PACIFIC JOURNAL OF MATHEMATICS
Vol. 28, No. 1, 1969

MEROMORPHIC MINIMAL SURFACES

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Meromorphic minimal surfaces are defined in this paper, and some of their differential-geometric properties are noted. The first fundamental theorem of Nevanlinna for meromorphic functions of a complex variable is extended so as to apply to these surfaces, as is the Ahlfors-Shimizu spherical version of this theorem. For these results, the classical proximity and enumerative functions of complex-variable theory are generalized, and a new visibility function is introduced. Convexity properties of some of these functions are established.

For plane meromorphic maps, the visibility function vanishes at all points on the plane but is positive at all other points of space. In general, in the present development, the sum of the enumerative function and the visibility function corresponds to the enumerative function in the classical theory.

Let a surface S be given by

$$S: x_j = x_j(u, v)\,, \qquad j = 1, 2, 3\,. \tag{1}$$

Then S is said to be given in terms of *isothermal parameters* (u, v) if and only if the representation (1) is such that

$$E = G = \lambda(u, v)\,, \qquad F = 0\,, \tag{2}$$

where

$$E = \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial u}\right)^2, \qquad F = \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial u}\right)\left(\frac{\partial x_j}{\partial v}\right), \qquad G = \sum_{j=1}^{3}\left(\frac{\partial x_j}{\partial v}\right)^2. \tag{3}$$

Such an *isothermal representation* is conformal, or angle-preserving, except at points where $\lambda(u, v) = 0$.

According to a theorem of Weierstrass [13, p. 27], a necessary and sufficient condition that a surface S, given in terms of isothermal parameters, be minimal is that the coordinate functions be harmonic, that is, that for all $(u, v) \in D$ the functions $x_j(u, v)$, $j = 1, 2, 3$, satisfy the equation

$$\Delta x_j(u, v) = 0\,, \tag{4}$$

where Δ denotes the Laplace operator,

$$\Delta = \frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\,. \tag{5}$$

Then in any simply connected part of D, the functions given by (1) are the real parts of analytic functions of a complex variable,

$$x_j = \mathscr{R} f_j(w) , \qquad w = u + iv , \tag{6}$$

and (2) is equivalent to

$$\sum_{j=1}^{3} \left[\frac{df_j(w)}{dw} \right]^2 = 0 . \tag{7}$$

If, in an isothermal representation (1) of a minimal surface S, one of the coordinate functions is identically zero, say $x_3(u, v) \equiv 0$, then the map lies on a plane, and either

$$x_1(u, v) + ix_2(u, v) \quad \text{or} \quad x_2(u, v) + ix_1(u, v)$$

is an analytic function of the complex variable $w = u + iv$. Then $x_1(u, v)$ and $x_2(u, v)$ are said to form a *couple of conjugate harmonic functions*. By analogy, the coordinate functions (1) of any minimal surface S in isothermal representation are called a *triple of conjugate harmonic functions* [7]. The generalization to μ-tuples of conjugate harmonic functions $x_j(u, v)$, $j = 1, 2, \cdots, \mu$, as isothermal coordinate functions of a minimal surface S in μ-dimensional Euclidean space, is rather direct and will not be pursued further in this paper.[1]

The analogy here indicated between analytic functions of a complex variable and isothermal representations of minimal surfaces has often been noted, and since the time of Weierstrass it has served as a guiding principle in the study of minimal surfaces. It is the purpose of the present paper, as announced earlier [6], to pursue this analogy in the direction of the classical Nevanlinna theory [10] of meromorphic functions of a complex variable. Applications [4] to rational minimal surfaces and a generalization [2, 3] of the second fundamental theorem of Nevanlinna to meromorphic minimal surfaces will appear elsewhere.

2. Meromorphic minimal surfaces. Let the real-valued function $x(u, v)$ be harmonic for (u, v) in a deleted circular neighborhood $\mathscr{U}_\varepsilon^*(P_0)$ of a point P_0: (u_0, v_0), that is, for (u, v) satisfying

$$0 < (u - u_0)^2 + (v - v_0)^2 < \varepsilon^2 .$$

Then $x(u, v)$ can be represented [12] in $\mathscr{U}_\varepsilon^*(P_0)$ by a series of the form

$$x(u, v) = c \log r + \sum_{k=-\infty}^{\infty} r^k (a_k \cos k\theta + b_k \sin k\theta) , \tag{8}$$

where (r, θ) are polar coordinates with pole P_0:

[1] By request, the results of this paper will be summarized elsewhere for minimal surfaces in μ-dimensional Euclidean space, $\mu \geqq 2$.

$$u - u_0 = r\cos\theta\ , \qquad v - v_0 = r\sin\theta\ .$$

The constant b_0 is arbitrary; throughout this paper, we shall assume that it has been assigned the value 0,

$$b_0 = 0\ . \tag{9}$$

Otherwise, the constants c, $a_k(k = 0, \pm 1, \pm 2, \cdots)$, and b_k $(k = \pm 1, \pm 2, \cdots)$ are uniquely determined by the function $x(u, v)$.

We then have, for $\omega = w - w_0 = (u + iv) - (u_0 + iv_0)$,

$$x(u, v) = \mathscr{R}[c\log\omega + f(\omega)]\ , \qquad \omega = r(\cos\theta + i\sin\theta)\ ,$$

where

$$f(\omega) = \sum_{k=-\infty}^{\infty}(a_k - ib_k)\omega^k$$

is an analytic function of ω in $\mathscr{U}_\varepsilon^*(P_0)$.

By (7), three such functions,

$$\begin{aligned} x_j(u, v) &= c_j\log r \\ &\quad + \sum_{k=-\infty}^{\infty} r_k(a_{j,k}\cos k\theta + b_{j,k}\sin k\theta) \\ &= \mathscr{R}[c_j\log\omega + f_j(\omega)]\ , \qquad j = 1, 2, 3, \end{aligned} \tag{10}$$

harmonic in $\mathscr{U}_\varepsilon^*(P_0)$, are a triple of conjugate harmonic functions there if and only if

$$\sum_{j=1}^{3}\left[c_j\omega^{-1} + \frac{df_j(\omega)}{d\omega}\right]^2 = 0\ . \tag{11}$$

Now

$$f_j(\omega) = \sum_{k=-\infty}^{\infty}(a_{j,k} - ib_{j,k})\omega^k\ ,$$

so that

$$c_j\omega^{-1} + \frac{df_j(\omega)}{d\omega} = \sum_{k=-\infty}^{\infty}(\alpha_{j,k} - i\beta_{j,k})\omega^{k-} \tag{12}$$

where for $j = 1, 2, 3$ we have

$$\alpha_{j,0} = c_j\ , \qquad \beta_{j,0} = 0\ , \tag{13}$$

and

$$\alpha_{j,k} = ka_{j,k}\ , \qquad \beta_{j,k} = kb_{j,k}\ , \qquad k = \pm 1, \pm 2, \cdots\ . \tag{14}$$

By (11) and (12), then, the functions (10) are a triple of conjugate harmonic functions in $\mathscr{U}_\varepsilon^*(P_0)$ if and only if

$$\sum_{j=1}^{3} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} (\alpha_{j,l} - i\beta_{j,l})(\alpha_{j,k-l} - i\beta_{j,k-l})\omega^{k-2} \equiv 0 .$$

Accordingly, the functions (10) are a triple of conjugate harmonic functions in $\mathscr{U}_\varepsilon^*(P_0)$ if and only if

$$\sum_{l=-\infty}^{\infty} \sum_{j=1}^{3} (\alpha_{j,l} - i\beta_{j,l})(\alpha_{j,k-l} - i\beta_{j,k-l}) = 0$$

for all k, $k = 0, \pm 1, \pm 2, \cdots$, that is, if and only if

$$\sum_{l=-\infty}^{\infty} \sum_{j=1}^{3} (\alpha_{j,l}\alpha_{j,k-l} - \beta_{j,l}\beta_{j,k-l}) = 0 \tag{15}$$

and

$$\sum_{l=-\infty}^{\infty} \sum_{j=1}^{3} (\alpha_{j,l}\beta_{j,k-l} + \alpha_{j,k-l}\beta_{j,l}) = 0 \tag{16}$$

for all k, $k = 0, \pm 1, \pm 2, \cdots$.

Condition (16) is equivalent to

$$\sum_{l=-\infty}^{\infty} \sum_{j=1}^{3} \alpha_{j,l}\beta_{j,k-l} = 0 , \tag{17}$$

so that the functions (10) are a triple of conjugate harmonic functions in $\mathscr{U}_\varepsilon^*(P_0)$ if and only if (15) and (17) hold for all k, $k = 0, \pm 1, \pm 2, \cdots$.

In terms of the original coefficients c_j, $a_{j,k}$, and $b_{j,k}$, by (13) and (14) the relation (15) can be written (cf. [5]) as

$$2k \sum_{j=1}^{3} a_{j,k}c_j + \sum_{l=-\infty}^{\infty} l(k-l) \sum_{j=1}^{3} (a_{j,l}a_{j,k-l} - b_{j,l}b_{j,k-l}) = 0 \tag{18}$$

for $k = \pm 1, \pm 2, \cdots$, and as

$$\sum_{j=1}^{3} c_j^2 - \sum_{l=-\infty}^{\infty} l^2 \sum_{j=1}^{3} (a_{j,l}a_{j,-l} - b_{j,l}b_{j,-l}) = 0 \tag{19}$$

for $k = 0$, and (17) can be written as

$$k \sum_{j=1}^{3} b_{j,k}c_j + \sum_{l=-\infty}^{\infty} l(k-l) \sum_{j=1}^{3} a_{j,l}b_{j,k-l} = 0 \tag{20}$$

for $k = 0, \pm 1, \pm 2, \cdots$.

Thus, (18), (19), *and* (20) *are necessary and sufficient conditions for the functions* (10) *to be a triple of conjugate harmonic functions.*

If for some $\varepsilon > 0$, the functions (10) are a triple of conjugate harmonic functions in $\mathscr{U}_\varepsilon^*(P_0)$, that is, if the functions (10) are the coordinate functions of a minimal surface S in isothermal representation for $(u, v) \in \mathscr{U}_\varepsilon^*(P_0)$, then [unless P_0 turns out to be a regular point of S (see p. 21)], we shall say that P_0 is an *isolated singular point* of S.

If S has an isolated singular point at P_0, and for an infinitude of negative indices l we have

$$\sum_{j=1}^{3} (a_{j,l}^2 + b_{j,l}^2) \neq 0 , \tag{21}$$

then we say that the singularity of S at P_0 is *essential*; otherwise, we say that it is *nonessential*.

If S has a nonessential isolated singularity at P_0, and the lowest index $l = t$ for which (21) holds is negative, then we say that S has a *pole* of order $|t|$ at P_0. By definition, then, the poles of S are isolated.

