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ON QUASI-NORMALITY OF FUNCTION RINGS

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ABSTRACT. An f-ring is called quasi-normal [**26**] if the sum of any two different minimal prime ℓ-ideals is either a maximal ℓ-ideal or the entire f-ring. Recall that the *zero-component* of a prime ideal P of a commutative ring A is the ideal

$$O_P = \{a \in A \mid ab = 0 \text{ for some } b \in A \setminus P\}.$$

Let $C(X)$ be the f-ring of continuous real-valued functions on a Tychonoff space X. Larson proved that $C(\beta X)$ is quasi-normal precisely when $C(X)$ is quasi-normal and the zero-component of every hyper-real ideal of $C(X)$ is prime. We show that this result is actually purely ring-theoretic and thus deduce its extension to the f-rings $\mathcal{R}L$ of continuous real-valued functions on a frame L. A subspace of X is called a 2-boundary subspace if it is of the form $\mathrm{cl}_X(C) \cap \mathrm{cl}_X(D)$ for some disjoint cozero-sets C and D of X. For normal spaces, Kimber [**25**] proved that $C(X)$ is quasi-normal precisely when every 2-boundary subspace of X is a P-space. By viewing spaces as locales, we obtain a characterization along similar lines which does not require normality, namely, for any Tychonoff space X, $C(X)$ is quasi-normal if and only if every 2-boundary sublocale of the Lindelöf reflection of X in the category of locales is a P-frame.

Introduction. All f-rings that play a significant role in this paper are reduced and have a multiplicative identity. The concept of quasi-normality was introduced by Larson [**26**]. She defined an f-ring A to be *quasi-normal* if the sum of any two different minimal prime ℓ-ideals is either a maximal ℓ-ideal or the entire f-ring. In order to clarify the meaning of quasi-normality somewhat, recall that the sum $I + J$ of

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two ℓ-ideals in an f-ring (actually, in any ℓ-ring) A is an ℓ-ideal, and, in fact, is the join $I \vee J$ in the frame of ℓ-ideals of A. It follows that $I+J=A$ if and only if there is no maximal ℓ-ideal that contains both I and J.

Larson gave several characterizations of quasi-normal f-rings. Subsequently, other authors such as Kimber **[25]** and Henriksen, Martínez and Woods **[22]** paid some attention to these f-rings. In particular, Kimber sharpened a number of Larson's results, including the characterization stated in the abstract. Henriksen, Martínez and Woods **[22]** improved a result of Larson stating that, if Y is a C-embedded subspace of X and $C(X)$ is quasi-normal, then $C(Y)$ is quasi-normal. They showed that, in fact, this is true for all z-embedded subspaces.

Throughout, we shall use the notation of Gillman and Jerison **[19]** regarding the ring $C(X)$ and its ideals. Our goal in this paper is to further extend and strengthen the results mentioned above, and also to make fully algebraic the result [**26,** Theorem 3.8] of Larson which states that, if X is a Tychonoff space, then $C(\beta X)$ is quasi-normal if and only if $C(X)$ is quasi-normal and the ideal $\boldsymbol{O}^p$ of $C(X)$ is a prime ideal for each $p \in \beta X \setminus \upsilon X$, where υX denotes the Hewitt real compactification of X. We shall show that this is actually an f-ring result; following is how we proceed. We recalled in the abstract the definition of the zero-component of a prime ideal. It is worth pointing out that, if A is reduced and P is a prime ideal in A, then the zero-component of P is also expressible as

$$O_P = \bigcap\{Q \in \operatorname{Spec} A \mid Q \subseteq P\},$$

where $\operatorname{Spec} A$ denotes, as usual, the set of prime ideals of A. We prove in Theorem 2.2 that, if A is an f-ring with bounded inversion and the sum of two minimal prime ideals of A is a prime ideal if it is proper (as is the case in $C(X)$), then A^*, the bounded part of A, is quasi-normal if and only if A is quasi-normal and O_M is a prime ideal for every maximal ideal M of A which is "hyper-real-like." We will make precise this adjective when we get to it. This result will be the content of Section 2, and, from it, we will derive as a corollary a localic version of [**26,** Theorem 3.8], from which Larson's result may actually be deduced.

It was mentioned in the abstract that Kimber proved in [**25,** Theorem 4.3] that, if X is a normal Tychonoff space, then the f-ring $C(X)$ is quasi-normal if and only if, for every pair C and D of disjoint cozero-sets of X, the subspace $\mathrm{cl}_X(C) \cap \mathrm{cl}_X(D)$ is a P-space. In Section 3, we extend this result to the f-rings $\mathcal{R}L$. Since λL is normal and $\mathcal{R}L \cong \mathcal{R}(\lambda L)$, we obtain a characterization (Theorem 3.6) along the lines of [**25,** Theorem 4.3] without the requirement that the frame L be normal.

As mentioned above in [**22,** Proposition 3.2], the authors showed that, if Y is a z-embedded subspace of X and $C(X)$ is quasi-normal, then $C(Y)$ is quasi-normal. We conclude the paper by extending this result to locales (Theorem 3.8). The proof in [**22**] uses results of Montgomery [**29, 30**]. Our proof is not modeled on that in [**22**] and does not use (frame-theoretic analogues of) Montgomery's results. Instead, it uses an artifact that is not available in the category of topological spaces, namely, the Lindelöf reflection.

1. Preliminaries.

1.1. Frames and their homomorphisms. We refer to [**24, 31**] for the theory of frames. As in those texts, we use the terms "frame" and "locale" interchangeably. Also, we shall at times speak of a sublocale of a frame. For instance, if L is a frame and $a \in L$, we shall write $\mathfrak{c}(a)$ (or $\mathfrak{c}_L(a)$) for the closed sublocale of L corresponding to a. We denote by $\prec\!\!\prec$ the familiar *completely below* relation. All of our frames are completely regular, which is to say, every element is the join of elements completely below it.

We write $\mathrm{Coz}\,L$ for the set of cozero elements of L. We denote by βL the Stone-Čech compactification of L, and we take it to be the frame of completely regular ideals of $\mathrm{Coz}\,L$. The coreflection map from compact completely regular frames to L will be denoted by

$$j_L \colon \beta L \longrightarrow L,$$

and its right adjoint by r_L. For any $a \in L$,

$$r_L(a) = \{c \in \mathrm{Coz}\,L \mid c \prec\!\!\prec a\};$$

and for any $c, d \in \mathrm{Coz}\,L$,

$$r_L(c \vee d) = r_L(c) \vee r_L(d).$$

By a *point* of a frame we mean a meet-irreducible element. We denote by $\mathrm{Pt}(L)$ the set of points of L. If

$$h\colon L \longrightarrow M$$

is a frame homomorphism, then $h_*(p) \in \mathrm{Pt}(L)$, for every $p \in \mathrm{Pt}(M)$, where h_* denotes the right adjoint of h. Every frame homomorphism $h\colon L \to M$ has a *Stone-extension*,

$$\beta h\colon \beta L \longrightarrow \beta M,$$

which is the unique frame homomorphism making the square

$$\begin{tikzcd} \beta L \arrow[r, "\beta h"] \arrow[d, "j_L"'] & \beta M \arrow[d, "j_M"] \\ L \arrow[r, "h"'] & M \end{tikzcd}$$

commute. Explicitly, for any $I \in \beta L$,

$$\beta h(I) = \{z \in \mathrm{Coz}\, M \mid z \leq h(u) \text{ for some } u \in I\}.$$

Completely regular Lindelöf frames form a coreflective subcategory of **CRFrm** **[28]**. The completely regular Lindelöf coreflection of L, denoted λL, is the frame of σ-ideals of $\mathrm{Coz}\, L$. The join map

$$e_L\colon \lambda L \longrightarrow L$$

is a dense onto frame homomorphism, and it is the coreflection map to L from Lindelöf completely regular frames. For any $a \in L$, let

$$[\![a]\!] = \{c \in \mathrm{Coz}\, L \mid c \leq a\}.$$

The right adjoint of $e_L\colon \lambda L \to L$ is given by $a \mapsto [\![a]\!]$, and the cozero part of λL is related to that of L as follows:

$$\mathrm{Coz}(\lambda L) = \{[\![c]\!] \mid c \in \mathrm{Coz}\, L\}.$$

1.2. f-Rings. We recall a few facts about f-rings that are relevant to our discussion. An ideal I of an f-ring A is called an *ℓ-ideal* if, for any $a, b \in A$, $|a| \leq |b|$ and $b \in I$ imply $a \in I$. An ideal is *semiprime* if, whenever it contains the square of an element, then it already contains the element. An f-ring is *reduced* (or *semiprime*) if it has no non-zero nilpotent elements. We write $\mathrm{Max}(A)$ for the space of maximal ideals of A with the Zariski topology.

