ON A GENERALIZED STIELTJES TRANSFORM

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1. Introduction. In a series of recent papers [1–4] I have discussed various properties and inversion theorems etc. for the transform

$$(1.1)\qquad F(x) = \frac{\Gamma(\beta+\eta+1)}{\Gamma(\alpha+\beta+\eta+1)} \times \int_0^\infty (xy)^\beta {}_1F_1(\beta+\eta+1;\ \alpha+\beta+\eta+1;\ -xy)f(y)dy$$

where $f(y) \in L(0, \infty)$, $\beta \geqq 0$, $\eta > 0$.

$$F(x) = A\int_0^\infty (xy)^\beta F(x, y)f(y)dy$$

where, for convenience, we denote $\Gamma(\beta+\eta+1)/\Gamma(\alpha+\beta+\eta+1)$ by A and ${}_1F_1(a; b; -xy)$ by $F(x, y)$, a and b standing respectively for $\beta+\eta+1$ and $\alpha+a$. For $\alpha=\beta=0$ (1.1) reduces to the well known Laplace Transform

$$(1.2)\qquad F(x) = \int_0^\infty e^{-xy}f(y)dy\ .$$

The transform (1.1), which may be called a generalization of Laplace Transform, arises when we apply Kober's [5] operators of Fractional Integration [6] to $x^\beta e^{-x}$.

The object of the present paper is to give a generalization of Stieltjes Transform, to give an inversion theorem for it and to use that inversion theorem to obtain an inversion theorem for the transform (1.1). In another paper (to appear elsewhere) I have found out inversion operators directly for (1.1).

2. Generalized Stieltjes transform. We prove

THEOREM 2.1. *If*

$$(2.1)\qquad \phi(s) = \int_0^\infty e^{-sx}F(x)dx$$

where $F(x)$ is given by the convergent integral (1.1), *then*

$$(2.2)\qquad \phi(s) = \frac{A\Gamma(\beta+1)}{s}\int_0^\infty \left(\frac{y}{s}\right)^\beta F\left(a,\ \beta+1; b; -\frac{y}{s}\right)f(y)dy$$

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provided that $\beta \geqq 0$, $\eta > 0$ and $f(y) \in L(0, \infty)$.

Proof. We have

$$\phi(s) = A\int_0^\infty e^{-sx}dx\int_0^\infty (xy)^\beta {}_1F_1(a; b; -xy)f(y)dy$$
$$= A\int_0^\infty y^\beta f(y)dy\int_0^\infty x^\beta e^{-sx} {}_1F_1(a; b; -xy)dx$$

on changing the order of integration, which is easily seen to be justified under the conditions stated, since [7, page 59]

$${}_1F_1(a; b; -x) = \frac{\Gamma(b)}{\Gamma(b-a)} x^{-a}\{1 + 0[|x)^{-1}\} \qquad (x \to \infty)$$

and

$${}_1F_1(a; b; -x) = 0(1) \qquad (x \to 0).$$

Therefore [7, page 43]

$$\phi(s) = \frac{A\Gamma(\beta+1)}{s}\int_0^\infty \left(\frac{y}{s}\right)^\beta F\left(a, \beta+1; b; -\frac{y}{s}\right)f(y)dy$$

under the conditions stated.

COROLLARY 2.1(a). *When $\beta = 0$, $\eta = 2m$, $\alpha = -m - k + (1/2)$, $\phi(s)$ reduces to the generalization of Stieltjes Transform*

$$\phi(s) = \frac{\Gamma(2m+1)}{\Gamma\left(m - k + \frac{3}{2}\right)} \times \frac{1}{s}\int_0^\infty F\left(2m+1, 1; m-k+\frac{3}{2}; -\frac{y}{s}\right)f(y)dy \tag{2.3}$$

introduced by Varma [8]