We note by (19) that if (21) does not hold for any negative value of l, that is, if for $j = 1, 2, 3$ we have

$$a_{j,l} = b_{j,l} = 0 , \qquad l = -1, -2, \cdots,$$

then

$$\sum_{j=1}^{3} c_j^2 = 0 , \tag{22}$$

or $c_1 = c_2 = c_3 = 0$. Hence, a minimal surface given in isothermal representation by functions $x_j(u, v)$ *cannot have an isolated singularity that is merely logarithmic.*

If (21) does not hold for any $l < 0$, then we say that S has a *removable singularity* at P_0. In this case, we adjoin to S the point

$$\boldsymbol{a}_0 = (a_{1,0}, a_{2,0}, a_{3,0}) \tag{23}$$

corresponding to P_0, if indeed this correspondence was not already given in the definition of S. Then the functions (10) determine an isothermal map of the neighborhood $\mathscr{U}_\varepsilon(P_0)$, that is, of the set of values (u, v) satisfying

$$(u - u_0)^2 + (v - v_0)^2 < \varepsilon^2 ,$$

onto the (extended) surface, which we again denote by S. We then say that S is *regular* at P_0.

If S is regular at P_0, then either each $x_j(u, v)$ satisfies

$$x_j(u, v) \equiv a_{j,0} , \qquad j = 1, 2, 3 ,$$

and S reduces to a point, or there is a lowest positive index $l = t$ for which (21) holds. In the former case, we say that S is a *constant minimal surface*. In the latter case, we say that S has an *$\boldsymbol{a}_0$-point of order t* at P_0; in particular, if $\boldsymbol{a}_0 = \boldsymbol{0} = (0, 0, 0)$ then we say that S has a *zero* of order t at P_0.

If S has a pole or order $-t > 0$ or an $\boldsymbol{a}_0$-point of order $t > 0$ at

P_0, then for $k = 2t$, (18) and (20) reduce respectively to

$$t^2 \sum_{j=1}^{3} (a_{j,t}^2 - b_{j,t}^2) = 0 \quad \text{and} \quad t^2 \sum_{j=1}^{3} a_{j,t} b_{j,t} = 0 \ ,$$

so that, since

$$\sum_{j=1}^{3} (a_{j,t}^2 + b_{j,t}^2) \neq 0 \ ,$$

$$\sum_{j=1}^{3} a_{j,t}^2 = \sum_{j=1}^{3} b_{j,t}^2 \neq 0 \ , \qquad \sum_{j=1}^{3} a_{j,t} b_{j,t} = 0 \ . \tag{24}$$

In what follows, we shall frequently use the notation

$$o(\psi(r)) \quad \text{or} \quad O(\psi(r))$$

to indicate a function (not always the same function) $\varphi(r, \theta)$ such that, uniformly with respect to θ, we have

$$\lim_{r \to 0} \frac{\varphi(r, \theta)}{\psi(r)} = 0 \quad \text{or} \quad \overline{\lim_{r \to 0}} \left| \frac{\varphi(r, \theta)}{\psi(r)} \right| < +\infty \ ,$$

respectively.

If S has a pole of order $-t > 0$ at P_0, then from (10) and (24) we obtain

$$\begin{aligned} \sum_{j=1}^{3} [x_j(u, v)]^2 &= r^{2t} \Big(\sum_{j=1}^{3} a_{j,t}^2 \cos^2 t\theta \\ &\quad + 2 \sum_{j=1}^{3} a_{j,t} b_{j,t} \cos t\theta \sin t\theta + \sum_{j=1}^{3} b_{j,t}^2 \sin^2 t\theta \Big) + o(r^{2t}) \\ &= r^{2t} \sum_{j=1}^{3} a_{j,t}^2 + o(r^{2t}) \ . \end{aligned} \tag{25}$$

Similarly, if S has an $\mathbf{a}_0$-point of order $t > 0$ at P_0, then

$$\sum_{j=1}^{3} [x_j(u, v) - a_{j,0}]^2 = r^{2t} \sum_{j=1}^{3} a_{j,t}^2 + o(r^{2t}) \ . \tag{26}$$

By (24) and (26) we thus see that if S does not reduce to a point, then *not only the poles but also the finite* $\mathbf{a}$*-points of* S *are isolated* [5].

In analogy with complex-variable theory, for the present development we extend Euclidean 3-space by postulating a single ideal point at ∞. In this space, the transformation

$$x_j^* = \frac{x_j}{\sum_{q=1}^{3} x_q^2} \ , \qquad j = 1, 2, 3 \ ,$$

effects an inversion in the unit sphere with center at the origin, and the transformation is isothermal (see § 6, below). If S has a pole of order $-t > 0$ at P_0, then the surface

$$S^* : x_j = x_j^*(u, v) = \frac{x_j(u, v)}{\sum_{q=1}^3 [x_q(u, v)]^2} , \qquad j = 1, 2, 3 ,$$

has a zero of order $|t|$ at P_0. The surface S^* will not ordinarily be a minimal surface; for example, if S is a plane not passing through the origin, then S^* will be a sphere. Since the transformation is isothermal, however, we say that angles between curves on S^* at the origin correspond to angles *of the same measure* on S at ∞.

Suppose that S has a pole of order $-t > 0$ at P_0: (u_0, v_0), let P_1: (u_1, v_1) and $P_2(u_2, v_2)$ be nearby points at which S does not have a zero, with

$$(u_j - u_0, v_j - v_0) = (r_j \cos \theta_j, r_j \sin \theta_j) , \qquad j = 1, 2 ,$$

and consider the vectors from P_0 to P_1 and from P_0 to P_2. An angle from the first of these to the second has measure $\theta_2 - \theta_1$. The corresponding space vectors joining points on S^* meet at an angle Θ, $0 \leqq \Theta \leqq \pi$, which, by (25), satisfies

$$\begin{aligned}
\cos \Theta &= \frac{\sum_{j=1}^{3} [x_j^*(u_1, v_1)][x_j^*(u_2, v_2)]}{\left\{\sum_{j=1}^{3} [x_j^*(u_1, v_1)]^2\right\}^{\frac{1}{2}} \left\{\sum_{j=1}^{3} [x_j^*(u_2, v_2)]^2\right\}^{\frac{1}{2}}} \\
&= \frac{\sum_{j=1}^{3} [x_j(u_1, v_1)][x_j(u_2, v_2)]}{\left\{\sum_{j=1}^{3} [x_j(u_1, v_1)]^2\right\}^{\frac{1}{2}} \left\{\sum_{j=1}^{3} [x_j(u_2, v_2)]^2\right\}^{\frac{1}{2}}} \\
&= \frac{\sum_{j=1}^{3} r_1^t(a_{j,t} \cos t\theta_1 + b_{j,t} \sin t\theta_1) r_2^t(a_{j,t} \cos t\theta_2 + b_{j,t} \sin t\theta_2) + o(r_1^t r_2^t)}{\left[r_1^{2t} \sum_{j=1}^{3} a_{j,t}^2 + o(r_1^{2t})\right]^{\frac{1}{2}} \left[r_2^{2t} \sum_{j=1}^{3} a_{j,t}^2 + o(r_2^{2t})\right]^{\frac{1}{2}}} .
\end{aligned}$$

By (24), this reduces to

$$\begin{aligned}
\cos \Theta &= \frac{\left(\sum_{j=1}^{3} a_{j,t}^2\right)(\cos t\theta_1 \cos t\theta_2 + \sin t\theta_1 \sin t\theta_2) + o(1)}{\sum_{j=1}^{3} a_{j,t}^2 + o(1)} \\
&= \cos t(\theta_2 - \theta_1) + o(1) ,
\end{aligned}$$

so that if $\theta_2 - \theta_1$ has a limit θ_0 as $r_1 \to 0$ and $r_2 \to 0$,

$$\lim_{\substack{r_1 \to 0 \\ r_2 \to 0}} (\theta_2 - \theta_1) = \theta_0 ,$$

then

$$\lim_{\substack{r_1 \to 0 \\ r_2 \to 0}} \cos \Theta = \cos t\theta_0 .$$

Hence the magnitudes of angles at P_0 are multiplied by $|t|$ in the map on S^* at the origin. Therefore, by the convention given above, they are multiplied by $|t|$ in the map on S at ∞.

Similarly, by (24) and (26), if S has an $\boldsymbol{a}$-point of order $t > 0$ at P_0, then the magnitudes of angles at P_0 are multiplied by t in the map on S at $\boldsymbol{a}$.

If, except for poles, S is a regular minimal surface given in isothermal representation by (1) for (u, v) in a finite domain D, then we say that S is a *meromorphic minimal surface for* (u, v) *in* D. In particular, if D is the entire finite plane, then we say simply that S is a *meromorphic minimal surface*. If D is the entire finite plane and S has no poles in D, then we say that S is an *entire minimal surface*.

For example, the functions

$$
\begin{aligned}
x_1 &= \mathscr{R}\left(\frac{1}{w} + w\right) = \left(\frac{1}{r} + r\right)\cos\theta\ , \\
x_2 &= \mathscr{R}i\left(\frac{1}{w} - w\right) = \left(\frac{1}{r} + r\right)\sin\theta\ , \\
x_3 &= \mathscr{R}(2\log w) = 2\log r\ , \qquad w = r(\cos\theta + i\sin\theta)\ ,
\end{aligned}
\tag{27}
$$

are the coordinate functions of a meromorphic minimal surface (actually a catenoid) in isothermal representation. Its single pole in the finite plane is at the origin and is of order 1.

The minimal surface of Enneper [11, p. 221] is given in isothermal representation by

$$
\begin{aligned}
x_1 &= \mathscr{R}(3w - w^3) = 3r\cos\theta - r^3\cos 3\theta\ , \\
x_2 &= \mathscr{R}i(3w + w^3) = -3r\sin\theta - r^3\sin 3\theta\ , \\
x_3 &= \mathscr{R}(3w^2) = 3r^2\cos 2\theta\ , \qquad w = r(\cos\theta + i\sin\theta)\ .
\end{aligned}
\tag{28}
$$

This is an entire minimal surface. Its single zero is at the origin and is of order 1.

The relations

$$
\begin{aligned}
x_1 &= \mathscr{R}(\log w - \tfrac{1}{2}w^2) = \log r - \tfrac{1}{2}r^2\cos 2\theta\ , \\
x_2 &= \mathscr{R}i(\log w + \tfrac{1}{2}w^2) = -i\theta - \tfrac{1}{2}r^2\sin 2\theta\ , \\
x_3 &= \mathscr{R}(2w) = 2r\cos\theta\ , \qquad w = r(\cos\theta + i\sin\theta)\ ,
\end{aligned}
\tag{29}
$$

give an isothermal representation of a minimal surface with a singularity of a different sort at the origin. The second of the relations (29) is not a (single-valued) function of $w = u + iv$, however, so that this surface is not included in the class of surfaces presently under consideration.

The subclass of meromorphic minimal surfaces (1) for which the

c_j are restricted to have the value 0 in the representation (10) for each P_0 is somewhat more tractable than the unrestricted class. The restriction is not needed for the validity of the results of this paper, however, and accordingly we shall not make it here.

3. Formulas. In this section, we shall develop some formulas that will be needed later. These formulas are concerned with the differential geometry of meromorphic minimal surfaces.

