent to M. In the sequel, the following convention on the range of indices is used, unless otherwised stated:

$$1 \leqq A, B, \cdots \leqq 3+p; \qquad 1 \leqq i, j, \cdots \leqq 3;$$

$$4 \leqq \alpha, \beta, \cdots \leqq 3+p.$$

And we agree that the repeated indices under a summation sign without indication are summed over the respective range. The connection forms $\{\omega_{AB}\}$ of $S^{3+p}(c)$ are characterized by the structure equations

$$(2.1) \qquad \begin{cases} d\omega_A - \sum \omega_{AB} \wedge \omega_B = 0, & \omega_{AB}+\omega_{AB}=0, \\ d\omega_{AB} - \sum \omega_{AC} \wedge \omega_{CB} = \Omega_{AB}, \\ \Omega_{AB} = -\dfrac{1}{2}\sum R'_{ABCD}\omega_C \wedge \omega_D, \end{cases}$$

$$(2.2) \qquad R'_{BBCD} = c(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}),$$

where Ω_{AB} (resp. R'_{ABCD}) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor) of $S^{3+p}(c)$. Therefore the components of Ricci curvature tensor Ric' and the scalar curvature r' are given as

$$R'_{AB} = c(n+p-1)\delta_{AB}, \qquad r' = (n+p)(n+p-1)c.$$

Restricting these forms to M, we have

$$(2.3) \qquad \omega_\alpha = 0 \quad \text{for } \alpha = 4, \cdots, 3+p.$$

We see that e_1, e_2 and e_3 is a local field of orthonormal frames on M and ω_1, ω_2 and ω_3 is a local field of its dual coframes on M. It follows from (2.1), (2.3) and Cartan's Lemma that

$$\omega_{\alpha i}=\sum h_{ij}^{\alpha}\omega_j\,, \qquad h_{ij}^{\alpha}=h_{ji}^{\alpha}\,. \tag{2.4}$$

The second fundamental form α and the mean curvature vector $\boldsymbol{h}$ of M are defined by

$$\alpha=\sum h_{ij}^{\alpha}\omega_i\omega_j e_\alpha\,, \qquad \boldsymbol{h}=\frac{1}{3}\sum(\sum_i h_{ii}^{\alpha})e_\alpha\,. \tag{2.5}$$

The mean curvature H is given by

$$H=|\boldsymbol{h}|=\frac{1}{3}\sqrt{\sum(\sum_i h_{ii}^{\alpha})^2}\,. \tag{2.6}$$

Let $S=\sum(h_{ij}^{\alpha})^2$ denote the squared norm of the second fundamental form of M. The connection forms $\{\omega_{ij}\}$ of M are characterized by the structure equations

$$\begin{cases} d\omega_i-\sum\omega_{ij}\wedge\omega_j=0\,, \qquad \omega_{ij}+\omega_{ji}=0\,, \\ d\omega_{ij}-\sum\omega_{ik}\wedge\omega_{kj}=\Omega_{ij}\,, \\ \Omega_{ij}=-\dfrac{1}{2}\sum R_{ijkl}\omega_k\wedge\omega_l\,, \end{cases} \tag{2.7}$$

where Ω_{ij} (resp. R_{ijkl}) denotes the Riemannian curvature form (resp. the components of the Riemannian curvature tensor) of M. Therefore the Gauss equation is given by, from (2.1) and (2.7),

$$R_{ijkl}=c(\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk})+\sum(h_{ik}^{\alpha}h_{jl}^{\alpha}-h_{il}^{\alpha}h_{jk}^{\alpha})\,. \tag{2.8}$$

The components of the Ricci curvature Ric and the scalar curvature r are given by

$$R_{jk}=2c\delta_{jk}+\sum h_{ii}^{\alpha}h_{jk}^{\alpha}-\sum h_{ik}^{\alpha}h_{ij}^{\alpha}\,, \tag{2.9}$$

$$r=6c+9H^2-\sum(h_{ij}^{\alpha})^2\,. \tag{2.10}$$

We also have

$$d\omega_{\alpha\beta}-\sum\omega_{\alpha\gamma}\wedge\omega_{\gamma\beta}=-\frac{1}{2}\sum R_{\alpha\beta ij}\omega_i\wedge\omega_j\,,$$

where

$$R_{\alpha\beta ij}=\sum(h_{il}^{\alpha}h_{jl}^{\beta}-h_{jl}^{\alpha}h_{il}^{\beta})\,. \tag{2.11}$$

