A NOTE ON A PAPER BY L. C. YOUNG

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1. Introduction. Suppose that $f(x)$ is a real- or complex-valued function defined for all real x. For $0 \leq \alpha \leq 1$, we define the α-variation of $f(x)$ over $a \leq x \leq b$ as the least upper bound of the sums

$$\{\sum |\Delta f|^{1/\alpha}\}^{\alpha}$$

taken over all finite subdivisions of $a \leq x \leq b$. (When $\alpha = 0$, we denote by the above sum simply the maximum $|\Delta f|$.) We say that $f(x)$ is in W_α if it has finite α-variation over the interval $0 \leq x \leq 1$. L. C. Young has proved the following result.

THEOREM 1. (See [2, Theorem 4.2].) *Suppose that* $0 < \beta < 1$ *and that* $f(x)$, *with period* 1, *satisfies the condition*

$$\int_0^1 |f\{\phi(t+h)\} - f\{\phi(t)\}|\, dt \leq h^{\beta} \qquad (h \geq 0)$$

for every monotone function $\phi(t)$ *such that*

$$\phi(t+1) = \phi(t) + 1$$

for all t. Then $f(x)$ *is in* W_α *for each* $\alpha < \beta$.

Young's argument does not suggest whether we can assert that $f(x)$ is in W_β. We present here an elementary proof for Theorem 1 and an example to show that this result is the best possible one in this direction.

2. Lemma. We require the following:

LEMMA 2. *Suppose that* $a_1, a_2, \cdots, a_N$ *and* $b_1, b_2, \cdots, b_N$ *are two sets of nonnegative numbers such that* $a_1 \geq a_2 \geq \cdots \geq a_N$ *and such that*

$$\sum_{\nu=1}^{n} a_\nu \leq \sum_{\nu=1}^{n} b_\nu$$

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for $n = 1, \cdots, N$. *Then for* $p > 1$,

$$\sum_{\nu=1}^{n} a_\nu^p \le \sum_{\nu=1}^{n} b_\nu^p$$

for $n = 1, \cdots, N$.

Let

$$S_n = \sum_{\nu=1}^{n} a_\nu \text{ and } T_n = \sum_{\nu=1}^{n} b_\nu .$$

With Abel's identity and Hölder's inequality, we have

$$\begin{aligned}
\sum_{\nu=1}^{n} a_\nu^p &= \sum_{\nu=1}^{n} a_\nu a_\nu^{p-1} \\
&= S_1(a_1^{p-1} - a_2^{p-1}) + \cdots + S_{n-1}(a_{n-1}^{p-1} - a_n^{p-1}) + S_n a_n^{p-1} \\
&\le T_1(a_1^{p-1} - a_2^{p-1}) + \cdots + T_{n-1}(a_{n-1}^{p-1} - a_n^{p-1}) + T_n a_n^{p-1} \\
&= \sum_{\nu=1}^{n} b_\nu a_\nu^{p-1}, \\
&\le \left\{\sum_{\nu=1}^{n} b_\nu^p\right\}^{1/p} \left\{\sum_{\nu=1}^{n} a_\nu^p\right\}^{(p-1)/p},
\end{aligned}$$

from which the lemma follows.

3. Proof of Theorem 1. For a subdivision $0 = x_0 < x_1 < \cdots < x_N = 1$, consider the numbers

$$|f(x_1) - f(x_0)|,\ |f(x_2) - f(x_1)|,\ \cdots,\ |f(x_N) - f(x_{N-1})|,$$

and label this set $a_1, a_2, \cdots, a_N$ so that $a_1 \ge a_2 \ge \cdots \ge a_N$. We say that the two points ξ' and ξ'' are *associated* with a_n if they are the two points of the subdivision for which

$$a_n = |f(\xi'') - f(\xi')|;$$

and, fixing n, we consider the union of points associated with $a_1, a_2, \cdots, a_n$. Labeling these $\xi_1 < \xi_2 < \cdots < \xi_{m_n}$, we define

