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A PHRAGMÉN-LINDELÖF THEOREM WITH APPLICATIONS TO $\mathscr{M}(u, v)$ FUNCTIONS

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A well-known theorem of Paley and Wiener asserts that if f is an entire function, its restriction to the real line belongs to the Hilbert space $\mathscr{F}^*L^2(-\tau, \tau)$ (where $\mathscr{F}$ is the Fourier-Plancherel operator) if and only if f is square integrable on the real axis and satisfies $|f(z)| \leqq Ke^{\tau|\mathrm{Im}\, z|}$ for some positive K. The "if" part of this result may be viewed as a Phragmén-Lindelöf type theorem. The pair $(e^{i\tau x}, e^{i\tau x})$ of inner functions can be associated with the above mentioned Hilbert space in a natural way. By replacing this pair by a more general pair (u, v) of inner functions it is possible to define a space $\mathscr{M}(u, v)$ of analytic functions similar to the Paley-Wiener space. For a certain class of inner functions (those of "type $\mathfrak{C}$") it is shown that membership in $\mathscr{M}(u, v)$ is implied by an inequality analogous to the exponential inequality above.

A second application of our results is to star-invariant subspaces of the Hardy space H^2. It is well known that if u is an inner function on the circle and f is in H^2, then in order for f to be in $(uH^2)^\perp$ it is necessary for f to have a meromorphic pseudocontinuation to $|z| > 1$ satisfying

$$|f(z)|^2 \leqq K\frac{1 - |u(z)|^2}{1 - |z|^2}, \quad |z| > 1 .$$

If u is inner of type $\mathfrak{C}$, it is proved that this necessary condition is also sufficient.

Let $\Gamma = \{e^{i\theta}: 0 < \theta < 2\pi\}$ be the unit circle and

$$R = \{x : -\infty < x < \infty\}$$

the real line considered as point sets in the complex plane $\boldsymbol{C}$. Let D and D_- be the interior and exterior of the unit circle and let Ω and Ω_- be the open upper and open lower half-planes in $\boldsymbol{C}$. A function Φ is *outer* on D or Ω if Φ is holomorphic on D or Ω and of the form

$$\Phi(z) = \exp\int_\Gamma \frac{e^{i\xi} + z}{e^{i\xi} - z} k_1(e^{i\xi})\,\sigma(d\xi), \quad z \in D ,$$

or

$$\Phi(z) = \exp\frac{1}{\pi i}\int_R \frac{1 + tz}{t - z} k_2(t)dt, \quad z \in \Omega ,$$

where k_1, k_2 are real with $k_1 \in L^1(\Gamma)$, $k_2 \in L^1(R)$, and σ is normalized Lebesgue measure on Γ. A function F on D or Ω is in $\mathfrak{N}^+$ if F is holomorphic on D or Ω and if there exists an outer function Φ that is not identically zero and such that ΦF is a bounded holomorphic function on D or Ω. If F is in $\mathfrak{N}^+$ on D or Ω, then $f(e^{i\theta}) = \lim F(re^{i\theta})$ exists for almost all $e^{i\theta} \in \Gamma$, or

$$f(x) = \lim_{y \downarrow 0} F(x + iy)$$

exists for almost all x in R. Such f form the class $\mathscr{N}^+$ of functions on Γ and R respectively. We shall systematically use capital letters $F, G, \cdots$ for functions in $\mathfrak{N}^+$ and lower case letters $f, g, \cdots$ for the corresponding functions in $\mathscr{N}^+$.

Every outer function is in $\mathfrak{N}^+$. A function U in $\mathfrak{N}^+$ is *inner* if $|u| = 1$ a.e.. Every function F in $\mathfrak{N}^+$ has a factorization of the form $F = UG$, where U is inner and G is outer.

