MEASURES DEFINED BY ABSTRACT L_p SPACES

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Let a linear space L of real-valued functions on a set E and a semi-norm on L be given. We shall consider when there exists a countably additive measure on E such that L is L_p with respect to this measure. We shall prove that certain conditions are sufficient for the measure to exist; it is obvious that these conditions are necessary. (We consider only the case where the constant function $1 \in L$.)

We need not assume that the elements of L are functions on a set. If we do not make this assumption, we use a theorem of Kakutani ([3], p. 998) to construct a representation for L as a space of continuous functions on a compact Hausdorff space. If, however, the elements of L are given as functions, we leave this preëstablished representation unchanged, even when it is not the one given by Kakutani's theorem.

The case where $p = 1$ and the elements of L are not given as functions was treated by Kakutani [2]. The case $p = 2$ will receive special attention at the end of the present paper. In this latter case, one may replace some of the hypotheses of the general case by the hypothesis that the semi-norm on L arises from a positive semi-definite bilinear form.

Let L be a Riesz space whose elements are functions on a set E. That is, let L be a set of real-valued functions on E which contains with f, g:

(a) $f + g$ defined by $(f + g)(x) = f(x) + g(x)$,

(b) αf defined by $(\alpha f)(x) = \alpha[f(x)]$, for each real number α,

(c) $f \wedge g$ defined by $(f \wedge g)(x) = \min(f(x), g(x))$,

and (d) $f \vee g$ defined by $(f \vee g)(x) = \max(f(x), g(x))$.

We denote $f \vee 0$ by f^+ and $(-f) \vee 0$ by f^-. (The case where L is an abstract Banach lattice will be considered shortly.)

Let p be a fixed real number ≥ 1. Throughout the paper, p will always stand for this fixed number. We suppose there is a semi-norm, which we denote by $\|\ \ \|$, defined on L. We further suppose:

(1) L is complete. That is, if $f_1, f_2, \cdots \in L$ are such that $\|f_n - f_m\|$ is small for large n, m; then there is a $g \in L$ such that $\|g - f_n\| \to 0$.

(2) For each $f \in L$, $\| |f| \| = \|f\|$.

(3) If f, g are positive, $\|f + g\|^p \geq \|f\|^p + \|g\|^p$.

(4) If f, g are positive and $f \wedge g = 0$, $\|f + g\|^p \leq \|f\|^p + \|g\|^p$.

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(5) $1 \in L$ and $\| 1 \| = 1$. (Here 1 denotes the constant function 1.)

We note that if $f, g \in L$ and $0 \leq f \leq g$, then $\| f \| \leq \| g \|$; since $\| f \|^p \leq \| f \|^p + \| g - f \|^p \leq \| f + g - f \|^p = \| g \|^p$ by (3) above. We also note that, for each $f \in L$, $\| f^+ \| \leq \| f \|$; since $\| f^+ \| \leq \| f^+ + f^- \| = \| \, | f | \, \| = \| f \|$ using (2) and the preceding remark.

We now briefly consider abstract L_p spaces. Let L be a Riesz space (i. e. a vector lattice), whose elements need not be functions. Suppose there is a norm on L. (If a semi-norm is given instead of a norm, we use, in place of L, the quotient space of L modulo the elements of norm 0. This quotient space will be a normed Riesz space provided the semi-norm satisfies (2) and (3) above.) Suppose, for some $p \geq 1$, that L has properties (1)-(4) above. Instead of (5), we suppose that L has a weak unit, i. e.:

(5′) There is a positive $e \in L$ such that $f \wedge e = 0$, $f \in L$ imply $f = 0$. (We suppose L is normalized so that $\| e \| = 1$.)

Under these conditions we may call L an abstract L_p space. (In the case $p = 1$, an abstract L_1 space is thus an abstract (L)-space in the sense of Kakutani [2].)