In a reduced f-ring, a prime ideal is minimal prime if and only if every member of the ideal is annihilated by some non-member [**21**]. Using this, it may easily be verified that, in a reduced f-ring, every minimal prime ideal is an ℓ-ideal. An f-ring *has bounded inversion* if every $a \geq 1$ is a unit. In a reduced f-ring with bounded inversion, maximal ℓ-ideals are precisely the maximal ring ideals.

Frequently, we shall use the extension and contraction notation. To recall, let B be a subring of A, I be an ideal of B and J an ideal of A. The *extension* of I, denoted I^e, is the (possibly improper) ideal of A generated by I. The *contraction* of J, denoted J^c, is the ideal $J \cap B$ of B. We always have

$$(I_1 + I_2)^e = I_1^e + I_2^e$$

and

$$J_1^c + J_2^c \subseteq (J_1 + J_2)^c.$$

We denote by A^* the *bounded part* of an f-ring A, and recall that

$$A^* = \{a \in A \mid |a| \leq n \cdot 1 \text{ for some } n \in \mathbb{N}\}.$$

It is proved in [**12**] that, if A is a reduced f-ring with bounded inversion, and

$$S = \{a \in A^* \mid a \text{ is invertible in } A\},$$

then $A = A^*[S^{-1}]$, that is, A is the ring of fractions of A^* with respect to S. An upshot of this is that, for any ideal I of A^*,

$$I^e = \{us^{-1} \mid u \in I \text{ and } s \in S\},$$

and every ideal of A is of the form I^e for some ideal I of A^*. If A has bounded inversion, so does A^*.

1.3. The f-ring $\mathcal{R}L$ and some of its ideals. Our approach to the f-ring $\mathcal{R}L$ follows that of [**2**]; thus, the elements of $\mathcal{R}L$ are frame homomorphisms

$$\mathfrak{L}(\mathbb{R}) \longrightarrow L,$$

where $\mathfrak{L}(\mathbb{R})$ is the frame of reals. The f-ring $\mathcal{R}L$ has bounded inversion, and every prime ideal in $\mathcal{R}L$ is an ℓ-ideal [**14,** Lemma 3.5]. For any L, the f-rings $\mathcal{R}L$ and $\mathcal{R}(\lambda L)$ are isomorphic. Every frame homomorphism

$$h\colon L \longrightarrow M$$

induces an ℓ-ring homomorphism

$$\mathcal{R}h\colon \mathcal{R}L \longrightarrow \mathcal{R}M,$$

given by $\mathcal{R}h(\alpha) = h \cdot \alpha$. Furthermore, $\operatorname{coz}(h \cdot \alpha) = h(\operatorname{coz}\alpha)$.

An ideal Q of $\mathcal{R}L$ is a *z-ideal* if, for any $\alpha, \beta \in \mathcal{R}L$, $\operatorname{coz}\alpha = \operatorname{coz}\beta$ and $\alpha \in Q$ imply $\beta \in Q$. It is shown in [**15**] that, exactly as in $C(X)$, the sum of z-ideals of $\mathcal{R}L$ is a z-ideal. For any $I \in \beta L$, the ideals $\boldsymbol{M}^{\boldsymbol{I}}$ and $\boldsymbol{O}^{\boldsymbol{I}}$ of $\mathcal{R}L$ are defined by

$$\boldsymbol{M}^{\boldsymbol{I}} = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \subseteq I\},$$

and

$$\boldsymbol{O}^{\boldsymbol{I}} = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \prec\!\!\prec I\} = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \in I\}.$$

Clearly, these ideals are z-ideals. We abbreviate $\boldsymbol{M}^{r_L(a)}$ as $\boldsymbol{M}_a$, and remark that

$$\boldsymbol{M}_a = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \leq a\}.$$

Maximal ideals of $\mathcal{R}L$ are precisely the ideals $\boldsymbol{M}^{\boldsymbol{I}}$, for $I \in \operatorname{Pt}(\beta L)$ [**6**]. At this juncture, it is proper to draw the reader's attention to the fact that, just as the ideal $\boldsymbol{O}^p$ is the zero-component of the maximal ideal $\boldsymbol{M}^p$ of $C(X)$ for any $p \in \beta X$ [**19**], the ideal $\boldsymbol{O}^{\boldsymbol{I}}$ is the zero-component of the maximal ideal $\boldsymbol{M}^{\boldsymbol{I}}$ of $\mathcal{R}L$ for every $I \in \operatorname{Pt}(\beta L)$ [**6,** Lemma 5.3].

A surjective frame homomorphism $h\colon M \to L$ is called a *C-quotient map* [**1**] if, for every $f \in \mathcal{R}L$, there is a (necessarily unique) $\widehat{f} \in \mathcal{R}M$ such that the triangle below commutes.

$$\begin{tikzcd} & \mathfrak{L}(\mathbb{R}) \arrow[dl, "\hat{f}"'] \arrow[dr, "f"] & \\ M \arrow[rr, "h"'] & & L. \end{tikzcd}$$

A completely regular frame L is *realcompact* if every ring homomorphism

$$\mathcal{R}L \longrightarrow \mathbb{R}$$

is a point evaluation, and it is *pseudocompact* if $\mathcal{R}^*L = \mathcal{R}L$.

2. Larson's result made algebraic. In [**26,** Theorem 3.8], Larson showed that $C(\beta X)$ is quasi-normal if and only if $C(X)$ is quasi-normal and $\boldsymbol{O}^p$ is a prime ideal for every $p \in \beta X \setminus \upsilon X$. We show that this holds in more general f-rings than rings of continuous functions. We then deduce the same result for $\mathcal{R}L$. In preparation for this, we recite the following result of Banaschewski [**3**] and then interpret it in a manner suitable for our purposes. Denote by $[a]$ the principal ℓ-ideal of an f-ring A generated by $a \in A$, and recall that, for any $x \in A$,

$$x \in [a] \iff |x| \le r|a| \quad \text{for some } r \ge 0.$$

The result from [**3**] we shall use states the following:

> For any f-ring A, the identical embedding $A^* \to A$ induces a homeomorphism $\mathrm{Max}(A^*) \to \mathrm{Max}(A)$ taking P to $\{a \in A \mid [a] \cap A^* \subseteq P\}$.

Given a maximal ideal P of A^*, let us put

$$\widetilde{P} = \{a \in A \mid [a] \cap A^* \subseteq P\},$$

and observe that $\widetilde{P}$ is the largest proper ideal J of A such that $J \cap A^* = P$. Thus, for every maximal ideal M of A, there is a unique maximal ideal M^* of A^* such that $M = \widetilde{M^*}$.