Let the functions (10) be the coordinate functions of a nonconstant meromorphic minimal surface S in isothermal representation in $\mathscr{U}_\varepsilon^*(P_0)$, and let τ denote the lowest index l for which we have

$$\sum_{j=1}^{3} a_{j,l}^2 \neq 0 \, . \tag{30}$$

By (9) and (24), τ is then also the lowest index for which (21) holds. Equation (10) can accordingly be written as

$$\begin{aligned} x_j(u, v) &= c_j \log r + \sum_{k=\tau}^{\infty} r^k (a_{j,k} \cos k\theta + b_{j,k} \sin k\theta) \\ &= \mathscr{R}[c_j \log \omega + f_j(\omega)] \, , \qquad j = 1, 2, 3 \, , \end{aligned} \tag{31}$$

where

$$\sum_{j=1}^{3} a_{j,\tau}^2 \neq 0 \tag{32}$$

and

$$f_j(\omega) = \sum_{k=\tau}^{\infty} (a_{j,k} - i b_{j,k}) \omega^k \, . \tag{33}$$

If $\tau = 0$, let t denote the lowest *positive* index l for which (21) holds; if $\tau \neq 0$, let $t = \tau$. Then $t \geqq \tau$, with inequality if and only if $\tau = 0$. Recalling that (22) holds if $\tau \geqq 0$, we see that:

> If $\tau < 0$, then $t = \tau$ and S has a pole of order $-t$ at P_0.
> If $\tau = 0$, then $t > \tau$ and S has an $\boldsymbol{a}_0$-point ($\boldsymbol{a}_0 \neq \boldsymbol{0}$) of order t at P_0.
> If $\tau > 0$, then $t = \tau$ and S has a zero of order t at P_0.

If $\tau < 0$, then $t = \tau$, so that (25) can be written as

$$\sum_{j=1}^{3} [x_j(u, v)]^2 = r^{2\tau} \sum_{j=1}^{3} a_{j,\tau}^2 + o(r^{2\tau}) \, . \tag{34}$$

If $\tau = 0$ then, because of (22), (34) follows from (31) by direct computation. If $\tau > 0$, then again $t = \tau$; since now $\boldsymbol{a}_0 = \boldsymbol{0}$, (26) can be written as (34) in this case. Hence (34) holds in all cases.

From (31) and the definition of t, by differentiating we obtain

$$\frac{\partial x_j}{\partial u} - i\frac{\partial x_j}{\partial v} = \frac{c_j}{\omega} + \sum_{k=t}^{\infty} k(a_{j,k} - ib_{j,k})\omega^{k-1} . \tag{35}$$

Equating real parts, and equating imaginary parts, in (35), we therefore have

$$\begin{aligned} \frac{\partial x_j}{\partial u} &= \frac{c_j}{r}\cos\theta \\ &+ \sum_{k=t}^{\infty} kr^{k-1}[a_{j,k}\cos(k-1)\theta + b_{j,k}\sin(k-1)\theta], \end{aligned} \tag{36}$$

$$\begin{aligned} \frac{\partial x_j}{\partial v} &= \frac{c_j}{r}\sin\theta \\ &+ \sum_{k=t}^{\infty} kr^{k-1}[b_{j,k}\cos(k-1)\theta - a_{j,k}\sin(k-1)\theta] . \end{aligned} \tag{37}$$

From (2), (3), (24), and (36), we obtain

$$\lambda(u, v) = t^2 r^{2t-2}\sum_{j=1}^{3} a_{j,t}^2 + O(r^{2t-1}) . \tag{38}$$

It follows from (24) and (38) (cf. [5]) that *for a nonconstant meromorphic minimal surface S given in terms of isothermal parameters (u, v), the zeros and infinities of the area-deformation ratio $\lambda(u, v)$ are isolated.*

At points where $\lambda(u, v) \neq 0$ and $\lambda(u, v) \neq \infty$, S has a tangent plane. The direction cosines $X_j(u, v)$ of its normal are given [11, p. 147] by

$$x_j(u, v) = \frac{\dfrac{\partial x_k(u, v)}{\partial u}\dfrac{\partial x_l(u, v)}{\partial v} - \dfrac{\partial x_l(u, v)}{\partial u}\dfrac{\partial x_k(u, v)}{\partial v}}{\lambda(u, v)} , \tag{39}$$

where $j, k, l = 1, 2, 3$ in cyclic order. For the functions (31), let $\varepsilon > 0$ be so small that $\lambda(u, v) \neq 0$ and $\lambda(u, v) \neq \infty$ in $\mathscr{U}_\varepsilon^*(P_0)$. Then for $(u, v) \in \mathscr{U}_\varepsilon^*(P_0)$, from (36), (37), (38), and (39) we obtain, by a computation,

$$\begin{aligned} X_j(u, v) &= \frac{t^2 r^{2t-2}(a_{k,t}b_{l,t} - a_{l,t}b_{k,t}) + O(r^{2t-1})}{t^2 r^{2t-2}\sum_{q=1}^{3} a_{q,t}^2 + O(r^{2t-1})} \\ &= \frac{a_{k,t}b_{l,t} - a_{l,t}b_{k,t}}{\sum_{q=1}^{3} a_{q,t}^2} + O(r) . \end{aligned} \tag{40}$$

By (40), we see that

$$\lim_{r \to 0} X_j(u, v) = \frac{a_{k,t} b_{l,t} - a_{l,t} b_{k,t}}{\sum_{q=1}^{3} a_{q,t}^2}, \tag{41}$$

$j, k, l = 1, 2, 3$ in cyclic order, even if $\lambda(u_0, v_0) = 0$ or $\lambda(u_0, v_0) = \infty$. We take this limiting value (41) as the definition of $X_j(u_0, v_0)$ if $\lambda(u_0, v_0) = 0$ or $\lambda(u_0, v_0) = \infty$. With this extended definition of the functions $X_j(u, v)$, we see by (24) that *a nonconstant meromorphic minimal surface S given in terms of isothermic parameters (u, v) has a continuous unit normal vector function*

$$X(u, v) = (X_1(u, v), X_2(u, v), X_3(u, v))$$

throughout the domain in which S is meromorphic.

The next formula we shall develop is fundamental for the present investigation. It is an expression [7] for the Laplacian of the logarithm of the distance function

$$\left\{\sum_{j=1}^{3} x_j(u, v)]^2\right\}^{\frac{1}{2}} \tag{42}$$

for a nonconstant meromorphic minimal surface given in isothermal representation.

For the isothermal coordinate functions (31) of a nonconstant meromorphic minimal surface S, let $\varepsilon > 0$ be so small that in $\mathscr{U}_\varepsilon^*(P_0)$ the distance function (42) and the area-deformation ratio $\lambda(u, v)$ have no zeros or infinities. They might or might not vanish or be infinite at P_0.

Using vector notation, for $(u, v) \in \mathscr{U}_\varepsilon^*(P_0)$ we obtain, by a computation,

$$\begin{aligned} &\Delta \log (\boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}} \\ &= \frac{(\boldsymbol{x}\cdot\boldsymbol{x})(\boldsymbol{x}\cdot\Delta\boldsymbol{x} + \boldsymbol{x}_u\cdot\boldsymbol{x}_u + \boldsymbol{x}_v\cdot\boldsymbol{x}_v) - 2[(\boldsymbol{x}\cdot\boldsymbol{x}_u)^2 + (\boldsymbol{x}\cdot\boldsymbol{x}_v)^2]}{(\boldsymbol{x}\cdot\boldsymbol{x})^2}, \end{aligned} \tag{43}$$

where the subscripts indicate partial differentiation. By (2) and (3) we have

$$\boldsymbol{x}_u\cdot\boldsymbol{x}_u = \boldsymbol{x}_v\cdot\boldsymbol{x}_v = \lambda(u, v),$$

and from (4) we obtain

$$\Delta\boldsymbol{x} = 0.$$

Hence (43) reduces to

$$\Delta \log (\boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}} = \frac{2[(\boldsymbol{x}\cdot\boldsymbol{x})\lambda - (\boldsymbol{x}\cdot\boldsymbol{x}_n)^2 - (\boldsymbol{x}\cdot\boldsymbol{x}_v)^2]}{(\boldsymbol{x}\cdot\boldsymbol{x})^2}. \tag{44}$$

Since $\lambda(u, v) \neq 0$ in $\mathscr{U}_\varepsilon^*(P_0)$, $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are nonnull vectors there.

Further, these vectors lie in the tangent plane to S and therefore are perpendicular to the unit normal vector $X(u, v)$; and since $F = 0$, they are perpendicular to each other. Accordingly, for any $(u, v) \in \mathscr{U}_\varepsilon^*(P_0)$, there are scalars α, β, γ such that

$$x = \alpha x_u + \beta x_v + \gamma X . \tag{45}$$

From (45), we obtain

$$\begin{aligned} x \cdot x &= \alpha^2 \lambda + \beta^2 \lambda + \gamma^2 , \\ x \cdot x_u &= \alpha \lambda , \\ x \cdot x_v &= \beta \lambda , \\ x \cdot X &= \gamma . \end{aligned}$$

Hence (44) can be written as

$$\Delta \log (x \cdot x)^{\frac{1}{2}} = \frac{2(\alpha^2\lambda^2 + \beta^2\lambda^2 + \gamma^2\lambda - \alpha^2\lambda^2 - \beta^2\lambda^2)}{(x \cdot x)^2} = \frac{2\gamma^2\lambda}{(x \cdot x)^2} ,$$

or, finally, the *fundamental formula*

$$\Delta \log (x \cdot x)^{\frac{1}{2}} = \frac{2(x \cdot X)^2 \lambda}{(x \cdot x)^2} . \tag{46}$$

In obtaining an estimate of the behavior of the right-hand member of (46) as $r \longrightarrow 0$, we can use the expressions (34) and (38) for $x \cdot x$ and λ, respectively. For $x \cdot X$, by (31) and (40) we have

$$x \cdot X = \frac{r^\tau \sum_{j,k,l} (a_{j,\tau} \cos \tau\theta + b_{j,\tau} \sin \tau\theta)(a_{k,t} b_{l,t} - a_{l,t} b_{k,t})}{\sum_{q=1}^{3} a_{q,t}^2} + o(r^\tau) , \tag{47}$$

where $j, k, l = 1, 2, 3$ in cyclic order in the sum in the numerator.

If $\tau = 0$, then by (24), (32), (34), (38), and (47) we have

$$\lim_{r \to 0} (x \cdot x)^2 = \left(\sum_{j=1}^{3} a_{j,0}^2 \right)^2 > 0 ,$$

$$\lim_{r \to 0} \lambda = t^2 \sum_{j=1}^{3} a_{j,t}^2 \lim_{r \to 0} r^{2t-2} \geqq 0 ,$$

with equality if and only if $t > 1$, and

$$\lim_{r \to 0} (x \cdot X)^2 = \left[\frac{\sum_{j,k,l} a_{j,0}(a_{k,t} b_{l,t} - a_{l,t} b_{k,t})}{\sum_{q=1}^{3} a_{q,t}^2} \right]^2 > 0 .$$

Since each of these three limits exists and is finite, and the limit of $(x \cdot x)^2$ is not 0, it follows from (46) that $\Delta \log (x \cdot x)^{\frac{1}{2}}$ is continuous at (u_0, v_0) in this case.