Suppose U and V are inner functions, say, on Ω. $\mathscr{M}(u, v, R)$ is the set of functions f on R such that uf and vf^* are in $\mathscr{N}^+$ on R. (f^* is the complex conjugate of f). $\mathscr{M}(u, v, \Gamma)$ is similarly defined. As shown in [5] one can associate with each f in $\mathscr{M}(u, v, R)$ a unique function F separately meromorphic in Ω and Ω_- such that $UF \in \mathfrak{N}^+$, $V\tilde{F} \in \mathfrak{N}^+$, and

$$f(x) = \lim_{y \downarrow 0} F(x + iy) = \lim_{y \downarrow 0} F(x - iy) \tag{1}$$

for almost all x in R, where $\tilde{F}(z) = F^*(z^*)$, $z \in \Omega$. If F is meromorphic in Ω, then an extension of F to a meromorphic function on $\Omega \cup \Omega_-$ satisfying (1) is said to be a *meromorphic pseudocontinuation* (relative to R) of F. Similarly, to each f in $\mathscr{M}(u, v, \Gamma)$ one associates a unique F meromorphic in $D \cup D_-$ such that $UF \in \mathfrak{N}^+$, $V\tilde{F} \in \mathfrak{N}^+$, and

$$f(e^{i\theta}) = \lim_{r \uparrow 1} F(re^{i\theta}) = \lim_{r \downarrow 1} F(re^{i\theta}) \tag{2}$$

for almost all $e^{i\theta} \in \Gamma$ where $\tilde{F}(z) = F^*(z^{*-1})$, $z \in D$. Meromorphic pseudocontinuation is defined relative to Γ in a manner analogous to the R definition.

Considerations about $\mathscr{M}(u, v, R)$ may be motivated by examining the special case when $U(z) = V(z) = e^{iz\tau}$, $\tau \geqq 0$. Then

$$\mathscr{M}(u, v, R) \cap L^2(R)$$

is the class of functions that are the restrictions to R of entire functions of exponential type $\leqq \tau$ such that $\int_R |F(x)|^2 dx < \infty$. Such entire F can be characterized by this integral condition together with the inequality

$$| F(z) |^2 < K | y |^{-1} | \sinh (2 \tau y) |$$

for all $z \in \Omega \cup \Omega_-$, where $K > 0$. The object of this paper is to extend this type of function-theoretic characterization to more general $\mathscr{M}(u, v)$ classes. The above mentioned application to star-invariant subspaces arises from the fact that $M(1, v) \cap L^2(R) = H^2(\Omega) \ominus vH^2(\Omega)$, where $H^2(\Omega)$ is the Hardy space of the upper half-plane. In § 3 and 4 applications are given to factorization problems for nonnegative operator-valued functions and to generalized Paley-Wiener representations.

1. A Phragmén-Lindelöf Theorem. In this section we shall derive a Phragmén-Lindelöf type theorem for certain functions holomorphic on D, and then transcribe the result to obtain a like theorem for functions on Ω. A rather different Phragmén-Lindelöf type theorem is discussed by Helson in [2, p. 33].

Recall that a Blaschke product B on D has a representation

$$B(z) = \prod_{j \geq 1} B_j(z),\ B_j(z) = \frac{z_j^*}{|z_j|} \frac{z_j - z}{1 - z_j^* z},\ z \in D\,, \tag{3}$$

where $\sum_{j \geq 1}(1 - | z_j |) < \infty$. We take $z_j^*/| z_j | = 1$ if $z_j = 0$. The *support* supp B of B is the intersection of Γ with the closure of $\{z_j\}_{j \geq 1}$. A singular inner function S has a representation

$$S(z) = \exp\left(-\int_\Gamma \frac{e^{i\xi} + z}{e^{i\xi} - z} \mu(d\xi)\right),\ z \in D\,, \tag{4}$$

where μ is a positive singular measure on Γ. The *support* supp S is the closed support of the measure μ.

Any inner function U on D can be factored in the form $U = cBS$, where $c \in C$, $| c | = 1$, B is a Blaschke product and S is a singular inner function. The *support* supp U of U is supp $B \cup$ supp S.

A closed set N on Γ is a *Carleson set* if N has zero Lebesgue measure and if the complement of N in Γ is a union of open arcs I_j of lengths ε_j such that $\sum_{j \geq 1} \varepsilon_j \log \varepsilon_j > -\infty$.