Define h_{ijk}^{α} and h_{ijkl}^{α} by

$$\sum h_{ijk}^{\alpha}\omega_k=dh_{ij}^{\alpha}+\sum h_{ik}^{\alpha}\omega_{kj}+\sum h_{jk}^{\alpha}\omega_{ki}-\sum h_{ij}^{\beta}\omega_{\alpha\beta}\,, \tag{2.12}$$

$$\sum h^{\alpha}_{ijkl}\omega_l = dh^{\alpha}_{ijk} + \sum h^{\alpha}_{ilk}\omega_{lj} + \sum h^{\alpha}_{ijl}\omega_{lk} + \sum h^{\alpha}_{ijk}\omega_{li} - \sum h^{\beta}_{ijk}\omega_{\alpha\beta} .$$

The Codazzi equation and the Ricci formula for the second fundamental form are given by

$$h^{\alpha}_{ijk} - h^{\alpha}_{ikj} = 0 ,$$

$$h^{\alpha}_{ijkl} - h^{\alpha}_{ijlk} = \sum h^{\alpha}_{im} R_{mjkl} + \sum h^{\alpha}_{mj} R_{mikl} + \sum h^{\beta}_{ij} R_{\beta\alpha kl} , \tag{2.13}$$

The Laplacian Δh^{α}_{ij} of the components h^{α}_{ij} of the second fundamental form α is given by

$$\Delta h^{\alpha}_{ij} = \sum_k h^{\alpha}_{ijkk} .$$

From (2.13) we get

$$\Delta h^{\alpha}_{ij} = \sum_k h^{\alpha}_{kkij} + \sum h^{\alpha}_{km} R_{mijk} + \sum h^{\alpha}_{mi} R_{mkjk} + \sum h^{\beta}_{ki} R_{\beta\alpha jk} . \tag{2.14}$$

In this paper, we assume that the mean curvature vector $\boldsymbol{h}$ of M is parallel. Hence the mean curvature H is constant. We choose e_4 such that $\boldsymbol{h} = He_4$, then

$$\sum_i h^4_{ii} = 3H , \qquad \sum_i h^{\alpha}_{ii} = 0 \qquad \text{for any } \alpha \neq 4 , \tag{2.15}$$

$$H_\alpha H_4 = H_4 H_\alpha \qquad \text{for any } \alpha , \tag{2.16}$$

where H_α denotes 3×3-matrix (h^{α}_{ij}). From (2.14), we have

$$\begin{aligned} \sum_{\alpha\neq 4} h^{\alpha}_{ij}\Delta h^{\alpha}_{ij} = & \sum_{\alpha\neq 4} h^{\alpha}_{ij} h^{\alpha}_{km} R_{mijk} \\ & + \sum_{\alpha\neq 4} h^{\alpha}_{ij} h^{\alpha}_{mi} R_{mkjk} + \sum_{\alpha\neq 4} h^{\alpha}_{ij} h^{\beta}_{ki} R_{\beta\alpha jk} , \end{aligned} \tag{2.17}$$

$$\sum h^4_{ij}\Delta h^4_{ij} = \sum h^4_{ij} h^4_{km} R_{mijk} + \sum h^4_{ij} h^4_{mi} R_{mkjk} . \tag{2.18}$$

Define $|\tau|^2 = \sum_{\alpha\neq 4} \mathrm{tr}(H^2_\alpha)$ and $|\sigma|^2 = \mathrm{tr}(H^2_4)$. Then $S = |\tau|^2 + |\sigma|^2$. A submanifold M is said to be *pseudo-umbilic* if it is umbilic with respect to the direction of the mean curvature vector $\boldsymbol{h}$, that is

$$h^4_{ij} = H\delta_{ij} .$$

3. Proofs of Theorems

In this section, we will give the proofs of Theorem 1 and Theorem 2. In order to prove Theorems, at first we give the following Propositions 1 and 2.

PROPOSITION 1. *Let M be a 3-dimensional complete pseudo-umbilical submanifold in $S^{3+p}(c)(p>1)$ with parallel mean curvature vector. If*

$$Ric(M) \geqq \frac{5p-9}{2(2p-3)}(c+H^2),$$

then M is a totally umbilical submanifold.