If $\tau \neq 0$, then $t = \tau$, and the sum in the numerator of the first term in the right-hand member of (47) vanishes *identically*. We then have, by (31) and (40),

$$x \cdot X = O(r^{\tau+1}) \tag{48}$$

unless

$$\tau = -1 \quad \text{and} \quad \sum_{j=1}^{3} c_j^2 \neq 0 \, . \tag{49}$$

In the exceptional case (49), we have

$$x \cdot X = O(\log r) \, . \tag{50}$$

If (48) holds, then from (34), (38), (46), and (48) we get

$$\Delta \log (x \cdot x)^{\frac{1}{2}} = \frac{O(r^{2t-2})[O(r^{t+1})]^2}{\left[r^{2t} \sum_{j=1}^{3} a_{j,t}^2 + o(r^{2t})\right]^2} = O(r^{2t-2+2t+2-4t}) = O(1) \, , \tag{51}$$

so that $\Delta \log(x \cdot x)^{\frac{1}{2}}$ is bounded in the neighborhood of P_0. Notice, however, that it is not necessarily continuous at P_0. For example, for the minimal surface of Enneper (28), we have

$$\Delta \log (x \cdot x)^{\frac{1}{2}} = 2 \cos^2 2\theta + o(1) \, ,$$

so that the limiting behavior of $\Delta \log (x \cdot x)^{\frac{1}{2}}$ depends on the limiting behavior of θ. Thus if $\theta \to 0$ as $r \to 0$ then $\Delta \log (x \cdot x)^{\frac{1}{2}} \to 2$, but if $\theta \to \pi/4$ as $r \to 0$ then $\Delta \log (x \cdot x)^{\frac{1}{2}} \to 0$.

In the exceptional case (49), in place of (51) we have, by (32) and (50),

$$\begin{aligned} \Delta \log (x \cdot x)^{\frac{1}{2}} &= \frac{O(r^{-4})[O(\log r)]^2}{\left[r^{-2} \sum_{j=1}^{3} a_{j,-1}^2 + o(r^{-2})\right]^2} \\ &= O[r^{-4+4}(\log r)^2] = O[(\log r)^2] \, . \end{aligned} \tag{52}$$

For example, for the catenoid (27) we have

$$\Delta \log (x \cdot x)^{\frac{1}{2}} = 8(\log r)^2[1 + o(1)] \, .$$

Thus in the exceptional case (49), we see that $\Delta \log (x \cdot x)^{\frac{1}{2}}$ becomes infinite as $r \to 0$. As we shall see in the next section, however, it does not become infinite too rapidly for the applications we shall be making.

4. An application of Green's theorem. Let $w = u + iv$; let $A_r(w_0)$ denote the closed circular disc $|w - w_0| \leqq r$; and let $\partial A_r(w_0)$ denote the boundary, $|w - w_0| = r$, of $A_r(w_0)$.

Let the functions (1) be the coordinate functions of a nonconstant meromorphic minimal surface S in isothermal representation for (u, v) in a finite domain D, and for a given fixed $R > 0$ let $A_R(0)$ be contained in D.

If S has a zero at the origin, let s_0 denote the order of this zero; otherwise, let $s_0 = 0$. Similarly, if S has a pole at the origin, let n_0 denote the order of this pole, and otherwise let $n_0 = 0$. Of course, at least one of s_0 and n_0 must be equal to 0, and both might be equal to 0.

Let s_0 be denoted by $n(0, \mathbf{0}; S)$, and n_0 by $n(0, \infty; S)$. Then by (34) and the definition of τ, if τ_0 is the value of τ, and t_0 that of t, for the functions (31) representing S when P_0 is the origin, we have

$$\boldsymbol{x}\cdot\boldsymbol{x} = r^{2\tau_0} \sum_{j=1}^{3} a_{j,\tau_0}^2 + o(r^{2\tau_0}) , \qquad r = |w| , \tag{53}$$

with

$$\sum_{j=1}^{3} a_{j,\tau_0}^2 \neq 0 \tag{54}$$

and

$$\tau_0 = s_0 - n_0 = n(0, \mathbf{0}; S) - n(0, \infty; S) . \tag{55}$$

For any ρ, $0 < \rho \leqq R$, there can be only a finite number of zeros and poles of S in $A_\rho(0)$, since the zeros and poles of a nonconstant meromorphic minimal surface are isolated. In the punctured disc $0 < |w| \leqq \rho$, let the zeros of S be at the points

$$w = z_1, z_2, \cdots, z_k , \qquad k = k(\rho) \geqq 0 ,$$

with

$$0 < |z_1| \leqq |z_2| \leqq \cdots \leqq |z_k| \leqq \rho ,$$

and let the poles be at

$$w = p_1, p_2, \cdots, p_l , \qquad l = l(\rho) \geqq 0 ,$$

with

$$0 < |p_1| \leqq |p_2| \leqq \cdots \leqq |p_l| \leqq \rho .$$

Let the orders of these zeros and poles be, respectively,

$$s_1, s_2, \cdots, s_k \quad \text{and} \quad n_1, n_2, \cdots, n_l ,$$

and denote the sum of the orders of the zeros and poles of S in $A_\rho(0)$ by $n(\rho, \mathbf{0}; S)$ and $n(\rho, \infty; S)$, respectively:

$$n(\rho, \mathbf{0}; S) = s_0 + s_1 + \cdots + s_k , \tag{56}$$

$$n(\rho, \infty; S) = n_0 + n_1 + \cdots + n_l . \tag{57}$$

Then $n(\rho, 0; S) \geqq 0$, with $s_j > 0$ for $j = 1, 2, \cdots, k$, and similarly $n(\rho, \infty; S) \geqq 0$, with $n_q > 0$ for $q = 1, 2, \cdots, l$.

Now let ρ, $0 < \rho \leqq R$, be chosen so that S does not have a zero or pole on $\partial A_\rho(0)$. Since there are only a finite number of zeros and poles of S in $A_\rho(0)$, we can choose $r > 0$ so small that the $k + l + 1$ closed circular discs $A_r(0), A_r(z_j)$, $j = 1, 2, \cdots, k$, and $A_r(p_q)$, $q = 1, 2, \cdots, l$, are disjoint from one another and interior to $A_\rho(0)$. Let Ω_r denote the domain interior to the circle $\partial A_\rho(0)$ and exterior to the circles $\partial A_r(0), \partial A_r(z_j)$, and $\partial A_r(p_q)$, $j = 1, \cdots, k$ and $q = 1, 2, \cdots, l$.

In Ω_r, the function

$$g(u, v) = \log (\boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}} \tag{58}$$

has continuous derivatives of all orders. Hence we can apply Green's theorem to $g(u, v)$ in Ω_r:

$$\int_{\partial\Omega_r} \frac{\partial g}{\partial\nu} ds = \iint_{\Omega_r} \Delta g(u, v) dA \ , \tag{59}$$

where ν refers to the normal directed outwardly from Ω_r.

By the definition of Ω_r, we have

$$\begin{aligned}\int_{\partial\Omega_r} \frac{\partial g(u, v)}{d\nu} ds = \int_{\partial A_\rho(0)} \frac{\partial g}{\partial\rho} \rho d\theta - \int_{\partial A_r(0)} \frac{\partial g}{\partial r} r d\theta \\ - \sum_{j=1}^{k} \int_{\partial A_r(z_j)} \frac{\partial g}{\partial r} r d\theta - \sum_{q=1}^{l} \int_{\partial A_r(p_q)} \frac{\partial g}{\partial r} r d\theta \ .\end{aligned} \tag{60}$$

From (31), (53), and (55), by a computation we find that on $\partial A_r(0)$ we have

$$\frac{\partial g}{\partial r} = \frac{\tau_0}{r} + o(r^{-1}) = \frac{s_0 - n_0}{r} + o(r^{-1}) \ ,$$

so that

$$\int_{\partial A_r(0)} \frac{\partial g}{\partial r} r d\theta = 2\pi\tau_0 + o(1) = 2\pi(s_0 - n_0) + o(1) \ .$$

Similarly, on the $\partial A_r(z_j)$ and $\partial A_r(p_q)$ we have, respectively,

$$\frac{\partial g}{\partial r} = \frac{s_j}{r} + o(r^{-1})$$

and

$$\frac{\partial g}{\partial r} = \frac{-n_q}{r} + o(r^{-1}) \ ,$$

so that

$$\int_{\partial A_r(z_j)} \frac{\partial g}{\partial r} r d\theta = 2\pi s_j + o(1)$$

and

$$\int_{\partial A_r(p_q)} \frac{\partial g}{\partial r} r d\theta = -2\pi n_q + o(1) .$$

By (56), (57), and (60), we accordingly have

$$\begin{aligned}\int_{\partial\Omega_r} \frac{\partial g(u, v)}{d\nu} ds &= \int_{\partial A_\rho(0)} \frac{\partial g}{\partial \rho} \rho d\theta - 2\pi \sum_{j=0}^{k} s_j \\ &\quad + 2\pi \sum_{q=0}^{l} n_q + o(1) \\ &= \int_{\partial A_\rho(0)} \frac{\partial g}{\partial \rho} \rho d\theta - 2\pi n(\rho, \mathbf{0}; S) \\ &\quad + 2\pi n(\rho, \infty; S) + o(1) ,\end{aligned}$$

whence

$$\begin{aligned}&\lim_{r\to 0} \int_{\partial\Omega_r} \frac{\partial g(u, v)}{d\nu} ds \\ &= \int_{\partial A_\rho(0)} \frac{\partial g}{\partial \rho} \rho d\theta - 2\pi n(\rho, \mathbf{0}; S) + 2\pi n(\rho, \infty; S) .\end{aligned} \tag{61}$$

By (51) and (52), for any $A_r(w) \subset D$ we have

$$\iint_{A_r(w)} \Delta g(u, v) dA = \int_0^{2\pi} \int_0^r O(1) \sigma \, d\sigma \, d\theta$$

or at worst, in the exceptional case (49),

$$\iint_{A_r(w)} \Delta g(u, v) dA = \int_0^{2\pi} O[(\log \sigma)^2] \sigma \, d\sigma \, d\theta .$$

Since

$$\sigma O(1) = o(1) \quad \text{and} \quad \sigma O[(\log \sigma)^2] = o(1) ,$$

in either case we have

$$\lim_{r\to 0} \iint_{A_r(w)} \Delta g(u, v) dA = 0 .$$

Therefore, by the definition of Ω_r,

$$\lim_{r\to 0} \iint_{\Omega_r} \Delta g(u, v) dA = \iint_{A_\rho(0)} \Delta g(u, v) dA . \tag{62}$$

From (59), (61), and (62), we obtain

$$\int_{\partial A_\rho(0)} \frac{\partial g}{\partial \rho} \rho d\theta - 2\pi n(\rho, \mathbf{0}; S) + 2\pi n(\rho, \infty; S) = \iint_{A_\rho(0)} \Delta g(u, v)\, dA ,$$

whence

$$\tag{63} \begin{aligned} &\frac{1}{2\pi}\int_{\partial A_\rho(0)} \frac{\partial g}{\partial \rho} d\theta - \frac{n(\rho, \mathbf{0}; S) - n(\rho, \infty; S)}{\rho} \\ &= \frac{1}{2\pi\rho}\iint_{A_\rho(0)} \Delta g(u, v) dA . \end{aligned}$$