Lemma 2.1. *Let A be an f-ring with bounded inversion, M a maximal ideal in A and M^* the unique maximal ideal in A^* such that $M = \widetilde{M^*}$.*

(a) *If M^* contains a unit of A, then $M^c \subset M^*$ (proper inclusion).*
(b) *If M^* contains no unit of A, then $M^c = M^*$.*

Proof.

(a) Let $a \in M^c$. Since $M = \widetilde{M^*}$, this implies $[a] \cap A^* \subseteq M^*$, and since $a \in [a] \cap A^*$, it follows that $a \in M^*$. Therefore, $M^c \subseteq M^*$. By hypothesis, M^* contains a unit of A, which cannot belong to M since M is a proper ideal of A. Thus, $M^c \subset M^*$.

(b) Exactly as above, we have $M^c \subseteq M^*$. Let $b \in M^*$. In order to show that $b \in M$, we must show that $[b] \cap A^* \subseteq M^*$. Thus, let $x \in [b] \cap A^*$. Choose $r \geq 0$ in A such that $|x| \leq r|b|$. Since A has a bounded inversion, $(1+r)^{-1}$ exists, and

$$|x| \cdot \frac{1}{1+r} \leq |b| \cdot \frac{r}{1+r} \in M^*.$$

Since M^* is an ℓ-ideal, it follows that

$$|x| \cdot \frac{1}{1+r} \in M^*,$$

and hence, by primeness, $|x| \in M^*$ since

$$\frac{1}{1+r} \notin M^*$$

as it is a unit of A. Therefore, $x \in M^*$, and we deduce from this that $b \in M$, showing that $M^* \subseteq M^c$, and hence, equality. $\square$

Recall the zero-component O_P of a prime ideal P. As previously mentioned, if A is a reduced ring, then O_P is the intersection of all minimal prime ideals of A contained in P [**23,** Lemma B]. In $\mathcal{R}L$, we have that, for any $I \in \mathrm{Pt}(\beta L)$, $O_{\boldsymbol{M}^I} = \boldsymbol{O}^I$ [**6,** Lemma 5.3(2)]. Recall that in $C(X)$ (and hence in $C^*(X)$) the sum of prime ideals is a prime ideal [**19,** Problem 14.B]; hence, the condition we impose in the next theorem holds in $\mathcal{R}L$ since $\mathcal{R}^*L$ is isomorphic to $C(X)$.

Theorem 2.2. *Let A be a reduced f-ring with bounded inversion. Suppose that the sum of two minimal prime ideals in A^* is a prime ideal if it is proper. Then, A^* is quasi-normal if and only if A is quasi-normal, and O_M is a prime ideal for every maximal ideal M of A for which M^* contains a unit of A.*

Proof.

($\Rightarrow$). Let M be a maximal ideal of A such that M^* contains a unit of A. We show that M contains exactly one minimal prime ideal, in which case it will follow that O_M is this minimal prime ideal. Suppose, by way of contradiction, that P and Q are distinct minimal prime ideals of A contained in M. Then, P^c and Q^c are minimal prime ideals of A^* contained in M^c, and hence, in the maximal ideal M^* of A^*. Since A^*

is quasi-normal, by the present hypothesis, we have

$$M^* = P^c + Q^c \subseteq (P+Q)^c \subseteq M^c,$$

which is a contradiction since $M^c \subset M^*$, by Lemma 2.1 (a). Therefore, O_M is a minimal prime ideal.

Now, to show that A is quasi-normal, let P and Q be distinct minimal prime ideals of A contained in some maximal ideal M of A, say. Then, by what we have just proved above, M^* contains no unit of A so that $M^c = M^*$, by Lemma 2.1 (b). Now, P^c and Q^c are minimal prime ideals of A^* each contained in $M^c = M^*$. Thus, since A^* is quasi-normal, $P^c + Q^c = M^*$. We argue from this that $P + Q = M$. Let $a \in M$. Then

$$\frac{a}{1+|a|} \in M^c = M^*,$$

so there exist $u \in P^c$ and $v \in Q^c$ such that

$$\frac{a}{1+|a|} = u + v,$$

which implies

$$a = u(1+|a|) + v(1+|a|) \in P + Q.$$

Thus, $M \subseteq P + Q$, and hence, $P + Q = M$. Therefore, A is quasi-normal.

($\Leftarrow$). We begin by showing that, under the current hypothesis, if M is a maximal ideal of A such that M^* contains a unit of A, then O_{M^*} is a prime ideal of A^*, which will then make it the only minimal prime ideal of A^* contained in M^*. By hypothesis, O_M is a prime ideal in A; thus, O_M^c is a prime ideal in A^* with $O_M^c \subseteq M^c \subset M^*$. Since O_{M^*} is the intersection of minimal prime ideals of A^* contained in M^*, we have $O_{M^*} \subseteq O_M^c$. We want to show that this inclusion is actually equality. Hence, let $a \in O_M^c$. Then, $a \in O_M$, and thus, there exists some $b \in A \setminus M$ such that $ab = 0$. Since $b \notin M = \widetilde{M^*}$, we must have $[b] \cap A^* \nsubseteq M^*$. Take $c \in [b] \cap A^*$ with $c \notin M^*$. Take $r \geq 0$ in A such that $|c| \leq r|b|$. Then,

$$|ca| = |c||a| \leq r|b||a| = 0,$$

which implies $ca = 0$. This shows that $a \in O_{M^*}$, so that $O_M^c \subseteq O_{M^*}$, and hence, equality. Thus, if M^* contains a unit of A, then there are no two distinct minimal prime ideals of A^* contained in M^*.

Now, let $\mathfrak{p}$ and $\mathfrak{q}$ be distinct minimal prime ideals of A^* contained in some maximal ideal $\mathfrak{m}$, say, of A^*. From the result of Banaschewski **[4]**, there is a maximal ideal M of A such that $\mathfrak{m} = M^*$. As just observed, M^* contains no unit of A. Since minimal prime ideals in reduced rings consist entirely of zero divisors, $\mathfrak{p}^e$ and $\mathfrak{q}^e$ are proper ideals in A. We show that they are minimal prime ideals. This is only done for $\mathfrak{p}^e$. Let S be the set of units of A that belong to A^*. Assume that $ab \in \mathfrak{p}^e$ for some $a, b \in A$. Choose $c \in \mathfrak{p}$ and $s \in S$ such that $ab = cs^{-1}$. Then,

$$\frac{s}{1+|s|} \cdot \frac{a}{1+|a|} \cdot \frac{b}{1+|b|} = \frac{c}{(1+|s|)(1+|a|)(1+|b|)} \in \mathfrak{p},$$

which implies that at least one of the factors on the left is in $\mathfrak{p}$. Since

$$\frac{s}{1+|s|} \notin \mathfrak{p}$$

as it is a non-zerodivisor, we may assume

$$\frac{a}{1+|a|} \in \mathfrak{p},$$

whence $a \in \mathfrak{p}^e$ since

$$a = \frac{a}{1+|a|}\left(\frac{1}{1+|a|}\right)^{-1}.$$

Therefore, $\mathfrak{p}^e$ is prime. In order to see that it is minimal prime, consider an arbitrary $ps^{-1} \in \mathfrak{p}^e$, with $p \in \mathfrak{p}$ and $s \in S$. Choose $r \in A^* \setminus \mathfrak{p}$ such that $rp = 0$. Then, $r(ps^{-1}) = 0$. It is easy to see that $r \notin \mathfrak{p}^e$; therefore, $\mathfrak{p}^e$ is minimal prime.

