PACIFIC JOURNAL OF MATHEMATICS
Vol. 38, No. 2, 1971

RADON-NIKODÝM THEOREMS FOR THE BOCHNER AND PETTIS INTEGRALS

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The first Radon-Nikodým theorem for the Bochner integral was proven by Dunford and Pettis in 1940. In 1943, Phillips proved an extension of the Dunford and Pettis result. Then in 1968–69, three results appeared. One of these, due to Metivier, bears a direct resemblence to the earlier Phillips theorem. The remaining two were proven by Rieffel and seem to stand independent of the others. This paper is an attempt to put these apparently diverse theorems in some perspective by showing their connections, by simplifying some proofs and by providing some modest extensions of these results. In particular, it will be shown that the Dunford and Pettis theorem together with Rieffel's theorem directly imply Phillips' result. Also, it will be shown that, with almost no sacrifice of economy of effort, the theorems here can be stated in the setting of the Pettis integral.

For ease of reference the theorems mentioned above are listed below in a form convenient for our purposes. Throughout this paper (Ω, Σ, μ) is a finite measure space and X is a Banach space.

I. (*Dunford-Pettis*) [2, *VI.* 8. 10] *Let* $t\colon L^1(\Omega, \Sigma, \mu) \to X$ *be a weakly compact operator whose range is separable. The there exists an essentially bounded strongly measurable* $g\colon \Omega \to X$ *such that*

$$t(f) = \text{Bochner} - \int_\Omega fgd\mu \quad f \in L^1(\Omega, \Sigma, \mu)\ .$$

II. (*Phillips*) [6, *p.* 134]. *A vector measure* $F\colon \to X$ *is of the form* $F(E) = \text{Bochner} - \int_E fd\mu$, $E \in \Sigma$, *for some Bochner integrable* $F\colon \Omega \to X$ *if* F *is* μ*-continuous,* F *is of bounded variation and for each* $\varepsilon > 0$ *there exists* $E_\varepsilon \in \Sigma$ *with* $\mu(\Omega - E_\varepsilon) < \varepsilon$ *such that*

$$\{F(E)/\mu(E)\colon E \subset E_\varepsilon,\ \mu(E) > 0,\ E \in \Sigma\}$$

is contained in a weakly compact subset of X.

III. *Metivier* [5]. *The converse of Phillips' theorem is true.*

IV. (*Rieffel*) [7]. *A vector measure* $F\colon \Sigma \to X$ *is of the form*

$F(E) = \text{Bochner} - \int_E f d\mu$, $E \in \Sigma$ *for some Bochner integrable* f *if and only if* F *is* μ*-continuous,* F *is of bounded variation, and for each* $\varepsilon > 0$ *there exists* $E_\varepsilon \in \Sigma$ *with* $\mu(\Omega - E_\varepsilon) < \varepsilon$ *such that*

$$\{F(E)/\mu(E): E \subset E_\varepsilon, E \in \Sigma, \mu(E) > 0\}$$

is contained in a compact subset of X.

Most will agree that of the results listed here the two most powerful are Phillips' theorem and the necessity part of Rieffel's theorem. This is not meant to obscure the fact that the sufficiency part of Rieffel's theorem is proved by elementary means and therein lies its beauty. The next two sections constitute the main part of the paper.

1. Necessary Conditions. Here a simplified version of Rieffel's necessity proof is given. As the proof shows, there is no extra effort needed to carry the proof through in the context of Pettis [4] integrals.

THEOREM 1. *Let* $f: \Omega \to X$ *be strongly measurable and Pettis integrable. If* $F(E) = \text{Pettis} - \int_E f d\mu$ *for* $E \in \Sigma$, *then*

(i) $F \ll \mu$

(ii) *for each* $\varepsilon > 0$, *there exists* $E_\varepsilon \in \Sigma$ *with* $\mu(\Omega - E_\varepsilon) < \varepsilon$ *such that*

$$\{F(E)/\mu(E): E \subset E_\varepsilon, \mu(E) > 0, E \in \Sigma\}$$

is contained in a compact subset of X.

