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LINEAR IDENTITIES IN GROUP RINGS

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Let $K[G]$ denote the group ring of a (not necessarily finite) group G and suppose that this ring satisfies a nontrivial polynomial identity of degree n. If Δ denotes the finite conjugate subgroup of G, then we show that $[G:\Delta] \leqq n!$. Furthermore, if $K[G]$ is semiprime, then G has an abelian subgroup of finite bounded index.

Several years ago this author worked on two seemingly unrelated group ring problems. In [9] I studied the question of the existence of nontrivial nilpotent ideals in group rings and the methods used were essentially combinatorial in nature. Later in [6] and [7], I. M. Isaacs and I studied group rings satisfying polynomial identities and the chief tool here was the ordinary character theory of finite groups. In her recent thesis [12] Martha Smith has observed that these two problems are in fact related and she applied the methods used in the first to obtain new results in the second. In this paper I take a more combinatorial and less ring theoretic approach than in [12] to the study of polynomial identities in group rings.

It occurred to me while writing this paper that I had the opportunity to include in one manuscript an elementary, essentially self-contained study of three distinct problems in group rings. These are the problems of finding necessary and sufficient conditions for $K[G]$ to be prime, semiprime and for $K[G]$ to satisfy a polynomial identity. I have availed myself of this opportunity, and therefore I have necessarily included here a number of results already in the literature. I hope that in doing this I have made this paper more enjoyable and interesting for the reader.

I would like to thank Miss Smith and her thesis advisor Professor I. N. Herstein for a number of stimulating conversations on this subject and for allowing me early access to [12].

1. First reduction. Let K be a field and let G be a (not necessarily finite) group. We let $K[G]$ denote the group ring of G over K. That is, $K[G]$ is a K-algebra with basis $\{x \mid x \in G\}$ and with multiplication defined distributively using the group multiplication in G.

If $\alpha = \Sigma k_x x \in K[G]$ we define the support of α to be

$$\operatorname{Supp} \alpha = \{x \in G \mid k_x \neq 0\} .$$

Then $\operatorname{Supp} \alpha$ is a finite subset of G.

Suppose for a moment that α is central in $K[G]$ and let $x \in \operatorname{Supp} \alpha$.

If $y \in G$ then

$$x^y = y^{-1}xy \in \operatorname{Supp} y^{-1}\alpha y = \operatorname{Supp} \alpha .$$

Since $\operatorname{Supp} \alpha$ is finite it follows that there are only a finite number of distinct x^y with $y \in G$. The set of all elements $x \in G$ with this property will be of great interest to us. We define

$$\Delta = \Delta(G) = \{x \in G \mid [G: \mathrm{C}_G(x)] < \infty\} .$$

Since the conjugates of x are in one to one correspondence with the right cosets of $\mathrm{C}_G(x)$ it follows that x has only finitely many conjugates if and only if $x \in \Delta$.

We can now observe that Δ is a normal subgroup of G. First $1 \in \Delta$ and since $\mathrm{C}_G(x) = \mathrm{C}_G(x^{-1})$ we see that $x \in \Delta$ implies $x^{-1} \in \Delta$. Finally, since a conjugate of xy is the product of a conjugate of x with one of y, it follows that if $x, y \in \Delta$ then $xy \in \Delta$. Thus Δ is a subgroup of G and it is clearly normal. It is called the F. C. (finite conjugate) subgroup of G.

The importance of Δ here is two-fold. First we are able to reduce the problems studied from $K[G]$ to $K[\Delta]$ and second we are able to handle the much simpler group Δ. In this section we consider the reduction to $K[\Delta]$ which will yield results on prime and semiprime group rings.

LEMMA 1.1. *Let $H_1, H_2, \cdots, H_n$ be subgroups of G of finite index. Then $H = H_1 \cap H_2 \cap \cdots \cap H_n$ has finite index in G and in fact*

$$[G: H] \leqq [G: H_1][G: H_2] \cdots [G: H_n] .$$

Proof. If Hx is a coset of H then clearly

$$Hx = H_1x \cap H_2x \cap \cdots \cap H_nx .$$

Since there are at most $[G: H_1][G: H_2] \cdots [G: H_n]$ choices for

$$H_1x, H_2x, \cdots, H_nx ,$$

the result follows.

LEMMA 1.2. *Let G be a group and let $H_1, H_2, \cdots, H_n$ be a finite number of subgroups. Suppose there exists a finite collection of elements $x_{ij} \in G$ $(i = 1, 2, \cdots, n;\ j = 1, 2, \cdots, f(i))$ with*

$$G = \bigcup\nolimits_{i,j} H_i x_{ij} ,$$

a set theoretic union. Then for some i, $[G: H_i] < \infty$.

Proof. By relabeling we can assume all the H_i to be distinct.

We prove the result by induction on n, the number of distinct H_i. The case $n = 1$ is clear.

If a full set of cosets of H_n appears among the $H_n x_{ni}$ then $[G: H_n] < \infty$ and we are finished. Otherwise if $H_n x$ is missing then

$$H_n x \subseteq \bigcup_{i,j} H_i x_{ij} .$$

But $H_n x \cap H_n x_{nj}$ is empty so $H_n x \subseteq \bigcup_{i\neq n,j} H_i x_{ij}$. Thus

$$H_n x_{nr} \subseteq \bigcup_{\substack{i\neq n \\ j}} H_i x_{ij} x^{-1} x_{nr}$$

and G can be written as a finite union of cosets of $H_1, H_2, \cdots, H_{n-1}$. By induction $[G: H_i] < \infty$ for some $i = 1, 2, \cdots, n-1$ and the result follows.

Let θ denote the projection $\theta: K[G] \to K[\Delta]$ given by

$$\alpha = \sum_{x \in G} k_x x \longrightarrow \theta(\alpha) = \sum_{x \in \Delta} k_x x .$$

Then θ is clearly a K-linear map but it is certainly not a ring homomorphism in general.

LEMMA 1.3. *Let $\alpha, \beta \in K[G]$ and suppose that for all $x \in G$ we have $\alpha x \beta = 0$. Then $\theta(\alpha)\theta(\beta) = 0$.*

Proof. We first show that $\theta(\alpha)\beta = 0$. Suppose, by way of contradiction, that $\theta(\alpha)\beta \neq 0$ and let $v \in \operatorname{Supp} \theta(\alpha)\beta$.

Suppose $\operatorname{Supp} \theta(\alpha) = \{u_1, u_2, \cdots, u_r\}$ and set $W = \bigcap \mathrm{C}_G(u_i)$. Since $u_i \in \Delta$, it follows from Lemma 1.1 that $[G: W] < \infty$.

Write $\alpha = \theta(\alpha) + \alpha'$ where $\operatorname{Supp} \alpha' \cap \Delta = \emptyset$ and then write the finite sums

$$\alpha' = \Sigma a_i y_i \qquad\qquad y_i \notin \Delta$$

$$\beta = \Sigma b_i z_i$$

with $a_i, b_i \in K$ and $y_i, z_i \in G$. If y_i is conjugate to some $v z_j^{-1}$ in G choose $h_{ij} \in G$ with $h_{ij}^{-1} y_i h_{ij} = v z_j^{-1}$. We show now that

$$(*) \qquad W \subseteq \bigcup_{i,j} \mathrm{C}_G(y_i) h_{ij} .$$

Let $x \in W$. Then

$$\begin{aligned} 0 = x^{-1}\alpha x \beta &= (x^{-1}\theta(\alpha)x + x^{-1}\alpha' x)\beta \\ &= \theta(\alpha)\beta + (x^{-1}\alpha' x)\beta \end{aligned}$$

since $x \in W$ implies that x centralizes $\theta(\alpha)$. Now v occurs in $\operatorname{Supp} \theta(\alpha)\beta$ and so this element must be cancelled by something from the second term. Thus there exists y_i, z_j with $v = x^{-1} y_i x z_j$ or

$$x^{-1}y_ix = vz_j^{-1} = h_{ij}^{-1}y_ih_{ij} \; .$$

Thus $xh_{ij}^{-1} \in \mathrm{C}_G(y_i)$ and $x \in \mathrm{C}_G(y_i)h_{ij}$ and (*) is proved.

Now $[G\colon W] < \infty$ so if $G = \cup Ww_k$ then by (*)

$$G = \bigcup\nolimits_{i,j,k} \mathrm{C}_G(y_i)h_{ij}w_k$$

a finite union of cosets. By Lemma 1.2, $[G\colon \mathrm{C}_G(y_i)] < \infty$ for some i, a contradiction since $y_i \notin \Delta$. Thus $\theta(\alpha)\beta = 0$.

Now Write $\beta = \theta(\beta) + \beta'$ where $\operatorname{Supp} \beta' \cap \Delta = \varnothing$. Then

$$0 = \theta(\alpha)\beta = \theta(\alpha)\theta(\beta) + \theta(\alpha)\beta' \; .$$

Since $\operatorname{Supp} \theta(\alpha)\theta(\beta) \subseteqq \Delta$ and $\operatorname{Supp} \theta(\alpha)\beta' \cap \Delta = \varnothing$ we have $\theta(\alpha)\theta(\beta) = 0$ and the result follows.

THEOREM 1.4. *(Passman [9]). Let A and B be ideals in $K[G]$ with $AB = 0$. Then $\theta(A)$ and $\theta(B)$ are ideals in $K[\Delta]$ and $\theta(A)\theta(B) = 0$.*

Proof. We show first that $\theta(A)$ is an ideal in $K[\Delta]$. Since

$$\theta(\alpha_1) + \theta(\alpha_2) = \theta(\alpha_1 + \alpha_2) \; ,$$

$\theta(A)$ is clearly closed under addition. Furthermore, if $\alpha \in A$ and $\gamma \in K[\Delta]$ then $\alpha\gamma \in A$, $\gamma\alpha \in A$ and we have easily

$$\theta(\alpha\gamma) = \theta(\alpha)\gamma,\ \theta(\gamma\alpha) = \gamma\theta(\alpha) \; .$$

Thus $\theta(A)$ is an ideal.

