HOMOMORPHISMS OF ANNIHILATOR BANACH ALGEBRAS

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Let A be a semi-simple annihilator Banach algebra, and let ν be a homomorphism of A into a Banach algebra. In this paper we describe various continuity properties of ν. Let (*) be the condition that $I \oplus \Re(I) = A$ for all closed two-sided ideals I, where $\Re(I) = \{x \mid Ix = (0)\}$. If (*) holds, then we show that there exists a constant K and a finite set of primitive ideals such that $|| \nu(x) || \leq K || x || \cdot || y ||$ whenever $yx = x$ and x is in the intersection of this finite set. If (*) does not hold, then essentially the same conclusion is true, but with the given norm replaced by one which is defined on a dense subset of A. If A is a dual B^*-algebra, then ν is continuous on the socle.

We also consider the existence of unconditional decompositions in A. We show that (*) holds if and only if the minimal-closed two-sided ideals of A form an unconditional decomposition for A.

Let A and B be Banach algebras, and let ν be a homomorphism of A into B. In this paper we are concerned with deducing various continuity properties of ν when restrictions are placed only on A, and B remains arbitrary. This problem has been considered by Bade and Curtis [2] for A a commutative, regular semi-simple Banach algebra, by Cleveland [6] for A a B^*-algebra, and by Johnson [9] for A the algebra of bounded operators on a Banach space. In this paper we consider this problem for A a semi-simple annihilator Banach algebra (see § 2 for definitions), and we obtain results analogous to those of Bade and Curtis. Since annihilator algebras are in general noncommutative and only have identity when finite-dimensional, techniques essentially different from those in [2] must be employed. The main technical device is a noncommutative version of the "Main Boundedness Theorem" of Bade and Curtis [6, Corollary 3.2].

The main theorems of this paper are found in § 5. There we show (Theorem 5.1) that if A is a semi-simple annihilator Banach algebra in which $I \oplus \Re(I) = A$ for all closed two-sided ideals I, e.g. if A is a proper H^*-algebra [1], a dual B^*-algebra, or l_p, $1 \leq p < \infty$, and if ν is a homomorphism of A into a Banach algebra, then there exists a constant K and a finite set of primitive ideals such that $|| \nu(x) || \leq K || x || \cdot || y ||$ whenever $yx = x$ and x is in the intersection of this finite set. If A is an arbitrary semi-simple annihilator Banach

algebra, then we show that essentially the same conclusion holds if one replaces the original norm by a new one which is defined on a dense subset of A. In § 7 we specialize to the case when A is a dual B^*-algebra. We show that any homomorphism of A into a Banach algebra is continuous on the socle and we obtain several decomposition theorems about the closure of the range of an isomorphism of A.

In § 3 the question of the existence of unconditional decompositions in a semi-simple annihilator Banach algebra A is considered. We show (Theorem 3.4) that the minimal-closed two-sided ideals form an unconditional decomposition for A if and only if $I \oplus \Re(I) = A$ for all closed two-sided ideals I. We obtain a similar result for one-sided ideals when A is assumed to have a proper involution. Both of these results follow from a technical theorem which is established about unconditional decompositions in Banach spaces. In § 4 the structure of ideals and idempotents in A is considered. We show that the socle is the set of elements possessing a left identity. In § 6 we establish several properties of the separating ideal of a homomorphism of A into a Banach algebra.

2. **Preliminaries.** For A an algebra, $S \subset A$, define $\mathfrak{L}(S) = \{x \mid xS = (0)\}$ and $\Re(S) = \{x \mid Sx = (0)\}$. Let I be a closed right ideal and let J be a closed left ideal in a Banach algebra A. Then A is an *annihilator algebra* if $\mathfrak{L}(I) = (0)$ only when $I = A$ and $\Re(J) = (0)$ only when $J = A$. The algebra is called *dual* if $\mathscr{R}\mathscr{L}(I) = I$ and $\mathscr{L}\mathscr{R}(J) = J$ for all such I and J. Dual algebras were introduced by Kaplansky [10] and annihilator algebras by Bonsall and Goldie [4]. For the basic structure theorems concerning these algebras, the reader is referred to the texts of Rickart [14] or Naimark [12]. The definitions of terms not defined in this paper are those of [14].

If A is a semi-simple annihilator Banach algebra, we will denote by $\varPi$ the set of primitive ideals, which Civin and Yood [5] have shown are the same as the maximal-closed two-sided ideals. We will denote the set of minimal-closed two-sided ideals by $\mathfrak{M}$, and the set of maximal-modular two-sided ideals by $\varXi$. The socle (the algebraic sum of the minimal left or right ideals) is denoted by F. We will always assume that our Banach algebras are complex.

3. **Unconditional decompositions.** In this section we consider the question of when certain families of ideals of a semi-simple annihilator Banach algebra form an unconditional decomposition. Since much of this problem can be stated in a Banach space framework, we will work in that more general area whenever possible.

Throughout this section, we will always assume that A is a

Banach space and that $\mathscr{F} = \{I_\alpha \mid \alpha \in C\}$ is a collection of closed subspaces with dense linear span such that $I_\alpha \cap \mathrm{cl}\,(\sum_{\alpha \neq \beta} I_\beta) = (0)$ for all $\alpha \in C$, where $\sum I_\beta$ denotes the algebraic sum of $\{I_\beta\}$ and "cl" means "closure".

DEFINITION 3.1. An element $x \in A$ is called $\mathscr{F}$-decomposable if $x = \sum_\alpha x_\alpha$, where $x_\alpha \in I_\alpha$ and convergence is with respect to the net of finite partial sums. We denote the set of $\mathscr{F}$-decomposable elements of A by $D(A, \mathscr{F})$ and we say that $\mathscr{F}$ is an unconditional decomposition for A if $A = D(A, \mathscr{F})$.

If an element x has an expansion as described above, then this expansion is necessarily unique, and also $\{\alpha \mid x_\alpha \neq 0\}$ is countable. If each I_α is one-dimensional, with generator b_α, then $\mathscr{F}$ is an unconditional decomposition for A if and only if $\{b_\alpha\}$ is an unconditional Schauder basis for A [7, p. 73].

DEFINITION 3.2. For $x \in D(A, \mathscr{F})$, define

$$| x |_{\mathscr{F}} = \sup_{J \in \mathscr{F}} \| \sum_{\alpha \in J} x_\alpha \| ,$$

where $\mathscr{F}$ denotes the collection of finite subsets of C.

It is straightforward to verify that $| \cdot |_{\mathscr{F}}$ defines a norm on $D(A, \mathscr{F})$, that $|x|_{\mathscr{F}} \geqq \| x \|, x \in D(A, \mathscr{F})$, and that $D(A, \mathscr{F})$ is complete in this new norm. *When there is no confusion, we write D for $D(A, \mathscr{F})$, $| \cdot |$ for $| \cdot |_{\mathscr{F}}$, and $\sum_J$ for $\sum_{\alpha \in J}$.* We first prove the following:

PROPOSITION 3.3. $D(D(A, \mathscr{F}), \mathscr{F}) = D(A, \mathscr{F})$.