(iii) *If* f *is also Bochner integrable, then* F *is of bounded variation.*

Proof. (i) and (iii) are standard facts. For (ii), choose a sequence $\{f_n\}$ of measurable simple functions converging almost everywhere to f. If $\varepsilon > 0$ is given, Egoroff's theorem establishes a set $E_\varepsilon \in \Sigma$ with $\mu(\Omega - E_\varepsilon) < \varepsilon$ such that f_n converges to f uniformly on E_ε. In particular f is bounded on E_ε, since all of the f_n's are bounded on E_ε. Hence the Bochner integral $t(g) = \int_{E_\varepsilon} g f d\mu$ and $t_n(g) = \int_{E_\varepsilon} g f_n d\mu$ exist for all $g \in L^1$ and define bounded linear operators of L^1 into X. Now note that is $g \in L^1$ and $\|g\|_1 \leqq 1$,

$$\|(t - t_n)g\|_X = \left\| \int_{E_\varepsilon} (gf - gf_n) d\mu \right\|$$
$$\leqq \int_{E_\varepsilon} |g| \, \|f - f_n\| \, d\mu \leqq \sup_{\omega \in E_\varepsilon} \|f(\omega) - f_n(\omega)\| \, .$$

Thus the uniform convergence of f_n to f on E_ε guarantees $\lim_n t_n = t$ in the uniform operator topology. But now note that each t_n has a finite dimensional range spanned in X by the finite set of values of f_n. Hence t is a compact operator. Since $\chi_E/\mu(E)$ has norm 1 for each $E \in \Sigma$ of positive μ-measure and $t(\chi_E) = F(E)$ for $E \in \Sigma$, $E \subset E_\varepsilon$, one sees immediately that $\{(F(E)/\mu(E)): E \in \Sigma, E \subset E_\varepsilon\}$ is norm conditionally compact in X.

2. Sufficient conditions. The program here is to use Rieffel's necessity condition together with the Dunford-Pettis theorem to show quickly that every weakly compact operator on $L^1(\Omega, \Sigma, \mu)$ has a separable range. This will prove that the Dunford Pettis theorem describes the arbitrary weakly compact operator on $L^1(\Omega, \Sigma, \mu)$. With this, it is not hard to recover Phillips' result. This line of reasoning has some interest because it shows that Phillips' theorem can be deduced directly, at the discretion of the reader, from the Dunford-Pettis theorem or from Rieffel's theorem.

As Rieffel emphasizes in his paper, the main hurdle between his theorem and Phillips' theorem is verifying a nontrivial separability condition. This separability condition is equivalent to showing that a weakly compact operator $t: L^1 \to X$ has a separable range. In the literature the standard proof of this is to show that such an operator maps weakly compact sets into norm compact sets. The following proof teams Rieffel's theorem with the Dunford-Pettis theorem to obtain

LEMMA 2. *A weakly compact $t: L^1(\Omega, \Sigma, \mu) \to X$ has a separable range.*

Proof. Consider $\{\chi_E: E \in \Sigma\} = S$. The closed linear span of this set in $L^1(\Omega, \Sigma, \mu)$ is all of $L^1(\Omega, \Sigma, \mu)$. Hence if it can be shown that $t(S)$ is separable, then the linearity and continuity of t will guarantee that t has a separable range. Now to show $t(S)$ is separable, it will be shown that $t(S)$ is conditionally compact in X and separable *a fortiori*.