Now (63) can be written as

$$\tag{64} \begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial g}{\partial \rho} d\theta - \frac{n(0, \mathbf{0}; S) - n(0, \infty; S)}{\rho} - \frac{n(\rho, \mathbf{0}; S) - n(0, \mathbf{0}; S)}{\rho} \\ &+ \frac{n(\rho, \infty; S) - n(0, \infty; S)}{\rho} = \frac{1}{2\pi\rho}\iint_{A_\rho(0)} \Delta g(u, v) dA . \end{aligned}$$

Nothing that, by (55),

$$\begin{aligned} -\frac{n(0, \mathbf{0}; S) - n(0, \infty; S)}{\rho} &= \frac{-\tau_0}{\rho} = \frac{\partial}{\partial \rho} \log \rho^{-\tau_0} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial \rho} \log \rho^{-\tau_0} d\theta , \end{aligned}$$

we see from (58) that

$$\tag{65} \begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial g}{\partial \rho} d\theta - \frac{n(0, \mathbf{0}; S) - n(\mathbf{0}, \infty; S)}{\rho} \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial \rho} \log [(\boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}}\rho^{-\tau_0}] d\theta . \end{aligned}$$

By (65), we can therefore rewrite (64) as

$$\tag{66} \begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial \rho} \log [(\boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}}\rho^{-\tau_0}] d\theta - \frac{n(\rho, \mathbf{0}; S) - n(0, \mathbf{0}; S)}{\rho} \\ &+ \frac{n(\rho, \infty; S) - n(0, \infty; S)}{\rho} = \frac{1}{2\pi\rho}\iint_{A_\rho(0)} \Delta g(u, v) dA . \end{aligned}$$

By (51) and (52), for any r, $0 < r \leqq R$, the right-hand member of (66) can be integrated from 0 to r with respect to ρ. The numerators in the second and third terms on the left vanish in an interval $0 \leqq \rho \leqq r_0$, $r_0 > 0$, so these terms also can be integrated. For the first term in the left-hand member of (66), by (53) we have

$$\begin{aligned} &\int_\varepsilon^r \left\{ \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial}{\partial \rho} \log [(\boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}}\rho^{-\tau_0}] d\theta \right\} d\rho \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left\{ \int_\varepsilon^r \frac{\partial}{\partial \rho} \log [(\boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}}\rho^{-\tau_0}] d\rho \right\} d\theta \end{aligned}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\log\,[(\boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}}\rho^{-\tau_0}]\,|_\varepsilon^r\,d\theta$$

$$= \frac{1}{2\pi}\int_{\partial A_\rho(0)}\log\,(\boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}}d\theta - \tau_0\log r$$

$$- \log\Big(\sum_{j=1}^{3} a_{j,\tau_0}^2\Big)^{\frac{1}{2}} + o(1)\;,$$

so that

$$\lim_{\varepsilon\to 0}\int_\varepsilon^r\Big\{\frac{1}{2\pi}\int_0^{2\pi}\frac{\partial}{\partial\rho}\log\,[(\boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}}\rho^{-\tau_0}]d\theta\Big\}d\rho$$

$$= \frac{1}{2\pi}\int_{\partial A_r(0)}\log\,(\boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}}d\theta - n(0, 0; S)\log r$$

$$+ n(0, \infty; S)\log r - \log\Big(\sum_{j=1}^{3} a_{j,\tau_0}^2\Big)^{\frac{1}{2}}\;.$$

Hence (66) yields

$$\begin{aligned}(67)\qquad &\frac{1}{2\pi}\int_{\partial A_r(0)}\log\,(\boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}}d\theta\\ &\quad - \int_0^r\frac{n(\rho, 0; S) - n(0, 0; S)}{\rho}\,d\rho - n(0, 0; S)\log r\\ &\quad + \int_0^r\frac{n(\rho, \infty; S) - n(0, \infty; S)}{\rho}\,d\rho + n(0, \infty; S)\log r\\ &\quad - \log\Big(\sum_{j=1}^{3} a_{j,\tau_0}^2\Big)^{\frac{1}{2}} = \frac{1}{\pi}\int_0^r\Big[\frac{1}{2\rho}\iint_{A_\rho(0)}\Delta g(u, v)dA\Big]d\rho\;.\end{aligned}$$

If for a nonnegative function φ the function $\log^+\varphi$ is defined by

$$\log^+\varphi = \begin{cases}\log\varphi & \text{for}\quad \varphi \geqq 1\;,\\ 0 & \text{for}\quad 0 \leqq \varphi \leqq 1\;,\end{cases}$$

then we have the identity

$$\log\varphi = \log^+\varphi - \log^+\frac{1}{\varphi}\;,$$

and (67) can be written, by (46) and (58), as

$$\begin{aligned}&\frac{1}{2\pi}\int_{\partial A_r(0)}\log^+(\boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}}d\theta\\ &\quad + \int_0^r\frac{n(\rho, \infty; S) - n(0, \infty; S)}{\rho}\,d\rho + n(0, \infty; S)\log r\\ (68)\qquad &\quad = \frac{1}{2\pi}\int_{\partial A_r(0)}\log^+(\boldsymbol{x}\cdot\boldsymbol{x})^{-\frac{1}{2}}d\theta\end{aligned}$$

$$+\int_0^r \frac{n(\rho, 0; S) - n(0, 0; S)}{\rho} d\rho + n(0, 0; S) \log r$$
$$+\int_0^r \left[\frac{1}{\rho} \iint_{A_\rho(0)} \frac{(x \cdot X)^2}{\pi (x \cdot x)^2} \lambda dA\right] d\rho + \log \left(\sum_{j=1}^3 a_{j,\tau_0}^2\right)^{\frac{1}{2}} .$$

5. The Nevanlinna characteristic function. For a nonconstant meromorphic minimal surface S given in isothermic representation by the functions (1), and for any given finite

$$a = (a_1, a_2, a_3) ,$$

consider the surface

(69) $$S - a: x_j = x_j(u, v) - a_j , \qquad j = 1, 2, 3 ,$$

for $(u, v) \in D$. This again is a nonconstant meromorphic minimal surface in isothermic representation.

Applying (68) to $S - a$, we obtain

(70)
$$\begin{aligned}
&\frac{1}{2\pi} \int_{\partial A_r(0)} \log^+ [(x - a) \cdot (x - a)]^{\frac{1}{2}} d\theta \\
&\quad + \int_0^r \frac{n(\rho, \infty; S - a) - n(0, \infty; S - a)}{\rho} d\rho \\
&\quad + n(0, \infty; S - a) \log r \\
&= \frac{1}{2\pi} \int_{\partial A_r(0)} \log^+ [(x - a) \cdot (x - a)]^{-\frac{1}{2}} d\theta \\
&\quad + \int_0^r \frac{n(\rho, 0; S - a) - n(0, 0; S - a)}{\rho} d\rho \\
&\quad + n(0, 0; S - a) \log r \\
&\quad + \int_0^r \left\{\frac{1}{\rho} \iint_{A_\rho(0)} \frac{[(x - a) \cdot X]^2}{\pi [(x - a) \cdot (x - a)]^2} \lambda dA\right\} d\rho \\
&\quad + \log \left(\sum_{j=1}^3 a_{j,\tau_a}^2\right)^{\frac{1}{2}} ,
\end{aligned}$$

where τ_a is the value of τ, and t_a that of t_0, for the functions (69), and the a_{j,τ_a} are leading coefficients of *these* functions.

Since the poles of S and the poles of $S - a$ occur at the same (u, v)-points, we have

(71) $$n(\rho, \infty; S) = n(\rho, \infty; S - a) .$$

Since the zeros of the surface $S - a$ are the a-points of the surface S, the function $n(\rho, a; S)$, defined by

(72) $$n(\rho, a; S) = n(\rho, 0; S - a) ,$$

gives an expression of the number of $\boldsymbol{a}$-points of the surface S in $|w| \leqq \rho$.

If the map of $A_\rho(0)$ on S is projected from $\boldsymbol{a}$ on the unit sphere with center at $\boldsymbol{a}$, then the function $h(\rho, \boldsymbol{a}; S)$, defined by

$$h(\rho, \boldsymbol{a}; S) = \iint_{A_\rho(0)} \frac{[(\boldsymbol{x} - \boldsymbol{a})\cdot X]^2}{\pi[(\boldsymbol{x} - \boldsymbol{a})\cdot(\boldsymbol{x} - \boldsymbol{a})]^2} \lambda dA , \tag{73}$$

gives a measure of the area of this projection, with the element of area of the projection weighted by

$$\frac{1}{\pi} |\cos(\boldsymbol{x} - \boldsymbol{a}, X)| .$$

Thus $h(\rho, \boldsymbol{a}; S)$ can be considered as a measure of the visibility of the surface S for $|w| \leqq \rho$, as viewed from $\boldsymbol{a}$.

In particular, we have

$$h(0, \boldsymbol{a}; S) = 0 . \tag{74}$$

Since

$$\lim_{\boldsymbol{a}\to\infty} h(\rho, \boldsymbol{a}; S) = 0 ,$$

we define $h(\rho, \infty; S)$ by

$$h(\rho, \infty; S) = 0 . \tag{75}$$

By analogy with the Nevanlinna theory of meromorphic functions of a complex variable, let us define a *proximity function* (*Schmiegungsfunktion*) for S by

$$\begin{aligned} m(r, \infty; S) &= \frac{1}{2\pi}\int_{\partial A_r(0)} \log^+ (\boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}} d\theta , \\ m(r, \boldsymbol{a}; S) &= \frac{1}{2\pi}\int_{\partial A_r(0)} \log^+ [(\boldsymbol{x} - \boldsymbol{a})\cdot(\boldsymbol{x} - \boldsymbol{a})]^{-\frac{1}{2}} d\theta , \end{aligned} \tag{76}$$

and an *enumerative function* (*Anzahlfunktion*) by

$$\begin{aligned} N(r, \infty; S) &= \int_0^r \frac{n(\rho, \infty; S) - n(0, \infty; S)}{\rho} d\rho + n(0, \infty; S) \log r , \\ N(r, \boldsymbol{a}; S) &= \int_0^r \frac{n(\rho, \boldsymbol{a}; S) - n(0, \boldsymbol{a}; S)}{\rho} d\rho + n(0, \boldsymbol{a}; S) \log r , \end{aligned} \tag{77}$$

for $\boldsymbol{a}$ finite.

To these we now adjoin a *visibility function* (*Sichbarkeitsfunktion*), defined by

$$\begin{aligned} H(r, \infty; S) &= 0 , \\ H(r, \boldsymbol{a}; S) &= \int_0^r \frac{h(\rho, \boldsymbol{a}; S)}{\rho} d\rho . \end{aligned} \tag{78}$$

By (74) and (75), the definition (78) of H is quite analogous to the definition (77) of N, with h in place of n.