Proof. One sees that $\mathrm{cl}(I_\alpha) = I_\alpha$ and that $I_\alpha \cap \mathrm{cl}(\sum_{\beta \neq \alpha} I_\beta) = (0), \alpha \in C$, where the closure in both cases is taken with respect to the norm $| \cdot |$. Hence $D(D(A, \mathscr{F}), \mathscr{F})$ is well defined. If $x \in D, \varepsilon > 0$, choose $J_0 \varepsilon \mathscr{F}$ such that $J \cap J_0 = \varnothing$ implies $\| \sum_J x_\alpha \| < \varepsilon$. If $J_1 \in \mathscr{F}, J_1 \supset J_0$, then $y = x - \sum_{J_1} x_\alpha \in D$ and it follows that $|y| = \sup_{J \in \mathscr{F}} \| \sum_{J-J_1} x_\alpha \| \leqq \varepsilon$. Thus $x = \sum x_\alpha (| \cdot |)$, so $x \in D(D(A, \mathscr{F}), \mathscr{F})$.

We now give necessary and sufficient conditions on $\mathscr{F}$ which are equivalent to $\mathscr{F}$ being an unconditional decomposition for A.

THEOREM 3.4.   *Let A be a Banach space, and let $\mathscr{F} = \{I_\alpha \mid \alpha \in C\}$ be a collection of closed subspaces with dense linear span such that $I_\alpha \cap \mathrm{cl}(\sum_{\beta \neq \alpha} I_\beta) = (0), \alpha \in C$. Then the following statements are equivalent:*

(1) $\mathscr{I}$ is an unconditional decomposition for A.

(2) The norms $|\cdot|_\mathscr{I}$ and $\|\cdot\|$ are equivalent.

(3) There exists a constant K such that, if $\alpha_1, \cdots, \alpha_n$ are distinct elements of C and $x_i \in I_{\alpha_i}$, then

$$\| \sum_{i=1}^{m} x_i \| \leq K \| \sum_{i=1}^{n} x_i \|, \; 1 \leq m \leq n \, .$$

(4) If $C = C_1 \cup C_2$, with $C_1 \cap C_2 = \varnothing$, then there exists a constant $K(C_1, C_2)$ such that $x \in \sum_{C_1} I_\alpha$, $y \in \sum_{C_2} I_\alpha$ imply $\|x\| \leq K(C_1, C_2) \|x + y\|$.

Proof.[1] Since D is dense in A, (1) is equivalent to D being closed in A. Therefore the equivalence of (1) and (2) is given by [15, Corollary I.11] and the fact that $\|\cdot\| \leq |\cdot|$. The equivalence of (1) and (3) follows from [15, Th. I.13].

Clearly (3) implies (4). Suppose that (4) does not imply (3). Then there exist disjoint finite subsets $\{B_1, C_1\} \subset C$, and elements $x_1 \in \sum_{B_1} I_\alpha$, $y_1 \in \sum_{C_1} I_\alpha$ such that $\|x_1\| > \|x_1 + y_1\|$. Assume that pairwise disjoint finite subsets $\{B_1, \cdots, B_n, C_1, \cdots, C_n\} \subset C$ and elements $x_1, \cdots, x_n$, $y_1, \cdots, y_n$ have been chosen so that $x_k \in \sum_{B_k} I_\alpha$, $y_k \in \sum_{C_k} I_\alpha$, $1 \leq k \leq n$ and such that $\|x_k\| > k \|x_k + y_k\|$, $1 \leq k \leq n$. Let $E = \bigcup_{k=1}^{n} (B_k \cup C_k)$. Since E is finite, $\sup_{G \subset E} K(G, C \sim G) < \infty$. Thus since (3) fails, there must exist disjoint subsets $\{B_{n+1}, C_{n+1}\} \subset C \sim E$ and elements $x_{n+1} \in \sum_{B_{n+1}} I_\alpha$, $y_{n+1} \in \sum_{C_{n+1}} I_\alpha$ such that $\|x_{n+1}\| > (n+1) \|x_{n+1} + y_{n+1}\|$. Now x_{n+1} and y_{n+1} are each contained in the linear span of a finite subset of $\{I_\alpha \mid \alpha \in C\}$, so B_{n+1} and C_{n+1} may be taken to be finite. Now let $H_1 = \bigcup_{k=1}^{\infty} B_k$, $H_2 = C \sim H_1$. Then $x_n \in H_1$, $y_n \in H_2$ for any n, so $K(H_1, H_2) > n$ for all n, which is a contradiction. Therefore (4) implies (3) and the theorem is established.

We now apply the preceding theorem to the case when A is a semi-simple annihilator Banach algebra and $\mathscr{I} = \mathfrak{M}$.

THEOREM 3.5. *Let A be a semi-simple annihilator Banach algebra. Then the following statements are equivalent.*

(1) *The minimal-closed two-sided ideals of A form an unconditional decomposition for A.*

(2) $I \oplus \mathfrak{R}(I) = A$ *for all closed two-sided ideals I.*

Proof. We note that it is always true that $\mathrm{cl}(I \oplus \mathfrak{R}(I)) = A$ for any closed two-sided ideal I.

(1) implies (2): Let I be a closed two-sided ideal in A and let $x \in A$. If $M \in \mathfrak{M}$, then $M \subset I$ or $M \subset \mathfrak{R}(I)$, and hence every summand in the infinite series expansion of x is either in I or $\mathfrak{R}(I)$. Thus $x \in I \oplus \mathfrak{R}(I)$.

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[1] The author is indebted to Professor R. C. James for suggestions which greatly shortened the original proof.

(2) implies (1): It suffices to show that (4) of the preceding theorem holds, with $\mathscr{F} = \mathfrak{M} = \{M_\alpha \mid \alpha \in C\}$. Suppose $C = C_1 \cup C_2$, with $C_1 \cap C_2 = \varnothing$, and let $J_k = \sum_{C_k} M_\alpha$, $k = 1, 2$. Then $\bar{J}_1 \oplus \mathfrak{R}(\bar{J}_1) = A$, so it follows from the Closed Graph theorem that there exists a constant K such that $\| u \| \leqq K \| u + v \|$, $u \in \bar{J}_1$, $v \in \mathfrak{R}(\bar{J}_1)$. Now $J_2 \subset \mathfrak{R}(\bar{J}_1)$, so (4) is established.

COROLLARY 3.6. *If A satisfies the conditions of the above theorem, and I is a closed two-sided ideal in A, then*
(1) $\mathfrak{L}\mathfrak{R}(I) = I$.
(2) *I is a semi-simple annihilator Banach algebra, with $\{M \in \mathfrak{M} \mid M \subset I\}$ its set of minimal-closed two-sided ideals, and I satisfies the conditions of the above theorem.*

Proof. If $x \in I$, then all of the summands in its infinite series expansion are in minimal-closed two-sided ideals which are contained in I. Hence $I = \mathrm{cl}(\sum_{M \subset I} M)$. The proof of the rest of the assertions is straightforward.

REMARK. The above theorem holds if A is a proper H^*-algebra, a dual B^*-algebra, or l_p, $1 \leqq p < \infty$.
We next apply Theorem 3.4 to the case when A is a semi-simple annihilator Banach algebra with involution $x \to x^*$ such that $xx^* = 0$ implies $x = 0$ (such an involution is called *proper*), and $\mathscr{F}$ is a family of right ideals generated by a family of orthogonal hermitian idempotents. We note that if $\mathscr{F} = \{e_\alpha A \mid \alpha \in C\}$, where $\{e_\alpha \mid \alpha \in C\}$ is an orthogonal family of idempotents, and if x is $\mathscr{F}$-decomposable, then the infinite series expansion of x is given by $x = \sum_\alpha e_\alpha x$.
The proof of the following lemma is routine and is omitted.