For this, let $\{\chi_{E_n}\} \subset S$, and let Σ_0 be the σ-algebra generated by $\{\chi_{E_n}\}$. Then since Σ_0 is countably generated, $L^1(\Sigma_0) = \{g \in L^1(\Omega, \Sigma, \mu): g$ is measurable with respect to $\Sigma_0\}$ is also separable. Hence

$$t: L^1(\Sigma_0) \longrightarrow X$$

is a weakly compact operator whose range is separable. By the Dunford Pettis Theorem I, there exists a strongly (Σ_0-) measurable essentially bounded function $f: \Omega \to X$ such that

$$t(g) = \text{Bochner} - \int_{\Omega} gfd\mu,\ g \in L^1(\Sigma_0)\ .$$

Now let $\varepsilon > 0$ be given. By Rieffel's Theorem IV there exists a set $E_\varepsilon \in \Sigma_0$ such that $\mu(\Omega - E_\varepsilon) < \varepsilon/(\text{ess sup}\ ||f|| + 1)$ and a norm compact set $A_\varepsilon \subset X$ such that

$$\left\{\int_E fd\mu/\mu(E) \colon E \in \Sigma_0,\ E \subset E_\varepsilon\right\} \subset A_\varepsilon\ .$$

Next note that the set $A'_\varepsilon = \{\alpha x\colon 0 \leqq \alpha \leqq \mu(\Omega);\ x \in A_\varepsilon\}$ is also compact since A_ε is compact. Also note that

$$t(\chi_{E_n}) = \int_{E_n} fd\mu = \int_{E_n \cap E_\varepsilon} fd\mu + \int_{E_n - E_\varepsilon} fd\mu\ .$$

Now

$$\int_{E_n \cap E_\varepsilon} fd\mu \in \mu(E_n \cap E_\varepsilon) A_\varepsilon \subset A'_\varepsilon$$

and

$$\left\|\int_{E_n - E_\varepsilon} fd\mu\right\| \leqq \text{ess sup}\ ||f||\ \mu(\Omega - E_\varepsilon) < \varepsilon\ .$$

Hence $t(\chi_{E_n})$ is within ε of a member of the compact set A'_ε. It follows that $\{t(\chi_{EEn})\}$ is totally bounded and therefore norm conditionally compact; i.e., $\{t(\chi_{En})\}$ has a convergent subsequence. But this proves $t(S)$ is conditionally compact and separable.

Another application of the Dunford-Pettis theorem yields

THEOREM 2. *Let $F\colon \Sigma \to X$ be a μ-continuous vector measure such that for each $\varepsilon > 0$ there exists $E_\varepsilon \in \Sigma$, with $\mu(\Omega - E_\varepsilon) < \varepsilon$ such that*

$$B_\varepsilon = \{F(E)/\mu(E)\colon E \subset E_\varepsilon,\ E \in \Sigma \mu(E) > 0\}$$

is contained in a weakly compact subset of X. Then there exists a strongly measurable Pettis integrable function $f\colon \Omega \to X$ such that for $E \in \Sigma$

$$F(E) = \text{Pettis} - \int_E fd\mu\ .$$

If, in addition, F is of bounded variation, then f is Bochner integrable and for $E \in \Sigma$,

$$F(E) = \text{Bochner} - \int_E fd\mu\ .$$

Proof. For each $\varepsilon = 1/n$, n a positive integer, choose E_ε as above. Define t on the simple functions of the form $\sum_{i=1}^{n} \alpha_i \chi_{E_i}$ α_i real, $\{E_i\} \subset \Sigma$, disjoint by

$$t\Big(\sum_{i=1}^{n} \alpha_i \chi_{E_i}\Big) = \sum_{i=1}^{n} \alpha_i F(E_i \cap E_\varepsilon) = \sum_{i=1}^{n} \alpha_i \mu(E_i \cap E) \frac{F(E_i \cap E_\varepsilon)}{\mu(E_i \cap E_\varepsilon)} .$$