PROOF. Because of $Ric(M) \geqq [(5p-9)/2(2p-3)](c+H^2)>0$, we know that M is a compact submanifold from Myers' theorem (2.17) implies

(3.1)
$$\frac{1}{2}\Delta|\tau|^2 = \sum_{\alpha\neq 4}(h^{\alpha}_{ijk})^2 + \sum_{\alpha\neq 4} h^{\alpha}_{ij}\Delta h^{\alpha}_{ij}$$
$$= \sum_{\alpha\neq 4}(h^{\alpha}_{ijk})^2 + \sum_{\alpha\neq 4}(h^{\alpha}_{km}R_{mijk} + h^{\alpha}_{mi}R_{mkjk})h^{\alpha}_{ij} + \sum_{\alpha,\beta\neq 4} h^{\alpha}_{ij}h^{\beta}_{ki}R_{\beta\alpha jk},$$
$$\sum_{\alpha\neq 4}(h^{\alpha}_{km}R_{mijk} + h^{\alpha}_{mi}R_{mkjk})h^{\alpha}_{ij}$$
$$= \sum_{\alpha,\beta\neq 4}\{\mathrm{tr}(H_\alpha H_\beta)^2 - \mathrm{tr}(H^2_\beta H^2_\alpha)\} - \sum_{\alpha,\beta\neq 4}\{\mathrm{tr}(H_\alpha H_\beta)\}^2$$
$$+3c|\tau|^2 + 3H\sum_{\alpha\neq 4}\mathrm{tr}(H_\alpha H_4 H_\alpha) - \sum_{\alpha\neq 4}\{\mathrm{tr}(H_\alpha H_4)\}^2$$
$$+\sum_{\alpha\neq 4}\mathrm{tr}(H_\alpha H_4)^2 - \sum_{\alpha\neq 4}\mathrm{tr}(H^2_\alpha H^2_4).$$

Since M is a pseudo-umbilical submanifold, we have $H_4=HI$, where I is the identity matrix. Hence

$$\sum_{\alpha\neq 4}\mathrm{tr}(H_\alpha H_4)^2 - \sum_{\alpha\neq 4}\mathrm{tr}(H^2_\alpha H^2_4) = 0,$$
$$\sum_{\alpha\neq 4}\mathrm{tr}(H_\alpha H_4 H_\alpha) = H|\tau|,$$
$$\sum_{\alpha\neq 4}\{\mathrm{tr}(H_\alpha H_4)\}^2 = 0 \qquad (\text{by } (2.15)).$$

Thus

(3.2)
$$\sum_{\alpha\neq 4}(h^{\alpha}_{km}R_{mijk} + h^{\alpha}_{mi}R_{mkjk})h^{\alpha}_{ij}$$
$$= \sum_{\alpha,\beta\neq 4}\{\mathrm{tr}(H_\alpha H_\beta)^2 - \mathrm{tr}(H^2_\beta H^2_\alpha)\} - \sum_{\alpha,\beta\neq 4}\{\mathrm{tr}(H_\alpha H_\beta)\}^2 + 3(c+H^2)|\tau|^2.$$

(3.3)
$$\sum_{\alpha,\beta\neq 4} h^{\alpha}_{ij}h^{\beta}_{ki}R_{\beta\alpha jk} = \sum_{\alpha,\beta\neq 4}\{\mathrm{tr}(H_\alpha H_\beta)^2 - \mathrm{tr}(H^2_\beta H^2_\alpha)\}.$$

According to (3.1), (3.2) and (3.3), we get

(3.4)
$$\frac{1}{2}\Delta|\tau|^2 = \sum_{\alpha\neq 4}(h^{\alpha}_{ijk})^2 - \sum_{\alpha,\beta\neq 4}\{\mathrm{tr}(H_\alpha H_\beta)\}^2$$
$$+3(c+H^2)|\tau|^2 + 2\sum_{\alpha,\beta\neq 4}\{\mathrm{tr}\{(H_\alpha H_\beta)^2 - \mathrm{tr}(H^2_\beta H^2_\alpha)\}.$$

For a suitable choice of $e_5, \cdots, e_{3+p}$, we can assume $(p-1)\times(p-1)$ matrix $(\mathrm{tr}(H_\alpha H_\beta))$ is diagonal. Hence

$$\sum_{\alpha,\beta\neq 4}\{\mathrm{tr}(H_\alpha H_\beta)\}^2=\sum_{\alpha\neq 4}\{\mathrm{tr}(H_\alpha^2)\}^2 \tag{3.5}$$

From Lemma 1 in [1], we have

$$\begin{aligned}&2\{\mathrm{tr}(H_\alpha H_\beta)^2-\mathrm{tr}(H_\beta^2 H_\alpha^2)\}\\ &=-\mathrm{tr}(H_\alpha H_\beta-H_\beta H_\alpha)^2\geqq -2\mathrm{tr}(H_\alpha^2)\,\mathrm{tr}(H_\beta^2),\end{aligned} \tag{3.6}$$

and equality holds for nonzero matrices H_α and H_β if and only if H_α and H_β can be transformed simultaneously by an orthogonal matrix into

$$H_\alpha^*=\lambda\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}\quad\text{and}\quad H_\beta^*=\mu\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}.$$