LEMMA 3.7. *Let A be a semi-simple annihilator Banach algebra with proper involution. Then*
(1) $\mathrm{cl}(I \oplus \mathfrak{L}(I)^*) = A$ *for all closed right ideals I.*
(2) $\mathrm{cl}(I \oplus \mathfrak{R}(I)^*) = A$ *for all closed left ideals I.*

THEOREM. 3.8. *Let A be a semi-simple annihilator Banach algebra with proper involution. Then the following statements are equivalent.*
(1) *If $\{e_\alpha\}$ is a maximal family of orthogonal hermitian idempotents, then $x \in A$ implies $x = \sum_\alpha e_\alpha x$.*
(2) *If $\{e_\alpha\}$ is as in (1), then $x \in A$ implies $x = \sum_\alpha x e_\alpha$.*
(3) *$I \oplus \mathfrak{L}(I)^* = A$ for all closed right ideals I.*
(4) *$I \oplus \mathfrak{R}(I)^* = A$ for all closed left ideals I.*
Moreover, the above conditions imply that A is dual.

Proof. (1) implies (3): Let I be a closed right ideal in A, and let $\{e_\alpha \mid \alpha \in C_1\}$ be a maximal family of hermitian idempotents in I. Extend this family to a maximal such family in A, $\{e_\alpha \mid \alpha \in C\}$. Let $J = \mathrm{cl}\left(\sum_{\beta \notin C_1} e_\beta A\right)$. If $J \cap I \neq (0)$, then $J \cap I$ contains a hermitian idempotent p. But then $e_\alpha p = 0$, $\alpha \in C_1$, which contradicts the maximality of C_1. Hence $x \in I$ implies $x - \sum_{C_1} e_\alpha x = \sum_{\alpha \notin C_1} e_\alpha x \in I \cap J = (0)$, so $e_\beta x = 0$, $\beta \notin C_1$. It follows that $\mathrm{cl}(\sum_{\beta \notin C_1} e_\beta A) \subset \mathfrak{L}(I)^*$. Therefore $x \in A$ implies $x = \sum_{C_1} e_\alpha x + \sum_{\alpha \notin C_1} e_\alpha x \in I \oplus \mathfrak{L}(I)^*$.

(3) implies (1): If $\{e_\alpha \mid \alpha \in C\}$ is a maximal family of orthogonal hermitian idempotents, and $\mathscr{I} = \{e_\alpha A \mid \alpha \in C\}$, then the hypotheses of Theorem 3.4 are satisfied. The proof of (1) now follows in a manner analogous to the proof of "(2) implies (1)" in Theorem 3.5.

The other equivalences now follow from left-right symmetry and properties of the involution.

Suppose that one of conditions (1) through (4) holds, and let I be a closed right ideal in A. Then $\mathfrak{L}(I)^* \cap \mathfrak{L}(\mathfrak{L}(I)^*)^* = (0)$. But $\mathfrak{L}(\mathfrak{L}(I)^*)^* = \mathfrak{R}\mathfrak{L}(I)$, and thus it follows that $I = \mathfrak{R}\mathfrak{L}(I)$. Similarly if I is a closed left ideal, then $I = \mathfrak{L}\mathfrak{R}(I)$. Thus A is dual.

REMARK. The above theorem holds if A is a proper H^*-algebra or a dual B^*-algebra.

We conclude this section with an analysis of $D(A, \mathscr{I})$ in case A is a Banach algebra and each I_α is a closed right ideal.

LEMMA. 3.9. (1) D *is a dense right ideal in* A *and* $|xy| \leqq |x| \cdot \|y\|$, $x \in D$, $y \in A$.

(2) *If each* I_α *is two-sided, then* D *is two-sided and* $|xy| \leqq \|x\| \cdot |y|$, $x \in A$, $y \in D$.

(3) $(D, |\cdot|)$ *is a Banach algebra which is semi-simple if* A *is semi-simple.*

Proof. Since $D \supset \sum I_\alpha$, D is dense in A. If $x \in D$, $y \in A$, then $xy = (\sum_\alpha x_\alpha)y = \sum_\alpha(x_\alpha y)$. Therefore $xy \in D$. Since $\|\sum_J x_\alpha y\| \leqq |x| \cdot \|y\|$, $J \in \mathscr{I}$, we have that $|xy| \leqq |x| \cdot \|y\|$. Statement (2) is established in a similar manner.

From (1) it follows that $(D, |\cdot|)$ is a Banach algebra. Let R be the radical of D. Then [8, p. 10] $R = \mathfrak{L}(D) \cap D \subset \mathfrak{L}(D) = \mathfrak{L}(\bar{D}) = \mathfrak{L}(A)$. Hence D is semi-simple if A is.

Using the preceding lemma and the fact that a semi-simple Banach algebra is an annihilator algebra if it can be expressed as the topological sum of annihilator algebras, we have the following theorem.

THEOREM. 3.10. *Let A be a semi-simple annihilator Banach algebra, and let $\mathscr{F} = \{I_\alpha \mid \alpha \in C\}$ be a collection of closed two-sided ideals of A. Suppose that either $\mathscr{F} = \mathfrak{M}$ or that A is dual and equals the topological direct sum of $\mathscr{F}$. Then $(D(A, \mathscr{F}), |\cdot|_{\mathscr{F}})$ is a semi-simple annihilator Banach algebra.*

We now give an explicit characterization of $D(A, \mathfrak{M})$ in case $A = C(G)$ for G a compact abelian topological group.

THEOREM. 3.11. *Let G be a compact abelian topological group, and let $A = C(G)$, with convolution for multiplication. Then $D(A, \mathfrak{M})$ is bi-continuously *-isomorphic to $L_1(\hat{G})$, where the latter has point-wise multiplication.*

Proof. Let $\hat{G} = \{e_\alpha\}$, and let $M_\alpha = \{ce_\alpha \mid c \text{ complex}\}$. Then $\mathfrak{M} = \{M_\alpha\}$. Let $x \in D$. By considering real and imaginary parts, one shows that

$$\sum_\alpha |x * e_\alpha(g)| \leqq 4 |x|_{\mathfrak{M}}, \qquad g \in G,$$

where $*$ denotes convolution multiplication. Now $\hat{G}$ is discrete, so it follows that the left hand side of the above equation is just the L_1 norm of $\hat{x}$, the Fourier transform of x. Thus $x \to \hat{x}$ is a continuous mapping of D into $L_1(\hat{G})$. It is straightforward to verify that the mapping is bicontinuous and a *-isomorphism onto $L_1(\hat{G})$.

COROLLARY. 3.12. *If A is as above, then $I \oplus \mathfrak{R}(I) = A$ for all closed ideals I if and only if G is finite.*

Proof. If the first statement holds, then $D(A, \mathfrak{M}) = A$. Thus the Fourier transform of every element of A is in $L_1(\hat{G})$. Thus, by the Inversion Theorem, every element of $C(G)$ is the Fourier transform of an element in $L_1(\hat{G})$, so by [16, Th. 4.6.8], G is finite.

4. **Structure and representation theorems.** *Throughout this section we assume that A is a semi-simple annihilator Banach algebra.* We investigate the structure of ideals and idempotents in A. The first lemma shows the relationship between $\mathfrak{M}$ and Π.