We seek to represent abstract L_p spaces as function spaces. We recall from [1], p. 248, that a norm on a Banach lattice is called uniformly monotone when, given $\varepsilon > 0$, one can find $\delta > 0$ so small that if $f \geq 0$, $g \geq 0$, $\| f \| = 1$ and $\| f + g \| - 1 \leq \delta$, then $\| g \| \leq \varepsilon$. It follows at once from (3) that the norm on L is uniformly monotone. Thus, since L is complete, it is completely reticulated ([1], p. 249); i. e. every non-empty subset of L bounded from above has a least upper bound. Hence, by a theorem of Kakutani ([3], p. 998) in the form given by Stone ([4], p. 85), L is isomorphic as a Riesz space to a space of continuous functions on a compact Hausdorff space, if we entirely ignore nowhere dense sets. If we do not ignore these sets, we obtain a space of functions with a semi-norm, defined by the norm on L, which satisfies the hypotheses given at the beginning of this section. Thus we may now return to these hypotheses without loss of generality.

We now define a collection N of functions, which we call null functions, by $f \in N$ if there are $f_1, f_2, \cdots \in L$ such that:

(a) $f_n \geq | f |$ for all n

and (b) $\| f_n \| \to 0$.

Clearly if $f \in N \cap L$, $\| f \| = 0$. It is also clear that N is a lattice ideal in the set of all functions on E; i. e. N is a linear subspace of this set with the property that $| f | \leq | g |$ and $g \in N$ imply $f \in N$.

We define $L' \supset L$ by $f \in L'$ if there are $g \in L$, $h \in N$ such that $f = g + h$. Clearly L' is a linear space. Suppose $f = g_1 + h_1 = g_2 + h_2$ with $g_i \in L$, $h_i \in N$ ($i = 1, 2$). Then $h_1 - h_2 = g_2 - g_1 \in L \cap N$. Thus $\| g_2 - g_1 \| = 0$. Hence $\| g_2 \| = \| g_1 + g_2 - g_1 \| \leq \| g_1 \| + \| g_2 - g_1 \| = \| g_1 \|$. Similarly $\| g_1 \| \leq \| g_2 \|$. Hence $\| g_1 \| = \| g_2 \|$. It follows that we may define a semi-norm on L' by defining $\| g + h \|$ to be $\| g \|$, where $g \in L$ and $h \in N$.

We next show that L' is a lattice; i. e. that $f_1 \wedge f_2 \in L'$ whenever $f_1, f_2 \in L'$. Let $f_1 = g_1 + h_1$, $f_2 = g_2 + h_2$ with $g_i \in L$, $h_i \in N$. Then $g_1 \wedge g_2 \in L$. We have $f_1 \wedge f_2 = (g_1 + h_1) \wedge (g_2 + h_2) \leq (g_1 + h_1^+) \wedge (g_2 + h_2^+) \leq g_1 \wedge g_2 + h_1^+ + h_2^+$. Thus $f_1 \wedge f_2 - g_1 \wedge g_2 \leq h_1^+ + h_2^+$. Similarly $f_1 \wedge f_2 - g_1 \wedge g_2 \geq - h_1^- - h_2^-$. Since N is a lattice ideal, $f_1 \wedge f_2 - g_1 \wedge g_2 \in N$. Hence $f_1 \wedge f_2 \in L'$.

It is easy to check that L' satisfies all the hypotheses imposed above on L. In addition, L' has the following property:

If $f_1, f_2, \cdots \in L'$ are positive, $f_n \uparrow f$ pointwise and $\| f_n \| < \alpha$ for all n, then $f \in L'$ and $\| f - f_n \| \to 0$. To see this we note that $\{\| f_n \|\}$ is an increasing sequence of real numbers bounded from above by α; hence it is a Cauchy sequence. Thus $\{\| f_n \|^p\}$ is also a Cauchy sequence. Whenever $n \geq m$ we have $\| f_n - f_m \|^p \leq \| f_n - f_m + f_m \|^p - \| f_m \|^p = \| f_n \|^p - \| f_m \|^p$ by (3) above. Thus there is an $f' \in L'$ such that $\| f' - f_n \| \to 0$ by (1) above. Since $f_n \leq f$ for all n, $f' - f_n \geq f' - f$ for all n. Since $f' - f_n \in L'$, $f' - f_n = g_n + h_n$ with $g_n \in L$, $h_n \in N$. By the definiton of N, we can find, for each n, a $g'_n \in L$ such that $g'_n \geq h_n$ and $\| g'_n \| \leq 1/n$. Let $f'_n = g_n + g'_n$. Then $f'_n \geq g_n + h_n = f' - f_n \geq f' - f$. Also $\| f'_n \| \leq \| g_n \| + \| g'_n \| \leq \| f' - f_n \| + 1/n \to 0$. By the definition of N, $f' - f \in N$. Thus $f \in L'$. Also $\| f - f_n \| \leq \| f - f' \| + \| f' - f_n \| \to 0$.