Now let $\alpha \in A, \beta \in B$. If $x \in G$ then $\alpha x \in A$ so $\alpha x\beta \in AB$ and $\alpha x\beta = 0$. By Lemma 1.3 we have $\theta(\alpha)\theta(\beta) = 0$ and hence $\theta(A)\theta(B) = 0$.

We remark that more generally if $A_1, A_2, \cdots, A_n$ are ideals in $K[G]$ with $A_1A_2 \cdots A_n = 0$, then $\theta(A_1)\theta(A_2) \cdots \theta(A_n) = 0$. A proof of this, in the more complicated context of twisted group rings, can be found in [11].

LEMMA 1.5. *Let A be an ideal in $K[G]$. Then $A \neq 0$ if and only if $\theta(A) \neq 0$.*

Proof. Certainly $\theta(A) \neq 0$ implies $A \neq 0$. Now suppose $A \neq 0$ and let $\alpha \in A, \alpha \neq 0$. If $x \in \operatorname{Supp} \alpha$ then since A is an ideal $x^{-1}\alpha \in A$ and $1 \in \operatorname{Supp} x^{-1}\alpha$. Thus $0 \neq \theta(x^{-1}\alpha) \in \theta(A)$ and $\theta(A) \neq 0$.

2. Prime rings.

A ring R is said to be prime if for any two ideals A, B in R, $AB = 0$ implies $A = 0$ or $B = 0$. In this section we consider the possibility of $K[G]$ being prime. We start by studying $\Delta(G)$.

LEMMA 2.1. *Let G be a group with a central subgroup Z of finite index. Then G', the commutator subgroup of G is finite.*

Proof. Let $(x, y) = x^{-1}y^{-1}xy$ denote commutators in G. Since $(x, y)^{-1} = (y, x)$ we see that G' is the set of all finite products of commutators and it is unnecessary to consider inverses.

Let $x_1, x_2, \cdots, x_n$ be coset representatives for Z in G and set $c_{ij} = (x_i, x_j)$. We observe first that these are all the commutators of G. Let $x, y \in G$ and say $x \in Zx_i$, $y \in Zx_j$. Then $x = ux_i$, $y = vx_j$ with u and v central in G. This yields easily $(x, y) = (x_i, x_j) = c_{ij}$.

Now let $x, y \in G$. Since Z is normal in G and G/Z has order n we have $(x, y)^n \in Z$. Thus

$$\begin{aligned}(x, y)^{n+1} &= x^{-1}y^{-1}xy(x, y)^n = x^{-1}y^{-1}x(x, y)^n y \\ &= x^{-1}y^{-1}x(x^{-1}y^{-1}xy)(x, y)^{n-1}y \\ &= x^{-1}y^{-2}xy^2 \cdot y^{-1}(x, y)^{n-1}y = (x, y^2)(y^{-1}xy, y)^{n-1}\end{aligned}$$

since conjugation by y being an automorphism of G implies that

$$y^{-1}(x, y)^{n-1}y = (y^{-1}xy, y^{-1}yy)^{n-1} = (y^{-1}xy, y)^{n-1} .$$

We show finally that every element of G' can be written as a product of at most n^3 commutators and this will yield the result. Suppose $u \in G'$ and $u = c_1c_2 \cdots c_m$ a product of m commutators. If $m > n^3$ then since there are at most n^2 distinct c_{ij} it follows that some c_{ij}, say $c = (x, y)$, occurs at least $n + 1$ times. We shift $n + 1$ of these successively to the left using

$$\begin{aligned}(x_r, x_s)(x, y) &= (x, y)c^{-1}(x_r, x_s)c \\ &= (x, y)(c^{-1}x_rc, c^{-1}x_sc)\end{aligned}$$

and obtain $u = (x, y)^{n+1}c'_{n+2}c'_{n+3} \cdots c'_m$ where each c'_i is a possibly new commutator. Using

$$(x, y)^{n+1} = (x, y^2)(y^{-1}xy, y)^{n-1}$$

we can then write u as a product of $m - 1$ commutators. Thus every element of G' is a product of at most n^3 of the c_{ij} and thus clearly G' is finite.

LEMMA 2.2. *Let H be a finitely generated subgroup of $\Delta(G)$. Then $[H\colon \mathbf{Z}(H)]$ and $|H'|$ are finite. Thus if $\Delta(G)$ contains no non-identity elements of finite order then $\Delta(G)$ is torsion free abelian.*

Proof. Let H be generated by $x_1, x_2, \cdots, x_n$. Since each x_i has only a finite number of conjugates in G, they have a finite number of conjugates in H. Hence $[H: \mathrm{C}_H(x_i)] < \infty$. By Lemma 1.1, $Z = \cap\, \mathrm{C}_H(x_i)$ has finite index in H. Since $x_1, x_2, \cdots, x_n$ generate H we see that Z is central in H. Thus by Lemma 2.1, H' is finite.

Now suppose $\Delta(G)$ has no nontrivial elements of finite order and let $x, y \in \Delta(G)$. Set $H = \langle x, y\rangle$. Since H is finitely generated the above implies that H' is finite and hence $H' = \langle 1\rangle$. Thus x and y commute and $\Delta(G)$ is abelian. By definition $\Delta(G)$ is torsion free.

LEMMA 2.3. *Group G has a finite normal subgroup H whose order is divisible by a prime p if and only if $\Delta(G)$ contains an element of order p.*

Proof. Let H be given. Since $p \mid |H|$, H contains an element x of order p. Since H is normal in G, all conjugates of x are contained in H and hence $x \in \Delta$.

Now let $x \in \Delta$ have order p. Let $x_1 = x, x_2, \cdots, x_n$ be the finite number of distinct conjugates of x. If $H = \langle x_1, x_2, \cdots, x_n\rangle$ then $H \subseteq \Delta$ and H is normal in G since conjugation by an element of G merely permutes the generators of H. By Lemma 2.2, H' is finite. Now H/H' is a finitely generated abelian group generated by elements of finite order. Thus H/H' is finite and H is finite. Since $x \in H$, $p \mid |H|$ and the result follows.

LEMMA 2.4. *Let H be a torsion free abelian subgroup of G and let $\alpha \in K[H] \subseteq K[G]$ with $\alpha \neq 0$. Then α is not a zero divisor in $K[G]$.*

Proof. We show that $\alpha\beta = 0$ implies that $\beta = 0$. An analogous proof works in the other direction. Suppose $\alpha\beta = 0$. We can choose $y_1, y_2, \cdots, y_k$ in distinct right cosets of H in G so that

$$\beta = \beta_1 y_1 + \beta_2 y_2 + \cdots + \beta_k y_k$$

with $\beta_i \in K[H]$. Then

$$0 = \alpha\beta = (\alpha\beta_1)y_2 + (\alpha\beta_2)y_2 + \cdots + (\alpha\beta_k)y_k$$

and since $\alpha\beta_i \in K[H]$ we have clearly $\alpha\beta_i = 0$. Thus it suffices to show that $\alpha\beta_i = 0$ implies $\beta_i = 0$ or equivalently we can assume that $G = H$ is a torsion free abelian group.

Assume then that $G = H$. Now there clearly exists a finitely generated subgroup $W \subseteq G$ with $\alpha, \beta \in K[W]$. Thus we may also assume that $G = W$ is finitely generated. By the fundamental theorem of abelian groups $G = \langle x_1\rangle \times \langle x_2\rangle \times \cdots \times \langle x_n\rangle$, a finite direct product of infinite cyclic groups. Then $K[G]$ is essentially a polynomial ring in the variables $x_1, x_2, \cdots, x_n$ except that negative exponents are also allowed. It is now obvious that $K[G]$ is an integral domain so $\alpha\beta = 0$ implies $\beta = 0$.

THEOREM 2.5. (*Connell* [4]). *The following are equivalent*:
(i) *$K[G]$ is prime.*
(ii) *$\Delta(G)$ is torsion free abelian.*
(iii) *G has no nonidentity finite normal subgroup.*

Proof. (i) $\Rightarrow$ (iii). Suppose G has a nonidentity finite normal subgroup H. Set

$$\alpha = \sum_{x \in H} x \in K[G] .$$

Since H is normal in G, $y^{-1}Hy = H$ for all $y \in G$ and thus $y^{-1}\alpha y = \alpha$. Hence α is central in $K[G]$ and clearly $\alpha \neq 0$.

If $y \in H$ then $yH = H$ so $y\alpha = \alpha$. This yields

$$\alpha^2 = \left(\sum_{x \in H} x\right)\alpha = |H|\alpha$$

and hence $(\alpha - |H|)\alpha = 0$. Since $H \neq \langle 1 \rangle$ we have clearly $\alpha - |H|^1 \neq 0$. Set

$$A = (\alpha - |H|)K[G] , \qquad B = \alpha K[G] .$$

Since α is central these are both nonzero ideals. Moreover, clearly $AB = 0$ so $K[G]$ is not prime, a contradiction. Hence H does not exist.

(iii) $\Rightarrow$ (ii). By Lemma 2.3, $\Delta(G)$ has no nonidentity elements of finite order and then by Lemma 2.2, $\Delta(G)$ is torsion free abelian.

(ii) $\Rightarrow$ (i). Let A and B be ideals in $K[G]$ with $AB = 0$. By Theorem 1.4 we have $\theta(A)\theta(B) = 0$ and hence by Lemma 2.4 either $\theta(A) = 0$ or $\theta(B) = 0$. The result follows from Lemma 1.5.

3. Semiprime rings. Let R be a ring. An ideal P of R is said to be prime if R/P is a prime ring. Thus P is prime if and only if for all ideals $A, B \subseteq R$ we have $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. R is said to be semiprime if the intersection of all prime ideals of R is 0. In particular, R is semiprime if and only if it is a subdirect product of prime rings.

LEMMA 3.1. *Ring R is semiprime if and only if R contains no nonzero ideal with square* 0.

Proof. Suppose R contains a nonzero ideal A of square 0. If P is any prime ideal in R then $A \cdot A = 0 \subseteqq P$ so $A \subseteqq P$. Hence A is contained in the intersection of all such prime ideals and R is not semiprime.