Moreover if $H_{\alpha_1},\cdots,H_{\alpha_s}$ satisfy

$$\mathrm{tr}(H_{\alpha_i}H_{\alpha_k}-H_{\alpha_k}H_{\alpha_i})^2+2\,\mathrm{tr}(H_{\alpha_i}^2)\,\mathrm{tr}(H_{\alpha_k}^2)\qquad\text{for } 1\leqq i,\ k\leqq s,$$

then at most two of the matrices H_{α}, are nonzero. Let

$$\begin{aligned}&(p-1)\sigma_1=|\tau|^2,\\ &(p-1)(p-2)\sigma_2=2\sum_{\alpha<\beta,\,\alpha,\,\beta\neq 4}\mathrm{tr}(H_\alpha^2)\,\mathrm{tr}(H_\beta^2).\end{aligned} \tag{3.7}$$

Then

$$(p-1)^2(p-2)(\sigma_1^2-\sigma_2)=\sum_{\alpha<\beta,\,\alpha,\,\beta\neq 4}\{\mathrm{tr}(H_\alpha^2)-\mathrm{tr}(H_\beta^2)\}^2. \tag{3.8}$$

Hence we obtain

$$\begin{aligned}\frac{1}{2}\Delta|\tau|^2&\geqq\sum_{\alpha\neq 4}(h_{ijk}^\alpha)^2+3(c+H^2)|\tau|^2\\ &\quad-2\{\sum_{\alpha\neq 4}\mathrm{tr}(H_\alpha^2)\}^2+\sum_{\alpha\neq 4}\{\mathrm{tr}(H_\alpha^2)\}^2\\ &\geqq-\{2(p-1)-1\}(p-1)\sigma_1^2+(p-1)(p-2)(\sigma_1^2-\sigma_2)+3(c+H^2)|\tau|^2\\ &\geqq-(p-1)(2p-3)\sigma_1^2+3(c+H^2)|\tau|^2\\ &=-\Big(2-\frac{1}{p-1}\Big)|\tau|^4+3(c+H^2)|\tau|^2.\end{aligned} \tag{3.9}$$

On the other hand, for each fixed $\alpha\neq 4$, we can choose a local field of orthonormal frames e_1, e_2 and e_3 such that, from (2.16),

$$h_{ij}^4=H\delta_{ij}\quad\text{and}\quad h_{ij}^\alpha=\lambda_i^\alpha\delta_{ij}.$$

Since $\mathrm{tr}\,H_\alpha=\sum\lambda_i^\alpha=0$, we have

$$\sum_i (\lambda_i^\alpha)^4 = \frac{1}{2}\{\sum_i (\lambda_i^\alpha)^2\}^2,$$

that is,

$$\operatorname{tr} H_\alpha^4 = \frac{1}{2}\{\operatorname{tr} H_\alpha^2\}^2.$$

Hence

$$\sum_{\alpha\neq 4} \operatorname{tr} H_\alpha^4 = \frac{1}{2}\sum_{\alpha\neq 4}\{\operatorname{tr} H_\alpha^2\}^2. \tag{3.10}$$

For any $\alpha\neq 4$,

$$\begin{aligned} &2\sum_{\beta=5}^{3+p}\{\operatorname{tr}(H_\alpha^2 H_\beta^2)-\operatorname{tr}(H_\alpha H_\beta)^2\} \\ &=\{\sum_{\beta=5}^{3+p}\sum_{ij}(h_{ij}^\beta)^2(\lambda_i^\alpha-\lambda_j^\alpha)^2 \\ &\leqq 4\sum_{\beta\neq 4,\,\beta\neq\alpha}\sum_{ij}(h_{ij}^\beta)^2(\lambda_i^\alpha)^2. \end{aligned} \tag{3.11}$$