LEMMA. 4.1. *$M \to \mathfrak{L}(M)$ is a one-to-one mapping of $\mathfrak{M}$ onto Π. If A is dual, then $P \to \mathfrak{R}(P)$ is a one-to-one mapping of Π onto $\mathfrak{M}$.*

Proof. Let $M \in \mathfrak{M}$. Then there exists an idempotent $e \in M$ such that $M = \mathrm{cl}(AeA)$. Let $P = \{x \mid xA \subset A(1-e)\}$. Then $P \in \Pi$. Now $\mathfrak{L}(M) \subset \mathfrak{L}(eA) = A(1-e)$, and $\mathfrak{L}(M)$ is a closed two-sided ideal containing P. Therefore $\mathfrak{L}(M) = P$.

If $P \in \Pi$, then $\mathfrak{R}(P)$ contains a minimal idempotent e. Thus $M = \mathrm{cl}(AeA) \in \mathfrak{M}$ and $\mathfrak{L}(M) = P$. Thus $M \to \mathfrak{L}(M)$ is onto. Since the minimal-closed two-sided ideals annihilate each other, it follows that the mapping is one-to-one.

Suppose now that A is dual. Let $P \in \Pi$ and let $M \in \mathfrak{M}$ such that $\mathfrak{L}(M) = P$. Then $\mathfrak{R}(P) = \mathfrak{R}\mathfrak{L}(M) = M$. Thus $P \to \mathfrak{R}(P)$ is the inverse mapping of $M \to \mathfrak{L}(M)$ and hence is necessarily one-to-one and onto.

We next establish a representation theorem which essentially reduces to the Gelfand representation theorem when A is commutative.

LEMMA. 4.2. *If* $P \in \Pi$, *then* A/P *with the quotient norm is a primitive, topologically simple, annihilator Banach algebra.*

Proof. The proof is straightforward and is omitted.

DEFINITION. 4.3. Let $\mathscr{P} A/P$ denote the direct product of $\{A/P \mid P \in \Pi\}$, and let $(\sum A/P)_0$ denote the set of $f \in \mathscr{P} A/P$ such that $\{P \mid \| f(P) \| \geqq \varepsilon\}$ is finite for all $\varepsilon > 0$. For $f \in (\sum A/P)_0$, define $\| f \| = \sup_{P \in \Pi} \| f(P) \|$. Now, for $x \in A$, define $\hat{x} \in \mathscr{P} A/P$ by $\hat{x}(P) = x + P$, and define the carrier of x, $\mathrm{car}(x) = \{P \mid \hat{x}(P) \neq 0\}$.

THEOREM 4.4. $(\sum A/P)_0$ *is a semi-simple annihilator Banach algebra, and* $x \to \hat{x}$ *is a norm-decreasing isomorphism of A onto a dense subset of* $(\sum A/P)_0$.

Proof. The proof of the first statement is sketched in [14, pp. 106-7]. It is easily seen that $x \to \hat{x}$ is an isomorphism of A into $\mathscr{P} A/P$. Let $x \in A, \varepsilon > 0$. Then there exist $M_i \in \mathfrak{M}$ and $x_i \in M_i$, $1 \leqq i \leqq n$, such that $\| x - \sum_{i=1}^{n} x_i \| < \varepsilon$. Let $P_i = \mathfrak{L}(M_i)$; then $\mathrm{car}(x_i) \subset \{P_i\}$. Thus $P \notin \{P_1, \cdots, P_n\}$ implies

$$\| \hat{x}(P) \| = \left\| \left(x - \sum_{i=1}^{n} x_i \right)^{\wedge}(P) \right\| < \varepsilon \ ,$$

so $\hat{x} \in (\sum A/P)_0$.

Since $\| \hat{x}(P) \| \leqq \| x \|$, the mapping is norm-decreasing. Thus it remains to show that the range of the mapping is dense in $(\sum A/P)_0$. It is sufficient to show that the set of $\hat{x}$ with finite carriers is dense in the direct sum of $\{A/P \mid P \in \Pi\}$, since this latter set is dense in $(\sum A/P)_0$. So, suppose $f \in (\sum A/P)_0$, that f vanishes except for $P \in \{P_1, \cdots, P_n\}$, and let $\varepsilon > 0$. Now choose $M_i \in \mathfrak{M}$ such that $\mathfrak{L}(M_i) = P_i$. Since $\mathrm{cl}(M_i + P_i) = A$, the image of M_i is dense under the natural map from A onto A/P_i. Thus there exist $x_i \in M_i$ such that $\| \hat{x}_i(P_i) - f(P_i) \| < \varepsilon, 1 \leqq i \leqq n$. If $x = \sum_{i=1}^{n} x_i$, then $\mathrm{car}(x) \subset \{P_1, \cdots, P_n\}$ and $\| \hat{x} - f \| = \sup_{1 \leqq i \leqq n} \| \hat{x}_i(P_i) - f(P_i) \| < \varepsilon$.

REMARK. In general, the mapping $x \to \hat{x}$ is not bicontinuous. However, if A is a B^*-algebra, then A is isometrically $*$-isomorphic to $(\sum A/P)_0$ [11, Th. 8.1].

We now turn our attention to the decomposability of idempotents.

THEOREM. 4.5. *Every idempotent in A can be expressed as the sum of orthogonal minimal idempotents.*

Proof. Let e be an idempotent in A. First, suppose that e is contained in some $M \in \mathfrak{M}$. Let I be a minimal left ideal in M and let $x \to L(x)$ be the left regular representation of M as compact operators on I (see [4, Th. 9 and 10]). If $S = \text{range } (L(e))$, then S is closed and $L(e)$ is the identity on S. Hence S is finite-dimensional. Thus there exist orthogonal bounded projections with one-dimensional range, $E_1, \cdots, E_n$ such that $E_1 + \cdots + E_n = L(e)$. Since the range of $x \to L(x)$ includes all bounded operators on I with finite-dimensional range, each E_k is the image of some $e_k \in M$. Thus $\{e_1, \cdots, e_n\}$ is a set of orthogonal minimal idempotents, and $e = e_1 + \cdots + e_n$.

Now, suppose that e is any idempotent in A. Then $\hat{e}$ is an idempotent in $(\sum A/P)_0$ and hence car (e) is finite, say car $(e) = \{P_1, \cdots, P_n\}$. Let $M_k \in \mathfrak{M}$, $\mathfrak{L}(M_k) = P_k$. It is straightforward to verify that every idempotent in A/P_k is the image of some idempotent $q_k \in M_k$ under the natural map from A onto A/P_k. Hence $\hat{q}_k(P_k) = \hat{e}(P_k)$, and by the first part each q_k is the sum of orthogonal minimal idempotents. Since $M_k M_j = (0)$, $k \neq j$, it follows that $e = q_1 + \cdots + q_n$ and hence that e is the sum of orthogonal minimal idempotents.

The following corollary has been shown by Barnes [3] for the more general class of modular annihilator algebras.

COROLLARY. 4.6. *Let e be an idempotent in A. Then eAe is finite-dimensional. Hence A is finite-dimensional if it has an identity.*

Proof. If e and f are minimal idempotents, then eAf is (0) or one-dimensional.

Using the above corollary, we now establish the following result about Ξ, the set of maximal-modular two-sided ideals.

PROPOSITION. 4.7. The following statements are equivalent.
(1) A is strongly semi-simple.
(2) $\Pi = \Xi$.
(3) Each $M \in \mathfrak{M}$ is finite-dimensional.
(4) Each $M \in \mathfrak{M}$ has identity.