$$\phi(t) = \xi_\nu \text{ for } \frac{\nu-1}{m_n} \leq t < \frac{\nu}{m_n} \qquad (\nu = 1, \cdots, m_n),$$

and we extend this function so that

$$\phi(t+1) = \phi(t) + 1.$$

Now $m_n \leq 2n$ and, if $0 < h < 1/m_n$,

$$h \sum_{\nu=1}^{n} a_\nu \leq h \sum_{\nu=2}^{m_n} |f(\xi_\nu) - f(\xi_{\nu-1})|,$$

$$\leq \int_0^1 |f\{\phi(t+h)\} - f\{\phi(t)\}| dt \leq h^\beta.$$

Letting h approach $1/m_n$, we have

$$\sum_{\nu=1}^{n} a_\nu \leq m_n^{1-\beta} \leq (2n)^{1-\beta}$$

for $n = 1, \cdots, N$. Finally selecting $b_1, b_2, \cdots, b_N$ so that

$$\sum_{\nu=1}^{n} b_\nu = (2n)^{1-\beta},$$

we have

$$b_1 = 2^{1-\beta} \text{ and } b_n < 2^{1-\beta}(n-1)^{-\beta} \qquad \text{for } n > 1,$$

and applying Lemma 2 we conclude that

$$\left\{\sum_{n=1}^{N} |\Delta_n f|^{1/\alpha}\right\}^\alpha \leq \left\{\sum_{n=1}^{N} b_n^{1/\alpha}\right\}^\alpha < 2\left\{\sum_{n=1}^{\infty} n^{-\beta/\alpha}\right\}^\alpha.$$

This completes the proof.

4. Further results. We now show that Theorem 1 is best possible.

THEOREM 3. *Suppose that* $0 < \beta < \gamma \leq 1$. *There exists a function* $f(x)$, *with period* 1, *which is not in* W_β *and which satisfies the condition*

$$\left\{ \int_0^1 |f\{\phi(t+h)\} - f\{\phi(t)\}|^{1/\gamma} \right\}^\gamma \leq h^\beta \qquad (h \geq 0)$$

for every monotone function $\phi(t)$ *such that*

$$\phi(t+1) = \phi(t) + 1.$$

Consider two increasing sequences, $\{x_n\}$ and $\{y_n\}$, such that

$$x_1 < y_1 < x_2 < \cdots < x_n < y_n < x_{n+1} < \cdots < x_1 + 1.$$

Define the function

$$g(x) = \begin{cases} n^{-\beta} & \text{for } x_n < x < y_n, \\ 0 & \text{everywhere else in } x_1 \leq x < x_1 + 1, \end{cases}$$

and extend $g(x)$ to have period 1.

LEMMA 4. *Suppose that* $0 < \beta < \gamma \leq 1$. *The function* $g(x)$ *defined above satisfies the condition*

$$\left\{ \int_0^1 |g(x+h) - g(x)|^{1/\gamma} dx \right\}^\gamma \leq \left(\frac{2\gamma}{\gamma - \beta} \right)^\gamma h^\beta \qquad (h \geq 0).$$

Fix h in the range $0 < h \leq 1/2$, and consider the finite sequence,

$$\xi_0 < \xi_1 < \cdots < \xi_N = \xi_0 + 1,$$

defined as follows.

A. Let $\xi_0 = x_1 - h$.

B. Suppose that $\xi_0 < \xi_1 < \cdots < \xi_{n-1} < \xi_0 + 1$ have been defined. Let $\xi_n = \text{Max}\{\xi_{n-1} + 2h, y_n\}$ if this does not exceed $\xi_0 + 1$. Otherwise let $\xi_n = \xi_0 + 1$.