THEOREM 1.1. (Carleson [1]). *A closed subset N of Γ is a Carleson set on Γ if and only if there exists an outer function G on D that satisfies a Lipschitz condition and such that*

$$g(e^{i\theta}) \overset{\text{def}}{=} \lim_{r \uparrow 1} G(re^{i\theta})$$

vanishes on N.

DEFINITION 1.2. An inner function U on D is of *type* $\mathfrak{C}$ if
(i) supp U is a Carleson set, and
(ii) $\sum_{j\geqq 1}$ [dist (z_j, supp U)] $< \infty$,
where $\{z_j\}_{j\leqq 1}$ are the zeros of U in D repeated according to multiplicity.

LEMMA 1.3. *Let B be the Blaschke product given by* (3) *and suppose B is of type $\mathfrak{C}$. If G is a Lipschitz outer function on D such that $g(e^{i\theta}) = \lim_{r\uparrow 1} G(re^{i\theta})$ vanishes on supp B, then*

$$(5) \qquad \sum_{j\geqq 1}(1 - |z_j|^2)\int |(1 - z_j^* e^{i\theta})^{-1} g(e^{i\theta})|^2 \sigma(d\theta) < \infty \,.$$

Proof. Since G is Lipschitz there exists $K > 0$ such that

$$|g(e^{i\theta})| \leqq K|e^{i\theta} - \lambda|$$

for all $e^{i\theta}$ in Γ and λ in supp B. Thus for λ in supp B,

$$\begin{aligned}&(1 - |z_j|^2)\int |(1 - z_j^* e^{i\theta})^{-1} g(e^{i\theta})|^2 \sigma(d\theta)\\ &\leqq (1 - |z_j|^2) K^2 \int |(1 - z_j^* e^{i\theta})^{-1}(e^{i\theta} - \lambda)|^2 \sigma(d\theta) \,.\end{aligned}$$

Applying Parseval's equality to the Fourier series for the function $(1 - z_j^* e^{i\theta})^{-1}(e^{i\theta} - \lambda)$ shows that this last expression is equal to

$$K^2(|z_j - \lambda|^2 + (1 - |z_j|^2)) \,.$$

Since $\sum_{j\geqq 1}(1 - |z_j|^2) < \infty$ and we are free to let λ vary over supp B this inequality implies (5).

The following theorem is our Phragmén-Lindelöf result for functions on D.

THEOREM 1.4. *Let U be an inner function of type $\mathfrak{C}$ on D. Suppose F is holomorphic in D and there exists $M > 0$ such that*

$$(6) \qquad |F(z)|^2 \leqq M(1 - |z|^2)^{-1}(1 - |U(z)|^2),\ z \in D \,.$$

Then $F \in \mathfrak{N}^+$.

Proof. U has the factorization $U = cBS$, where $|c| = 1$, B is a Blaschke product of type $\mathfrak{C}$ and S is a singular inner function of type $\mathfrak{C}$. We have

$$\begin{aligned}(7) \qquad &(1 - |z|^2)^{-1}(1 - |U(z)|^2)\\ &= (1 - |z|^2)^{-1}(1 - |B(z)|^2) + |B(z)|^2(1 - |z|^2)^{-1}(1 - |S(z)|^2)\\ &\leqq (1 - |z|^2)^{-1}(1 - |B(z)|^2) + (1 - |z|^2)^{-1}(1 - |S(z)|^2),\ z \in D \,.\end{aligned}$$

If B is given by (3), then

$$1 - |B(z)|^2 = 1 - |B_1(z)|^2 + \sum_{n \geqq 2} \left| \prod_{j=1}^{n-1} B_j(z) \right|^2 (1 - |B_n(z)|^2)$$
$$\leqq \sum_{j \geqq 1} (1 - |B_j(z)|^2) .$$

Thus

$$(8) \qquad (1 - |z|^2)^{-1} (1 - |B(z)|^2) \leqq \sum_{j \geqq 1} (1 - |z_j|^2) |1 - z_j^* z|^{-2} .$$