Proof. If $P_0 \in \Pi \sim \Xi$, and $M \in \mathfrak{M}$, $\mathfrak{L}(M) = P_0$, then $M \subset \cap \{P \mid P \in \Xi\}$. Therefore (1) implies (2). If (2) holds and $M \in \mathfrak{M}$, then $\mathfrak{L}(M) \in \Xi$, so $A/\mathfrak{L}(M)$ has identity and hence is finite-dimensional. The mapping $x \to \hat{x}(\mathfrak{L}(M))$ is one-to-one on M, so M is finite-dimensional. Therefore (2) implies (3). If (3) holds and $M \in \mathfrak{M}$, then (see Theorem 4.5) M is isomorphic to all the operators on any of its minimal left ideals. Hence M has identity. Thus (3) implies (4). If $M \in \mathfrak{M}$, then $\mathfrak{L}(M)$ is maximal-closed. If (4) holds, then $\mathfrak{L}(M)$ is also modular, so $\Pi = \{\mathfrak{L}(M) \mid M \in \mathfrak{M}\} = \Xi$. Therefore (4) implies (2). Clearly (2) implies (1).

The rest of this section consists of structure theorems for certain kinds of one-sided ideals in A. Although the theorems are stated and proved only for right ideals, the corresponding statements for left ideals also hold.

THEOREM. 4.8. *Let $e_1, \cdots, e_n$ be idempotents in A. Then there exists an idempotent $e \in A$ such that $eA = e_1 A + \cdots + e_n A$.*

Proof. We note that if $e_1, \cdots, e_n$ are orthogonal, then e can be taken as $e_1 + \cdots + e_n$.

It suffices to prove the theorem for $n = 2$, since the general case then follows by induction. First, suppose that e_1 and e_2 are both contained in the same $M \in \mathfrak{M}$. Let $x \to L(x)$ be as in the proof of Theorem 4.5 and let $S_k = \text{range}(L(e_k))$. Then S_1, S_2 are finite-dimensional, so there exist bounded projections P_1, P_2 such that $\text{range}(P_1) = S_1$ and $\text{range}(P_2) = S_2 \sim S_1$. Thus $E = P_1 + P_2$ is a bounded projection, so there exists an idempotent $e \in M$ such that $L(e) = E$. Now $L(e)L(e_k) = L(e_k)$, so $e_1 A + e_2 A \subset eA$. Also, $L(e_k)P_k = P_k$, so it follows that $eA \subset e_1 A + e_2 A$.

Now suppose that e_1 and e_2 are any idempotents in A. Since any minimal idempotent is contained in some $M \in \mathfrak{M}$, it follows from Theorem 4.5 that $e_1 = p_{1,1} + \cdots + p_{n,1}$ and $e_2 = p_{1,2} + \cdots + p_{n,2}$, where each $p_{k,j}$ is an idempotent in $M_k \in \mathfrak{M}$, $j = 1, 2, 1 \leqq k \leqq n$. By applying the previous case to each pair $\{p_{k,1}, p_{k,2}\}$ and using the fact that $M_i M_j = (0)$, $i \neq j$, the theorem is established.

As a consequence of Theorems 4.5 and 4.8, we obtain the following:

COROLLARY. 4.9. *The socle of A is the union of the idempotent-generated right ideals of A.*

We conclude this section with an analysis of a special class of right ideals which arise in the study of homomorphisms. If $y \in A$ we denote by $J(y)$ the closed right ideal $\{x \mid yx = x\}$. The following fact about $J(y)$ will be used in establishing Theorem 5.1.

LEMMA. 4.10.   *Let* $y \in A$ *and let* $G = \{P \mid \|\hat{y}(P)\| \geqq 1\}$.   *Then* G *is a finite set, and* car $(x) \subset G$ *for all* $x \in J(y)$.

Proof.   If $0 \neq x \in J(y)$, $P \in$ car (x),   then $\hat{y}(P)\hat{x}(P) = \hat{x}(P) \neq 0$. Therefore $\|\hat{y}(P)\| \geqq 1$, so $P \in G$.   Since $\hat{y} \in (\sum A/P)_0$, G is finite.

THEOREM. 4.11. $J(y)$ *is idempotent-generated for all* $y \in A$.

Proof.   First, suppose that $J(y)$ is contained in some $M \in \mathfrak{M}$, and let $x \rightarrow L(x)$ be as in the proof of Theorem 4.5.   If $S = \{x \mid L(y)x = x\}$, then S is closed and $L(y)$ is the identity on S, so S is finite-dimensional. Thus there exists a bounded projection E with range S, and so there exists an idempotent $e \in M$ such that $L(e) = E$.   It follows that $ye = e$, and hence $eA \subset J(y)$.   If $u \in J(y)$, then $L(y)L(u) = L(u)$, so $L(e)L(u) = L(u)$.   Therefore $u \in eA$.

If y is any element of A, let $\{P \mid \|\hat{y}(P)\| \geqq 1\} = \{P_1, \cdots, P_n\}$. Since A/P_k is topologically simple, by the previous case there exist idempotents $q_k \in A/P_k$ such that $q_k(A/P_k) = J(\hat{y}(P_k))$.   Thus there exists an idempotent $e \in A$ with carrier $\{P_1, \cdots, P_n\}$ such that $\hat{e}(P_k) = q_k$, $1 \leqq k \leqq n$.   It follows that $eA = J(y)$.

COROLLARY 4.12.   *The socle of* A *is the set of elements possessing a left identity.*

5.   **Continuity properties of homomorphisms.**   We now apply the results of the previous two sections to the study of continuity properties of homomorphisms of semi-simple annihilator Banach algebras. Our first objective is to establish the following analog for annihilator algebras of a theorem by Bade and Curtis [2, Th. 3.7].

THEOREM 5.1.   *Let* A *be a semi-simple annihilator Banach algebra in which* $I \oplus \mathfrak{R}(I) = A$ *for all closed two-sided ideals* I, *and let* ν *be a homomorphism of* A *into a Banach algebra.   Then there exists a finite set of primitive ideals,* Π_0, *and a constant* K *such that*

$$\|\nu(x)\| \leqq K\|x\| \cdot \|y\|$$

for all x *and* y *in* A *such that* $yx = x$ *and* car$(x) \subset \Pi \sim \Pi_0$.

The proof of this theorem will be given after several lemmas have been established.   If A is *any* semi-simple annihilator Banach algebra, then it will be possible to make a slightly weaker conclusion, but with $\|\cdot\|$ replaced by $|\cdot|_{\mathfrak{M}}$.

We will make repeated use of the following theorem, which was proved in the commutative case by Bade and Curtis [2].

THEOREM 5.2. *Let A and B be Banach algebras, and let ν be a homomorphism of A into B. If $\{x_n\}$ and $\{y_n\}$ are sequences in A such that*

(1) $$y_n x_n = x_n$$

and

(2) $$y_n x_m = 0 , \qquad n \neq m ,$$

then

$$\sup_n \| \nu(x_n) \|/\| x_n \| \cdot \| y_n \| < \infty .$$

Proof. This is proved in [6, Corollary 3.2]. The hypotheses in [2] and [6] have (2) replaced by the stronger condition $y_n y_m = 0$, $n \neq m$. However, this is not necessary.

In proving Theorem 5.1, we will use a construction which is basically the same as that in [2]. *Through Lemma 5.8, we will assume that A is a semi-simple annihilator Banach algebra in which $I \oplus \Re(I) = A$ for all closed two-sided ideals I, and that ν is a homomorphism of A into a Banach algebra.* By Theorems 3.4 and 3.5, the norms $\|\cdot\|$ and $|\cdot|_{\mathfrak{M}}$ are equivalent. Since we are dealing with continuity properties of ν, there is no loss of generality in assuming that these norms are numerically equal. We will do so as it simplifies the computations.