Substituting from (76), (77), and (78) into (68) and (70), and using (71) and (72), we obtain, respectively,

$$\begin{aligned} &m(r, \infty; S) + N(r, \infty; S) + H(r, \infty; S) \\ &\quad = m(r, \mathbf{0}; S) + N(r, \mathbf{0}; S) + H(r, \mathbf{0}; S) + \log\left(\sum_{j=1}^{3} a_{j,\tau_0}^2\right)^{\frac{1}{2}} \end{aligned} \tag{79}$$

and

$$\begin{aligned} &m(r, \infty, S - \boldsymbol{a}) + N(r, \infty; S) + H(r, \infty; S) \\ &\quad = m(r, \boldsymbol{a}; S) + N(r, \boldsymbol{a}; S) + H(r, \boldsymbol{a}; S) + \log\left(\sum_{j=1}^{3} a_{j,\tau_a}^2\right)^{\frac{1}{2}} . \end{aligned} \tag{80}$$

It is well known that for any two nonnegative numbers, say γ_1 and γ_2 with $\gamma_1 \geqq \gamma_2 \geqq 0$, we have

$$\log^+ (\gamma_1 + \gamma_2) \leqq \log^+ \gamma_1 + \log^+ \gamma_2 + \log 2 . \tag{81}$$

To establish (81), notice that for $\gamma_1 \geqq 1$ we have

$$\begin{aligned} \log^+ (\gamma_1 + \gamma_2) &= \log (\gamma_1 + \gamma_2) \leqq \log 2\gamma_1 \\ &= \log\gamma_1 + \log 2 \leqq \log^+ \gamma_1 + \log^+ \gamma_2 + \log 2 , \end{aligned}$$

while for $\gamma_1 < 1$ we have

$$\log^+ (\gamma_1 + \gamma_2) \leqq \log 2 = \log^+ \gamma_1 + \log^+ \gamma_2 + \log 2 .$$

Hence (81) holds in any case.

By the triangle inequality, we have

$$[(\boldsymbol{x} - \boldsymbol{a})\cdot(\boldsymbol{x} - \boldsymbol{a})]^{\frac{1}{2}} \leqq (\boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}} + (\boldsymbol{a}\cdot\boldsymbol{a})^{\frac{1}{2}}$$

and

$$\begin{aligned} (\boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}} &= [(\boldsymbol{x} - \boldsymbol{a} + \boldsymbol{a})\cdot(\boldsymbol{x} - \boldsymbol{a} + \boldsymbol{a})]^{\frac{1}{2}} \\ &\leqq [(\boldsymbol{x} - \boldsymbol{a})\cdot(\boldsymbol{x} - \boldsymbol{a})]^{\frac{1}{2}} + (\boldsymbol{a}\cdot\boldsymbol{a})^{\frac{1}{2}} . \end{aligned}$$

Accordingly, from (76) and (81) we obtain

$$m(r, \infty; S - \boldsymbol{a}) \leqq m(r, \infty; S) + \log^+ (\boldsymbol{a}\cdot\boldsymbol{a})^{\frac{1}{2}} + \log 2 \tag{82}$$

and

$$m(r, \infty; S) \leqq m(r, \infty; S - \boldsymbol{a}) + \log^+ (\boldsymbol{a}\cdot\boldsymbol{a})^{\frac{1}{2}} + \log 2 . \tag{83}$$

From (82) and (83), we have

$$m(r, \infty; S) - m(r, \infty; S - \boldsymbol{a}) = B(r, \boldsymbol{a}; S) , \tag{84}$$

with

$$|B(r, \boldsymbol{a}; S)| \leqq \log^{+} (\boldsymbol{a} \cdot \boldsymbol{a})^{\frac{1}{2}} + \log 2 . \tag{85}$$

Substituting for $m(r, \infty; S - \boldsymbol{a})$ from (84) into (80), we get

$$\begin{aligned} & m(r, \infty; S) + N(r, \infty; S) + H(r, \infty; S) \\ & \quad = m(r, \boldsymbol{a}; S) + N(r, \boldsymbol{a}; S) + H(r, \boldsymbol{a}; S) + C(r, \boldsymbol{a}; S) , \end{aligned} \tag{86}$$

where

$$C(r, \boldsymbol{a}; S) = \log \left(\sum_{j=1}^{3} a_{j,\tau_{\boldsymbol{a}}}^{2} \right)^{\frac{1}{2}} + B(r, \boldsymbol{a}; S) . \tag{87}$$

By (85) and (87), we have

$$|C(r, \boldsymbol{a}; S)| \leqq \left| \log \left(\sum_{j=1}^{3} a_{j,\tau_{\boldsymbol{a}}}^{2} \right)^{\frac{1}{2}} \right| + \log^{+} (\boldsymbol{a} \cdot \boldsymbol{a})^{\frac{1}{2}} + \log 2 . \tag{88}$$

We define the *total affinity* of S to $\boldsymbol{a}$ in $|w| \leqq r$, or the *affinity function for* S, by

$$\mathfrak{A}(r, \boldsymbol{a}; S) = m(r, \boldsymbol{a}; S) + N(r, \boldsymbol{a}; S) + H(r, \boldsymbol{a}; S) . \tag{89}$$

In particular, we call the total affinity of S to ∞ the *Nevanlinna characteristic function* of S and denote it by $T(r; S)$, so that

$$\begin{aligned} T(r; S) & = \mathfrak{A}(r, \infty; S) \\ & = m(r, \infty; S) + N(r, \infty; S) + H(r, \infty; S) \\ & = m(r, \infty; S) + N(r, \infty; S) . \end{aligned} \tag{90}$$

The first fundamental theorem of R. Nevanlinna [10] concerning meromorphic functions of a complex variable is generalized by means of the inequality (88) to meromorphic minimal surfaces:

THEOREM 1. *If the functions*

$$x_j = x_j(u, v) , \qquad j = 1, 2, 3 ,$$

are the coordinate functions of a nonconstant meromorphic minimal surface S in isothermal representation for $u^2 + v^2 < \infty$, then for each finite $\boldsymbol{a}$ we have

$$T(r, S) = \mathfrak{A}(r, \boldsymbol{a}; S) + C(r, \boldsymbol{a}; S) , \tag{91}$$

where $C(r, \boldsymbol{a}; S)$ is a bounded function of r for each $\boldsymbol{a}$:

$$|C(r, \boldsymbol{a}; S)| \leqq \left| \log \left(\sum_{j=1}^{3} a_{j,\tau_{\boldsymbol{a}}}^{2} \right)^{\frac{1}{2}} \right| + \log^{+} (\boldsymbol{a} \cdot \boldsymbol{a})^{\frac{1}{2}} + \log 2 . \tag{92}$$

Thus S has essentially the same affinity for all points $\boldsymbol{a}$ in space, in the sense that for any two given points $\boldsymbol{a}$ and $\boldsymbol{b}$ the difference

$$\mathfrak{A}(r, \boldsymbol{a}; S) - \mathfrak{A}(r, \boldsymbol{b}; S)$$

is a bounded function of r:

$$\begin{aligned} |\mathfrak{A}(r, \boldsymbol{a}; S) - \mathfrak{A}(r, \boldsymbol{b}; S)| &\leqq \left|\log\left(\sum_{j=1}^{3} a_{j,\tau_a}^2\right)^{\frac{1}{2}}\right| \\ &+ \left|\log\left(\sum_{j=1}^{3} a_{j,\tau_b}^2\right)^{\frac{1}{2}}\right| + \log^+(\boldsymbol{a}\cdot\boldsymbol{a})^{\frac{1}{2}} + \log^+(\boldsymbol{b}\cdot\boldsymbol{b})^{\frac{1}{2}} + \log 4 \,. \end{aligned} \tag{93}$$

This is true in particular if S lies on a plane (the complex-variable case) and the point $\boldsymbol{a}$, or the point $\boldsymbol{b}$, or both, are not on the plane.

6. The hyperspherical characteristic function. Because of the $\log^+$ function in the formulas (76), the value of the proximity function $m(r, \boldsymbol{a}; S)$ is affected only by the portion of the map of $\partial A_r(0)$ on S that lies at distance $\leqq 1$ from $\boldsymbol{a}$ for $\boldsymbol{a}$ finite, or at distance $\geqq 1$ from $\boldsymbol{0}$ for $\boldsymbol{a} = \infty$.

In the Ahlfors-Shimizu theory [1, 14; 9, 15] for the complex-variable case, the plane of the map S is projected stereographically [8, pp. 119, 120] onto a spherical surface of radius $\frac{1}{2}$, and then the chordal distance is used as a metric. In this metric, each point of the plane is at distance $\leqq 1$ from each other point of the plane, and accordingly all of the map of $\partial A_r(0)$ on S contributes to the proximity function for each point $\boldsymbol{a}$ of the plane.

An analogous treatment can be given for isothermal maps on nonconstant meromorphic minimal surfaces.

In the four-dimensional (x_1, x_2, x_3, x_4)-space, let $\mathscr{S}_0$ be the hypersphere with center

$$\boldsymbol{x}^0 = (x_1^0, x_2^0, x_3^0, x_4^0)$$

and radius δ_0. Then, as in three-dimensional inversion, the points

$$\boldsymbol{x} = (x_1, x_2, x_3, x_4) \quad \text{and} \quad \boldsymbol{x}' = (x_1', x_2', x_3', x_4')$$

are said to be *inverses* of each other with respect to $\mathscr{S}_0$ if and only if $\boldsymbol{x}$ and $\boldsymbol{x}'$ are on the same ray with endpoint $\boldsymbol{x}^0$ and are such that

$$\delta\delta' = \delta_0^2 \,,$$

where

$$\delta = [(\boldsymbol{x} - \boldsymbol{x}^0)\cdot(\boldsymbol{x} - \boldsymbol{x}^0)]^{\frac{1}{2}} \quad \text{and} \quad \delta' = [(\boldsymbol{x}' - \boldsymbol{x}^0)\cdot(\boldsymbol{x}' - \boldsymbol{x})]^{\frac{1}{2}}$$

are the Euclidean distances in four-dimensional space from $\boldsymbol{x}^0$ to $\boldsymbol{x}$ and from $\boldsymbol{x}^0$ to $\boldsymbol{x}'$, respectively.

By similar triangles, then, $\boldsymbol{x}$ and $\boldsymbol{x}'$ are inverses of each other with respect to $\mathscr{S}_0$ if and only if

$$\frac{x'_j - x_j^0}{x_j - x_j^0} = \frac{\delta'}{\delta} = \frac{\delta\delta'}{\delta^2} = \frac{\delta_0^2}{\delta^2}\,, \qquad j = 1, 2, 3, 4\,,$$

or

$$x'_j - x_j^0 = \frac{\delta_0^2}{\delta^2}(x_j - x_j^0)\,, \qquad j = 1, 2, 3, 4\,. \tag{94}$$

As in the three-dimensional case [8, pp. 117–120], inversion in $\mathscr{S}_0$ maps four-dimensional space in a one-to-one way onto itself, with $\boldsymbol{x}^0$ corresponding to a unique ideal point, ∞, at infinity. A hypersphere or hyperplane is mapped onto a hypersphere or hyperplane, according as the given hypersphere or hyperplane does not or does pass through $\boldsymbol{x}^0$.