If S is given by (4), then

$$|S(z)|^2 = \exp\left\{-2 \int_\Gamma (1 - |z|^2) |e^{i\xi} - z|^{-2} \mu(d\xi)\right\}, \quad z \in D .$$

Applying the elementary inequality $(1 - e^{-ah})/h) \leqq a$ if $a, h \geqq 0$, with $h = 1 - |z|^2$ and $a = 2 \int_\Gamma |e^{i\xi} - z|^{-2} \mu(d\xi)$ yields

$$(9) \qquad (1 - |z|^2)^{-1} (1 - |S(z)|^2) \leqq 2 \int_\Gamma |e^{i\xi} - z|^{-2} \mu(d\xi), \quad z \in D .$$

Suppose now that (6) holds and let G be a Lipschitz outer function such that $g(e^{i\theta}) = \lim_{r \uparrow 1} G(re^{i\theta})$ vanishes on supp U. We have from (6) – (9) that

$$|G(z)F(z)|^2 \leqq M \sum_{j \geqq 1} (1 - |z_j|^2) |1 - z_j^* z|^{-2} |G(z)|^2$$
$$+ 2M \int_\Gamma |e^{i\xi} - z|^{-2} |G(z)|^2 \mu(d\xi), \quad z \in D .$$

But for some $K > 0$

$$|G(z)|^2 \leqq K^2 |e^{i\xi} - z|^2 \text{ if } e^{i\xi} \in \text{supp}\, U ,$$

and μ is supported on supp $S \subseteqq$ supp U. Thus for all $z \in D$

$$|G(z)F(z)|^2 \leqq M \sum_{j \geqq 1} (1 - |z_j|^2) |1 - z_j^* z|^{-2} |G(z)|^2 + 2MK^2 \mu(\Gamma) .$$

It now follows from Lemma 1.3 that

$$\sup_{0 \leqq r < 1} \int_\Gamma |G(re^{i\theta})F(re^{i\theta})|^2 \sigma(d\theta) \qquad < \infty \qquad ,$$

so $GF \in H^2$. It is easy to multiply G by an outer function G_1 and obtain G_1GF bounded, and so F is in $\mathfrak{N}^+$.

We shall next recast Theorem 1.4 for functions holomorphic on Ω. Any inner function U on Ω has a factorization $U = cBSV^a$, where $c \in C$, $|c| = 1$, B is a Blaschke product on Ω, S is a singular function on Ω, and $V^a(z) = e^{iaz}$, where $0 \leqq a \in R$. Then supp B is defined to be the set of limit points on $R \cup \{\infty\}$ of the zeros of B, and supp S is defined to be the support of the singular measure in the representation for S analogous to (4), (Hoffman [3] p. 132–133). We define supp V^a to be empty if $a = 0$, and $\{\infty\}$ if $a > 0$. The *support* supp U of U is supp $B \cup$ supp $S \cup$ supp V^a.

A closed subset N of the extended real line $R \cup \{\infty\}$ is a *Carleson set* if $N \cap R$ has Lebesgue measure zero, $\infty \in N$, and the complement of N in $R \cup \{\infty\}$ is a union of open intervals

$$I_j = (a_j, b_j), \ -\infty \leqq a_j < b_j \leqq \infty, \ j = 1, 2, \cdots$$

such that $\sum_{j>1} \delta_j \log \delta_j > -\infty$, where

$$\delta_j = \frac{b_j - a_j}{(1 + b_j^2)^{1/2} (1 + a_j^2)^{1/2}}, \quad j = 1, 2, \cdots .$$

We understand in the above that $\infty/\infty = 1$.