Next, we show that $\mathfrak{p}^e$ and $\mathfrak{q}^e$ are distinct. Take $a \in \mathfrak{p} \setminus \mathfrak{q}$. If a were in $\mathfrak{q}^e$, there would exist $q \in \mathfrak{q}$ and $s \in S$ such that $a = qs^{-1}$, whence we would have

$$a \cdot \frac{s}{1+|s|} = q \cdot \frac{1}{1+|s|} \in \mathfrak{q},$$

implying either $a \in \mathfrak{q}$ or

$$\frac{s}{1+|s|} \in \mathfrak{q},$$

neither of which is true. Therefore, $a \in \mathfrak{p}^e \setminus \mathfrak{q}^e$, and thus, these minimal prime ideals are distinct. Furthermore, they are contained in M since $\mathfrak{p} \subseteq M^*$ implies

$$\mathfrak{p}^e \subseteq (M^*)^e = M^{ce} \subseteq M.$$

Since A is quasi-normal and $\mathfrak{p}^e$ and $\mathfrak{q}^e$ are distinct minimal prime ideals of A contained in M, we have $\mathfrak{p}^e + \mathfrak{q}^e = M$. Let $m \in M^* = M^c$. Then,

$$m \in M = \mathfrak{p}^e + \mathfrak{q}^e = (\mathfrak{p} + \mathfrak{q})^e,$$

so that $m = as^{-1}$ for some $a \in \mathfrak{p} + \mathfrak{q}$ and $s \in S$. Thus, sm is an element of the prime ideal $\mathfrak{p} + \mathfrak{q}$, which implies $m \in \mathfrak{p} + \mathfrak{q}$ since $s \notin \mathfrak{p} + \mathfrak{q}$ as it is a unit in A. So, $M^* \subseteq \mathfrak{p} + \mathfrak{q}$, and hence, equality. Therefore, A^* is quasi-normal. □

We shall deduce the equivalence of (1) and (2) in Theorem 2.7 below from this foregoing theorem. We require more background to prove the other two statements in Theorem 2.7 that are equivalent to the quasi-normality of $\mathcal{R}(\beta L)$. Observe that, since, for any $\alpha \in \mathcal{R}L$ and $I \in \beta L$, $\alpha \in \boldsymbol{O}^I$ if and only if $\operatorname{coz} \alpha \in I$, we have that $\boldsymbol{O}^I$ is a prime ideal in $\mathcal{R}L$ if and only if I is a prime ideal in the regular σ-frame $\operatorname{Coz} L$. Hence, by [**18,** Lemma 3.8], $\boldsymbol{O}^I$ is prime if and only if, for any $\alpha, \beta \in \mathcal{R}L$, $\alpha\beta = \mathbf{0}$ implies $\alpha \in \boldsymbol{O}^I$ or $\beta \in \boldsymbol{O}^I$. In fact, any z-ideal of $\mathcal{R}L$ is prime if and only if it contains at least one of any two functions that annihilate each other, exactly as in $C(X)$ [**19,** Theorem 2.9].

Next, recall that an ideal Q of $\mathcal{R}L$ is called *free* in the case

$$\bigvee \{\operatorname{coz} \alpha \mid \alpha \in Q\} = 1,$$

and *fixed* otherwise. It is shown in [**8,** Lemma 4.7] that a frame L is compact if and only if every maximal ideal in $\mathcal{R}L$ is fixed. In [**5,** Lemma 4.4] it is shown that, for any $I \in \beta L$,

$$\bigvee \{\operatorname{coz} \alpha \mid \alpha \in \boldsymbol{M}^I\} = \bigvee I.$$

Consequently, *L is compact if and only if $\bigvee I \neq 1$ for every $I \in \operatorname{Pt}(\beta L)$.* In [**17,** Lemma 5.1], it is proved that, if $h\colon L \to M$ is a frame homomorphism, then

$$(\beta h)_*(I) = \bigvee \{r_L h_*(u) \mid u \in I\} = \bigcup \{r_L h_*(u) \mid u \in I\},$$

for any $I \in \beta M$. That the join is a union is not mentioned in **[17]**, but it holds since the set whose join is indicated is upward directed. In particular, if $a \in L$ and $\kappa_a \colon L \to {\uparrow}a$ is the homomorphism

$$x \longmapsto a \vee x,$$

then

$$(\beta(\kappa_a))_*(I) = \bigcup\{r_L(u) \mid u \in I\}$$

since $(\kappa_a)_*$ is the inclusion map ${\uparrow}a \to L$.

In **[25]**, Kimber calls a subspace of a Tychonoff space X a 2-*boundary subspace* if it is of the form $\mathrm{cl}_X(C) \cap \mathrm{cl}_X(D)$ for some disjoint cozero-sets C and D of X. We wish to adapt her terminology to locales. Denote the *pseudocomplement* of an element a in a frame by a^*, and recall that a^* is the largest element disjoint from a. We say that a sublocale of L is a 2-*boundary sublocale* if it is of the form $\mathfrak{c}(c^*) \cap \mathfrak{c}(d^*)$ for some $c, d \in \mathrm{Coz}\, L$ with $c \wedge d = 0$. Observe that, as a frame,

$$\mathfrak{c}(c^*) \cap \mathfrak{c}(d^*) = {\uparrow}(c^* \vee d^*).$$

The next lemma should be compared with **[22,** Theorem 1.4].

Lemma 2.3. *The following are equivalent for a normal frame* L:

(1) $\boldsymbol{O}^I$ *is prime for every* $I \in \mathrm{Pt}(\beta L)$ *with* $\bigvee I = 1$.
(2) *Every* 2*-boundary sublocale of* L *is compact.*

Proof.

(1) $\Rightarrow$ (2). Assume that (1) holds, and let $c \wedge d = 0$ in $\mathrm{Coz}\, L$. In order to prove that ${\uparrow}(c^* \vee d^*)$ is compact, we use the compactness criterion quoted above. For brevity, write $a = c^* \vee d^*$. Suppose, by way of contradiction, that ${\uparrow}a$ is not compact. Then, there exists an $I \in \mathrm{Pt}(\beta({\uparrow}a))$ such that $\bigvee I = 1$. Suppress a, and write

$$\overline{\kappa} \colon \beta L \longrightarrow \beta({\uparrow}a)$$

for the Stone extension of the homomorphism

$$\kappa_a \colon L \longrightarrow {\uparrow}a.$$

The argument employed in the proof of [**8,** Proposition 4.8] shows that $\bigvee \overline{\kappa}_*(I) = 1$. Now, $\overline{\kappa}_*(I) \in \mathrm{Pt}(\beta L)$, and, as remarked above,

$$\overline{\kappa}_*(I) = \bigcup\{r_L(u) \mid u \in I\}.$$

By hypothesis, $\boldsymbol{O}^{\bar{\kappa}_*(I)}$ is a prime ideal in $\mathcal{R}L$; thus, $\overline{\kappa}_*(I)$ is a prime ideal in $\operatorname{Coz} L$. Since $c \wedge d = 0$, we may assume that $c \in \overline{\kappa}_*(I)$, which implies $c \prec\!\!\prec u$ for some $u \in I$. However, this implies $c^* \vee u = 1$, and hence, $(c^* \vee d^*) \vee u = 1$. Since I is an ideal in the frame $\uparrow(c^* \vee d^*)$, it contains the bottom of this frame, which then implies $1 \in I$. This is false since I is a point in $\beta(\uparrow a)$. This contradiction proves that $\uparrow(c^* \vee d^*)$ is compact.