Now suppose that R contains no nonzero ideal of sequare 0. Let $\alpha \in R$, $\alpha \neq 0$. We define a sequence $T = \{\alpha_1, \alpha_2, \cdots, \alpha_n, \cdots\}$ or nonzero elements of R inductively as follows. First $\alpha_1 = \alpha$. Second given $\alpha_n \neq 0$ then the ideal $R\alpha_n R$ does not have square 0. Thus for some $\beta_n \in R$ we have $\alpha_n \beta_n \alpha_n \neq 0$. Set $\alpha_{n+1} = \alpha_n \beta_n \alpha_n$. Since $0 \notin T$ it follows that T is disjoint from some ideal of R namely 0. By Zorn's lemma there exists an ideal P of R maximal with respect to $P \cap T = \emptyset$. We show that P is prime. Let A and B be ideals of R with $A \not\subseteqq P$, $B \not\subseteqq P$. Then $P + A$ and $P + B$ properly contain P so by the maximality of P, it follows that for some i, j we have $\alpha_i \in P + A$, $\alpha_j \in P + B$. If $m = \max(i, j)$ then clearly $\alpha_m \in P + A$, $\alpha_m \in P + B$ so

$$\alpha_{m+1} = \alpha_m \beta_m \alpha_m \in (P + A)(P + B) \subseteqq P + AB .$$

Since $\alpha_{m+1} \notin P$ we have $AB \not\subseteqq P$ and P is prime. Since $\alpha = \alpha_1 \notin P$ the result follows.

An element $\alpha \in R$ is said to be nilpotent if $\alpha^n = 0$ for some positive integer n. An ideal I of R is nil if all elements of I are nilpotent.

THEOREM 3.2. (*Pascual Jordan*). *Suppose that K is a subfield of the complex numbers which is closed under complex conjugation. Then $K[G]$ contains no nonzero nil ideal.*

Proof. Let * denote complex conjugation and extend * to a map of $K[G]$ to itself by

$$\alpha = \sum_{x \in G} k_x x \longrightarrow \alpha^* = \sum_{x \in G} k_x^* x^{-1} .$$

Clearly $(\alpha^*)^* = \alpha$ and $(\alpha\beta)^* = \beta^*\alpha^*$. In addition, the coefficient of $1 \in G$ in $\alpha\alpha^*$ is $\sum_{x \in G} |k_x|^2$ and thus $\alpha\alpha^* = 0$ if and only if $\alpha = 0$.

Let I be a nil ideal in $K[G]$ and let $\alpha \in I$. Since I is an ideal we have $\alpha\alpha^* \in I$ and hence for some $n \geqq 1$, $(\alpha\alpha^*)^n = 0$. Let n be minimal with this property. Suppose that $n > 1$ and set $\beta = (\alpha\alpha^*)^{n-1}$. Clearly $\beta^* = \beta$ so we have $\beta\beta^* = (\alpha\alpha^*)^{2n-2} = 0$ since $2n - 2 \geqq n$. Thus $\beta = 0$ by the above, contradicting the minimality of n. This shows that $n = 1$, $\alpha\alpha^* = 0$ and hence $\alpha = 0$. Thus $I = 0$.

We remark that $K[G]$ has no nonzero nil ideals if K is any field of characteristic 0 (see [9], Th. II). However, the above is quite sufficient for our purposes.

THEOREM 3.3. *Let K be a field of characteristic* 0. *Then $K[G]$ is semiprime.*

Proof. Suppose $K[G]$ is not semiprime. Then by Lemma 3.1, $K[G]$ contains a nonzero ideal A with $A^2 = 0$. Let $\alpha = \sum_{i=1}^{n} k_i x_i \in A$, $\alpha \neq 0$ and let F be a subfield of K generated over the rationals by $k_1, k_2, \cdots, k_n$. Then $F[G] \subseteq K[G]$ and $A \cap F[G]$ is a nonzero ideal of $F[G]$ of square zero. Thus it clearly suffices to assume that $K = F$ or equivalently that K is finitely generated over the rationals. This implies that K is contained in the complex numbers C and we fix an imbedding. Then $K[G] \subseteq C[G]$ and AC is a nonzero ideal of $C[G]$ with square zero. This is a contradiction by Theorem 3.2 and the result follows.

We now consider fields of characteristic $p > 0$. Let R be a ring. We set $[R, R]$ equal to the set of all finite sums of Lie products

$$[\alpha, \beta] = \alpha\beta - \beta\alpha$$

with $\alpha, \beta \in R$.

LEMMA 3.4. *Let E be an algebra over a field K of characteristic $p > 0$ and let k and n be positive integers. If $\alpha_1, \alpha_2, \cdots, \alpha_n \in E$ then*

$$(\alpha_1 + \alpha_2 + \cdots + \alpha_n)^{p^k} = \alpha_1^{p^k} + \alpha_2^{p^k} + \cdots + \alpha_n^{p^k} + \beta$$

for some $\beta \in [E, E]$.

Proof. Observe that

$$(\alpha_1 + \alpha_2 + \cdots + \alpha_n)^{p^k} = \alpha_1^{p^k} + \alpha_2^{p^k} + \cdots + \alpha_n^{p^k} + \beta$$

where β is the sum of all words $\alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_{p^k}}$ with at least two distinct subscripts occurring. If words ω_1 and ω_2 are cyclic permutations of each other, that is, if

$$\omega_1 = \alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_{p^k}}$$

$$\omega_2 = \alpha_{i_j}\alpha_{j+1} \cdots \alpha_{i_{p^k}}\alpha_{i_1} \cdots \alpha_{i_{j-1}}$$

then $\omega_1 - \omega_2 = \gamma\delta - \delta\gamma \in [E, E]$ where

$$\gamma = \alpha_{i_1}\alpha_{i_2} \cdots \alpha_{i_{j-1}} \quad \text{and} \quad \delta = \alpha_{i_j}\alpha_{i_{j+1}} \cdots \alpha_{i_{p^k}} .$$

Hence modulo $[E, E]$ all cyclic permutations of a word ω are equal. For convenience we let the cyclic group Z_{p^k} act on the set of these words by performing the cyclic shifts. Then the number of formally distinct permutations of a word ω occurring in β is the size of a nontrivial orbit of Z_{p^k} and hence is divisible by p. Since K has characteristic p, the result follows.

THEOREM 3.5. (*Passman* [9], *Connell* [4]). *Let K be a field of characteristic $p > 0$ and let G have no elements of order p. Then $K[G]$ has no nonzero nil ideals.*

Proof. If $\alpha = \Sigma k_x x \in K[G]$ we set $\tau(\alpha) = k_1$, the coefficient of 1. τ is clearly a K-linear map of $K[G]$ onto K. Now $[K[G], K[G]]$ is spanned over K by all Lie products of the form $[x, y]$ with $x, y \in G$. Furthermore, if $\tau([x, y]) \neq 0$ then certainly $y = x^{-1}$ and then

$$[x, y] = xx^{-1} - x^{-1}x = 0 ,$$

a contradiction. Hence $\tau([K[G], K[G]]) = 0$.

Let I be a nontrivial nil ideal in $K[G]$ and let $\alpha = \Sigma k_x x \in I - \{0\}$. Then for some x, $k_x \neq 0$. Since I is an ideal $x^{-1}\alpha \in I$ and clearly $\tau(x^{-1}\alpha) = k_x \neq 0$. Thus we may assume that $\tau(\alpha) \neq 0$. Say

$$\alpha = k_1 1 + k_2 x_2 + \cdots + k_n x_n$$

where $k_i \in K$, $k_1 \neq 0$ and the x_i are distinct nonidentity elements of G. Since $\alpha^m = 0$ for some $m > 0$ it follows that $\alpha^{p^k} = 0$ for some integer $k > 0$. By Lemma 3.4

$$0 = \alpha^{p^k} = (k_1 1)^{p^k} + (k_2 x_2)^{p^k} + \cdots + (k_n x_n)^{p^k} + \beta$$

where $\beta \in [K[G], K[G]]$. Since $0 = \tau(0) = \tau(\beta)$ and

$$\tau((k_1 1)^{p^k}) = k_1^{p^k} \neq 0$$

we conclude that for some $i = 2, 3, \cdots, n$, $\tau((k_i x_i)^{p^k}) \neq 0$. Thus $x_i \neq 1$, $x_i^{p^k} = 1$ and G has an element of order p, a contradiction.

The converse to Theorem 3.5 is decidedly false. Namely, there are many examples of groups G with elements of order p such that $K[G]$ has no nontrivial nil ideals. (See, for example, [9] and [10].)

THEOREM 3.6. (*Passman* [9]). *Let K be a field of characteristic $p > 0$. The following are equivalent.*

(i) *$K[G]$ is semiprime.*

(ii) *$\Delta(G)$ has no elements of order p.*

(iii) *G has no finite normal subgroups with order divisible by p.*

Proof. (i) $\Rightarrow$ (iii) Suppose G has a finite normal subgroup H with $p \mid |H|$. Set

$$\alpha = \Sigma_{x \in H} x \in K[G] .$$

As in the proof of Theorem 2.5 we see that $\alpha \neq 0$, α is central in $K[G]$ and $\alpha^2 = |H|\alpha$. Now $p \,|\, |H|$ and K has characteristic p so $|H| = 0$ in K. Thus if $A = \alpha K[G]$, then A is a nonzero ideal of $K[G]$ and $A^2 = 0$. By Lemma 3.1 $K[G]$ is not semiprime, a contradiction. Hence H does not exist.

(iii) $\Rightarrow$ (ii). This follows from Lemma 2.3.

(ii) $\Rightarrow$ (i). Let A be an ideal in $K[G]$ with $A^2 = 0$. Then by Theorem 1.4, $\theta(A)$ is an ideal in $K[\Delta]$ with $\theta(A)^2 = 0$. Now Δ has no elements of order p so by Theorem 3.5, $\theta(A) = 0$. Hence by Lemma 1.5 we have $A = 0$ and $K[G]$ is semiprime by Lemma 3.1.