Now let α: $\bar{D} \to \bar{\Omega} \cup \{\infty\}$ be the mapping defined by

$$\alpha(z) = i(1 + z)(1 - z)^{-1}$$

if $z \neq 1$ and $\alpha(1) = \infty$, and let β be the inverse of α. Then if $z_1, z_2 \in \bar{\Omega}$,

$$|\beta(z_1) - \beta(z_2)|^2 = 4 \frac{|z_1 - z_2|^2}{|z_1 + i|^2 |z_2 + i|^2} .$$

Moreover β maps $(-\infty, \infty]$ onto Γ and N is a Carleson set on $R \cup \{\infty\}$ if and only if $\beta(N) \cup \{1\}$ is a Carleson set on Γ. If U is inner on Ω then $U \circ \alpha$ is inner on D and supp $(U \circ \alpha) = \beta$ (Supp U). Furthermore if $\{z_j\}_{j>1}$ is the sequence of zeros of U, then $\{\beta(z_j)\}_{j \geqq 1}$ is the sequence of zeros of $U \circ \alpha$.

DEFINITION 1.5. Let U be an inner function on Ω. U is of *type* $\mathfrak{C}$ if supp $U \cup \{\infty\}$ is a Carleson set on $R \cup \{\infty\}$ and

$$\sum_{j \geqq 1} \left(\inf_{\lambda \in \operatorname{supp} U} \frac{|z_j - \lambda|^2}{(1 + \lambda^2)(1 + |z_j|^2)} \right) < \infty ,$$

where $\{z_j\}_{j \geqq 1}$ is the sequence of zeros of U in Ω repeated according to multiplicity.

The following lemma follows from the above discussion.

LEMMA 1.6. *Let U be inner on Ω. Then U is of type $\mathfrak{C}$ if and only if $U \circ \alpha$ is of type $\mathfrak{C}$ on D.*

We can now recast Theorem 1.4 for the half-plane.

THEOREM 1.7. *Let F be holomorphic in Ω and suppose that U is inner of type $\mathfrak{E}$ in Ω. Suppose that there exists $K > 0$ such that*

$$|F(z)|^2 \leqq K(\operatorname{Im} z)^{-1}(1 + |z|^2)(1 - |U(z)|^2) \text{ for } z \in \Omega. \tag{10}$$

Then $F \in \mathfrak{N}^+$ on Ω.

Proof. Set $G = F \circ \alpha$, so G is meromorphic on D and

$$|G(z)|^2 \leqq K[\operatorname{Im} \alpha(z)]^{-1}(1 + |\alpha(z)|^2)(1 - |U(\alpha(z))|^2), \quad z \in D\,.$$

We can replace $1 + |\alpha(z)|^2$ by $|i + \alpha(z)|^2$ and the inequality still holds but for a different constant K. Now

$$\operatorname{Im} \alpha(z) = (1 - |z|^2)\,|1 - z|^{-2}$$

and

$$|i + \alpha(z)|^2 = 4\,|1 - z|^{-2}\,,$$

so

$$|G(z)|^2 \leqq K'(1 - |z|^2)^{-1}(1 - |U(\alpha(z)|^2), \; z \in D\,.$$

But by Lemma 1.6 $U \circ \alpha$ is of type $\mathfrak{E}$, and thus Theorem 1.4 implies that $G \in \mathfrak{N}^+$ on D. We then deduce that $F = G \circ \beta$ is in $\mathfrak{N}^+$ on Ω.

2. The classes $\mathscr{M}(u, v, \Gamma)$ and $\mathscr{M}(u, v, R)$. Suppose U is inner in D. Then U has a meromorphic pseudocontinuation to a function U on $D \cup D_-$ that is given by

$$U(z) = \begin{cases} U(z), & z \in D \\ 1/U^*(z^{*-1}), & z \in D_-\,. \end{cases} \tag{11}$$

If supp $U \neq \Gamma$, then U on D has a single valued meromorphic continuation to D_- that coincides with U as given by (11). If F is meromorphic on D_- then $\tilde{F}(z) = F^*(z^{*-1})$ defines $\tilde{F}$ to be meromorphic on D. Of course $\tilde{F}$ need not be a pseudocontinuation of F.