According to (2.9), we get

$$R_{ij}=2(c+H^2)-(\lambda_i^\alpha)^2-\sum_{\beta\neq 4,\,\beta\neq\alpha}\sum_j (h_{ij}^\beta)^2, \tag{3.12}$$

$$\begin{aligned} &\sum_{\beta\neq 4,\,\beta\neq\alpha}\sum_{ij}(h_{ij}^\beta)^2(\lambda_i^\alpha)^2 \\ &=2(c+H^2)\sum_i(\lambda_i^\alpha)^2-\sum_i(\lambda_i^\alpha)^4-\sum_i R_{ii}(\lambda_i^\alpha)^2 \\ &\leqq 2(c+H^2)\operatorname{tr}(H_\alpha^2)-\frac{1}{2}\{\operatorname{tr}(H_\alpha^2)\}^2-\delta_1\operatorname{tr}(H_\alpha^2), \end{aligned} \tag{3.13}$$

where δ_1 is the infimum of the Ricci curvature of M. Hence

$$\begin{aligned} &2\sum_{\beta=5}^{3+p}\{\operatorname{tr}(H_\alpha^2 H_\beta^2)-\operatorname{tr}(H_\alpha H_\beta)^2\} \\ &\leqq \{8(c+H^2)-4\delta_1\}\operatorname{tr}(H_\alpha^2)-2\{\operatorname{tr}(H_\alpha^2)\}^2. \end{aligned} \tag{3.14}$$

The terms at the both ends of the inequality above do not depend on the choice of the frame fields. Hence

$$\begin{aligned} &2\sum_{\alpha,\,\beta\neq 4}\{\operatorname{tr}(H_\alpha^2 H_\beta^2)-\operatorname{tr}(H_\alpha H_\beta)^2\} \\ &\leqq \{8(c+H^2)-4\delta_1\}|\tau|^2-2\sum_{\alpha\neq 4}\{\operatorname{tr}(H_\alpha^2)\}^2. \end{aligned} \tag{3.15}$$

(3.4), (3.5) and (3.15) yield

$$\frac{1}{2}\Delta|\tau|^2 \geqq -\{8(c+H^2)-4\delta_1\}|\tau|^2+\sum_{\alpha\neq 4}\{\mathrm{tr}(H_\alpha^2)\}^2+3(c+H^2)|\tau|^2 \tag{3.16}$$

$$\geqq\{-5(c+H^2)+4\delta_1\}|\tau|^2+\frac{1}{p-1}|\tau|^4 .$$

$(3.9)\times 1/(2p-3)+(3.16)$ implies

$$\frac{1}{2}\left[1+\frac{1}{2p-3}\right]\Delta|\tau|^2 \geqq \left\{4\delta_1-\left(5-\frac{3}{2p-3}\right)(c+H^2)\right\}|\tau|^2 . \tag{3.17}$$

Since δ_1 is the infimum of the Ricci curvature, we have

$$\delta_1 \geqq \frac{5p-9}{2(2p-3)}(c+H^2) .$$

If $\delta_1>((5p-3)/2(2p-3))(c+H^2)$, from (3.17) and Hopf's maximum principle, we obtain $|\tau|^2=0$. If $\delta_1=((5p-3)/2(2p-3))(c+H^2)$, (3.16) and Hopf's maximum principle yield $|\tau|^2=$constant and all inequalities above become actually equalites.

If $|\tau|^2=0$, then M is totally umbilic. If $|\tau|^2\neq 0$, from (3.6) and (3.9), we have

$$h_{ijk}^{\alpha}=0 , \tag{3.18}$$

$$|\tau|^2=\frac{3}{2-\dfrac{1}{p-1}}(c+H^2) , \tag{3.19}$$

$$\mathrm{tr}(H_\alpha H_\beta-H_\beta H_\alpha)^2=2\,\mathrm{tr}(H_\alpha^2)\,\mathrm{tr}(H_\beta^2) \qquad \text{for } \alpha\neq\beta ,$$

$$(p-1)(p-2)(\sigma_1^2-\sigma_2)=0 . \tag{3.20}$$

From Lemma 1 in [1], we know that at most two of the matrices H_α are non-zero, say H_{α_1} and H_{β_1}, and we can suppose

$$H_{\alpha_1}=\lambda\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix} \quad\text{and}\quad H_{\beta_1}=\mu\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix} .$$

From (2.16), we have

$$H_{\alpha_1}H_4=H_4H_{\alpha_1}, \qquad H_{\beta_1}H_4=H_4H_{\beta_1}, \qquad \mathrm{tr}\, H_4=3H . \tag{3.21}$$