An ideal A is said to be nilpotent if $A^n = A \cdot A \cdot \cdots \cdot A = 0$ for some integer $n \geqq 1$. If A is such a nonzero ideal, then certainly a suitable power of A is a nonzero ideal of square zero. Thus if K has characteristic $p > 0$ then by Lemma 3.1 and Theorem 3.6 we see that $K[G]$ has a nonzero nilpotent ideal if and only if $\Delta(G)$ contains an element of order p. It is shown in [11] that $K[G]$ has a unique maximal nilpotent ideal if and only if $\Delta(G)$ contains just finitely many elements whose order is a power of p.

4. **Examples.** Let $K[\zeta_1, \zeta_2, \cdots]$ be the polynomial ring over K in the noncommuting indeterminates $\zeta_1, \zeta_2, \cdots$. An algebra E over K is said to satisfy a polynomial identity if there exists

$$f(\zeta_1, \zeta_2, \cdots, \zeta_n) \in K[\zeta_1, \zeta_2, \cdots] ,$$

$f \neq 0$ with

$$f(\alpha_1, \alpha_2, \cdots, \alpha_n) = 0$$

for all $\alpha_1, \alpha_2, \cdots, \alpha_n \in E$. For example, any commutative algebra satisfies $f(\zeta_1, \zeta_2) = \zeta_1\zeta_2 - \zeta_2\zeta_1$.

The standard polynomial of degree n is defined by

$$[\zeta_1, \zeta_2, \cdots, \zeta_n] = \sum_{\sigma \in S_n} (-1)^\sigma \zeta_{\sigma(1)}\zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)} .$$

Here S_n is the symmetric group of degree n and $(-1)^\sigma$ is 1 or -1 according as σ is an even or an odd permutation.

LEMMA 4.1. *Let E be a commutative algebra over a field K and let E_n denote the ring of $n \times n$ matrices over E. Then E_n satisfies the standard polynomial identity of degree $n^2 + 1$.*

Proof. Now E_n has a basis $\{\beta_1, \beta_2, \cdots, \beta_{n^2}\}$ over E of size n^2. Since E is central in E_n and since $[\zeta_1, \zeta_2, \cdots, \zeta_{n^2+1}]$ is linear in each variable it clearly suffices to verify that

$$[\beta_{i_1}, \beta_{i_2}, \cdots, \beta_{i_{n^2+1}}] = 0 .$$

However, since here there are only n^2 distinct β_i we must have two of the above variables equal. The result now follows since it is obvious from the form of the standard polynomial, that if two variables are equal then the polynomial vanishes.

It is in fact true that E_n satisfies the standard polynomial identity of degree $2n$ (see [2]) and by using this stronger result we could strengthen the next theorem.

THEOREM 4.2. (*Kaplansky* [8], *Amitsur* [1]). *Let G have an abelian subgroup A with $[G: A] = n < \infty$. Then $K[G]$ satisfies the standard polynomial identity of degree $n^2 + 1$.*

Proof. Let $x_1, x_2, \cdots, x_n$ be a set of right coset representatives of A in G. Let $E = K[A]$ and $V = K[G]$. Then clearly V is a left E-module with basis $\{x_1, x_2, \cdots, x_n\}$. Now V is also a right $K[G]$-module and as such it is faithful. Since right and left multiplication commute as operators on V, it follows that $K[G]$ is a set of E-linear transformations on a n-dimensional free E-module V. Thus $K[G] \subseteq E_n$ and the result follows from Lemma 4.1.

We will see later that a reasonable converse to the above holds. However we consider some examples now to show that a converse need not hold in all situations.

LEMMA 4.3. *Let E be an algebra over K and suppose that $[E, E]^n = 0$. Then E satisfies the standard polynomial identity of degree $2n$.*

Proof. Let $\alpha_1, \alpha_2, \cdots, \alpha_{2n} \in E$ and consider

$$[\alpha_1, \alpha_2, \cdots, \alpha_{2n}] = \sum_\sigma (-1)^\sigma \alpha_{\sigma(1)}\alpha_{\sigma(2)} \cdots \alpha_{\sigma(2n)} .$$

Consider all such terms on the right hand side with

$$\{\sigma(1), \sigma(2)\} = \{i_1, i_2\}, \{\sigma(3), \sigma(4)\} = \{i_3, i_4\}, \cdots ,$$

$\{\sigma(2n-1), \sigma(2n)\} = \{i_{2n-1}, i_{2n}\}$ where of course

$$\{i_1, i_2, \cdots, i_{2n}\} = \{1, 2, \cdots, 2n\} .$$

Then the subsum Σ' of all these terms is easily seen to be equal to

$$\Sigma' = \pm[\alpha_{i_1}, \alpha_{i_2}][\alpha_{i_3}, \alpha_{i_4}] \cdots [\alpha_{i_{2n-1}}, \alpha_{i_{2n}}] = 0$$

since $[E, E]^n = 0$. Thus the result clearly follows.

LEMMA 4.4. *Let K be a field of characteristic $p > 0$ and let G be a group with $|G'| = p$ and G' central in G. Then $K[G]$ satisfies* *the standard polynomial identity of degree* $2p$.

Proof. Since $|G'| = p$, $G' = \langle z \rangle$ is cyclic. We show first that

$$[K[G], K[G]] \subseteqq (1 - z)K[G] .$$

Now $[K[G], K[G]]$ is spanned over K by elements of the form $[x, y]$ with $x, y \in G$. For $x, y \in G$ we have

$$\begin{aligned}[x, y] &= xy - yx = (1 - yxy^{-1}x^{-1})xy \\ &= (1 - z^i)xy = (1 - z)(1 + z + \cdots + z^{i-1})xy\end{aligned}$$

for some $i > 0$ since $yxy^{-1}x^{-1} \in G' = \langle z \rangle$. Thus $[x, y] \in (1 - z)K[G]$ and this fact follows.

Now K has characteristic p and $z^p = 1$ so $(1 - z)^p = 1 - z^p = 0$. Since z is central in G we have $((1 - z)K[G])^p = 0$ and the result follows from Lemma 4.3.

THEOREM 4.5. *Let K be a field of characteristic $p > 0$. Then there exists a sequence of finite p-groups $P_1, P_2, \cdots, P_n, \cdots$ and an infinite p-group P_∞ such that*

(i) *For all $\nu = 1, 2, \cdots, \infty$, $K[P_\nu]$ satisfies the standard polynomial identity of degree $2p$.*

(ii) *P_n has no abelian subgroup of index $< p^n$.*

(iii) *P_∞ has no abelian subgroup of finite index.*

Proof. Let Q be a nonabelian group of order p^3. Then Z, the center of Q, has order p, Q/Z is abelian of type (p, p) and $Q' = Z$. Let $Q_1, Q_2, Q_3, \cdots$ be copies of Q with centers $Z_1, Z_2, Z_3, \cdots$ and say $Z_i = \langle z_i \rangle$. For each integer n set

$$G_n = Q_1 \times Q_2 \times \cdots \times Q_n$$

and set

$$G_\infty = Q_1 \times Q_2 \times \cdots \times Q_n \times \cdots .$$

We have clearly $G'_\nu = \mathbf{Z}(G_\nu) = Z_1 \times Z_2 \times \cdots$. Now let N_ν be the subgroup of $\mathbf{Z}(G_\nu)$ generated by the elements $z_2z_1^{-1}, z_3z_1^{-1}, z_4z_1^{-1}, \cdots$. Then N_ν is a central and hence a normal subgroup of G_ν and we set

$$P_n = G_n/N_n , \qquad P_\infty = G_\infty/N_\infty .$$

Clearly $P'_\nu \subseteqq \mathbf{Z}(G_\nu)/N_\nu$ and the latter group has order p. Thus $|P'_\nu| \leqq p$ and P'_ν is central so (i) follows by Lemma 4.4. We observe now that $\mathbf{Z}(P_\nu) = \mathbf{Z}(G_\nu)/N_\nu$. For suppose $x = x_1x_2 \cdots \in G_\nu - \mathbf{Z}(G_\nu)$. Then for some i, $x_i \notin Z_i$ and hence there exists $y_i \in Q_i$ which does not centralize x_i. Then $y_i \in G_\nu$ and

$$(x, y_i) = x^{-1}y_i^{-1}xy_i = x_i^{-1}y_i^{-1}x_iy_i \ .$$

is a nonidentity element of Z_i. Since clearly $Z_i \cap N_\nu = \langle 1 \rangle$ we see that the images of x and of y_i do not commute in P_ν. This yields $[P_n: \mathbf{Z}(P_n)] = p^{2n}$ and $[P_\infty: \mathbf{Z}(P_\infty)] = \infty$.

Suppose A is an abelian subgroup of P_ν of finite index p^t and set $B = A\mathbf{Z}(P_\nu)$. Then B is abelian of index $\leqq p^t$ and B is normal in P_ν since $B \supseteqq \mathbf{Z}(P_\nu) = P'_\nu$. Now P_ν/B is clearly elementary abelian and we can choose $w_1, w_2, \cdots, w_t \in P_\nu$ with $P_\nu = \langle B, w_1, w_2, \cdots, w_t \rangle$. If $y \in P_\nu$ then $y^{-1}w_iy = w_i(w_i, y) \in w_iP'_\nu$. Hence since $|P'_\nu| = p$ we see that w_i has at most p conjugates in P_ν and $[P_\nu: \mathbf{C}_{P_\nu}(w_i)] \leqq p$. Thus by Lemma 1.1 if

$$W = B \cap \mathbf{C}_{P_\nu}(w_1) \cap \mathbf{C}_{P_\nu}(w_2) \cap \cdots \cap \mathbf{C}_{P_\nu}(w_t)$$

then $[P_\nu: W] \leqq p^t \cdot p \cdot p \cdot \cdots \cdot p = p^{2t}$. Now B is abelian so W centralizes B and all the w_i and hence $W = \mathbf{Z}(P_\nu)$. Since $[P_\infty: \mathbf{Z}(P_\infty)] = \infty$, (iii) follows and since $[P_n: \mathbf{Z}(P_n)] = p^{2n}$ we have $t \geqq n$ and (ii) follows. This completes the proof.