Analogous definitions are made for Ω. Suppose U is inner on Ω. Then U has a meromorphic pseudocontinuation on $\Omega \cup \Omega_-$ given by

$$U(z) = \begin{cases} U(z) & z \in \Omega \\ 1/U^*(z^*) & z \in \Omega_-\,. \end{cases} \tag{12}$$

If F is meromorphic on Ω, then $\tilde{F}(z) = F^*(z^*)$ defines $\tilde{F}$ to be meromorphic on Ω_-.

We say that F is $\mathfrak{N}_0^+$ on D if $F \in \mathfrak{N}^+$ on D and $F(0) = 0$. $\mathscr{N}_0^+$ is defined to be the set of all f such that $f(e^{i\theta}) = \lim_{r \uparrow 1} F(re^{i\theta})$ a.e., where $F \in \mathfrak{N}_0^+$ on D.

Suppose U, V are inner functions on D. $\mathscr{M}_0(u, v, \Gamma)$ is the set of all functions f on Γ such that $uf \in \mathscr{N}^+$ and $vf^* \in \mathscr{N}_0^+$. $\mathscr{M}_0(u, v, \Gamma)$ can be characterized as follows: $f \in \mathscr{M}_0(u, v, \Gamma)$ if and only if there exists a function F separately meromorphic in D and D_- and such that

$$f(e^{i\theta}) = \lim_{r\uparrow 1} F(re^{i\theta}) = \lim_{r\downarrow 1} F(re^{i\theta}) \quad \text{a.e.,} \tag{13}$$

with

$$UF \in \mathfrak{N}^+ \text{ on } D \text{ and } V\tilde{F} \in \mathfrak{N}_0^+ \text{ on } D\,. \tag{14}$$

In case U and V are of type $\mathfrak{C}$ we can deduce (14) from an inequality involving F, U and V.

THEOREM 2.1. *Suppose U and V are of type $\mathfrak{C}$, and F is meromorphic in D and has a meromorphic pseudocontinuation to a function F on $D \cup D_-$. Further suppose there exists $K > 0$ such that*

$$|F(z)|^2 \leqq K(1 - |z|^2)^{-1}(|U(z)|^{-2} - |V(z)|^2), \quad |z| \neq 1. \tag{15}$$

Then $f(e^{i\theta}) = \lim_{r\uparrow 1} F(re^{i\theta}) \in \mathscr{M}_0(u, v, \Gamma)$.

Proof. If F satisfies (15) on D then

$$|U(z)F(z)|^2 \leqq K(1 - |z|^2)^{-1}(1 - |U(z)V(z)|^2)\,,$$

so $UF \in \mathfrak{N}^+$ by Theorem 1.4.

If F satisfies (15) on D_-, then for all $z \in D$,

$$|V(z)\tilde{F}(z)|^2 \leqq K|z|^2(1 - |z|^2)^{-1}(1 - |U(z)V(z)|^2)$$

so $V\tilde{F} \in \mathfrak{N}^+$ by 1.4. But we also deduce that $V(0)\tilde{F}(0) = 0$, so $V\tilde{F} \in \mathfrak{N}_0^+$. It therefore follows from the characterization of $\mathscr{M}_0(u, v, \Gamma)$ given in (13) and (14) that $f \in \mathscr{M}_0(u, v, \Gamma)$.

In case $f \in L^2(\Gamma)$, i.e., in case $\int |f|^2 d\sigma < \infty$, we have a stronger result.

THEOREM 2.2. *Assume that U, V are inner of type $\mathfrak{C}$ on D and $f \in L^2(\Gamma)$. Then $f \in \mathscr{M}_0(u, v, \Gamma)$ if and only if there exists a function F satisfying the hypotheses of Theorem 2.1 such*

$$f(e^{i\theta}) = \lim_{r\uparrow 1} F(re^{i\theta}) \text{ a.e..}$$

Proof. It follows from Theorem 2.1 that if F satisfies (15) then $f \in \mathscr{M}_0(u, v, \Gamma)$. Conversely, suppose $f \in \mathscr{M}_0(u, v, \Gamma) \cap L^2(\Gamma)$. Then $uf \in \mathscr{N}^+ \cap L^2(\Gamma) = H^2$ and $vf^* \in \mathscr{N}_0^+ \cap L^2(\Gamma) \subseteqq H^2$ with $\int vf^* d\sigma = 0$.