Hence under this local field of orthonormal frames, we also have

$$h_{ij}^4=H\delta_{ij} .$$

a) Case $p=2$. (2.16) implies for a suitable choice of the orthonormal frame field

$$h_{ij}^4=H\delta_{ij} ,$$

$$h_{ij}^{\alpha}=\lambda_i^{\alpha}\delta_{ij}\,,$$

$$\sum_i \lambda_i^{\alpha}=0\,. \tag{3.22}$$

If $\lambda_i^{\alpha}\neq 0$, from (3.12) and (3.13), we have

$$R_{ii}=\delta_1=\frac{1}{2}(c+H^2) \qquad \text{for } i=1,\,2,\,3,$$

$$3\delta_1=\frac{3}{2}(c+H^2)=6(c+H^2)-|\tau|^2=3(c+H^2) \qquad \text{(from (3.19))}.$$

This is a contradiction. Hence at least one of λ_i^{α} is zero, say $\lambda_3^{\alpha}=0$. Thus $\lambda_1^{\alpha}=-\lambda_2^{\alpha}$ from (3.22).

$$|\tau|^2=(\lambda_1^{\alpha})^2+(\lambda_2^{\alpha})^2=3(c+H^2),$$

$$(\lambda_1^{\alpha})^2=(\lambda_2^{\alpha})^2=\frac{3}{2}(c+H^2),$$

$$R_{ii}=\frac{1}{2}(c+H^2)=\text{constant}>0, \qquad i=1,\,2,$$

$$R_{33}=2(c+H^2)=\text{constant}>0\,,$$

$$r=\sum_i R_{ii}=3(c+H^2)>0\,.$$

$$\sum_{ij} R_{ij}^2=\frac{9}{2}(c+H^2)^2=\text{constant}.$$

Hence $\nabla_k R_{ij}=0$. Thus M is a 3-dimensional conformally flat submanifold with positive definite Ricci curvature. From Theorem 2 due to Goldberg [3], we know that M is a space form. Hence M is totally umbilic. This is a contradiction.

b) Case $p\geqq 3$. In this cases, (3.20) implies

$$\sigma_1^2=\sigma_2\,.$$

We obtain that at most two of H_{α}, $\alpha=5,\cdots,3+p$, are different from zero. Suppose that only one of them, say H_{α_1}, is different from zero. Then we have $\sigma_1^2=(1/p-1)|\tau|^2$ and $\sigma_2=0$, which is a contradiction. Therefore we can suppose that

$$H_5=\lambda\begin{pmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 0\end{pmatrix} \quad \text{and} \quad H_6=\mu\begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},$$

$$H_{\alpha}=0 \qquad \text{for } \alpha\geqq 7\,.$$

In this case,

$$H_4=HI, \qquad \operatorname{tr} H_5^2=2\lambda^2, \qquad \operatorname{tr} H_6^2=2\mu^2\,,$$

$$2\lambda^2+2\mu^2=|\tau|^2=3(c+H^2)\,. \tag{3.23}$$

(2.3) implies

$$\omega_{4i}=H\omega_i, \qquad \omega_{51}=\lambda\omega_1, \qquad \omega_{52}=-\lambda\omega_2, \qquad \omega_{53}=0,$$

$$\omega_{61}=\mu\omega_2, \qquad \omega_{62}=\mu\omega_1, \qquad \omega_{63}=0, \qquad \omega_{\alpha i}=0 \quad \text{for } \alpha=2, \cdots, 3+p\,.$$

Since $h_{ijk}^{\alpha}=0$ from (3.9), we have, for $\alpha=5, \cdots, 3+p$,

$$-dh_{ij}^{\alpha}=\sum h_{ik}^{\alpha}\omega_{kj}+\sum h_{kj}^{\alpha}\omega_{ki}+\sum h_{ij}^{\beta}\omega_{\beta\alpha}\,.$$

Setting $\beta=6$, $i=1$ and $j=2$, we have

$$d\mu=dh_{12}^{6}=0\,.$$

Hence μ is constant. Thus λ is also constant from (3.23).

$$R_{11}=R_{22}=2(c+H^2)-\lambda^2-\mu^2=\frac{1}{2}(c+H^2)=\text{constant}>0\,.$$

$$R_{33}=2(c+H^2)=\text{constant}>0\,.$$

Making use of the same proof as in case $p=2$, we obtain $|\tau|^2=0$. This is a contradiction. Thus we complete the proof of Proposition 1.

COROLLARY. *Let M be a 3-dimensional minimal submanifold in a sphere $S^{3+p}(c)$. If*

$$Ric(M)\geqq\frac{5p-4}{2(2p-1)}c\,,$$

then M is totally geodesic.