DEFINITION 5.3. Let $\mathscr{H}$ be the set of $E \subset \Pi$ such that $\sup \{ \| \nu(x) \|/\| x \| \cdot \| y \| : yx = x, \ \mathrm{car}\,(x) \cup \mathrm{car}\,(y) \subset E \} \equiv K(E) < \infty$.

We will show that $\mathscr{H}$ has a largest element whose complement is finite.

LEMMA 5.4. *If $\{E_\delta\}$ is a pairwise disjoint family of subsets of Π, then there exists a constant K such that $K(E_\delta) \leq K$, and hence $E_\delta \in \mathscr{H}$, for all but finitely many δ.*

Proof. Suppose the conclusion is false. Then there exists a distinct sequence $\{E_{\delta_n}\}$ and sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n x_n = x_n$, $\mathrm{car}(x_n) \cup \mathrm{car}(y_n) \subset E_{\delta_n}$ and $\| \nu(x_n) \|/\| x_n \| \cdot \| y_n \| > n$. But $\mathrm{car}(y_n x_m) \subset E_{\delta_n} \cap E_{\delta_m} = \phi$, $n \neq m$, so that $y_n x_m = 0$, $n \neq m$, which contradicts Theorem 5.2.

LEMMA 5.5. *Suppose that E_1 and E_2 are disjoint subsets of Π and that x and y are elements of A such that $yx = x$ and $\mathrm{car}\,(x) \cup \mathrm{car}\,(y) \subset E_1 \cup E_2$. Then for $k = 1,2$ there exist $x_k, y_k \in A$ such that*

(1) $$y_k x_k = x_k \,, \qquad \text{car } (x_k) \cup \text{car } (y_k) \subset E_k$$

(2) $$x_1 + x_2 = x, \qquad y_1 + y_2 = y$$

(3) $$\| \, x_k \, \| \leqq \| \, x \, \| \,, \qquad \| \, y_k \, \| \leqq \| \, y \, \| \,.$$

Proof. Since x is $\mathfrak{M}$-decomposable, and car $(x) \subset E_1 \cup E_2$, x has an infinite series expansion in which each summand has its one element carrier either in E_1 or E_2. Hence x can be factored into $x_1 + x_2$ where car $(x_k) \subset E_k$, and similarly for y. Since $yx = x$ and $E_1 \cap E_2 = \phi$, it follows that $y_k x_k = x_k$. Statement (3) follows from the fact that $\| \cdot \| = | \cdot |$.

LEMMA 5.6. *If* $\{E_1, \cdots, E_n\} \subset \mathscr{H}$, *then* $\bigcup_{k=1}^{n} E_k \in \mathscr{H}$ *and* $K(\bigcup_{k=1}^{n} E_k) \leqq \sum_{k=1}^{n} K(E_k)$.

Proof. It suffices to establish the conclusion for $n = 2$, since the general case then follows by induction. Without loss of generality, we may assume $E_1 \cap E_2 = \phi$.

Suppose that $yx = x$ and car $(x) \cup$ car $(y) \subset E_1 \cup E_2$. Now chose x_k and y_k as in the previous lemma. Then

$$\frac{\| \, \nu \, (x) \, \|}{\| \, x \, \| \cdot \| \, y \, \|} \leqq \frac{\| \, \nu (x_1) + \nu (x_2) \, \|}{\| \, x \, \| \cdot \| \, y \, \|} \leqq \frac{\| \, \nu (x_1) \, \|}{\| \, x_1 \, \| \cdot \| \, y_1 \, \|} + \frac{\| \, \nu (x_2) \, \|}{\| \, x_2 \, \| \cdot \| \, y_2 \, \|}$$
$$\leqq K(E_1) + K(E_2),$$

and hence the conclusion follows.

LEMMA 5.7. *There exists a constant* K *such that* $K(E) \leqq K$ *for all* $E \in \mathscr{H}$.

Proof. Suppose the conclusion is false. Then there exists a sequence $\{E_k\} \subset \mathscr{H}$ such that $K(E_1) > 1$ and $K(E_k) > (k+1)K(E_{k-1})$. Let $A_1 = E_1$ and let $A_n = E_n \sim \bigcup_{k=1}^{n-1} E_k$. Then $\{A_n\}$ is a pairwise disjoint sequence in $\mathscr{H}$. Now $\bigcup_{k=1}^{n-1} E_k \in H$ and $(\bigcup_{k=1}^{n-1} E_k) \cup A_n = E_n$. Thus, using the preceding lemma, it follows that $K(A_n) > 2K(E_{n-1}) \geqq 2K(A_{n-1})$, so $K(A_n) > 2^{n-1}$, which contradicts Lemma 5.4.

LEMMA 5.8. $\cup \mathscr{H}$ *is an element of* $\mathscr{H}$.

Proof. Let E_0 be the union of everything in $\mathscr{H}$. Suppose that $yx = x$ and that car $(x) \cup$ car $(y) \subset E_0$. Then by Lemma 4.10, car (x) is finite, so it is contained in the union of a finite number of elements of $\mathscr{H}$, say E_1. Now $E_1 \in \mathscr{H}$, and in view of Lemma 5.5 there is no loss of generality in assuming that car $(y) \subset E_1$. Thus if K is as in

Lemma 5.7, then $\| \nu(x) \| / \|x\| \cdot \| y \| \leq K(E_1) \leq K$. Thus $K(E_0) \leq K$ and $E_0 \in \mathscr{H}$.

Proof of Theorem 5.1. Let E_0 and K be as in previous lemmas and let Π_0 be the complement of E_0. By Lemma 5.4, Π_0 has only finitely many elements. Suppose that $yx = x$ and car $(x) \subset E_0$. Now choose y_1 such that $y_1 x = x$, car $(y_1) \subset E_0$ and $\| y_1 \| \leq \| y \|$. Then $\| \nu(x) \| \leq K \| x \| \cdot \| y_1 \| \leq K \| x \| \cdot \| y \|$.

COROLLARY 5.9. *If A is strongly semi-simple, then*

$$\| \nu(x) \| \leq K \| x \| \cdot \| y \|$$

for all x and y such that $yx = x$.

Proof. Let $P \in \Pi = \Xi$, and choose $M \in \mathfrak{M}$ such that $\mathfrak{L}(M) = P$. Then M is finite-dimensional, so $\nu | M$ is continuous. Therefore $\{P\} \in \mathscr{H}$. Thus $\Pi \in \mathscr{H}$.

EXAMPLE. If ν is the discontinuous isomorphism of l_2 constructed in [2], and if p_n is the sequence whose first n terms are one and the rest zero, then $p_n = p_n^2$ and $\| \nu(p_n) \| = n = \| p_n \|^2$.

The next lemma shows that is A if any semi-simple annihilator Banach algebra, and ν is a homomorphism of A into a Banach algebra, then we can apply Theorem 5.1 to ν by considering it as a homomorphism of $D(A, \mathfrak{M})$.