Thus uf and $v\chi^* f^*$ are in $(uvH^2)^\perp \cap H^2$, where $\chi(e^{i\theta}) = e^{i\theta}$.

Now any $g \in (uvH^2)^\perp \cap H^2$ is the boundary value function of

$$G(z) = \int (1 - ze^{-i\xi})^{-1} (1 - u^*(e^{i\xi}) v^*(e^{i\xi}) U(z) V(z)) g(e^{i\xi}) \sigma(d\xi), \quad z \in D .$$

But then it follows from the Schwarz inequality that

$$(16) \qquad | G(z) |^2 \leqq K(1 - | z |^2)^{-1} (1 - | U(z) V(z) |^2), \quad z \in D ,$$

where $K = \int | g |^2 d\sigma$.

By applying (16) to $g = uf$ and $g = v\chi^* f^*$ we obtain

$$(17) \qquad | U(z) F(z) |^2 \leqq K(1 - | z |^2)^{-1} (1 - | U(z) V(z) |^2), \quad z \in D ,$$

and

$$(18) \qquad | V(z) \tilde{F}(z) |^2 \leqq K | z |^2 (1 - | z |^2)^{-1} (1 - | U(z) V(z) |^2), \quad z \in D ,$$

where $K = \int_\Gamma | f |^2 d\sigma$.

It is easily seen that (17) and (18) together is equivalent to (15).

COROLLARY 2.3. *Assume that V is inner of type $\mathfrak{C}$ on D and $f \in H^2$ on Γ. Then $f \in (vH^2)^\perp$ if and only if there exists a meromorphic function F on $D \cup D_-$ such that*

$$(19) \qquad f(e^{i\theta}) = \lim_{r \uparrow 1} F(re^{i\theta}) = \lim_{r \downarrow 1} F(re^{i\theta}) \text{ a.e.,}$$

for which there exists $K > 0$ with

$$| F(z) |^2 \leqq K (1 - | z |^2)^{-1} (1 - | V(z) |^2), \quad z \in D \cup D_- .$$

Proof. Note that $(vH^2)^\perp \cap H^2 = \mathscr{M}_0(1, v, \Gamma)$, and use 2.2.

COROLLARY 2.4. *Assume that U, V are inner of type $\mathfrak{C}$ on D and $f \in L^2(\Gamma)$. Then $f \in \mathscr{M}(u, v, \Gamma)$ if and only if there exists a function F meromorphic in D with pseudocontinuation F such that* (19) *holds and there exists $K > 0$ such that*

$$| F(z) |^2 \leqq K (1 - | z |^2)^{-1} (| U(z) |^{-2} - | zV(z) |^2), \quad z \in D .$$

Proof. Note that $\mathscr{M}(u, v, \Gamma) = \mathscr{M}_0(u, \chi v, \Gamma)$.

The same kind of problem can be considered on Ω with minor modifications in the proofs.

THEOREM 2.5. *Suppose F is meromorphic on Ω and has a mero-*

morphic pseudocontinuation to a function F *on* $\Omega \cup \Omega_-$. *Assume that* U *and* V *are inner functions of type* $\mathfrak{C}$ *on* Ω. *Further suppose that there exists* $K > 0$ *such that*

$$| F(z) |^2 \leqq K(\operatorname{Im} z)^{-1} (1 + | z |^2) (| U(z) |^{-2} - | V(z) |^2), \quad z \in \Omega \cap \Omega_- .$$

Then $f(x) = \lim_{y \downarrow 0} F(x + iy) \in \mathscr{M}(u, v, R)$.