Further, from (94) we obtain

$$dx'_j = \frac{\delta_0^2}{\delta^2}dx_j - \frac{2\delta_0^2}{\delta^3}(x_j - x_j^0)d\delta\,,$$

whence

$$\begin{aligned} ds'^2 &= d\boldsymbol{x}'\cdot d\boldsymbol{x}' \\ &= \frac{\delta_0^4}{\delta^4}d\boldsymbol{x}\cdot d\boldsymbol{x} - \frac{4\delta_0^4}{\delta^5}d\delta(\boldsymbol{x} - \boldsymbol{x}^0)\cdot d\boldsymbol{x} \\ &\quad + \frac{4\delta_0^4}{\delta^6}(d\delta)^2(\boldsymbol{x} - \boldsymbol{x}^0)\cdot(\boldsymbol{x} - \boldsymbol{x}^0) \\ &= \frac{\delta_0^4}{\delta^4}d\boldsymbol{x}\cdot d\boldsymbol{x} = \frac{\delta_0^4}{\delta^4}ds^2\,, \end{aligned}$$

or

$$ds' = \frac{\delta_0^2}{\delta^2}ds\,, \tag{95}$$

so that the transformation is an isothermal one.

For the particular choice

$$\boldsymbol{x}^0 = (0, 0, 0, 1) \quad \text{and} \quad \delta_0 = 1\,,$$

(94) yields

$$\begin{aligned} x'_j &= \frac{x_j}{x_1^2 + x_2^2 + x_3^2 + (x_4 - 1)^2}\,, \qquad j = 1, 2, 3, \\ x'_4 &= \frac{x_1^2 + x_2^2 + x_3^2 + x_4(x_4 - 1)}{x_1^2 + x_2^2 + x_3^2 + (x_4 - 1)^2}\,. \end{aligned}$$

Under this inversion, the coordinates of the image of a point $(x_1\, x_2, x_3, 0)$ on the hyperplane $x_4 = 0$ are given by

$$
\begin{aligned}
x'_j &= \frac{x_j}{x_1^2 + x_2^2 + x_3^2 + 1} \\
&= \frac{x_j}{1 + \boldsymbol{x}\cdot\boldsymbol{x}}, \qquad j = 1, 2, 3, \\
x'_4 &= \frac{x_1^2 + x_2^2 + x_3^2}{x_1^2 + x_2^2 + x_3^2 + 1} \\
&= \frac{\boldsymbol{x}\cdot\boldsymbol{x}}{1 + \boldsymbol{x}\cdot\boldsymbol{x}}.
\end{aligned} \tag{96}
$$

The coordinates (96) satisfy

$$x_1'^2 + x_2'^2 + x_3'^2 + (x_4' - \tfrac{1}{2})^2 = (\tfrac{1}{2})^2 ,$$

so that the image of $(x_1, x_2, x_3, 0)$ lies on the sphere $\mathscr{S}$ with center $(0, 0, 0, \frac{1}{2})$ and radius $\frac{1}{2}$. In fact, the stereographic projection of the hypersphere $\mathscr{S}$ from its "north" pole $(0, 0, 0, 1)$ onto the hyperplane $x_4 = 0$ tangent to $\mathscr{S}$ at its "south" pole $(0, 0, 0, 0)$ coincides with the mapping of $\mathscr{S}$ onto this hyperplane under inversion in $\mathscr{S}_0$.

We shall henceforth call the hyperplane $x_4 = 0$ the (x_1, x_2, x_3)-space. For points

$$\boldsymbol{x} = (x_1, x_2, x_3,) \quad \text{and} \quad \boldsymbol{y} = (y_1, y_2, y_3)$$

in the finite (x_1, x_2, x_3)-space, the line segment joining their images

$$\boldsymbol{x}' = (x_1', x_2', x_3', x_4') \quad \text{and} \quad \boldsymbol{y}' = (y_1', y_2', y_3', y_4')$$

under the inversion in $\mathscr{S}_0$ described above is a chord of the sphere $\mathscr{S}$. If we let $\chi(\boldsymbol{x}, \boldsymbol{y})$ denote the length of this chord,

$$\chi(\boldsymbol{x}, \boldsymbol{y}) = \text{distance } (\boldsymbol{x}', \boldsymbol{y}') = [(\boldsymbol{x}' - \boldsymbol{y}')\cdot(\boldsymbol{x}' - \boldsymbol{y}')]^{\frac{1}{2}}$$

then we have

$$0 \leqq \chi(\boldsymbol{x}, \boldsymbol{y}) \leqq 1 . \tag{97}$$

From (96), we obtain

$$
\begin{aligned}
(1 &+ \boldsymbol{x}\cdot\boldsymbol{x})^2(1 + \boldsymbol{y}\cdot\boldsymbol{y})^2[\chi(\boldsymbol{x}, \boldsymbol{y})]^2 \\
&= [(1 + \boldsymbol{y}\cdot\boldsymbol{y})\boldsymbol{x} - (1 + \boldsymbol{x}\cdot\boldsymbol{x})\boldsymbol{y}]\cdot[(1 + \boldsymbol{y}\cdot\boldsymbol{y})\boldsymbol{x} - (1 + \boldsymbol{x}\cdot\boldsymbol{x})\boldsymbol{y}] \\
&\quad + [(1 + \boldsymbol{y}\cdot\boldsymbol{y})(\boldsymbol{x}\cdot\boldsymbol{x}) - (1 + \boldsymbol{x}\cdot\boldsymbol{x})(\boldsymbol{y}\cdot\boldsymbol{y})]^2 \\
&= (1 + \boldsymbol{x}\cdot\boldsymbol{x})(1 + \boldsymbol{y}\cdot\boldsymbol{y})[(1 + \boldsymbol{y}\cdot\boldsymbol{y})(\boldsymbol{x}\cdot\boldsymbol{x}) + (1 + \boldsymbol{x}\cdot\boldsymbol{x})(\boldsymbol{y}\cdot\boldsymbol{y}) \\
&\quad - 2(\boldsymbol{x}\cdot\boldsymbol{y}) - 2(\boldsymbol{x}\cdot\boldsymbol{x})(\boldsymbol{y}\cdot\boldsymbol{y})] ,
\end{aligned}
$$

so that

$$
\begin{aligned}
(1 + \boldsymbol{x}\cdot\boldsymbol{x})(1 + \boldsymbol{y}\cdot\boldsymbol{y})[\chi(\boldsymbol{x}, \boldsymbol{y})]^2 &= \boldsymbol{x}\cdot\boldsymbol{x} - 2\boldsymbol{x}\cdot\boldsymbol{y} + \boldsymbol{y}\cdot\boldsymbol{y} \\
&= (\boldsymbol{x} - \boldsymbol{y})\cdot(\boldsymbol{x} - \boldsymbol{y}) ,
\end{aligned}
$$

or

$$\chi(\boldsymbol{x}, \boldsymbol{y}) = \frac{[(\boldsymbol{x} - \boldsymbol{y})\cdot(\boldsymbol{x} - \boldsymbol{y})]^{\frac{1}{2}}}{(1 + \boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}}(1 + \boldsymbol{y}\cdot\boldsymbol{y})^{\frac{1}{2}}} . \tag{98}$$

In the limit, as $y \to \infty$, (98) gives

$$\chi(\boldsymbol{x}, \infty) = \frac{1}{(1 + \boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}}} . \tag{99}$$

We now define a *hyperspherical proximity function* for S, analogous to (76), in terms of the chordal distance:

$$m^\circ(r, \boldsymbol{a}; S) = \frac{1}{2\pi}\int_{\partial A_r(0)} \log \frac{1}{\chi(\boldsymbol{x}, \boldsymbol{a})} d\theta . \tag{100}$$

Notice that here the integrand, which in (76) was restricted to nonnegative values by use of the $\log^+$ function, takes on only nonnegative values by virtue of (97).

By (98), (99), and (100), we have

$$\begin{aligned} m^\circ(r, \infty; S) &= \frac{1}{2\pi}\int_{\partial A_r(0)} \log (1 + \boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}}\, d\theta , \\ m^\circ(r, \boldsymbol{a}; S) &= \frac{1}{2\pi}\int_{\partial A_r(0)} \log \frac{(1 + \boldsymbol{x}\cdot\boldsymbol{x})^{\frac{1}{2}}(1 + \boldsymbol{a}\cdot\boldsymbol{a})^{\frac{1}{2}}}{[(\boldsymbol{x} - \boldsymbol{a})\cdot(\boldsymbol{x} - \boldsymbol{a})]^{\frac{1}{2}}} d\theta \end{aligned} \tag{101}$$

for $\boldsymbol{a} = (a_1, a_2, a_3)$ finite.

From (76), (101), and the fact that

$$\log \varphi = \log^+ \varphi - \log^+ \frac{1}{\varphi} ,$$

for $\boldsymbol{a}$ finite we obtain

$$\begin{aligned} m^\circ(r, \boldsymbol{a}; S) &= m^\circ(r, \infty; S) + m(r, \boldsymbol{a}; S) \\ &\quad - m(r, \infty; S - \boldsymbol{a}) + \log (1 + \boldsymbol{a}\cdot\boldsymbol{a})^{\frac{1}{2}} . \end{aligned} \tag{102}$$

Substituting from (102) into (80), we get

$$\begin{aligned} m^\circ(r, \infty; S) &+ N(r, \infty; S) + H(r, \infty; S) \\ &= m^\circ(r, \boldsymbol{a}; S) + N(r, \boldsymbol{a}; S) + H(r, \boldsymbol{a}; S) \\ &\quad + \log \frac{(\sum_{j=1}^{3} a_{j,\tau a}^2)^{\frac{1}{2}}}{(1 + \boldsymbol{a}\cdot\boldsymbol{a})^{\frac{1}{2}}} . \end{aligned} \tag{103}$$

We can write (103) as

$$\begin{aligned} m^\circ(r, \infty; S) &+ N(r, \infty; S) + H(r, \infty; S) + C(\infty; S) \\ &= m^\circ(r, \boldsymbol{a}; S) + N(r, \boldsymbol{a}; S) + H(r, \boldsymbol{a}; S) + C(\boldsymbol{a}; S) , \end{aligned} \tag{104}$$

where the constants $C(\infty; S)$ and $C(\boldsymbol{a}; S)$ are such that in (104) both the left-hand member and the right-hand member $\to 0$ as $r \to 0$. Namely, if there is not a pole of S at $r = 0$ then we have

$$\begin{aligned} C(\infty; S) &= \log \frac{1}{(1 + \sum_{j=1}^{3} a_{j,0}^2)^{\frac{1}{2}}} , \\ C(\boldsymbol{a}; S) &= \log \frac{(\sum_{j=1}^{3} a_{j,\tau_a}^2)^{\frac{1}{2}}}{(1 + \sum_{j=1}^{3} a_{j,0}^2)^{\frac{1}{2}}(1 + \boldsymbol{a}\cdot\boldsymbol{a})^{\frac{1}{2}}} , \end{aligned} \tag{105}$$

while if there is a pole of S at $r = 0$ then $t_a = t_0 < 0$ is independent of $\boldsymbol{a}$,

$$a_{j,\tau_a} = a_{j,t_0} , \qquad j = 1, 2, 3 ,$$

and we have

$$\begin{aligned} C(\infty, S) &= \log \frac{1}{(\sum_{j=1}^{3} a_{j,t_0}^2)^{\frac{1}{2}}} , \\ C(\boldsymbol{a}; S) &= \log \frac{1}{(1 + \boldsymbol{a}\cdot\boldsymbol{a})^{\frac{1}{2}}} . \end{aligned} \tag{106}$$

It might be noted that in complex-variable theory the equation analogous to (104) does not include the terms $H(r, \infty; S)$ and $H(r, \boldsymbol{a}; S)$, for then these terms are identically zero. Neither does the complex-variable equation ordinarily include constant terms $C(\infty; S)$ and $C(\boldsymbol{a}; S)$; here, however, the distinction is only notational, for the constants are then included in the definitions either [9] of $m^\circ(r, \infty, S)$ and $m^\circ(r, \boldsymbol{a}; S)$ or [15] of $N(r, \infty; S)$ and $N(r, \boldsymbol{a}; S)$.