Usual arguments show t is linear. Moreover, if $\sum_{i=1}^{n} |\alpha_i| \mu(E_i) \leqq 1$, then the above computation shows $t(\sum_{i=1}^{n} \alpha_i \mu(E_i)) \subset$ closed convex hull of $(B_\varepsilon - B_\varepsilon)$[1] which is weakly compact by the Krein-Smulian theorem [2, p. 434]. Hence t maps a dense subset of the unit ball of $L'(\Omega, \Sigma, \mu)$ into a weakly compact set and thus has a weakly compact extension to all of $L^1(\Omega, \Sigma, \mu)$. By the above lemma, t has a separable range and by the Dunford-Pettis theorem there exists a strongly measurable f_ε vanishing off E_ε such that

$$t(g) = \text{Bochner} - \int_{E_\varepsilon} g f_\varepsilon d\mu$$

for all $g \in L^1(\Omega, \Sigma, \mu)$. Now if this is done for each $\varepsilon = 1/n$, one can produce an increasing sequence of measurable sets $\{E_n\}$ such that $\mu(\Omega - E_n) \to 0$ and a sequence of strongly measurable Bochner integrable functions $\{f_n\}$ such that $F(E \cap E_n)$ is given by the Bochner integral $\int_{E \cap E_n} f_n d\mu$ for $E \in \Sigma$. Clearly $f_n \chi_{E_m} = f_m$ a.e., for $n \geqq m$ since $E_n \uparrow$. Also since $E_n \uparrow \Omega$, it is evident that there exists a strongly measurable function f such that $f_n \chi_{E_n} = f \chi_{E_n}$ a.e., Now note that if $E \in \Sigma$ is arbitrary, the μ-continuity of F and the fact that $\lim_n \mu(\Omega - E_n) = 0$ imply

$$F(E) = \lim_n F(E \cap E_n) = \lim_n \int_{E \cap E_n} f_n d\mu = \lim_n \int_{E \cap E_n} f d\mu$$

strongly in X. Thus $x^* F(E) = \lim_n \int_{E \cap E_n} x^* f d\mu$ for all $x^* \in X^*$, the dual space of X. Next note that for $x^* \in X^*$, $x^* F$ is a bounded signed measure on Σ. Hence it is of finite variation $|x^* F|$, and

$$\int_\Omega |x^* f| \, d\mu = \lim_n \int_{E_n} |x^* f| \, d\mu = \lim_n |x^* F| \, (E_n) = |x^* F|(\Omega) < \infty .$$

Therefore $x^* f \in L^1(\Omega, \Sigma, \mu)$ and by the dominated convergence theorem,

$$x^* F(E) = \lim_n \int_{E \cap E_n} x^* f d\mu = \int_E x^* f d\mu .$$

This shows that F is the indefinite Pettis integral of f.

[1] An obvious modification holds if X is a complex B-space.

To prove (b), suppose in addition that F is of bounded variation. Replacing $|x^*f|$ by $||f||$ and $|x^*F|$ by $|F|$, the variation of F, in the above paragraph shows that $||f|| \in L^1(\mu)$, and hence that f is Bochner integrable; i.e., $F(E) = \int_E f d\mu$ for all $E \in \Sigma$.

4. Concluding remarks.

Here some observations and theorems are given without full proofs.

Fact 1. *If $f: \Omega \rightarrow X$ is strongly measurable and Pettis integrable then Pettis $- \int_E f d\mu$ can be realized as an "improper" Bochner integral = strong* $\lim_n$ *Bochner* $- \int_{E \cap E_n} f d\mu$ *for some sequence $\{E_n\} \subset \Sigma$ such that $E_n \uparrow \Omega$.*

Fact 2. *If X is reflexive and $F: \Sigma \rightarrow X$ is a vector measure whose variation is σ-finite, then F has a separable range.*

Fact 3. *If X is reflexive and $F: \Sigma \rightarrow X$ is μ-continuous, then F is representable as a μ-Pettis integral if and only if its variation is σ-finite and as a μ-Bochner integral if and only if its variation is finite.*

Finally it is noted that Rieffel's interesting Radon-Nikodým theorem [8] dealing with dentable subsets of Banach spaces does not seem to fit conveniently into the treatment of this paper. It would be interesting to see how this theorem relates to the others.

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Received January 28, 1971. The second author was supported in part by NSF Grant GP-14592.

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