COROLLARY 2.1(b). *When $\alpha = \beta = 0$, then $\phi(s)$ reduces to the well known Stieltjes Transform* [9, page 323]

$$\phi(s) = \int_0^\infty (s+y)^{-1}f(y)dy\,. \tag{2.4}$$

COROLLARY 2.1(c). *When $\beta = 0$, $\alpha = -\eta = 1 - \sigma$, $\phi(s)$ reduces to another generalization of Stieltjes Transform* [9, page 328]

$$\chi(s) = \frac{\phi(s)}{\Gamma(\sigma)s^{\sigma-1}} = \int_0^\infty \frac{f(y)}{(s+y)^\sigma}dy\,. \tag{2.5}$$

3. Generalized Stieltjes transform as convolution transform. In this section we will find out an inversion operator for the generalized Stieltjes Transform (2.2) by putting it into the form of Convolution Transform. The Convolution Transform with kernel $G(x)$ of the function $\phi(x)$ into $f(x)$ is defined as [10, page 4]

$$f(x) = \int_{-\infty}^{\infty} G(x-t)\phi(t)dt \, . \tag{3.1}$$

The corresponding inversion function $E(x)$, which serves to invert the transform, is defined by the equation

$$[E(x)]^{-1} = \int_{-\infty}^{\infty} G(y)e^{-xy}dy \, .$$

If $\phi(s)$ be defined as in (2.2), we have

$$-\phi'(s) = \frac{A\Gamma(\beta+1)}{s^{\beta+2}}\int_0^{\infty} F\left(a, \beta+2; b; -\frac{y}{s}\right)(sy)^{\beta} f(y)dy$$

because, by Euler's theorem on homogeneous functions,

$$S\left(\frac{\partial}{\partial s}\right)\left[\left(\frac{s}{y}\right)^{-\beta-1} F\left(a, \beta+1; b; -\frac{y}{s}\right)\right]$$
$$= -y\left(\frac{\partial}{\partial y}\right)\left[\left(\frac{s}{y}\right)^{-\beta-1} F\left(a, \beta+1; b; -\frac{y}{s}\right)\right]$$

or

$$\left(\frac{\partial}{\partial s}\right)\left[\left(\frac{y}{s}\right)^{\beta}\cdot\frac{1}{s}F\left(a, \beta+1; b; -\frac{y}{s}\right)\right]$$
$$= -\frac{1}{s^{\beta+2}}\left(\frac{\partial}{\partial y}\right)\left[y^{\beta+1}F\left(a, \beta+1; b; -\frac{y}{s}\right)\right]$$

and

$$\left(\frac{\partial}{\partial y}\right)[y^{\beta+1}F(a, \beta+1; b; y)] = y^{\beta}F(a, \beta+2; b; y) \, .$$

Therefore

$$-e^s\phi'(e^s) = A\Gamma(\beta+1)\int_{-\infty}^{\infty} e^{-(s-y)(\beta+1)}F(a, \beta+2; b; e^{-(s-y)})f(e^y)dy$$

or

$$\xi(s) = A\Gamma(\beta+1)\int_{-\infty}^{\infty} e^{-(s-y)(\beta+1)}F(a, \beta+2; 2; e^{-(s-y)})\zeta(y)dy$$

where

$$\xi(s) \equiv -e^s\phi'(e^s)$$

and

$$\zeta(s) \equiv f(e^s) .$$

Therefore the inversion function $E(x)$ is given by the equation

$$\begin{aligned}\frac{1}{E(x)} &= A\Gamma(\beta+1)\int_{-\infty}^{\infty} e^{-y(\beta+x+1)}F(a, \beta+2; b; -e^{-y})dy \\ &= \frac{\Gamma(\eta-x)\Gamma(\beta+x+1)\Gamma(1-x)}{\Gamma(\alpha+\eta-x)}\end{aligned}$$

provided that

$$b \neq 0, -1, -2, \cdots, Re(1-x) > 0, Re(\eta-x) > 0$$

and

$$Re(\beta+x+1) > 0$$

since [11, page 79]

$$\int_0^\infty z^{-s-1}F(a, b; d; -z)dz = \frac{\Gamma(a)\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(a)\Gamma(b)\Gamma(d+s)}$$

if

$$Res < 0, Re(a+s) > 0, Re(b+s) > 0 ,$$

and $d \neq 0$ or a negative integer.