(2) $\Rightarrow$ (1). Assume that (2) holds, and let $I \in \mathrm{Pt}(\beta L)$ be such that $\bigvee I = 1$. In order to show that $\boldsymbol{O}^I$ is prime, consider any two functions $\alpha, \beta \in \mathcal{R}L$ such that $\alpha\beta = \mathbf{0}$ and $\alpha \notin \boldsymbol{O}^I$. We must show that $\beta \in \boldsymbol{O}^I$. Set $a = \operatorname{coz} \alpha$ and $b = \operatorname{coz} \beta$. Then, a and b are cozero elements with $a \wedge b = 0$. Thus, by (2), $\uparrow(a^* \vee b^*)$ is compact, and therefore, the cover

$$\{a^* \vee b^* \vee s \mid s \in I\}$$

of $\uparrow(a^* \vee b^*)$ has a finite subcover. In light of I being an ideal, this implies that there exists some $c \in I$ such that $a^* \vee b^* \vee c = 1$. Since L is normal, there are cozero elements $u \leq a^*$ and $v \leq b^*$ such that $u \vee v \vee c = 1$, see [**1,** Corollary 8.3.2]. Choose $\rho \geq \mathbf{0}$ and $\tau \geq \mathbf{0}$ in $\mathcal{R}L$ with $u = \operatorname{coz} \rho$ and $v = \operatorname{coz} \tau$. Observe that $\alpha\rho = \mathbf{0}$ since $a \wedge u = 0$ such that $\operatorname{coz}(\alpha\rho) = 0$. Since $\alpha \notin \boldsymbol{O}^I$ and $\boldsymbol{O}^I = O_{\boldsymbol{M}^I}$, it follows that $\rho \in \boldsymbol{M}^I$. Since $\operatorname{coz}(\gamma + \rho + \tau) = 1$, the function $\gamma + \rho + \tau$ is invertible, and is therefore not in $\boldsymbol{M}^I$. Observe that $\gamma \in \boldsymbol{M}^I$ because $c \in I$ implies $r_L(\operatorname{coz} \gamma) \subseteq I$. Thus, $\gamma + \rho \in \boldsymbol{M}^I$, which then implies $\tau \notin \boldsymbol{M}^I$. However, $\tau\beta = \mathbf{0}$ as $v \wedge b = 0$; thus, it follows that $\beta \in \boldsymbol{O}^I$ since $\boldsymbol{O}^I$ is the zero-component of $\boldsymbol{M}^I$. Therefore, $\boldsymbol{O}^I$ is a prime ideal. □

Next, we interpret Banaschewski's result quoted above in $\mathcal{R}L$. Recall from [**10**] that maximal ideals of $\mathcal{R}^*L$ are in a one-to-one correspondence with the points of βL and are denoted by $\boldsymbol{M}^{*I}$ for $I \in \mathrm{Pt}(\beta L)$. We do not need their description here; all we need is how each $\boldsymbol{M}^{*I}$ relates to $\boldsymbol{M}^I$ with regard to contraction, and for that we refer to [**10,** Lemma 4.1].

Lemma 2.4. *For any $I \in \mathrm{Pt}(\beta L)$, $\widetilde{\boldsymbol{M}^{*I}} = \boldsymbol{M}^I$.*

Proof. From Banaschewski's result, there is a $J \in \mathrm{Pt}(\beta L)$ such that $\boldsymbol{M}^I = \widetilde{\boldsymbol{M}^{*J}}$. By Lemma 2.1, $(\boldsymbol{M}^I)^c \subseteq \boldsymbol{M}^{*J}$. From [**10,** Lemma 4.1], $(\boldsymbol{M}^I)^c \subseteq \boldsymbol{M}^{*I}$. Since $(\boldsymbol{M}^I)^c$ is a prime ideal in $\mathcal{R}^*L$ contained in the maximal ideals $\boldsymbol{M}^{*I}$ and $\boldsymbol{M}^{*J}$, we must have $\boldsymbol{M}^{*I} = \boldsymbol{M}^{*J}$ because every prime ideal in $\mathcal{R}^*L$ is contained in a unique maximal ideal. Now, by [**10,** Proposition 3.8], we conclude that $I = J$. □

A point I of βL is *σ-proper* in the case for every countable $S \subseteq I$, $\bigvee S < 1$. Otherwise, it is *σ-improper*. Collecting results from [**9,** Corollary 3.7] and [**10,** Proposition 4.2], we have the following facts that we shall use below.

Facts 2.5. *The following are equivalent for a maximal ideal $\boldsymbol{M}^I$ of $\mathcal{R}L$.*

(1) *$\boldsymbol{M}^I$ is hyper-real.*
(2) *$\boldsymbol{M}^{*I}$ contains a unit of $\mathcal{R}L$.*
(3) *I is σ-improper.*

We also recite the following result from [**11,** Lemma 3.9].

Lemma 2.6. *Let $h\colon M \to L$ be a dense C-quotient map with M realcompact. Then, for the ring isomorphism $\mathcal{R}h\colon \mathcal{R}M \to \mathcal{R}L$, the map*

$$Q \longmapsto \mathcal{R}h[Q]$$

is a bijective correspondence between the free maximal ideals of $\mathcal{R}M$ and the hyper-real maximal ideals of $\mathcal{R}L$.

We are now equipped to state the following theorem regarding when the f-ring $\mathcal{R}(\beta L)$ is quasi-normal. The reader will recall that $\mathcal{R}(\beta L) \cong \mathcal{R}^*L$.

Theorem 2.7. *The following are equivalent for any completely regular frame L.*

(1) *$\mathcal{R}(\beta L)$ is quasi-normal.*

(2) *$\mathcal{R}L$ is quasi-normal, and O_M is prime for every hyper-real maximal ideal M of $\mathcal{R}L$.*
(3) *$\mathcal{R}L$ is quasi-normal, and $\boldsymbol{O}^I$ is prime for every σ-improper $I \in \mathrm{Pt}(\beta L)$.*
(4) *$\mathcal{R}L$ is quasi-normal, and every 2-boundary sublocale of λL is compact.*
(5) *$\mathcal{R}L$ is quasi-normal, and every 2-boundary sublocale of λL is pseudocompact.*

Proof.

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3). Since $\mathcal{R}^*L$ is isomorphic to a $C(X)$, the sum of two prime ideals in $\mathcal{R}^*L$ is a prime ideal if it is proper; thus, we may apply Theorem 2.2. Now, if $M = \boldsymbol{M}^I$ is a maximal ideal of $\mathcal{R}L$, the maximal ideal M^* of $\mathcal{R}^*L$ for which $M = \widetilde{M^*}$ is $\boldsymbol{M}^{*I}$, by Lemma 2.4. Therefore, the maximal ideals P of $\mathcal{R}L$, for which the corresponding maximal ideals P^* of $\mathcal{R}^*L$, contain a unit of $\mathcal{R}L$, are precisely those which are hyper-real. The stated equivalences therefore follow from Theorem 2.2 and Facts 2.5.

(2) $\Rightarrow$ (4). Assume that (2) holds. We only need prove that $\uparrow(C^* \vee D^*)$ is compact for all $C, D \in \mathrm{Coz}(\lambda L)$ with $C \wedge D = 0$. Denote by ϕ the ring isomorphism

$$\mathcal{R}(e_L)\colon \mathcal{R}(\lambda L) \longrightarrow \mathcal{R}L$$

induced by the frame homomorphism

$$e_L\colon \lambda L \longrightarrow L.$$

Since λL is realcompact (as it is Lindelöf), the free maximal ideals of $\mathcal{R}(\lambda L)$ are precisely its hyper-real maximal ideals ([**9,** Proposition 4.1]). Now, let M be a free maximal ideal of $\mathcal{R}(\lambda L)$. Then, by Lemma 2.6, $\phi[M]$ is a hyper-real maximal ideal of $\mathcal{R}L$. Thus, by (2), $O_{\phi[M])}$ is prime, which clearly implies O_M is a prime ideal in $\mathcal{R}L$. We have thus shown that the zero-component of every free maximal ideal in the ring $\mathcal{R}(\lambda L)$ is prime; therefore, in light of λL being normal, Lemma 2.3 proves the desired result.