LEMMA 5.10. *Let A be a semi-simple annihilator Banach algebra. Let Π_D denote the primitive ideals and $\mathfrak{M}_D$ the minimal-closed two-sided ideals of $D(A, \mathfrak{M})$. Then*

(1) $$\mathfrak{M}_D = \mathfrak{M} ,$$

(2) $$\Pi_D = \{P \cap D(A, \mathfrak{M}) \mid P \in \Pi\} ,$$

and

(3) *$(D(A, \mathfrak{M}), |\cdot|_{\mathfrak{M}})$ is a semi-simple annihilator Banach algebra in which $I \oplus R(I) = D(A, \mathfrak{M})$ for all closed two-sided ideals I.*

Proof. The first assertion follows from the fact that $\|\cdot\| = |\cdot|$ on each $M_\alpha \in \mathfrak{M}$ and that cl $(\sum_\alpha M_\alpha) = A$. By Theorem 3.10, $(D, |\cdot|)$ is a semi-simple annihilator Banach algebra. Hence (2) follows from (1) and Lemma 4.1. Statement (3) follows from (1), Proposition 3.3, and Theorem 3.5.

THEOREM 5.11. *Let A be a semi-simple annihilator Banach algebra, and let ν be a homomorphism of A into a Banach algebra. Then* (1) *If $yx = x$, then $x \in D(A, \mathfrak{M})$ and there exists $y_1 \in D(A, \mathfrak{M})$ such that $y_1 x = x$.*

(2) *There exists a constant K and a finite set of primitive ideals Π_0 such that*

$$\| \nu(x) \| \leqq K |x|_{\mathfrak{M}} \cdot | y |_{\mathfrak{M}}$$

for all x and y in A such that $yx = x$, $\mathrm{car}\,(x) \subset \Pi \sim \Pi_0$ and $y \in D(A, \mathfrak{M})$.

Proof. The first assertion follows from Theorem 4.11 and its corollary. By the preceding lemma, we can apply Theorem 5.1 to ν considered as a homomorphism of D. Statement (2) now follows from this and the fact that primitive ideals in D are just the intersection with D of primitive ideals in A.

6. The separating ideal. If A and B are Banach algebras, and ν is a homomorphism of A onto a dense subset of B, than define *the separating ideal of ν in B,*

$$S = \{ \mathrm{y} \in B \,|\, \inf_{x \in A} \{\| x \| + \| \nu(x) - y \|\} = 0 \} \;.$$

The separating ideal was first introduced by Rickart [13]. For basic properties of S, see [18] or [6]. If A is a semi-simple annihilator Banach algebra, then we have the following.

PROPOSITION 6.1. Let $S_1 = \nu^{-1}(S)$. Then
(1) $S_1 \cdot A \subset \mathrm{cl}\,(\nu^{-1}(0))$.
(2) If A is dual, then $S_1 = \mathrm{cl}\,(\nu^{-1}(0))$.
(3) If ν is an isomorphism, then $S_1 = (0)$.

Proof. Let $K = \nu^{-1}(0)$. Let $M \in \mathfrak{M}$. If $M \cap S_1 = (0)$ then $S_1 M \subset M \cap S_1 \subset K$. Otherwise, $M \cap S_1 = M$, so $M \subset S_1$: If e is an idempotent in M then $e \in S_1$, so by a theorem of Yood [6, Th. 4.8] $\nu(e) = 0$. Thus the socle of M is contained in K, so $M \subset \bar{K}$. Thus $S_1 M \subset \bar{K}$ for all $M \in \mathfrak{M}$ so (1) follows.

If A is dual, then S_1 is a semi-simple dual Banach algebra. Now the socle of S_1 is contained in K and hence $S_1 \subset \bar{K}$. Clearly $\bar{K} \subset S_1$, so (2) holds.

If ν is an isomorphism, then by (1), $S_1 \subset \mathfrak{A}(A) = (0)$.

7. Homomorphisms of dual B*-algebras. In this section we specialize to the case when A is a dual B*-algebra. By using a theorem of Johnson [9] which says that any homomorphism of the

compact operators on Hilbert space into a Banach algebra is continuous, we can strengthen Theorem 5.1 in this case. Throughout this section, F denotes the socle of A.

THEOREM 7.1. *Let A be a dual B^*-algebra, and let ν be a homomorphism of A into a Banach algebra. Then ν is continuous on F.*

Proof. By Theorem 5.1 there exists a constant K and a finite set of primitive ideals Π_0 such that $\| \nu(x) \| \leqq K \| x \| \cdot \| y \|$ whenever $yx = x$ and car $(x) \subset \Pi \sim \Pi_0$.

Let $x \in F$. Then by Corollary 4.9 there exists an idempotent e such that $ex = x$. By Theorem 4.5 there exist orthogonal minimal idempotents $e_1, \cdots, e_n$ such that $e = e_1 + \cdots + e_n$. Now any minimal left ideal in A is a Hilbert space in the given norm and $x \to L(x)$ is a $*$-isomorphism. Thus an examination of the proof of Theorem 4.5 in this special case shows that the one-dimensional subspaces, range $(L(e_k))$, can be taken as mutually orthogonal, and hence the e_k can be taken as hermitian. Thus we may (and do) assume that e is a hermitian idempotent, and hence that $\| e \| = 1$.

Now, let $\Pi_0 = \{ P_1, \cdots, P_m \}$, and choose $M_k \in \mathfrak{M}$ such that $\mathfrak{L}(M_k) = P_k$. Then $x = x_0 + \cdots + x_m$, where $x_k \in M_k$, $1 \leqq k \leqq m$, and x_0 is in the sum of the remaining members of $\mathfrak{M}$. Thus $ex_0 = x_0$ and car $(x_0) \subset \Pi \sim \Pi_0$, so $\| \nu(x_0) \| \leqq K \| x_0 \|$. Each M_k is isometrically $*$-isomorphic to the compact operators on a Hilbert space, so by the theorem of Johnson, ν is continuous on each M_k. Since $\| x_k \| \leqq \| x \|$, $0 \leqq k \leqq m$, it follows that there exists a constant K' independent of x such that $\| \nu(x) \| \leqq K' \| x \|$.

REMARK. Since $\nu \mid F$ is continuous and $\bar{F} = A$, there exists a unique continuous homomorphism of A onto a (not necessarily dense) subset of cl $(\nu(A))$ which agrees with ν on F.

We now establish two decomposition theorems about the range of an isomorphism of a dual B^*-algebra. These theorems are analogous to a theorem of Bade and Curtis on homomorphisms of commutative B^*-algebras [2, Th. 4.3].

THEOREM 7.2. *Let A be a dual B^*-algebra, and let ν be an isomorphism of A onto a dense subset of a Banach algebra B. Let μ be the continuous homomorphism of A into B which agrees with ν on F, let $\lambda = \nu - \mu$, and let S be the separating ideal for ν in B. Then*

(1) μ is an isomorphism

(2) $S = $ cl $(\lambda(A))$

(3) $\mu(A)$ *is a closed two-sided ideal in B and* $B = \mu(A) \oplus S$

(4) $\mu(A)S = S\mu(A) = (0)$ *and* λ *is a homomorphism.*

Proof. (1) By [6, Lemma 5.3] there exists a constant L such that $\| x \| \leqq L \| \nu(x) \|$, $x \in A$. Suppose $\mu(x) = 0$. Then there exists a sequence $\{x_n\} \subset F$ such that $x_n \to x$. Hence $\nu(x_n) = \mu(x_n) \to \mu(x) = 0$, so $x_n \to 0$. Therefore μ is an isomorphism.