Therefore,

$$E(D)\{\xi(s)\} = \zeta(s)$$

or

$$\frac{\Gamma(\alpha+\eta-D)}{\Gamma(\beta+1+D)\Gamma(1-D)}\{-e^s\phi'(e^s)\} = f(e^s) , \qquad D \equiv \frac{d}{ds}$$

and we shall give definite meaning to the operations involved. Now

$$\frac{1}{\Gamma(1-x)} = \lim_{n\to\infty} n^x \prod_{k=1}^{n}\left(1-\frac{x}{k}\right)$$

and

$$\frac{\Gamma(\alpha+\eta-x)}{\Gamma(\eta-n)\Gamma(\beta+x+1)} = \lim_{n\to\infty}\frac{n^{\alpha-\beta-x}}{\Gamma(n+2)}\prod_{k=0}^{n}\frac{(D-\eta-k)(D+\beta+1+k)}{(D-\alpha-\eta-k)} .$$

Also we have [10, page 66]

$$\prod_{k=1}^{n-1}\left(1-\frac{D'}{k}\right)[e^{x}F(e^{x})]=\frac{(-)^{n-1}}{(n-1)!}e^{nx}F^{(n-1)}(e^{x})$$

and

$$\prod_{k=0}^{n}(D'+a+k)[e^{-(a+n)x}F(e^{x})]=e^{-(a-1)x}F^{(n+1)}(e^{x})$$

$$\prod_{k=0}^{n}(D'+a-k)[e^{-ax}F(e^{x})]=e^{(n+1-a)x}F^{(n+1)}(e^{x})$$

$$\prod_{k=0}^{n}(D'+a-k)^{-1}[e^{(n+1-a)x}F(e^{x})]=e^{-ax}F^{(-n-1)}(e^{x})$$

where $F^{(-n-1)}(x)$ denotes a function $\psi(x)$ such that

$$\left(\frac{d}{dx}\right)^{n+1}[\psi(x)]=F(x)\,,\qquad D'\equiv\frac{d}{dx}\,.$$

Using the above relations,

$$\begin{aligned}&E(D)\{-e^{s}\phi'(e^{s})\}\\&\quad\equiv(-)^{n}n^{\alpha-\beta}e^{(\alpha+\eta)s}D_1^{-n-1}e^{-\alpha s}D_1^{n+1}e^{-(\eta+\beta)s}\\&\quad\quad\times D_1^{n+1}e^{(2n+\beta+1)s}\phi^{(n)}(e^{s})=f(e^{s})\,,\qquad D_1\equiv\frac{d}{de^{s}},\ (n\to\infty)\,.\end{aligned}$$

Returning to original variables, we have.

$$\begin{aligned}\text{(3.2)}\qquad&\lim_{n\to\infty}(-)\frac{n\Gamma(n+\alpha)}{\Gamma(n+\beta)\Gamma(n)\Gamma(n+2)}\\&\quad\times S^{\alpha+\eta}D^{-n-1}s^{-\alpha}D^{n+1}s^{-(\eta+\beta)}D^{n+1}s^{2n+\beta+1}\phi^{(n)}(s)\,.\end{aligned}$$

We thus have.

THEOREM 3.1. *$f(s)\in C\cdot B$ on $0<s<\infty$ and if the integral* (2.2) *converges, then* (3.2) *holds for* $s>0$.

COROLLARY 3.1(a). *When $\beta=0$, $\alpha=-m-k+(1/2)$, $\eta=2m$ we have the corresponding result for Varma's Transform.*

COROLLARY 3.1(b). *When $\alpha=\beta=0$ we have the Theorem* 9.4 *of Hirschman and Widder* [10, page 69].

COROLLARY 3.1(c). *Similarly for $\alpha=-\eta=1-\sigma$ and $\beta=0$ we have a theorem for* (2.5).