At this point, we replace L by L'; i. e. we write L for L'.

LEMMA. *Let $f \in L$ be positive. Let g be the characteristic function of the set on which f differs from 0. Then $g \in L$.*

Proof. Clearly $nf \wedge 1 \uparrow g$ pointwise. Since $\| nf \wedge 1 \| \leq \| 1 \| = 1$ for all n, $g \in L$ by what has just been proved.

LEMMA. *Let $f \in L$ be positive. Then there are positive $f_1, f_2, \cdots \in L$ such that $f_n \uparrow f$ pointwise, $\| f - f_n \| \to 0$, and each f_n assumes only finitely many values.*

Proof. For each positive integer n, let f_n be defined by: $f_n(x) = 2^{-n}[2^n(f \wedge n)(x)]$ for all $x \in E$. (By $[\alpha]$ we mean the largest integer $\leq \alpha$.) For each $x \in E$, $f_n(x) = 2^{-n}[2^n f(x)]$ for large n; thus clearly $f_n(x) \to f(x)$. Hence $f_n \to f$ pointwise. We note

$$f_{n+1}(x) = \frac{1}{2^{n+1}}[2^{n+1}(f \wedge (n+1))(x)] \geq \frac{1}{2^{n+1}}[2^{n+1}(f \wedge n)(x)]$$
$$\geq \frac{1}{2^n}[2^n(f \wedge n)(x)] = f_n(x)$$

for each $x \in E$. Hence $f_n \uparrow f$ pointwise. If we show $f_n \in L$ for all n, we shall know $\| f - f_n \| \to 0$ and the lemma will be proved.

Let n be fixed. Let $g_1, g_2, \cdots$ be functions on E defined by:

$$g_i(x) = 1 \text{ if } x \text{ is such that } 2^n(f \wedge n)(x) \geq i$$

$$g_i(x) = 0 \text{ if } x \text{ is such that } 2^n(f \wedge n)(x) < i\,.$$

We note that $f_n(x) = 2^{-n} \sum_{i=1}^{\infty} g_i(x)$. Since $2^n(f \wedge n)(x) \leq 2^n n$, $g_i(x) = 0$ for all x when $i \geq 2^n n$. Thus $f_n(x) = 2^{-n} \sum_{i=1}^{2^n n} g_i(x)$. Clearly each $1 - g_i$ is the characteristic function of the set on which $(2^n(f \wedge n) - i)^-$ differs from 0. Since $(2^n(f \wedge n) - i)^- \in L$, $1 - g_i \in L$ by the previous lemma; hence $g_i \in L$. We note $f_n = 2^{-n} \sum_{i=1}^{2^n n} g_i$ which shows $f_n \in L$ and completes the proof.

We now define a measure μ on the set E. Let A be a subset of E. If f_A, the characteristic function of A, is in L, we call A measurable and put $\mu(A) = \| f_A \|^p$. The verification that μ is a countably additive measure is trivial, making use of conditions (3) and (4) of our hypothesis, except for the following: Let $A_1, A_2, \cdots \subset E$ be pairwise disjoint and measurable. Let f_n be the characteristic function of $A_1 \cup \cdots \cup A_n$ $(n = 1, 2, \cdots)$. Then $f_n \uparrow f$ pointwise, where f is the characteristic function of $\bigcup_{n=1}^{\infty} A_n$. By what has been shown above, $f \in L$ and $\| f - f_n \| \to 0$. Thus $\mu(A_1) + \cdots + \mu(A_n) = \| f_n \|^p \to \| f \|^p = \mu(\bigcup_{n=1}^{\infty} A_n)$.