It is not difficult to show that

$$\int_{\xi_{n-1}}^{\xi_n} |g(x+h) - g(x)|^{1/\gamma} dx \leq 2h\, n^{-\beta/\gamma}$$

for $n = 1, \cdots, N$. Since $\xi_n - \xi_{n-1} \geq 2h$ for $n = 1, \cdots, N-1$, we have $Nh < 1$ and

$$\int_0^1 |\Delta g|^{1/\gamma}dx = \sum_{n=1}^{N}\int_{\xi_{n-1}}^{\xi_n} |\Delta g|^{1/\gamma}dx \leq 2h\sum_{n=1}^{N} n^{-\beta/\gamma},$$

$$< \frac{2}{1-\beta/\gamma}\, hN^{1-\beta/\gamma} < \frac{2\gamma}{\gamma-\beta}\, h^{\beta/\gamma}.$$

This completes the proof of Lemma 4.

Take any *strictly increasing continuous* function $\phi(t)$ such that

$$\phi(t+1) = \phi(t) + 1.$$

If ϕ^{-1} is the inverse function, and

$$u_n = \phi^{-1}(x_n) \text{ and } v_n = \phi^{-1}(y_n),$$

then $u_1 < v_1 < u_2 < \cdots < u_n < v_n < u_{n+1} < \cdots < u_1 + 1$ and

$$g\{\phi(t)\} = \begin{cases} n^{-\beta} & \text{for } u_n < t < v_n, \\ 0 & \text{everywhere else in } u_1 \leq t < u_1 + 1. \end{cases}$$

Now $g\{\phi(t)\}$ has period 1 in t, and, by Lemma 4,

$$\left\{\int_0^1 |g\{\phi(t+h)\} - g\{\phi(t)\}|^{1/\gamma}dt\right\}^{\gamma} \leq \left(\frac{2\gamma}{\gamma-\beta}\right)^{\gamma} h^{\beta} \qquad (h \geq 0).$$

The Lebesgue limit theorem allows us to conclude this holds for all *nondecreasing* $\phi(t)$ such that

$$\phi(t+1) = \phi(t) + 1.$$

To complete the proof of Theorem 3, observe that $g(x)$ is *not* in W_β and let

$$f(x) = \left(\frac{\gamma-\beta}{2\gamma}\right)^{\gamma} g(x).$$

In the proof of Theorem 3, the fact that $\beta < \gamma$ plays an important role. We have a different situation when $\beta = \gamma$.

THEOREM 5. *Suppose that* $0 \leq \beta \leq 1$ *and that* $f(x)$ *is measurable and real-valued with period* 1. *The* β*-variation of* $f(x)$ *over any interval of length* 1 *does not exceed* 1 *if and only if*

$$\left\{\int_0^1 |f\{\phi(t+h)\} - f\{\phi(t)\}|^{1/\beta} dt\right\}^{\beta} \leq h^{\beta} \qquad (h \geq 0)$$

for each monotone function $\phi(t)$ such that $\phi(t+1) = \phi(t) + 1$.

For the sufficiency, let $x_0 < \cdots < x_N = x_0 + 1$ be a subdivision of some interval of length 1. Define the function

$$\phi(t) = x_n, \qquad \frac{n}{N} \leq t < \frac{n+1}{N} \qquad (n = 0, \cdots, N-1),$$

and extend $\phi(t)$ so that

$$\phi(t+1) = \phi(t) + 1;$$

for $0 < h < 1/N$ we get

$$\left\{\sum_{n=1}^{N} |\Delta_n f|^{1/\beta}\right\}^{\beta} \leq \left\{\frac{1}{h}\int_0^1 |f\{\phi(t+h)\} - f\{\phi(t)\}|^{1/\beta} dt\right\}^{\beta} \leq 1.$$

For the necessity, we see that the β-variation for $f\{\phi(t)\}$ over any interval of length 1 does not exceed 1, and we can apply the following:

THEOREM 6. (See [**1**, Theorem 1.3.3].) *Suppose that $0 \leq \beta \leq 1$, that $f(x)$ is measurable and real-valued with period 1, and that the β-variation of $f(x)$ over any interval of length 1 does not exceed 1. Then*

$$\left\{\int_0^1 |f(x+h) - f(x)|^{1/\beta} dx\right\}^{\beta} \leq h^{\beta} \qquad (h \geq 0).$$

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