5. **Second reduction.** We now obtain a refinement of the reduction of § 1 which is applicable to studying polynomial identities.

LEMMA 5.1. *Let G be a group and suppose that G can be written as $G = \cup H_ix_{ij}$ a finite union of cosets. Then $G = \cup' H_ix_{ij}$ where the union is restricted to those H_i with $[G: H_i] < \infty$.*

Proof. Let $\mathscr{S} = \{i \mid [G: H_i] < \infty\}$ and let $\mathfrak{T} = \{i \mid [G: H_i] = \infty\}$. By Lemma 1.2, $\mathscr{S} \neq \varnothing$. Let $W = \bigcap_{i \in \mathscr{S}} H_i$. Then $[G: W] < \infty$ by Lemma 1.1 and each coset H_ix_{ij} with $i \in \mathscr{S}$ is a finite union of cosets of W. Thus

$$\cup' H_ix_{ij} = \bigcup_{i \in \mathscr{S}} H_ix_{ij} = \cup\, Wy_k$$

a finite union of cosets of W. If $G \neq \cup' H_ix_{ij}$ then $G \neq \cup\, Wy_k$ and some coset Wy is missing. Then

$$Wy \subseteqq (\cup\, Wy_k) \cup \Big(\bigcup_{i \in \mathfrak{T}} H_ix_{ij} \Big)$$

and since $Wy \cap Wy_k$ is empty we have $Wy \subseteqq \bigcup_{i \in \mathfrak{T}} H_ix_{ij}$. Thus all cosets of W are contained in finite unions of cosets of those H_i with $i \in \mathfrak{T}$. Since $[G: W] < \infty$ this yields a representation of G as a finite union of cosets of those H_i with $i \in \mathfrak{T}$. This contradicts Lemma 1.2 and thus $G = \cup' H_ix_{ij}$.

LEMMA 5.2. *Let $G \neq \cup H_mg_{mn}$, a finite union of cosets. Let*

$$\alpha_1, \alpha_2, \cdots, \alpha_s, \beta_1, \beta_2, \cdots, \beta_s \in K[G]$$

and suppose that for all $x \in G - \cup H_m g_{mn}$ *we have*

$$\alpha_1 x\beta_1 + \alpha_2 x\beta_2 + \cdots + \alpha_s x\beta_s = 0 .$$

Then there exists $y \in G$ *with*

$$\theta(\alpha_1)^y\beta_1 + \theta(\alpha_2)^y\beta_2 + \cdots + \theta(\alpha_s)^y\beta_s = 0 .$$

Proof. Let W be the intersection of the centralizers of all elements in $\mathrm{Supp}\,\theta(\alpha_i)$ for $i = 1, 2, \cdots, s$. By Lemma 1.1, $[G: W] = t < \infty$. Clearly if $x \in W$ then x centralizes $\theta(\alpha_1), \theta(\alpha_2), \cdots, \theta(\alpha_s)$. Let $\{u_i\}$ be a set of coset representatives for W in G. Let us suppose by way of contradiction that for $i = 1, 2, \cdots, t$

$$\gamma_i = \theta(\alpha_1)^{u_i}\beta_1 + \theta(\alpha_2)^{u_i}\beta_2 + \cdots + \theta(\alpha_s)^{u_i}\beta_s \neq 0$$

and let $v_i \in \mathrm{Supp}\,\gamma_i$.

Write $\alpha_j = \theta(\alpha_j) + \alpha'_j$ where $\mathrm{Supp}\,\alpha'_j \cap \Delta = \varnothing$ and then write the finite sums

$$\alpha'_j = \Sigma a_{jk}y_k , \qquad y_k \notin \Delta$$
$$\beta_j = \Sigma b_{jk}z_k .$$

If y_j is conjugate to some $v_i z_k^{-1}$ in G choose $h_{ijk} \in G$ with $h_{ijk}^{-1}y_j h_{ijk} = v_i z_k^{-1}$.

Let $x \in G$ and suppose that $x \notin \cup H_m g_{mn}$. Then we must have

$$\begin{aligned} 0 &= x^{-1}\alpha_1 x\beta_1 + x^{-1}\alpha_2 x\beta_2 + \cdots + x^{-1}\alpha_s x\beta_s \\ &= [\theta(\alpha_1)^x\beta_1 + \theta(\alpha_2)^x\beta_2 + \cdots + \theta(\alpha_s)^x\beta_s] \\ &\quad + [\alpha_1'^x\beta_1 + \alpha_2'^x\beta_2 + \cdots + \alpha_s'^x\beta_s] . \end{aligned}$$

Since $\{u_i\}$ is a full set of coset representatives of W in G we have $x \in Wu_i$ for some i. Since W centralizes $\theta(\alpha_1), \theta(\alpha_2), \cdots, \theta(\alpha_s)$ the first expression above is equal to γ_i. Hence

$$0 = \gamma_i + [\alpha_1'^x\beta_1 + \alpha_2'^x\beta_2 + \cdots + \alpha_s'^x\beta_s] .$$

Now v_i occurs in the support of γ_i and so this element must be cancelled by something from the second term. Thus there exists y_j, z_k with $v_i = y_j^x z_k$ or

$$x^{-1}y_j x = v_i z_k^{-1} = h_{ijk}^{-1}y_j h_{ijk} .$$

Thus $x \in \mathrm{C}_G(y_j)h_{ijk}$. We have therefore shown that

$$G = (\cup H_m g_{mn}) \cup (\cup \mathrm{C}_G(y_j)h_{ijk})$$

a finite union of cosets. Now $y_j \notin \Delta$ so $[G: \mathrm{C}_G(y_j)] = \infty$. Since, by

Lemma 5.1, we can delete subgroups of infinite index from the above we have $G = \cup H_m g_{mn}$, a contradiction. The lemma is proved.

It is obvious from the above that we can handle linear identities in $K[G]$. Thus we need the following.

LEMMA 5.3. *Suppose E is an algebra over a field K which satisfies a nontrivial polynomial identity of degree n. Then E satisfies the polynomial identity $f \in K[\zeta_1, \zeta_2, \cdots, \zeta_n]$ with*

$$f(\zeta_1, \zeta_2, \cdots, \zeta_n) = \sum_{\sigma \in S_n} \alpha_\sigma \zeta_{\sigma(1)} \zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)}$$

where $a_\sigma \in K$ and they are not all zero.

Proof. A monomial in $K[\zeta_1, \zeta_2, \cdots]$ is an element of the form $\zeta_{i_1} \zeta_{i_2} \cdots \zeta_{i_r}$. These of course form a basis for $K[\zeta_1, \zeta_2, \cdots]$ over K.

Let $g = g(\zeta_1, \zeta_2, \cdots)$ be the given polonomial of degree n satisfied by E. Suppose some variable ζ_i occurs in some but not all of the monomials in the expression for g. Then $g = g' + g''$ where ζ_i occurs in all the monomials of g' and in none of g''. Then $g'' \neq 0$, degree $g'' \leqq n$ and $g''(\zeta_1, \zeta_2, \cdots, \zeta_i, \cdots) = g(\zeta_1, \zeta_2, \cdots, 0, \cdots)$ so g'' is also clearly a polynomial identity for E. We continue in this manner reducing the number of variables involved until we obtain a nonzero polynomial h of degree $\leqq n$ with the property that each variable ζ_i which occurs in h in fact occurs in each monomial. Since degree $h \leqq n$ we see that h is a function of at most n variables. By changing notation if necessary we may assume that $h \in K[\zeta_1, \zeta_2, \cdots, \zeta_n]$.

Let $\mathscr{H}$ be the set of all $h \in K[\zeta_1, \zeta_2, \cdots, \zeta_n]$, $h \neq 0$ which are polynomial identities for E of degree $\leqq n$ and for which all variables which are involved in h occur in each monomial. We choose $f \in \mathscr{H}$ to be a function of the maximal number of variables possible. Say f is a function of $t \leqq n$ variables. We show now that f has the desired property.

Suppose that some monomial in f is not linear in say ζ_1. Since degree $f \leqq n$ and $f \in \mathscr{H}$ this implies that f cannot be a function of all ζ_i so say ζ_n is missing. Set

$$f' = f(\zeta_1 + \zeta_n, \zeta_2, \cdots) - f(\zeta_1, \zeta_2, \cdots) - f(\zeta_n, \zeta_2, \cdots) .$$

It follows easily that $f' \neq 0$ and that $f' \in \mathscr{H}$. Furthermore f' is a function of $t + 1$ variables, a contradiction. Hence all monomials in f are linear in each variable and thus they all have degree $t \leqq n$. If $t < n$ then say ζ_n is missing and setting $f'' = \zeta_n f$ yields a contradiction. Thus $t = n$ and f has the desired form.

6. Polynomial identity rings. Suppose A is an abelian subgroup of G with $[G: A] < \infty$. Then every element of A has only a finite number of conjugates in G and thus $\Delta(G) \supseteq A$ and $[G: \Delta] < \infty$. Therefore, according to the observation of [12], a first step in finding a converse to Theorem 4.2 is to show that $[G: \Delta]$ is finite. That is the goal of this section.

Let $K[\zeta_1, \zeta_2, \cdots]$ be the polynomial ring over K in the noncommuting indeterminates $\zeta_1, \zeta_2, \cdots$. A linear monomial is an element $\mu \in K[\zeta_1, \zeta_2, \cdots]$ of the form $\mu = \zeta_{i_1}\zeta_{i_2} \cdots \zeta_{i_r}$ with all i_j distinct and with $r \geqq 1$. Thus μ is linear in each variable.

LEMMA 6.1. *The number of linear monomials in* $K[\zeta_1, \zeta_2, \cdots, \zeta_m]$ *is* $\leqq (m+1)!$.

Proof. The number of linear monomials in $K[\zeta_1, \zeta_2, \cdots, \zeta_m]$ of degree m is of course $m!$. Now any other linear monomial is clearly just an initial segment of one of these. This yields a bound of

$$m \cdot m! \leqq (m+1)! \,.$$

We remark that a more precise upper bound here is $e \cdot m! = (2.718\ldots)m!$. We now come to the first main theorem of this paper.