PROOF. Since M is a minimal submanifold in $S^{3+p}(c)$ and $S^{3+p}(c)$ is a totally umbilical hypersurface in $S^{3+p+1}(c-H^2)$, then M can be seen as a submanifold in $S^{3+p+1}(c-H^2)$. It is a pseudo-umbilical submanifold with parallel mean curvature vector $\boldsymbol{h}$. According to Proposition 1, we know that Corollary is true.

REMARK. The result in Corollary is better than one due to Shen [5].

PROPOSITION 2. *Let M be a 3-dimensional complete submanifold in $S^{3+p}(c)$ with parallel mean curvature vector. If*

$$Ric(M)\geqq\frac{3}{4}c+\frac{39}{64}H^2+\frac{1}{8}\sqrt{\frac{1521}{64}H^4+\frac{45}{2}H^2c}\,,$$

then M is a pseudo-umblical submanifold.

PROOF. Because of

$$Ric(M) \geqq \frac{3}{4}c + \frac{39}{64}H^2 + \frac{1}{8}\sqrt{\frac{1521}{64}H^4 + \frac{45}{2}H^2 c} > 0,$$

we conclude that M is compact from Myers' theorem. We choose a frame field in such a way that

$$h_{ij}^4 = \lambda_i \delta_{ij}.$$

Let $\mu_i = \lambda_i - H$, we have

$$\sum_i \mu_i = 0, \qquad \sum \mu_i^2 = |\sigma|^2 - 3H^2,$$

$$\sum_i \mu_i^3 = 6H^3 - 3H|\sigma|^2 + \sum \lambda_i^3,$$

$$(3.26) \qquad -\frac{1}{\sqrt{6}}\sqrt{(|\sigma|^2 - 3H^2)^3} \leqq \sum \mu_i^3 \leqq \frac{1}{\sqrt{6}}\sqrt{(|\sigma|^2 - 3H^2)^3},$$

and equality holds if and only if two of μ_i are equal (cf. [4]). Because of

$$(3.27) \qquad \sum_{\alpha \neq 4}\Big(\sum_i \lambda_i h_{ii}^\alpha\Big)^2 = \sum_{\alpha \neq 4}\Big\{\sum_i (\lambda_i - H)h_{ii}^\alpha\Big\}^2$$

$$\leqq (|\sigma|^2 - 3H^2)|\tau|^2,$$

from (2.18), (3.26) and (3.27), we obtain

$$(3.28) \qquad \frac{1}{2}\Delta|\sigma|^2 = \sum (h_{ijk}^4)^2 + \sum h_{ij}^4 \Delta h_{ij}^4$$

$$= \sum (h_{ijk}^4)^2 + \sum (h_{km}^4 R_{mijk} + h_{mi}^4 R_{mkjk}) h_{ij}^4$$

$$= \sum (h_{ijk}^4)^2 + 3c(|\sigma|^2 - 3H^2) - |\sigma|^4 + 3H\sum \lambda_i^3 - \sum_{\alpha \neq 4}\Big(\sum_i \lambda_i h_{ii}^\alpha\Big)^2$$

$$\geqq \sum (h_{ijk}^4)^2 + 3c(|\sigma|^2 - 3H^2) + 9H^2(|\sigma|^2 - 2H^2)$$

$$-\frac{3H}{\sqrt{6}}\sqrt{(|\sigma|^2 - 3H^2)^3} - (|\sigma|^2 - 3H^2)|\tau|^2 - |\sigma|^4$$

$$= \sum (h_{ijk}^4)^2 + (|\sigma|^2 - 3H^2)$$

$$\times \Big\{3(c + H^2) - \frac{3H}{\sqrt{6}}\sqrt{(|\sigma|^2 - 3H^2)} - |\sigma|^2 + 3H^2 - |\tau|^2\Big\}.$$

On the other hand, since M is a 3-dimensional submanifold, its Weyl conformally curvature tensor vanishes, i.e.,

$$R_{ijkm} = R_{ik}\delta_{jm} - R_{im}\delta_{jk} + R_{jm}\delta_{ik} - R_{jk}\delta_{im} - \frac{1}{2}r(\delta_{ik}\delta_{jm} - \delta_{im}\delta_{jk}),$$

$$\sum (h_{km}^4 R_{mijk} + h_{mi}^4 R_{mkjk}) h_{ij}^4$$

$$= \frac{1}{2}\sum_{i \neq j} (\lambda_i - \lambda_j)^2 R_{ijij}$$

$$=\frac{1}{2}\sum_{i\neq j}(\lambda_i-\lambda_j)^2\left(R_{ii}+R_{jj}-\frac{r}{2}\right)$$

$$\geqq 3\left(2\delta_1-\frac{r}{2}\right)(|\sigma|^2-3H^2).$$

From (2.18), we have

$$\frac{1}{2}\Delta|\sigma|^2\geqq\sum(h^4_{ijk})^2+3\left(2\delta_1-\frac{r}{2}\right)(|\sigma|^2-3H^2) \tag{3.29}$$

$$\geqq 3\left(2\delta_1-3c-\frac{9}{2}H^2+\frac{1}{2}|\tau|^2+\frac{1}{2}|\sigma|^2\right)(|\sigma|^2-3H^2).$$