(4) $\Rightarrow$ (1). Assume that (4) holds. Then $\mathcal{R}(\lambda L)$ is quasi-normal. The latter condition in (4) assures, by Lemma 2.3, that the zero-component of every free maximal ideal of $\mathcal{R}(\lambda L)$ is prime, and hence, that the zero-component of every hyper-real maximal ideal of $\mathcal{R}(\lambda L)$ is prime.

As observed in the proof of the first three statements, this implies, by Theorem 2.2, that $\mathcal{R}(\beta(\lambda L))$ is quasi-normal. Since βL is (isomorphic to) $\beta(\lambda L)$, the result follows.

(4) $\Leftrightarrow$ (5). A closed sublocale of a Lindelöf frame is Lindelöf, and a Lindelöf (in fact, a realcompact) frame is pseudocompact if and only if it is compact. $\square$

In the special case of normal realcompact frames, we have the following extension of [**22,** Theorem 1.6].

Corollary 2.8. *The following are equivalent for a normal realcompact frame L:*

(1) *$\mathcal{R}(\beta L)$ is quasi-normal.*
(2) *$\mathcal{R}L$ is quasi-normal and O_M is prime for every hyper-real maximal ideal M of $\mathcal{R}L$.*
(3) *$\mathcal{R}L$ is quasi-normal and every 2-boundary sublocale of L is compact.*

Remark 2.9. The requirement that, in a reduced f-ring A with bounded inversion, the sum of two minimal prime ideals in A^* be prime if it is proper forces the same for A. For, if P and Q are minimal prime ideals in A with $P+Q \neq A$, then P^c and Q^c are minimal prime ideals in A^* with $P^c + Q^c \neq A^*$. Thus, $(P+Q)^c$ is prime, and, if $ab \in P+Q$ for $a, b \in A$, then

$$\frac{a}{1+|a|} \cdot \frac{b}{1+|b|} \in (P+Q)^c,$$

so that we may assume

$$\frac{a}{1+|a|} \in (P+Q)^c,$$

which implies $a \in P+Q$, showing that $P+Q$ is prime.

We end this section with a curious observation regarding compactness of 2-boundary sublocales. We show that, if this condition holds in L, then it holds in λL, but not conversely. It is not difficult to check that, for any $I \in \lambda L$, $I^* = [\![(\bigvee I)^*]\!]$. Recall the characterization that L is pseudocompact if and only if every sequence (s_n) in L for which

$s_n \prec\!\!\prec s_{n+1}$ for every n and $\bigvee s_n = 1$ terminates, which is to say, there is an index m for which $s_m = 1$.

Proposition 2.10. *If every* 2*-boundary sublocale of* L *is compact, then every* 2*-boundary sublocale of* λL *is compact. The converse fails.*

Proof. Let $C \wedge D = 0$ in $\mathrm{Coz}(\lambda L)$. Choose $c, d \in \mathrm{Coz}\, L$ with $C = [\![c]\!]$ and $D = [\![d]\!]$, and observe that $c \wedge d = 0$, so that $\uparrow(c^* \vee d^*)$ is compact, by hypothesis. Observe that $C^* = [\![c^*]\!]$ and $D^* = [\![d^*]\!]$. Write $I = C^* \vee D^*$. In order to show that $\uparrow(C^* \vee D^*)$ is compact, it suffices to show that it is pseudocompact. Therefore, consider a sequence (J_n) in $\mathrm{Coz}(\uparrow I)$ with

$$J_n \prec\!\!\prec J_{n+1} \quad \text{and} \quad \bigvee_n J_n = 1_{\lambda L}.$$

We must show that this sequence terminates. Since the homomorphism $\kappa_I \colon \lambda L \to \uparrow I$ is a C-quotient map, and, in the language of [**1**], (J_n) is a regular cozero tower in $\uparrow I$, we can apply [**1,** Theorem 7.2.7] to find cozero elements c_n in L such that

$$I \vee [\![c_n]\!] \leq J_n, \qquad [\![c_n]\!] \prec\!\!\prec [\![c_{n+1}]\!], \qquad \bigvee_n [\![c_n]\!] = 1_{\lambda L}.$$

For each n, put $s_n = \bigvee J_n$. Applying the join map $e_L \colon \lambda L \to L$ to the equality $\bigvee_n J_n = 1_{\lambda L}$ yields $\bigvee_n s_n = 1$, which shows that $\{s_n \mid n \in \mathbb{N}\}$ is a cover of the frame $\uparrow(c^* \vee d^*)$ since $c^* \vee d^* = \bigvee I \leq s_n$ for each n. Thus, by compactness, there is an index k such that $s_k = 1$. However, now $J_k \prec\!\!\prec J_{k+1}$ implies $J_k^* \vee J_{k+1} = 1_{\lambda L}$, that is, $[\![s_k^*]\!] \vee J_{k+1} = 1_{\lambda L}$, which implies $J_{k+1} = 1_{\lambda L}$, as required. □

Example 2.11. Let L be the completely regular pseudocompact frame without points constructed in [**16**]. Since L is pseudocompact, λL is isomorphic to βL, and is therefore compact, which makes every closed sublocale of λL compact, and hence, every 2-boundary sublocale of λL, compact. Since L has no points, and since points of a sublocale are points in the ambient locale, L has no non-void compact sublocale because every non-trivial compact locale has at least one point [**24,** Lemma 1.9 III]. Note that, being non-trivial and completely regular, L does have disjoint cozero elements, and thus, the condition fails legitimately, and not vacuously, for L.

3. When L is quasinormal. In this section, we adopt the terminology of [**22**] and say a space X is *quasinormal* (note the absence of a hyphen) if the f-ring $C(X)$ is quasi-normal. Similarly, we say a frame L is quasinormal if $\mathcal{R}L$ is quasi-normal. We aim to give a characterization of quasinormal frames which does not require the imposition of normality on the frames, that is, we shall remove normality from quasi-normality, so to speak. We need some other tools that we now recall. We shall somewhat paraphrase. In [**25,** Theorem 5.5], Kimber proves the following result. If A is a ring and $a \in A$, we write $a^{\perp}$ for the annihilator of a.

Theorem 3.1 ([**25**])**.** *The following are equivalent for a commutative reduced f-ring A with identity in which the sum of two distinct minimal prime ideals is a prime ℓ-ideal*:

(1) *A is quasi-normal.*
(2) *For any positive $a, b \in A$ with $a \wedge b = 0$, every prime ℓ-ideal of A containing $a^{\perp} + b^{\perp}$ is a maximal ℓ-ideal.*

We interpret this result for the rings $\mathcal{R}L$, keeping in mind that in $\mathcal{R}L$ prime ideals are ℓ-ideals. First, a lemma regarding sums of prime ideals in $\mathcal{R}L$ is required.