(2) Let $x \in A$ and let $\{x_n\} \subset F$, $x_n \to x$. Then $\nu(x_n) = \mu(x_n) \to \mu(x)$. Thus $x - x_n \to 0$, $\nu(x - x_n) \to \lambda(x)$, so $\lambda(A) \subset S$. If $y \in S$ and $x_n \to 0$, $\nu(x_n) \to y$, then $\lambda(x_n) = (\nu - \mu)(x_n) \to y - 0 \in S$, so $S \subset \mathrm{cl}\,(\lambda(A))$.

(3) By [6, p. 1104], $\mu(A)$ is closed and $B = \nu(A) \oplus S$. If $x \in A$, $y \in F$, then $\nu(x)\mu(y) = \nu(xy) = \mu(xy) \in \mu(A)$. Therefore $\nu(A)\mu(A) \subset \mathrm{cl}(\mu(A)) = \mu(A)$, and similarly $\mu(A)\nu(A) \subset \mu(A)$, so $\mu(A)$ is a two-sided ideal in B. If $\mu(x) \in S$, then $\nu(x) = (\lambda + \mu)(x) \in S$, so $\nu(x) = 0 = x$. Therefore $\mu(A) \cap S = (0)$. Thus $B = \nu(A) \oplus S = (\lambda + \mu)(A) \oplus S \subset \mu(A) \oplus S$, so $B = \mu(A) \oplus S$.

(4) The first statement follows directly from (3). If $x, y \in A$ then $\lambda(x)\mu(y) = \mu(x)\lambda(y) = 0$, so

$$\lambda(x)\lambda(y) = ((\mu + \lambda)(x)((\mu + \lambda)(y)) - \mu(xy)$$
$$= (\nu - \mu)(xy) = \lambda(xy)\,.$$

Therefore λ is a homomorphism.

Before proving the second decomposition theorem, we digress to establish the following theorem about homomorphisms of a large class of B*-algebras. This was proved in the commutative case by Yood [17, Corollary 5.5].

THEOREM 7.3. *Let A be a B*-algebra in which every primitive ideal is modular. Let ν be a homomorphism of A into a Banach algebra B such that $\mathrm{cl}\,(\nu(A)) = B$ and $\nu^{-1}(0)$ is closed. Let R denote the strong radical of B.*
Then $R = S$ and $B = \nu(A) \oplus R$.

Proof. We note that the hypotheses for A will be satisfied if A has identity, if A is commutative, or if A is a strongly semi-simple dual algebra.

First, suppose that ν is one-to-one. By a theorem of Cleveland [6, Th. 5.5] $B = \nu(A) \oplus S$, and by a theorem of Yood [18, Th. 3.5] $R \supset S$. Hence to complete the proof for this case we only need show that $\nu(A) \cap R = (0)$.

Now there exists a constant L such that $\| x \| \leqq L \| \nu(x) \|$, $x \in A$. Let $P \in \Xi$ and let e be a two-sided identity for A modulo P. If $\mathrm{cl}\,(\nu(P)) = B$ then there exists a sequence $\{x_n\} \subset P$ such that $\nu(x_n) \to \nu(e)$.

Thus $x_n \rightarrow e$, so $e \in P$, which is a contradiction. Thus cl $(\nu(P))$ is a proper modular two-sided ideal and hence is contained in a maximal such ideal, J_P. It follows that $\nu^{-1}(J_P) = P$ and hence that

$$(0) = \cap \{\nu^{-1}(J_P) \mid P \in \Xi\} \supset \nu^{-1}(R) \ .$$

Hence the conclusion follows if ν is one-to-one.

If $\nu^{-1}(0)$ is closed, then $A_1 = A/\nu^{-1}(0)$ is a B^*-algebra. Let π be the natural map from A onto A_1. By considering inverse images under π, we see that every primitive ideal of A_1 is modular. Also, it is straightforward to show that the mapping ν_1 defined by $\nu_1 \circ \pi = \nu$ is an isomorphism of A_1 onto $\nu(A)$. By the first case,

$$B = \nu_1(A_1) \oplus R = \nu(A) \oplus R \ .$$

Let S_1 be the separating ideal for ν_1 in B. Since π is continuous, $S \subset S_1$. On the other hand, suppose there exists a sequence $\{x_n\} \subset A$ such that $\pi(x_n) \rightarrow 0$ and $\nu_1(\pi(x_n)) \rightarrow y \in B$. Then there exists a sequence $\{w_n\} \subset \nu^{-1}(0)$ such that $\| x_n + w_n \| \rightarrow 0$. But $\nu(x_n + w_n) = \nu(x_n) \rightarrow y$, so $y \in S$. Thus $S = S_1 = R$, and the theorem is established.

As a direct consequence of the preceding two theorems, we have the following.

THEOREM 7.4. *Let A be a strongly semi-simple dual B^*-algebra, and let ν be an isomorphism of A onto a dense subset of a Banach algebra B. Let μ be the continuous isomorphism of A into B which agrees with ν on F, let $\lambda = \nu - \mu$, and let R denote the strong radical of B. Then*
  (1) cl $(\lambda(A)) = R$.
  (2) $\mu(A)$ *is a closed two-sided ideal in B and $B = \mu(A) \oplus R$.*
  (3) $R\mu(A) = \mu(A)R = (0)$, *and λ is a homomorphism.*

REFERENCES

1. W. Ambrose, *Structure theorems for a special class of Banach algebras*, Trans. Amer. Math. Soc. **57** (1945), 364-386.
2. W. G. Bade and P. C. Curtis, Jr., *Homomorphisms of commutative Banach algebras*, Amer. J. Math. **82** (1960), 589-608.
3. B. A. Barnes, *Modular annihilator algebras*, Canad. J. Math. **18** (1966), 566-578.
4. F. F. Bonsall and A. W. Goldie, *Annihilator algebras*, Proc. London Math. Soc. (3) **4** (1954), 154-167.
5. P. Civin and B. Yood, *Lie and Jordan structures in Banach algebras*, Pacific J. Math. **15** (1965), 775-797.
6. S. B. Cleveland, *Homomorphisms of non-commutative *-algebras*, Pacific J. Math. **13** (1963), 1097-1109.
7. M. M. Day, *Normed Linear Spaces*, Academic Press Inc., New York, 1962.

8. N. Jacobson, *Structure of Rings*, Amer. Math. Soc. Colloq. Publ. no. 37, Providence, 1956.

9. B. E. Johnson, *Continuity of homomorphisms of algebras of operators*, J. London Math. Soc. **42** (1967), 537–541.

10. I. Kaplansky, *Dual rings*, Ann. of Math. (2) **49** (1948), 689–701.

11. ———, *Normed algebras*, Duke Math. J. **16** (1949), 399–418.

12. M. A. Naimark, *Normed Rings*, P. Noordhoff, Groningen, 1959.

13. C. E. Rickart, *The uniqueness of norm problem in Banach algebras*, Ann. of Math. (2) **51** (1950), 615–628.

14. ———, *General Theory of Banach Algebras*, Van Nostrand, New York, 1960.

15. W. H. Ruckle, *The infinite sum of closed subspaces of an F-space*, Duke Math. J. **31** (1964), 543–554.

16. W. Rudin, *Fourier Analysis on Groups*, Interscience Publishers, New York, 1962.

17. B. Yood, *Topological properties of homomorphisms between Banach algebras*, Amer. J. Math. **76** (1954), 155–167.

18. ———, *Homomorphisms on normed algebras*, Pacific J. Math. **8** (1958), 373–381.

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