THEOREM 6.2. *Let* $K[G]$ *satisfy a nontrivial polynomial identity of degree* n. *Then* $[G: \Delta] \leqq n!$.

Proof. We assume by way of contradiction that $[G: \Delta] > n!$ By Lemma 5.3 we may assume that $K[G]$ satisfies the polynomial identity

$$f(\zeta_1, \zeta_2, \cdots, \zeta_n) = \zeta_1\zeta_2 \cdots \zeta_n + \sum_{\substack{\sigma \in S_n \\ \sigma \neq 1}} a_\sigma \zeta_{\sigma(1)}\zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)}$$

so that clearly $n > 1$. For $j = 1, 2, \cdots, n$ define

$$f_j \in K[\zeta_j, \zeta_{j+1}, \cdots, \zeta_n]$$

by

$$f = \zeta_1\zeta_2 \cdots \zeta_{j-1} f_j + \text{terms not starting with } \zeta_1\zeta_2 \cdots \zeta_{j-1} \,.$$

Then clearly $f_1 = f, f_n = \zeta_n$ and f_j is a homogeneous multilinear polynomial of degree $n - j + 1$. In particular, for all j, ζ_j occurs in each monomial of f_j. We clearly have

$$f_j = \zeta_j f_{j+1} + \text{terms no starting with } \zeta_j \,.$$

For $j = 2, 3, \cdots, n$ let $\mathscr{M}_j$ denote the set of all linear monomials in $K[\zeta_j, \zeta_{j+1}, \cdots, \zeta_n]$ and let $\mathscr{M}_1$ be empty. Then by Lemma 6.1 we have for all j, $|\mathscr{M}_j| \leqq |\mathscr{M}_2| \leqq n!$ We show now by induction on

$j = 1, 2, \cdots, n$ that for any $x_j, x_{j+1}, \cdots, x_n \in G$ then either

$$f_j(x_j, x_{j+1}, \cdots, x_n) = 0$$

or $\mu(x_j, x_{j+1}, \cdots, x_n) \in \Delta$ for some $\mu \in \mathscr{M}_j$. Since $f = f_1$ is a polynomial identity satisfied by $K[G]$, the result for $j = 1$ is clear.

Suppose the result holds for some $j < n$. Fix

$$x_{j+1}, x_{j+2}, \cdots, x_n \in G$$

and let $x \in G$ play the role of the j-th variable. Let $\mu \in \mathscr{M}_{j+1}$. If $\mu(x_{j+1}, x_{j+2}, \cdots, x_n) \in \Delta$ we are done. Thus we may assume that

$$\mu(x_{j+1}, x_{j+2}, \cdots, x_n) \notin \Delta$$

for all $\mu \in \mathscr{M}_{j+1}$. Set $\mathscr{M}_j - \mathscr{M}_{j+1} = \mathfrak{F}_j$.

Now let $\mu \in \mathfrak{F}_j$ so that μ involves the variable ζ_j. Write $\mu = \mu'\zeta_j\mu''$ where μ' and μ'' are monomials in $K[\zeta_{j+1}, \zeta_{j+2}, \cdots, \zeta_n]$. Then $\mu(x, x_{j+1}, \cdots, x_n) \in \Delta$ if and only if

$$x \in \mu'(x_{j+1}, \cdots, x_n)^{-1}\Delta\mu''(x_{j+1}, \cdots, x_n)^{-1} = \Delta h_\mu$$

a fixed coset of Δ, since μ' and μ'' do not involved ζ_j and since Δ is normal in G. Thus it follows that for all $x \in G - \bigcup_{\mu \in \mathfrak{F}_j} \Delta h_\mu$ we have $\mu(x, x_{j+1}, \cdots, x_n) \notin \Delta$ for all $\mu \in \mathscr{M}_j$ since $\mathscr{M}_j \subseteqq \mathscr{M}_{j+1} \cup \mathfrak{F}_j$. Since the inductive result holds for j we conclude that for all $x \in G - \bigcup_{\mu \in \mathfrak{F}_j} \Delta h_\mu$ we have $f_j(x, x_{j+1}, \cdots, x_n) = 0$. Note that

$$|\mathfrak{F}_j| \leqq |\mathscr{M}_j| \leqq n!$$

and $[G: \Delta] > n!$ by assumption so $G - \bigcup_{\mu \in \mathfrak{F}_j} \Delta h_\mu$ is nonempty.

Write

$$f_j(\zeta_j, \zeta_{j+1}, \cdots, \zeta_n) = \zeta_j f_{j+1} + \Sigma_r \eta_r \zeta_j \eta'_r$$

where $\eta_r, \eta'_r \in K[\zeta_{j+1}, \zeta_{j+2}, \cdots, \zeta_n]$ and η_r is a linear monomial. Hence $\eta_r \in \mathscr{M}_{j+1}$. Now by the above we have

$$\begin{aligned} 0 = 1 \cdot x \cdot f_{j+1}(x_{j+1}, \cdots, x_n) \\ + \Sigma_r \eta_r(x_{j+1}, \cdots, x_n) x \eta'_r(x_{j+1}, \cdots, x_n) \end{aligned}$$

for all $x \in G - \bigcup_{\mu \in \mathfrak{F}_j} \Delta h_\mu \neq \varnothing$. Hence by Lemma 5.2 there exists $y \in G$ with

$$0 = \theta(1)^y f_{j+1}(x_{j+1}, \cdots, x_n) + \Sigma_r \theta(\eta_r(x_{j+1}, \cdots, x_n))^y \eta'_r(x_{j+1}, \cdots, x_n) .$$

Clearly $\theta(1)^y = 1$. Also $\eta_r(x_{j+1}, \cdots, x_n) \in G - \Delta$ since $\eta_r \in \mathscr{M}_{j+1}$ and hence $\theta(\eta_r(x_{j+1}, \cdots, x_n)) = 0$. Thus

$$0 = 1 \cdot f_{j+1}(x_{j+1}, \cdots, x_n) = f_{j+1}(x_{j+1}, \cdots, x_n)$$

and the induction step is proved.

In particular, the inductive result holds for $j = n$. Here $f_n(\zeta_n) = \zeta_n$ and $\mathscr{M}_n = \{\zeta_n\}$. Thus we conclude that for all $x \in G$ that either $x = 0$ or $x \in \Delta$, a contradiction since $G \neq \Delta$. Therefore the assumption $[G: \Delta] > n!$ is false and the theorem is proved.

7. Corollaries.

LEMMA 7.1. *Let G a finitely generated group and let H be a subgroup of finite index. Then H is finitely generated.*

Proof. By adding inverses if necessary we can assume that G is generated by $x_1, x_2, \cdots, x_t$ as a semigroup. Let $y_1, y_2, \cdots, y_n$ be a set of right coset representatives for H in G. For each i, j, Hy_ix_j is a coset of H say $Hy_ix_j = Hy_{i'}$. Then there exists $h_{ij} \in H$ with

$$y_ix_j = h_{ij}y_{i'} .$$

Let $\bar{H}$ be the subgroup of H generated by $\{h_{ij}\}$, and set $W = \cup \bar{H}y_i$. Since $h_{ij} \in H$ we have $(\bar{H}y_i)x_j = \bar{H}h_{ij}y_{i'} = \bar{H}y_{i'} \subseteq W$ and hence $Wx_j = W$. Thus since the x_j generate G as a semigroup we have $WG = W$ and hence clearly $W = G$. This yields easily $H = \bar{H}$ and the result follows.

COROLLARY 7.2. *Let G be a finitely generated group and suppose that $K[G]$ satisfies a polynomial identity. Then G has a normal abelian subgroup of finite index.*

Proof. By Theorem 6.2, $[G: \Delta] < \infty$ and hence by the previous lemma Δ is finitely generated. Hence by Lemma 2.2, $[\Delta: \mathbf{Z}(\Delta)] < \infty$ so $\mathbf{Z}(\Delta)$ is an abelian subgroup of G of finite index. Since $\mathbf{Z}(\Delta)$ is characteristic in Δ, it is normal in G.

We remark that even if we know the degree of the polynomial identity we cannot, in general, bound the index of the abelian subgroup in the above as the finite examples of Theorem 4.5 indicate. Furthermore, the example of the group P_∞ shows that if G is not finitely generated then G need not have an abelian subgroup of finite index.

LEMMA 7.3. *Let $E = K_m$ be the ring of $m \times m$ matrices over K. Then E does not satisfy a polynomial identity of degree $< 2m$.*

Proof. Suppose by way of contradiction that E satisfies a polynomial identity of degree $n < 2m$. By Lemma 5.3 we may assume that E satisfies

$$f(\zeta_1, \zeta_2, \cdots, \zeta_n) = \zeta_1\zeta_2 \cdots \zeta_n + \sum_{\substack{\sigma \in S_n \\ \sigma \neq 1}} a_\sigma \zeta_{\sigma(1)}\zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)} \,.$$

Let $\{e_{ij}\}$ denote the set of matrix units in E, that is e_{ij} is the matrix whose only nonzero entry is a 1 in the (i, j)-th position. Since $n < 2m$ we may set

$$\zeta_1 = e_{11}, \zeta_2 = e_{12}, \zeta_3 = e_{22}, \zeta_4 = e_{23}, \zeta_5 = e_{33}, \cdots \,.$$

Then $\zeta_1\zeta_2 \cdots \zeta_n$ at these values is not zero but clearly for all $\sigma \neq 1$, $\zeta_{\sigma(1)}\zeta_{\sigma(2)} \cdots \zeta_{\sigma(n)}$ at these values is zero. Thus E does not satisfy f, a contradiction.

Under certain circumstances we can improve the bound on $[G: \Delta]$ given in Theorem 6.2. The following result can be found in [12]. The proof here retains the basic flavor of the original, namely the formation of a suitable ring of quotients, but it does not require the use of deep ring theoretic machinery. Amazingly enough we apply some elementary Galois theory.