Lemma 3.2. *The sum of two prime ideals in $\mathcal{R}L$ is a prime ideal.*

Proof. Recall that Henriksen [**20**] calls an ideal I of an f-ring A *square dominated* if

$$I = \{a \in A \mid |a| \leq x^2 \text{ for some } x \in A \text{ with } x^2 \in I\}.$$

It is clear that an ℓ-ideal in an f-ring in which positive elements are squares (as is the case in $\mathcal{R}L$ [**2,** Proposition 11]) is square dominated. Since prime ideals in $\mathcal{R}L$ are ℓ-ideals, it follows from Corollary 3.11 of [**20**] that the sum of two prime ideals in $\mathcal{R}L$ is a prime ideal. $\square$

In [**7,** Lemma 3.1] it is shown that, for any $\alpha \in \mathcal{R}L$, $\alpha^{\perp} = \boldsymbol{M}_{(\operatorname{coz}\alpha)^*}$. The lattice $\mathsf{Z}(\mathcal{R}L)$ of z-ideals of $\mathcal{R}L$ is a frame [**13**]. It is shown in [**13,** Lemma 3.6] that, for any $c, d \in \operatorname{Coz} L$, $\boldsymbol{M}_c \vee \boldsymbol{M}_d = \boldsymbol{M}_{c \vee d}$, where the join is calculated in $\mathsf{Z}(\mathcal{R}L)$. If L is normal, then the equality $\boldsymbol{M}_a \vee \boldsymbol{M}_b = \boldsymbol{M}_{a \vee b}$ holds for all $a, b \in L$, [**15,** Lemma 4.2], in other words, since the sum of z-ideals in $\mathcal{R}L$ is a z-ideal, $\boldsymbol{M}_c + \boldsymbol{M}_d = \boldsymbol{M}_{c\vee d}$ for any $c, d \in \operatorname{Coz} L$, and, if L is normal, then $\boldsymbol{M}_a + \boldsymbol{M}_b = \boldsymbol{M}_{a\vee b}$ for all $a, b \in L$. The result in Theorem 3.1 now states the following:

Corollary 3.3. *A normal completely regular frame L is quasinormal if and only if, for any $c, d \in \operatorname{Coz} L$ with $c \wedge d = 0$, every prime ideal of $\mathcal{R}L$ containing $\boldsymbol{M}_{c^*\vee d^*}$ is a maximal ideal.*

In [**25,** Theorem 4.3], Kimber proves that a normal space X is quasinormal if and only if every 2-boundary subspace of X is a P-space. We seek a similar characterization which, however, relaxes the normality requirement. Our pattern of proof follows that of Kimber.

Lemma 3.4. *Let L be a frame and $a \in L$. Let $\kappa_a \colon L \to {\uparrow}a$ be the frame homomorphism $x \mapsto x \vee a$, and denote by $\phi\colon \mathcal{R}L \to \mathcal{R}({\uparrow}a)$ the ring homomorphisms induced by κ_a. Then,* $\ker(\phi) = \boldsymbol{M}_a$.

Proof. For any $\alpha \in \mathcal{R}L$ we have

$$\begin{aligned}\alpha \in \ker(\phi) &\iff \phi(\alpha) = \mathbf{0} \iff \kappa_a \cdot \alpha = \mathbf{0}\\ &\iff \operatorname{coz}(\kappa_a \cdot \alpha) = 0_{\uparrow a} \iff \kappa_a(\operatorname{coz}\alpha) = a\\ &\iff a \vee \operatorname{coz}\alpha = a \iff \operatorname{coz}\alpha \le a,\end{aligned}$$

which shows that $\ker(\phi) = \boldsymbol{M}_a$. ☐

A consequence of this lemma is that, if L is normal (so that ϕ is onto), then we have an isomorphism $\mathcal{R}L/\boldsymbol{M}_a \cong \mathcal{R}({\uparrow}a)$. By a standard algebraic fact, we then have

$$\operatorname{Spec}(\mathcal{R}L/\boldsymbol{M}_a) = \{P/\boldsymbol{M}_a \mid P \in \operatorname{Spec}(\mathcal{R}({\uparrow}a))\}.$$

Recall that a frame M is a *P-frame* if $\operatorname{Coz} L$ is a Boolean algebra. This is so precisely when the ring $\mathcal{R}M$ is von Neumann regular. This, in turn, is the case precisely when every prime ideal in $\mathcal{R}M$ is maximal (or, equivalently, minimal). Now, every prime ideal in $\mathcal{R}({\uparrow}a)$ is maximal if and only if every prime ideal in $\mathcal{R}L$ containing $\boldsymbol{M}_a$ is maximal. Thus, given $c, d \in \operatorname{Coz} L$ with $c \wedge d = 0$, and setting $a = c^* \vee d^*$, we deduce from Corollary 3.3 the following result.

Corollary 3.5. *A normal frame is quasinormal if and only if each of its* 2*-boundary sublocales is a* P*-frame.*

Since $\mathcal{R}L \cong \mathcal{R}(\lambda L)$, and since λL is a normal frame, we have the following result.

Theorem 3.6. *A completely regular frame* L *is quasinormal if and only if every* 2*-boundary sublocale of* λL *is a* P*-frame.*

As mentioned in the last paragraph of the introduction, Henriksen, Martínez and Woods proved in [**22,** Proposition 3.2] that a z-embedded subspace of a quasinormal space is quasinormal. Recall that a subspace Y of X is z-embedded (in X) if every zero-set of Y is a trace on Y of some zero-set of X. This naturally may be extended to locales by saying a sublocale M of a frame L is z-embedded if, for every $c \in \operatorname{Coz} M$, there is a $d \in \operatorname{Coz} L$ such that $\mathfrak{c}_M(c) = M \cap \mathfrak{c}_L(d)$. In frame language, this states that, if $h\colon L \to M$ is a quotient map, then h is coz-*onto*, meaning that, for every $c \in \operatorname{Coz} M$, there is a $d \in \operatorname{Coz} L$ such that $h(d) = c$.

In preparation for the following result we recall a few facts, the first of which is [**18,** Proposition 3.3]. The other two are folklore.

Facts 3.7. *Let* $h\colon L \to M$ *be a frame homomorphism.*

(1) *h is coz-onto if and only if, for all* $a, b \in \operatorname{Coz} M$ *with* $a \wedge b = 0$*, there exist* $c, d \in \operatorname{Coz} L$ *such that* $c \wedge d = 0$*,* $h(c) = a$*, and* $h(d) = b$*.*

(2) *The induced map* $\lambda h\colon \lambda L \to \lambda M$ *has the property that, for any* $c \in \operatorname{Coz} L$*,* $(\lambda h)(c) = [\![h(c)]\!]$ *since, for any* $I \in \lambda L$ *and* $z \in \operatorname{Coz} M$*,*

$$z \in (\lambda h)(I) \iff z \leq h(a) \text{ for some } a \in I.$$

(3) $h(a^*) \leq h(a)^*$ *for every* $a \in L$*.*

Observe that, for any $a \in L$, $[\![a^*]\!] = [\![a]\!]^*$ since $[\![a]\!] = (e_L)_*(a)$, for the dense onto homomorphism $e_L\colon \lambda L \to L$, and the right adjoint of a dense onto homomorphism commutes with pseudocomplements. Hence, if $c \in \operatorname{Coz} L$, then

$$(\dagger) \qquad \lambda h([\![c^*]\!]) = \lambda h([\![c]\!]^*) \leq (\lambda h([\![c]\!]))^* = [\![h(c)]\!]^* = [\![h(c)^*]\!].$$