4. Application to generalized Laplace transform. We may now use inversion formula derived above to obtain a new inversion of the Generalized Laplace Transform (1.1). For we have, as above

$$(4.1) \qquad \phi(s) = \frac{A\Gamma(\beta+1)}{s}\int_0^\infty \left(\frac{y}{s}\right)^\beta F\left(a_1\beta+1; b; -\frac{y}{s}\right) f(y)dy \, .$$

Therefore if we invert the integral (4.1) we get $f(y)$. But

$$\phi(s) = \int_0^\infty e^{-sn} F(x)dx \, .$$

Therefore

$$\begin{aligned} \phi^{(n-1)}(s) &= (-)^{n-1}\int_0^\infty e^{-sx} x^{n-1} F(x)dx \\ &= \frac{(-)n-1}{s^n}\int_0^\infty e^{-y} y^{n-1} F\left(\frac{y}{s}\right)dy \end{aligned}$$

by a simple change of variable. But the repeated use of the theorem

$$\left(\frac{\partial}{\partial x}\right)\left[\frac{x^{n+\beta-1}}{y^{n+\beta}} f\left(\frac{y}{x}\right)\right] = -\left(\frac{\partial}{\partial y}\right)\left[\frac{x^{n+\beta-2}}{y^{n+\beta-1}} f\left(\frac{y}{x}\right)\right]$$

gives

$$\left(\frac{\partial}{\partial x}\right)^n\left[\frac{x^{n+\beta 1}}{y^{n+\beta}} f\left(\frac{y}{x}\right)\right] = (-)^n\left(\frac{\partial}{\partial y}\right)^n\left[\frac{x^{\beta-1}}{y^{\beta}} f\left(\frac{y}{x}\right)\right].$$

Therefore,

$$D^n s^{2n+\beta-1}\varphi^{(n-1)}(s) = (-)^{2n-1} s^{n-1+\beta}\int_0^\infty e^{-sx} D^n[x^\beta F(x)]dx \, .$$

Similarly,

$$\begin{aligned} &D^n s^{-(\eta+\beta)} D^n s^{2n+\beta-1}\phi^{(n-1)}(s) \\ &\qquad = (-)^n\int_0^\infty e^{-sx} s^{-(\eta+1)} f_1(x)dx \end{aligned}$$

where, for convenience, we write

$$f_1(x) = x^{-n-\eta-1} D'^n x^{\eta+\beta+2n} D'^n(x^\beta F(x)) \, .$$

Then

$$\begin{aligned} &D^{-n} s^{-d} D^n s^{-(\eta+\beta)} D^n s^{2n+\beta-1}\varphi^{(n-1)}(s) \\ &\qquad = \int_0^\infty e^{-sn} s^{n-1}(sn)^{-\eta-\alpha-1} D'^{-n}\{x^{\eta+\alpha+1-n} f_1(x)\}dx \, . \end{aligned}$$

Therefore finally we have,

$$\lim_{n\to\infty}(-)^n \frac{\Gamma(n-1+\alpha)}{\Gamma(n-1+\beta)\Gamma(n+1)\Gamma(n-1)} \times s^{\alpha+\eta}D^{-n}s^{-\alpha}D^n s^{-(\eta+\beta)}D^n s^{2n+\beta-1}\varphi^{(n-1)}(s)$$
$$= \lim_{n\to\infty}(-)^n \frac{\Gamma(n-1+\alpha)}{\Gamma(n-1+\beta)\Gamma(n+1)\Gamma(n-1)} \times \int_0^\infty e^{-sx}s^{n-1}x^{-\eta-\alpha-}D'^{-n}x^{\alpha} \times D'^n x^{\eta+\beta+2n}D'^n\{x^\beta F(x)\}dx = f(s) \cdots (A) .$$

We have thus proved

THEOREM 4.1. *If $f(x)\in L$ in $0<x<\infty$ and if $F(x)$ is defined by the convergent integral* (1.1) *then the result* (A) *holds for almost all positive values of s.*

COROLLARY 4.1. *When $\alpha=\beta=0$ we have Theorem* 25(a) *of Widder* [9, page 385].

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