THEOREM 7.4. (*Smith* [12]). *Let $K[G]$ be prime and suppose that $K[G]$ satisfies a polynomial identity of degree n. Then Δ is a torsion free abelian group and $[G: \Delta] \leqq n/2$.*

Proof. By Theorem 2.5, Δ is torsion free abelian and by Theorem 6.2, $[G: \Delta] = k < \infty$. Hence by Lemma 2.4, no nonzero element of $K[\Delta]$ is a zero divisor in $K[G]$ and in particular $K[\Delta]$ is an integral domain. Set $\bar{G} = G/\Delta$. Then $\bar{G}$ acts faithfully by conjugation on Δ since if $x \in G$ and x centralizes Δ, then $[G: \mathrm{C}(x)] < \infty$ and $x \in \Delta$. Thus $\bar{G}$ acts faithfully by conjugation as ring automorphisms on $K[\Delta]$. Let $x_1, x_2, \cdots, x_k$ be a complete set of coset representatives of Δ in G with $x_1 = 1$.

Let Z denote the center of $K[G]$. As we observed in § 1, $Z \subseteqq K[\Delta]$ and thus no nonzero element of Z is a zero divisor in $K[G]$. Since Z is central it is then trivial to form the ring of quotients $Z^{-1}K[G]$. This is the set of all formal fractions $\eta^{-1}\alpha$ with $\eta \in Z - \{0\}$, $\alpha \in K[G]$ and with the usual identifications made.

Let $L = Z^{-1}K[\Delta] \subseteqq Z^{-1}K[G]$ and let $F = Z^{-1}Z \subseteqq L$. Clearly F is a field and L is an integral domain. Suppose $\alpha \in K[\Delta]$, $\alpha \neq 0$. Then $\alpha(\alpha^{x_2}\alpha^{x_3} \cdots \alpha^{x_k}) \in Z - \{0\}$ since $K[\Delta]$ is commutative. Thus α is invertible in L and L is a field. Now $\bar{G}$ acts on L and in fact we see that $\bar{G}$ is a group of field automorphisms of L with fixed field precisely F. The latter follows since if $\eta^{-1}\alpha \in L$ is fixed by all elements of $\bar{G}$, then $\alpha \in Z$ and $\eta^{-1}\alpha \in F$. Thus by Galois theory ([3], Th. 14)

$$(L: F) = |\bar{G}| = k \,.$$

Since $K[G]$ is free over $K[\Delta]$ of rank k, this shows that $E = Z^{-1}K[G]$ is a finite dimensional algebra over F and $\dim_F E = k^2$.

We observe now that E is prime. Suppose A and B are ideals of E with $AB = 0$. Let $\eta_1^{-1}\alpha \in A$, $\eta_2^{-1}\beta \in B$. Then since η_1 and η_2 are central we have clearly $(K[G]\alpha K[G])(K[G]\beta K[G]) = 0$ and since $K[G]$ is prime we conclude that either $\alpha = 0$ or $\beta = 0$. Thus if $B \neq 0$ we can assume that $\beta \neq 0$ and conclude that $A = 0$. This implies that E is a full matrix ring over some division algebra over F. It is clear that F is the center of E so E is central simple over F. Thus if $\tilde{F}$ denotes the algebraic closure of F then $\tilde{F} \otimes_F E \cong \tilde{F}_m$, the ring of $m \times m$ matrices over F. Since

$$m^2 = \dim_{\tilde{F}} \tilde{F}_m = \dim_F E = k^2$$

we see that $m = k$.

Now by Lemma 5.3 we can assume that $K[G]$ satisfies a multilinear polynomial identity of degree n. Since Z is central it follows that E also satisfies this identity viewed as a polynomial over F. Then clearly $\tilde{F} \otimes_F E = \tilde{F}_k$ satisfies this identity viewed as a polynomial over $\tilde{F}$. Thus by Lemma 7.3, $n > 2k$ or $n/2 \geqq k$. The result follows.

LEMMA 7.5. *Suppose $K[G]$ satisfies a polynomial identity f of degree n. Let H be a subgroup of G. Then $K[H]$ also satisfies f. Furthermore if H is normal in G, then $K[G/H]$ satisfies f.*

Proof. The first statement is clear since $K[H] \subseteqq K[G]$. Suppose H is normal in G. Then the homomorphism $G \to G/H$ induces an epimorphism $K[G] \to K[G/H]$ so the second result follows.

COROLLARY 7.6. *Suppose G is finitely generated and $K[G]$ satisfies a polynomial identity of degree n. Then $[G: \Delta] \leqq n/2$.*

Proof. By Theorem 6.2, $[G: \Delta] < \infty$ and hence by Lemma 7.1, Δ is finitely generated. Thus by Lemma 2.2, Δ' is finite. Since Δ/Δ' is a finitely generated abelian group and Δ' is finite we conclude that H, the set of all elements of finite order in Δ, is in fact a finite subgroup of Δ. Clearly H is normal in G.

Set $\bar{G} = G/H$ and $\bar{\Delta} = \Delta/H$ so that clearly $\bar{\Delta} \subseteqq \Delta(\bar{G})$. On the other hand suppose $\bar{x} = Hx \in \Delta(\bar{G})$. Then the conjugates of x are contained in only finitely many cosets of H and since H is finite, $x \in \Delta$. Thus $\bar{\Delta} = \Delta(\bar{G})$. Since $\bar{\Delta}$ is clearly torsion free abelian we see that $K[\bar{G}]$ is prime by Theorem 2.5. Furthermore by Lemma 7.5, $K[\bar{G}]$ satisfies a polynomial identity of degree n. Hence by Theorem 7.4, $[\bar{G}: \bar{\Delta}] \leqq n/2$ and since $[G: \Delta] = [\bar{G}: \bar{\Delta}]$, the result follows.

8. **Finite groups.** At this point we can no longer keep this paper self contained. We will need Theorem 8.2 below which is a result on finite groups. In characteristic 0, in a slightly different form, this is due to Isaacs and Passman in [7]. Our proof will merely translate the statement here to its original form in [7] and then quote that result. The characteristic $p > 0$ case is shown to follow from the characteristic 0 one, but the proof requires a certain amount of character theory. The reader who is not familiar with these techniques should just skip the proof. The remainder of this paper will again be self contained.

LEMMA 8.1. *Let G be a finite group and suppose that $K[G]$ satisfies a polynomial identity of degree n. Let K_0 denote the prime subfield of K and let $\tilde{K}_0$ be the algebraic closure of K_0. Then $\tilde{K}_0[G]$ satisfies a polynomial identity of degree n and all irreducible representations of $\tilde{K}_0[G]$ have degree $\leqq n/2$.*

Proof. Let f be the given polynomial identity for $K[G]$ of degree n and write $f = \Sigma a_i f_i$ where the f_i are polynomials over K_0 and the $a_i \in K$ are linearly independent over K_0. If we evaluate f at elements of $K_0[G]$ then each f_i evaluated is in $K_0[G]$. Since the a_i are also linearly independent over $K_0[G]$ we conclude that each f_i is an identity for $K_0[G]$. Clearly for some i, f_i has degree n.

Thus $K_0[G]$ satisfies a polynomial identity of degree n and thus by Lemma 5.3 it satisfies a multilinear polynomial g of degree n. Clearly g is also an identity for $\tilde{K}_0[G]$. Since $\tilde{K}_0$ is algebraically closed, an irreducible representation of $\tilde{K}_0[G]$ of degree m yields a homomorphism of $\tilde{K}_0[G]$ onto $(\tilde{K}_0)_m$, the ring of $m \times m$ matrices over $\tilde{K}_0$. This ring must therefore also satisfy g so by Lemma 7.3, $n \geqq 2m$ and $n/2 \geqq m$.

THEOREM 8.2. *There exists a finite valued function J with the following property. Let G be a finite group and let $K[G]$ satisfy a polynomial identity of degree n. Suppose that either K has characteristic 0 or K has characteristic $p > 0$ and $p \nmid |G'|$ where G' is the commutator subgroup of G. Then G has an abelian subgroup A with $[G: A] \leqq J(n)$.*

Proof. Let $\tilde{Q}$ denote the algebraic closure of the rational numbers. If K has characteristic 0 then by Lemma 8.1 we conclude that all irreducible representations of $\tilde{Q}[G]$ have degree $\leqq n/2$. Hence the result follows from Theorem 5.3 of [7].

Now let K have characteristic p. Since $p \nmid |G'|$ by assumption, it follows easily that $G = HP$ where H is a normal p-complement and

P is an abelian Sylow p-subgroup. We consider the irreducible $\tilde{Q}$-characters of G. Let χ be such a character of G and let φ be an irreducible constituent of χ_H, the restriction of χ to H. Let T denote the inertia group of φ in G so that $G \supseteqq T \supseteqq H$. By Satz V. 17.11.b of [5], $\chi = \zeta^G$ where ζ is an irreducible character of T which is a constituent of φ^T. Now $|T/H|$ is prime to $|H|$ so that Satz V. 17.12.c of [5] yields $\varphi^T = \Sigma_i \lambda_i(1)\eta\lambda_i$ where η is an irreducible character of T with $\eta_H = \varphi$ and the λ_i are irreducible characters of T/H. Since T/H is abelian all λ_i have degree 1 and by Satz V. 17.12.b of [5] we must have $\zeta = \eta\lambda$ for some $\lambda = \lambda_i$. Hence

$$\zeta_H = \eta_H \lambda_H = \eta_H = \varphi .$$

This shows that

$$\chi(1) = \zeta^G(1) = [G: T]\zeta(1) = [G: T]\varphi(1) .$$

Now by Hauptsatz V. 17.3.g of [5] we have $\chi_H = e\Sigma_1^t \varphi^{x_i}$ where $t = [G: T]$ and $\{x_i\}$ is a complete set of coset representations of T in G. Thus evaluating at 1 yields $t\varphi(1) = \chi(1) = et\varphi(1)$ so $e = 1$ and $\chi_H = \sum_1^t \varphi^{x_i}$.