(3.28)×3/2+(3.29) implies

$$\frac{5}{2}\Delta|\sigma|^2\geqq\left\{6\delta_1-\frac{9}{2}(c+H^2)-\frac{3\sqrt{6}}{4}H\sqrt{(|\sigma|^2-3H^2)}\right\}(|\sigma|^2-3H^2).$$

Because of

$$3\delta_1\leqq\sum R_{ii}=r=6c+9H^2-|\sigma|^2-|\tau|^2,$$

we have

$$|\sigma|^2-3H^2\leqq 6c+6H^2-3\delta_1.$$

Hence

$$\frac{5}{4}\Delta(|\sigma|^2-3H^2)=\frac{5}{4}\Delta|\sigma|^2 \tag{3.30}$$

$$\geqq\left\{6\delta_1-\frac{9}{2}(c+H^2)-\frac{9\sqrt{2}}{4}H\sqrt{2(c+H^2)-\delta_1}\right\}(|\sigma|^2-3H^2).$$

By a straightforward calculation, we can easily verify that if

$$\delta_1>\frac{3}{4}c+\frac{39}{64}H^2+\frac{1}{8}\sqrt{\frac{1521}{64}H^4+\frac{45}{2}H^2c},$$

we have

$$\left\{6\delta_1-\frac{9}{2}(c+H^2)-\frac{9\sqrt{2}}{4}H\sqrt{2(c+H^2)-\delta_1}\right\}(|\sigma|^2-3H^2)>0.$$

According to (3.30) and Hopf's maximum principle, we conclude

$$|\sigma|^2-3H^2=0.$$

Hence, M is pseudo-umbilic. If

$$\delta_1=\frac{3}{4}c+\frac{39}{64}H^2+\frac{1}{8}\sqrt{\frac{1521}{64}H^4+\frac{45}{2}H^2c},$$

then,

$$\left\{6\delta_1-\frac{9}{2}(c+H^2)-\frac{9\sqrt{2}}{4}H\sqrt{2(c+H^2)-\delta_1}\right\}(|\sigma|^2-3H^2)=0.$$

Therefore from (3.30), we obtain that $|\sigma|^2-3H^2=$constant and all inequalities above are equalities. If $|\sigma|^2-3H^2=0$, then M is pseudo-umbilic. If $|\sigma|^2-3H^2>0$, we get that two of μ_i are equal. Without loss of generality, we can suppose $\mu_1=\mu_2$, then $\mu_3=-2h_1$. From (2.18), we have

$$\begin{aligned}0&=\sum(h^4_{km}R_{mijk}+h^4_{mi}R_{mkjk})h^4_{ij}\\&=\frac{1}{2}\sum_{i\neq j}(\lambda_i-\lambda_j)^2\Big(R_{ii}+R_{jj}-\frac{r}{2}\Big)\\&=9\mu_1^2R_{33}\,.\end{aligned}$$

Therefore $R_{33}=0$. On the other hand,

$$R_{33}\geqq\delta_1=\frac{3}{4}c+\frac{39}{64}H^2+\frac{1}{8}\sqrt{\frac{1521}{64}H^4+\frac{45}{2}H^2c}>0\,.$$

This is a contradiction. Hence M is pseudo-umbilic.

Proof of Theorem 1. When $p=2$, $((5p-9)/2(2p-3))=1/2$. Hence

$$\frac{3}{4}c+H^2+\frac{1}{8}\sqrt{\frac{1521}{64}H^4+\frac{45}{2}H^2c}>\frac{1}{2}(c+H^2)\,.$$

According to Propositions 1 and 2, we conclude easily that M is a 3-dimensional small sphere. When $p=1$, Proposition 1 implies that Theorem 1 is true.

Proof of Theorem 2. According to Propositions 1 and 2, Theorem 2 holds good obviously.

Authors would like to express their deep thanks for the referee for his suggestion on Corollary in this section.

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Keywords. Parallel mean curvature vector, Ricci curvature and totally umbilic submanifolds.

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