Theorem 3.8. *A z-embedded sublocale of a quasinormal frame is quasinormal.*

Proof. Let $h\colon L \to M$ be a coz-onto frame homomorphism with L quasinormal. We must show that M is quasinormal, and for that, we show that every 2-boundary sublocale of λM is a P-frame. Consider then $u, v \in \operatorname{Coz} M$ with $[\![u]\!] \wedge [\![v]\!] = 0$. Then, $u \wedge v = 0$. Since h is coz-onto, there exist $a, b \in \operatorname{Coz} L$ with $a \wedge b = 0$, $h(a) = u$, and $h(b) = v$. Then, $[\![a]\!] \wedge [\![b]\!] = 0$ in $\operatorname{Coz}(\lambda L)$, which, by hypothesis and Theorem 3.6, implies $\uparrow([\![a^*]\!] \vee [\![b^*]\!])$ is a P-frame. Let J be a cozero element in $\uparrow([\![u^*]\!] \vee [\![v^*]\!])$. Since λM is normal, the closed-quotient map $\lambda M \to \uparrow([\![u^*]\!] \vee [\![v^*]\!])$ is coz-onto (see [**1,** Theorem 8.3.3]), and thus, there exists a $w \in \operatorname{Coz} M$ such that

$$J = [\![u^*]\!] \vee [\![v^*]\!] \vee [\![w]\!].$$

Since h is coz-onto, there is a $c \in \operatorname{Coz} L$ such that $h(c) = w$. Now

$$[\![a^*]\!] \vee [\![b^*]\!] \vee [\![c]\!]$$

is a cozero element in the P-frame $\uparrow([\![a^*]\!] \vee [\![b^*]\!])$, and thus, it is complemented in this frame. Its complement is a cozero element in this frame, which means that there exists a $d \in \operatorname{Coz} L$ such that

$$\text{(A)} \qquad ([\![a^*]\!] \vee [\![b^*]\!] \vee [\![c]\!]) \vee ([\![a^*]\!] \vee [\![b^*]\!] \vee [\![d]\!]) = 1_{\lambda L}$$

and

$$\text{(B)} \quad ([\![a^*]\!] \vee [\![b^*]\!] \vee [\![c]\!]) \wedge ([\![a^*]\!] \vee [\![b^*]\!] \vee [\![d]\!]) = 0_{\uparrow([\![a^*]\!] \vee [\![b^*]\!])} = [\![a^*]\!] \vee [\![b^*]\!].$$

From (A), we have $[\![a^*]\!] \vee [\![b^*]\!] \vee [\![c]\!] \vee [\![d]\!] = 1_{\lambda L}$, which, on applying the map λh and taking into account the inequality in (†), yields $[\![h(a)^*]\!] \vee [\![h(b)^*]\!] \vee [\![h(c)]\!] \vee [\![h(d)]\!] = 1_{\lambda M}$, and consequently,

$$\text{(C)} \qquad ([\![u^*]\!] \vee [\![v^*]\!] \vee [\![w]\!]) \vee ([\![u^*]\!] \vee [\![v^*]\!] \vee [\![h(d)]\!]) = 1_{\lambda L}.$$

From (B), we have $([\![a^*]\!] \vee [\![b^*]\!]) \vee ([\![c]\!] \wedge [\![d]\!]) = [\![a^*]\!] \vee [\![b^*]\!]$, whence $[\![c]\!] \wedge [\![d]\!] \leq [\![a^*]\!] \vee [\![b^*]\!]$. Applying the map λh to this inequality gives

$$[\![h(c)]\!] \wedge [\![h(d)]\!] \leq [\![h(a)^*]\!] \vee [\![h(b)^*]\!] = [\![u^*]\!] \vee [\![v^*]\!].$$

Calculating the join $([\![u^*]\!] \vee [\![v^*]\!]) \vee ([\![h(c)]\!] \wedge [\![h(d)]\!])$, and keeping in mind that $h(c) = w$, gives

$$([\![u^*]\!] \vee [\![v^*]\!] \vee [\![w]\!]) \wedge ([\![u^*]\!] \vee [\![v^*]\!] \vee [\![h(d)]\!]) = [\![u^*]\!] \vee [\![v^*]\!],$$

which, together with (C), shows that J is complemented in the frame $\uparrow([\![u^*]\!] \vee [\![v^*]\!])$, thus making this frame a P-frame. Therefore, M is quasinormal by Theorem 3.6. $\square$

As in spaces, this tells us that every cozero-sublocale, and every Lindelöf sublocale, of a quasinormal frame is quasinormal.

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REFERENCES

1. R.N. Ball and J. Walters-Wayland, *C- and C^*-quotients in pointfree topology*, Dissert. Math. (Rozprawy Mat.) **412** (2002), 62 pages.

2. B. Banaschewski, *The real numbers in pointfree topology*, Text. Matem. **12** Dept. Mat., Univ. Coimbra, 1997.

3. ______, *Functorial maximal spectra*, J. Pure Appl. Alg. **168** (2002), 327–346.

4. ______, *A new aspect of the cozero lattice in pointfree topology*, Topol. Appl. **156** (2009), 2028–2038.

5. T. Dube, *Some algebraic characterizations of F-frames*, Alg. Univ. **62** (2009) 273–288.

6. ______, *Some ring-theoretic properties of almost P-frames*, Alg. Univ. **60** (2009), 145–162.

7. ______, *Contracting the socle in rings of continuous functions*, Rend. Sem. Mat. Univ. Padova **123** (2010), 37–53.

8. ______, *On the ideal of functions with compact support in pointfree function rings*, Acta Math. Hungar. **129** (2010), 205–226.

9. ______, *Real ideals in pointfree rings of continuous functions*, Bull. Australian Math. Soc. **83** (2011), 338–352.

10. ______, *Extending and contracting maximal ideals in the function rings of pointfree topology*, Bull. Math. Soc. Sci. Math. Roum. **103** (2012), 365–374.

11. ______, *Pseudocompact supports in pointfree topology*, Houston J. Math. **40** (2014), 601–620.

12. T. Dube and O. Ighedo, *Comments regarding d-ideals of certain f-rings*, J. Alg. Appl. **12** (2013), 16 pages.

13. ______, *On z-ideals of pointfree function rings*, Bull. Iran. Math. Soc. **40** (2014), 655–673.

14. ______, *More ring-theoretic characterizations of P-frames*, J. Alg. Appl. **14** (2015), 8 pages.

15. ______, *On lattices of z-ideals of function rings*, Math. Slovaca, accepted.

16. T. Dube, S. Iliadis, J. van Mill and I. Naidoo, *A pseudocompact completely regular frame which is not spatial*, Order **31** (2014), 115–120.

17. T. Dube and I. Naidoo, *Round squares in the category of frames*, Houston J. Math. **392** (2013), 453–473.

18. T. Dube and J. Walters-Wayland, *Coz-onto frame maps and some applications*, Appl. Categ. Struct. **15** (2007), 119–133.

19. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, 1960.

20. M. Henriksen, *Semiprime ideals of f-rings*, Sympos. Math. **21** (1977), 401–409.

21. M. Henriksen and M. Jerison, *The space of minimal prime ideals of a commutative ring*, Trans. Amer. Math. Soc. **115** (1965), 110–130.

22. M. Henriksen, J. Martínez and R.G. Woods, *Spaces X in which all prime z-ideals of $C(X)$ are minimal or maximal*, Comment. Math. Univ. Carolin. **44** (2003), 261–294.

23. C.B. Huijsmans, *Some analogies between commutative rings, Riesz spaces and distributive lattices with smallest element*, Indag. Math. **36** (1974), 132–147.

24. P.T. Johnstone, *Stone spaces*, Cambridge University Press, Cambridge, 1982.

25. C.M. Kimber, *m-Quasinormal f-rings*, J. Pure Appl. Alg. **158** (2001), 197–223.

26. S. Larson, *A characterization of f-rings in which the sum of semiprime ℓ-ideals is semiprime and its consequences*, Comm. Algebra **23** (1995), 5461–5481.

27. ______, *Quasi-normal f-rings*, in *Ordered algebraic structures*, Kluwer, Dordrecht, 1997.

28. J. Madden and J. Vermeer, *Lindelöf locales and realcompactness*, Math. Proc. Cambr. Philos. Soc. **99** (1986), 473–480.

29. R. Montgomery, *Structures determined by prime ideals of rings of functions*, Trans. Amer. Math. Soc. **147** (1970), 367–380.

30. ______, *The mapping of prime z-ideals*, Symp. Math. **17** (1973), 113–124.

31. J. Picado and A. Pultr, *Frames and locales*: *Topology without points*, Frontiers Math., Springer, Basel, 2012.

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