Let $*$ denote a fixed homomorphism from the multiplicative group of $|G|$-th roots of unity in $\tilde{Q}$ onto the group of $|G|$-th roots of unity in $\widetilde{GF(p)}$, the algebraic closure of $GF(p)$. If $x \in G$ then $\chi(x)$ is a sum of $|G|$-th roots of unity and hence we can speak of χ^*, a function from G to $\widetilde{GF(p)}$. The map $\chi \to \chi^*$ is then essentially the map of § V. 12 of [5] and χ^* is the character of some representation of $\widetilde{GF(p)}[G]$. Clearly

$$(\chi^*)_H = \sum_1^t (\varphi^{x_i})^* = \sum_1^t (\varphi^*)^{x_i} .$$

Since $p \nmid |H|$ it follows from Hauptsatz V. 12.9 of [5] that the $(\varphi^{x_i})^*$ are all characters of distinct, irreducible, G-conjugate representations of $\widetilde{GF(p)}[H]$. Thus Hauptsatz V. 17.3 of [5] implies easily that χ^* is the character of an irreducible representation of $\widetilde{GF(p)}[G]$.

Now $K[G]$ satisfies a polynomial identity of degree n and hence by Lemma 8.1 we see that

$$\text{degree}\,\chi = \text{degree}\,\chi^* \leqq n/2 .$$

We have therefore shown that all irreducible $\tilde{Q}[G]$ representations have degree $\leqq n/2$. The result now follows from Theorem 5.3 of [7].

We remark that the function J is actually the function associated with Jordan's theorem on finite complex linear groups.

9. Semiprime polynomial identity rings. In this final section we consider semiprime group rings which satisfy a polynomial identity.

LEMMA 9.1. *Let G be a finitely generated group and let m be an integer. Then there exist only finitely many subgroups H of G with $[G: H] \leqq m$.*

Proof. Let H be a subgroup of G with $[G: H] = t \leqq m$. Then G permutes the t right cosets of H by right multiplication and this yields a homomorphism $\varphi: G \rightarrow S_t \subseteqq S_m$ where S_m is the symmetric group on m letters. It is clear that the kernel of φ is contained in H so that $H = \varphi^{-1}(W)$ for some subgroup W of S_m. Now there are only finitely many choices for W and furthermore there are only finitely many φ since φ is determined by the images of the finite number of generators of G. Thus there are only finitely many possibilities for H.

LEMMA 9.2. *Let G be an arbitrary group and let m be an integer. Then G has an abelian subgroup with index at most m if and only if every finitely generated subgroup of G has such an abelian subgroup.*

Proof. If A is abelian with $[G: A] \leqq m$ then for any subgroup H of G we have

$$m \geqq [G: A] \geqq [G \cap H: A \cap H] = [H: A \cap H] .$$

Hence $A \cap H$ is an abelian subgroup of H with index at most m.

Conversely, let us assume that every finitely generated subgroup of G has an abelian subgroup of index at most m. For each finite subset α of G let $G_\alpha = \langle \alpha \rangle$ be the group generated by the elements in α. Let m_α be the minimum index of abelian subgroups of G_α. By assumption $1 \leqq m_\alpha \leqq m$ for each α. Choose α_0 such that $m_0 = m_{\alpha_0}$ is the largest of the m_α's and set $G_0 = G_{\alpha_0}$.

Let $A_1, A_2, \cdots, A_r$ be the abelian subgroup of G_0 with $[G_0: A_i] = m_0$. By Lemma 9.1 there are only finitely many of these. We show that for some $i = 1, 2, \cdots, r$ both $[G: \mathrm{C}(A_i)] \leqq m_0$ and $\mathrm{C}(A_i)$ is abelian. This will, of course, yield the result. Suppose this is not the case. Then for each i choose α_i to consist of two noncommuting elements of $\mathrm{C}(A_i)$ if the latter is nonabelian or choose α_i to consist of $m_0 + 1$ elements in distinct right cosets of $\mathrm{C}(A_i)$ if $[G: \mathrm{C}(A_i)] > m_0$. Let

$$\alpha = \alpha_0 \cup \alpha_1 \cup \cdots \cup \alpha_r .$$

This is a finite set so let A_α be an abelian subgroup of G_α with

$$[G_\alpha : A_\alpha] = m_\alpha \; .$$

Now

$$\begin{aligned} m_0 \geqq m_\alpha = [G_\alpha : A_\alpha] &\geqq [G_\alpha \cap G_0 : A_\alpha \cap G_0] \\ &= [G_0 : A_\alpha \cap G_0] \; . \end{aligned}$$

On the other hand $A_\alpha \cap G_0$ is an abelian subgroup of G_0 and

$$m \geqq m_0 \geqq [G_0 : A_\alpha \cap G_0]$$

so we must have $[G_0 : A_\alpha \cap G_0] = m_0$ by definition of m_0. Thus $m_0 = m_\alpha$ and $A_\alpha \cap G_0 = A_i$ for some i. Say $A_\alpha \cap G_0 = A_1$.

Since A_α is abelian we have $A_\alpha \subseteqq \mathrm{C}_{G_\alpha}(A_1)$. On the other hand

$$\begin{aligned} [G_\alpha : \mathrm{C}_{G_\alpha}(A_1)] &\geqq [G_\alpha \cap G_0 : \mathrm{C}_{G_\alpha}(A_1) \cap G_0] \\ &= [G_0 : A_1] = m_0 = m_\alpha \end{aligned}$$

since A_1 is clearly its own centralizer in G_0. Thus $A_\alpha = \mathrm{C}_{G_\alpha}(A_1)$. Now $\alpha_1 \subseteqq G_\alpha$. Hence if $\mathrm{C}_G(A_1)$ were nonabelian then α_1 would contain noncommuting elements in $\mathrm{C}_{G_\alpha}(A_1) = A_\alpha$. Since A_α is abelian, this is not the case. On the other hand, if $[G : \mathrm{C}_G(A_1)] > m_0$ then G_α would contain $m_0 + 1$ elements in different right cosets of $\mathrm{C}_G(A_1)$ and hence in different right cosets of

$$G_\alpha \cap \mathrm{C}_G(A_1) = \mathrm{C}_{G_\alpha}(A_1) = A_\alpha \; .$$

But $[G_\alpha : A_\alpha] = m_0$ so we have a contradiction here and the result follows.

LEMMA 9.3. *Let G be a finitely generated group and let K be any field. Suppose that $K[G]$ satisfies a polynomial identity. Then G is residually finite, that is $\cap N = \langle 1 \rangle$ where N runs over all normal subgroups of G of finite index.*

Proof. By Corollary 7.2, G has a normal abelian subgroup A with $[G : A] < \infty$. Moreover A is finitely generated by Lemma 7.1. For each integer m set $A_m = \{x^m \mid x \in A\}$. Then A_m is a characteristic subgroup of A and hence a normal subgroup of G. Since A is finitely generated we have clearly $[A : A_m] < \infty$ and $\bigcap_{m=1}^{\infty} A_m = \langle 1 \rangle$.

We now come to the second main theorem of this paper. Let J' be the finite valued function on the set of integers given by

$$J'(n) = (n!)J(n)$$

where J is the function of Theorem 8.2. The following result in characteristic 0 is due to Isaacs and Passman in [7].

THEOREM 9.4. *Let $K[G]$ be a semiprime group ring which satisfies a polynomial identity of degree n. Then G has an abelian subgroup A with $[G: A] \leqq J'(n)$.*

Proof. Set $m = J(n)$. By Theorem 6.2 $[G: \Delta(G)] \leqq n!$ and thus it suffices to show that $\Delta = \Delta(G)$ has an abelian subgroup A with $[\Delta: A] \leqq m$. Note that since $K[G]$ is semiprime either K has characteristic 0 or by Theorem 3.6 K has characteristic $p > 0$ and Δ has no elements of order p.

Suppose by way of contradiction that Δ does not have an abelian subgroup of index $\leqq m$. Then by Lemma 9.2 there exists a finitely generated subgroup H of Δ which has no abelian subgroup of index $\leqq m$. Now H has only finitely many subgroups of index $\leqq m$ by Lemma 9.1 and say these are $L_1, L_2, \cdots, L_t$. By assumption each is nonabelian so we can choose $x_i \in L_i'$, $x_i \neq 1$. Now by Lemma 9.3, H is residually finite and thus for each i we can choose N_i normal in H with $[H: N_i] < \infty$ and $x_i \notin N_i$. Let $N = \cap N_i$. Then N is normal in H, $[H: N] < \infty$ by Lemma 1.1 and $x_i \notin N$ for all i.

By Lemma 7.5 $K[H/N]$ satisfies a polynomial identity of degree n. We consider $\bar{H} = H/N$. If K has characteristic 0 then $\bar{H}$ has an abelian subgroup $\bar{B}$ with $[\bar{H}: \bar{B}] \leqq J(n) \leqq m$ by Theorem 8.2. Suppose K has characteristic $p > 0$. Then by Lemma 2.2, H' is a finite p'-group. Since $\bar{H}' = H'N/N$ we conclude that $\bar{H}'$ is also a p'-group and thus by Theorem 8.2, $\bar{H}$ has an abelian subgroup $\bar{B}$ of index $\leqq m$ in this case too.

Let B be the complete inverse image of $\bar{B}$ in H. Then $H \supseteqq B \supseteqq N$ and $B/N = \bar{B}$. Since $[H: B] = [\bar{H}: \bar{B}] \leqq m$ we have $B = L_i$ for some i. Thus $L_i/N = B/N$ is abelian and this is a contradiction since $x_i \in L_i'$, $x_i \neq 1$ and $x_i \notin N$. The result follows.

We remark in closing that the study of group rings satisfying polynomial identities is far from complete. We have seen in Theorem 4.2, Corollary 7.2 and Theorem 9.4 that if either G is finitely generated or if $K[G]$ is semiprime, then $K[G]$ satisfies a polynomial identity if and only if G has an abelian subgroup of finite index. While the examples of Theorem 4.5 are suggestive, it is still too early to venture a guess at the answer in the remaining cases.

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