We now define the *hyperspherical affinity* of S to $\boldsymbol{a}$ in $|w| \leqq r$, or the *hyperspherical affinity function* for S, by

$$\begin{aligned} \mathfrak{A}^\circ(r, \boldsymbol{a}; S) = m^\circ(r, \boldsymbol{a}; S) &+ N(r, \boldsymbol{a}; S) \\ &+ H(r, \boldsymbol{a}; S) + C(\boldsymbol{a}; S) . \end{aligned} \tag{107}$$

In particular, we call the hyperspherical affinity of S to ∞ the *hyperspherical characteristic function* of S and denote it by $T^0(r; S)$:

$$\begin{aligned} T^\circ(r; S) &= \mathfrak{A}^\circ(r, \infty; S) \\ &= m^\circ(r, \infty; S) + N(r, \infty; S) + H(r; \infty; S) + C(\infty; S) . \end{aligned} \tag{108}$$

Substituting from (107) and (108) in (104), we have the following generalization of the Ahlfors-Shimizu spherical form of the first fundamental theorem of Nevanlinna to meromorphic minimal surfaces:

THEOREM 2. *If the functions*

$$x_j = x_j(u, v) , \qquad j = 1, 2, 3 ,$$

are the coordinate functions of a nonconstant meromorphic minimal surface in isothermal representation for $u^2 + v^2 < \infty$, *then for each finite* $\boldsymbol{a} = (a_1, a_2, a_3)$ *we have*

$$T^\circ(r; S) = \mathfrak{A}^\circ(r, \boldsymbol{a}; S) , \tag{109}$$

where $\mathfrak{A}^\circ(r, \boldsymbol{a}; S)$ *is the hyperspherical affinity of* S *to* $\boldsymbol{a}$, *and* $T^0(r; S) = \mathfrak{A}^\circ(r, \infty; S)$ *is the hyperspherical characteristic function of* S.

7. Convexity properties. For

$$u^2 + v^2 = |w|^2 = |u + iv|^2 < \infty ,$$

let the functions (1) be the coordinate functions of a nonconstant meromorphic minimal surface S in isothermal representation. For $\boldsymbol{a}$ finite or infinite, let the $\boldsymbol{a}$-points of S in $0 < |w| \leqq r \leqq r$ be at the points $w = w_j$, $j = 1, 2, \cdots, k$, of moduli $r_1 \leqq r_2 \leqq \cdots \leqq r_k$, and let the respective orders of these $\boldsymbol{a}$-points be α_j, $j = 1, 2, \cdots, k$. Then

$$n(r, \boldsymbol{a}; S) = \alpha_0 + \alpha_1 + \cdots + \alpha_k ,$$

where $\alpha_0 \geqq 0$ is the order of the $\boldsymbol{a}$-point, if any, at $w = 0$.

Evaluating the integral in (77), we obtain

$$N(r, \boldsymbol{a}; S) = \log \frac{r^{n(r,\boldsymbol{a};S)}}{r_1^{\alpha_1} r_2^{\alpha_2} \cdots r_k^{\alpha_k}} . \tag{110}$$

From either (77) or (110), we see that $N(r, \boldsymbol{a}; S)$ is a continuous function of $\log r$. Further, by differentiating either (77) or (110), we get

$$\frac{dN(r, \boldsymbol{a}; S)}{d \log r} = n(r, \boldsymbol{a}; S) \tag{111}$$

except at the points of discontinuity of $n(r, \boldsymbol{a}; S)$. Accordingly, since $n(r, \boldsymbol{a}; S)$ is a nondecreasing, nonnegative function of $\log r$, we have the following result:

The function $N(r, \boldsymbol{a}; S)$ *is a nondecreasing, piecewise linear, convex function of* $\log r$.

Similarly, for $\boldsymbol{a}$ finite, by differentiating the second equation in (78) we obtain

$$\frac{dH(r, \boldsymbol{a}; S)}{d \log r} = h(r, \boldsymbol{a}; S) . \tag{112}$$

By (73), we have

$$h(r, \boldsymbol{a}; S) \geqq 0 ,$$

with equality for $r > 0$ if and only if S is a plane surface and $\boldsymbol{a}$ lies in the plane. Further, by (78), we have

$$H(r, \boldsymbol{a}; S) > 0 \quad \text{for} \quad r > 0 \quad \text{if} \quad h(r, \boldsymbol{a}; S) \not\equiv 0 \,.$$

Hence we have the following result:

The function $H(r, \boldsymbol{a}; S)$ vanishes identically if $\boldsymbol{a}$ is infinite or if S is a plane surface and $\boldsymbol{a}$ lies in the plane. Otherwise, for $r > 0$, $H(r, \boldsymbol{a}; S)$ is a positive, increasing, strictly convex function of $\log r$.

To determine the behavior of $T^\circ(r; S)$, we integrate (109) with respect to $\boldsymbol{a}$ over the three-dimensional hyperspherical "surface" $\mathscr{S}$ and divide by the content

$$V = 2\pi^2\left(\frac{1}{2}\right)^3 = \frac{\pi^2}{4}$$

of $\mathscr{S}$ to obtain

$$\begin{aligned} \frac{1}{V}\iiint_{\mathscr{S}} T^\circ(r; S)dV_{\boldsymbol{a}} &= \frac{1}{V}\iiint_{\mathscr{S}} m^\circ(r, \boldsymbol{a}; S)dV_{\boldsymbol{a}} \\ &+ \frac{1}{V}\iiint_{\mathscr{S}} N(r, \boldsymbol{a}; S)dV_{\boldsymbol{a}} + \frac{1}{V}\iiint_{\mathscr{S}} H(r, \boldsymbol{a}; S)dV_{\boldsymbol{a}} \\ &+ \frac{1}{V}\iiint_{\mathscr{S}} C(\boldsymbol{a}; S)dV_{\boldsymbol{a}} \,. \end{aligned} \tag{113}$$

The integrand $T^\circ(r, S)$ of the integral in the left-hand member of (113) does not vary with $\boldsymbol{a}$, and accordingly the value of this integral is $T^\circ(r; S)$.

The first integral in the right-hand member of (113) is

$$\begin{aligned} &\frac{1}{V}\iiint_{\mathscr{S}} m^\circ(r, \boldsymbol{a}; S)dV_{\boldsymbol{a}} \\ &= \frac{1}{V}\iiint_{\mathscr{S}}\left[\frac{1}{2\pi}\int_{\partial A_r(0)} \log\frac{1}{\chi(\boldsymbol{x}, \boldsymbol{a})}d\theta\right]dV_{\boldsymbol{a}} \\ &= \frac{1}{2\pi}\int_{\partial A_r(0)}\left[\frac{1}{V}\iiint_{\mathscr{S}} \log\frac{1}{\chi(\boldsymbol{x}, \boldsymbol{a})}dV_{\boldsymbol{a}}\right]d\theta \,, \end{aligned} \tag{114}$$

and here by geometric symmetry the inner integral in the last expression is the same for all $\boldsymbol{x} \in \mathscr{S}$. Accordingly, we can replace $\chi(\boldsymbol{x}, \boldsymbol{a})$ by $\chi(\boldsymbol{x}(0, 0), \boldsymbol{a})$ in (114). Except at $\boldsymbol{a} = \boldsymbol{x}(0, 0)$, by (105) and (106), we have

$$C(\boldsymbol{a}; S) = \log \chi(\boldsymbol{x}(0, 0), \boldsymbol{a}) \,.$$

Therefore the sum of the first and fourth integrals in the right-hand member of (113) is 0.

The function $N(r, \boldsymbol{a}; S)$ has value 0 for all $\boldsymbol{a}$ not on S. It therefore has value 0 everywhere except at most on a set of three-dimensional measure 0 on $\mathscr{S}$. Hence the second integral in the right-hand member of (113) has value 0.

It follows, accordingly, that (113) can be written as

$$T^{\circ}(r; S) = \frac{1}{V}\int_{\mathscr{S}} H(r, \boldsymbol{a}; S) dV_{\boldsymbol{a}} . \tag{115}$$

By (78), then, we have

$$\begin{aligned} T^{\circ}(r; S) &= \frac{1}{V}\iiint_{\mathscr{S}}\left[\int_0^r \frac{h(\rho, \boldsymbol{a}; S)}{\rho}\, d\rho\right] dV_{\boldsymbol{a}} \\ &= \int_0^r\left[\frac{1}{V}\iiint_{\mathscr{S}} h(\rho, \boldsymbol{a}; S) dV_{\boldsymbol{a}}\right]\frac{d\rho}{\rho} \\ &= \int_0^r \mathscr{P}(\rho, S)\frac{d\rho}{\rho} . \end{aligned} \tag{116}$$

By (73), the function

$$\mathscr{P}(\rho, S) = \frac{1}{V}\iiint_{\mathscr{S}} h(\rho, \boldsymbol{a}; S) dV_{\boldsymbol{a}}$$

is positive for $\rho > 0$ and is a strictly increasing function of $\log \rho$. It can be given a quasi-geometric interpretation, as indicated in the discussion of $h(\rho\, \boldsymbol{a}; S)$ in § 5.

From (116) we obtain

$$\frac{dT^{\circ}(r; S)}{d \log r} = \mathscr{P}(r; S) . \tag{117}$$

Since $\mathscr{P}(r; S)$ is positive for $r > 0$ and is a strictly increasing function of $\log r$, and since $T^{\circ}(0; S) = 0$, we therefore have the following result:

The function $T^{\circ}(r; S)$ *is positive for* $r > 0$ *and is an increasing, strictly convex function of* $\log r$.

It follows from (76), (81), and (101) that

$$0 \leqq m(r, \infty; S) < m^{\circ}(r, \infty, S) < m(r, \infty; S) + \log 2^{\frac{1}{2}} .$$

Therefore, by (90) and (108), *the difference* $T^{\circ}(r; S) - T(r; S)$ *is a bounded function of* r. Actually, it can be shown that, like $T^{\circ}(r; S)$, *the function* $T(r; S)$ *is an increasing, strictly convex function of* $\log r$.

The foregoing convexity properties are useful, in particular, in the study of problems of order and type [4] in the theory of meromorphic minimal surfaces.

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Received September 8, 1967. The preparation of this paper was sponsored in part by the National Science Foundation Grant GP-5279.

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