Next we consider the space L_p defined by μ. The functions in L which assume only finitely many values are precisely the measurable functions which assume only finitely many values. Clearly the given semi-norm on L coincides with the L_p norm for such functions. It follows, by considering pointwise limits of increasing sequences of such functions, that the functions in L_p are precisely those in L and that the norms agree. Remembering that we modified the original L by introducing null functions, we have the following theorem:

THEOREM. *Let L be a Riesz space of functions on a set E. Suppose there is a semi-norm on L which satisfies conditions* (1)-(5) *above. Then there is a countably additive measure μ on E such that L is essentially L_p with respect to μ; i. e. such that:*

(a) *For every $f \in L$, $\| f \|^p = \int | f |^p \, d\mu$.*

and (b) *If $f \geq 0$ and $\int f^p \, d\mu < \infty$, then there is a $g \in L$ such that $f(x) = g(x)$ for almost all $x \in E$.*

In the case $p = 2$, we can modify the hypotheses above. We suppose that H is a Riesz space of functions. We also suppose that there is a positive semi-definite bilinear form defined on H and that H is complete in the semi-norm determined by this form. We also assume that $\| f \| \leq \| g \|$ whenever $f, g \in H$ and $0 \leq f \leq g$. Next suppose that $\| f^+ \| \leq \| f \|$ for all $f \in H$. Finally we suppose $1 \in H$ and $\| 1 \| = 1$. We prove the following lemmas to show that H satisfies, with $p = 2$, the hypotheses given at the beginning of the paper.

LEMMA. *If $f, g \in H$ are positive, then $(f, g) \geq 0$.*

Proof. We note $f + \alpha g \geq f \geq 0$ for all $\alpha > 0$. Thus $\| f + \alpha g \| \geq \| f \|$. Hence we have $0 \leq \| f + \alpha g \|^2 - \| f \|^2 = 2\alpha(f, g) + \alpha^2 \| g \|^2$. It follows that $2(f, g) \geq -\alpha \| g \|^2$ for all $\alpha > 0$. Hence $(f, g) \geq 0$.

LEMMA. *If $f, g \in H$ are positive and $f \wedge g = 0$, then $(f, g) = 0$.*

Proof. We note $f \wedge (\alpha g) = 0$ for all $\alpha > 0$. Hence $(f - \alpha g)^+ = f$. Thus we have $\| f \|^2 = \|(f - \alpha g)^+ \|^2 \leq \| f - \alpha g \|^2 = \| f \|^2 - 2\alpha(f, g) + \alpha^2 \| g \|$. Hence $\alpha \| g \|^2 \geq 2(f, g)$ for all $\alpha > 0$. Thus $(f, g) \leq 0$. By the previous lemma, $(f, g) \geq 0$. Therefore $(f, g) = 0$.

LEMMA. *$\| f \| = \| \, | f | \, \|$ for all $f \in H$.*

Proof. We have $\| f \|^2 = \| f^+ - f^- \|^2 = \| f^+ \|^2 - 2(f^+, f^-) + \| f^- \|^2 = \| f^+ + f^- \|^2 - 4(f^+, f^-) = \| \, | f | \, \|^2 - 4(f^+, f^-)$. But $(f^+, f^-) = 0$ by the previous lemma.

LEMMA. *$\| f + g \|^2 \geq \| f \|^2 + \| g \|^2$ whenever $f, g \in H$ are positive.*

Proof. $\| f + g \|^2 = \| f \|^2 + 2(f, g) + \| g \|^2 \geq \| f \|^2 + \| g \|^2$ since $(f, g) \geq 0$.

LEMMA. *If $f, g \in H$ are positive and $f \wedge g = 0$, then $\| f + g \|^2 = \| f \|^2 + \| g \|^2$.*

Proof. We have $\| f + g \|^2 = \| f \|^2 + 2(f, g) + \| g \|^2 = \| f \|^2 + \| g \|^2$.
Thus we have verified that H satisfies the hypotheses for L with $p = 2$. On this basis we prove:

THEOREM. *Let H be as described above. Then there is a countably additive measure μ on E such that H is essentially L_2 with respect to μ; i. e. such that:*

(a) *For every $f, g \in H$, $(f, g) = \int fg \, d\mu$.*

and (b) *If $f \geq 0$ and $\int f^2 \, d\mu < \infty$, then there is a $g \in H$ such that $f(x) = g(x)$ for almost all $x \in E$.*

Proof. In addition to what has been proved above, it is enough to note that the inner product may be expressed in terms of the norm in the usual way.

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