THEOREM 2.6. *Assume that* U, V *are inner of type* $\mathfrak{C}$ *on* Ω *and* $f \in L^2(R)$. *Then* $f \in \mathscr{M}(u, v, R)$ *if and only if there exists a function satisfying the hypotheses of Theorem* 2.5 *such that*

$$f(x) = \lim_{y \downarrow 0} F(x + iy) \text{ a.e.}.$$

3. Factorization of nonnegative functions. In this section we shall reformulate an operator factorization theorem of the type set down in [5] in terms of inequalities of the type discussed in § 1 and 2. Throughout $\mathscr{C}$ is a complex separable Hilbert space and $B(\mathscr{C})$ the space of bounded operators on $\mathscr{C}$. We shall restrict ourselves to considerations involving Ω rather than D in order to simplify the exposition. Following [5] we say that a holomorphic function F on Ω taking values in $B(\mathscr{C})$ is in $\mathfrak{N}^+_{B(\mathscr{C})}$ if there exists a nonzero complex-valued outer function Φ such that ΦF is a bounded holomorphic function on Ω that takes values in $B(\mathscr{C})$. Any F in $\mathfrak{N}^+_{B(\mathscr{C})}$ has strong boundary values a.e., that is, the limit $\lim_{y \downarrow 0} F(x + iy) = f(x)$ exists a.e. in the strong operator topology.

We say that a holomorphic function G in $\mathfrak{N}^+_{B(\mathscr{C})}$ has a *meromorphic pseudocontinuation* G if G is meromorphic in Ω_- and the strong limits $\lim_{y \uparrow 0} G(x - iy)$ and $\lim_{y \uparrow 0} G(x + iy)$ exist and are a.e. equal. For such G we define $\tilde{G}$ by $\tilde{G}(z) = G^*(z^*)$, $z \in \Omega \cup \Omega_-$.

THEOREM 3.1. *Let* U *be a complex-valued inner function on* Ω *and* F *a meromorphic function on* Ω *taking values in* $B(\mathscr{C})$ *such that* $UF \in \mathfrak{N}^+_{B(\mathscr{C})}$. *Then* $F(x + iy)$ *has strong boundary values* $f(x)$ a.e. *as* $y \downarrow 0$. *Assume that* $\langle f(x)c, c\rangle \geqq 0$ a.e. *for each* c *in* $\mathscr{C}$.

Then F *has a factorization* $F(z) = \tilde{G}(z)G(z)$, $z \in \Omega$, *where* G *is in* $\mathfrak{N}^+_{B(\mathscr{C})}$ *and has a meromorphic pseudocontinuation* G *such that* $U\tilde{G} \in \mathfrak{N}^+_{B(\mathscr{C})}$. *If there is real interval* I *such that* $f(.)$ *is a.e. bounded on* I *and* U *is analytically continuable across* I, *then* G *is analytically continuable across* I.

Proof. This theorem is a summary of results proved in [5].

THEOREM 3.2. *Theorem* 3.1 *may be modified as follows*:

(i) *The hypothesis "*$UF \in \mathfrak{N}^+_{B(\mathscr{C})}$*" may be replaced by the stronger hypothesis "there exists* $K > 0$ *such that*

$$(20) \qquad \| F(z) \|^2 \leqq K(\operatorname{Im} z)^{-1} (1 + |z|^2) (| U(z) |^{-2} - | U(z) |^2)$$

for all z *in* Ω*".*

(ii) *If in addition one assumes that* $\int_{-\infty}^{\infty} \langle f(x)c, c\rangle \, dx < \infty$ *for all* c *in* $\mathscr{C}$*, then* G *can be chosen to in addition satisfy*

$$(21) \qquad | \langle G(z)c, c\rangle |^2 \leqq K_c(\operatorname{Im} z)^{-1} (1 + |z|^2) (1 - | U(z) |^2), \ c \in \mathscr{C}$$

for some $K_c > 0$ *(*K_c *depends on* c*) and all* $z \in \Omega \cup \Omega_-$*.*

Proof. The proof of 1.4 shows that (20) implies that $UF \in \mathfrak{N}^+_{B(\mathscr{C})}$.

Assume the hypotheses of (ii). Now $f = g^*g$, where $g(x)$ are the strong boundary values of $G(x + iy)$ as $y \downarrow 0$ and $y \uparrow 0$. We have $| \langle g(\cdot)c, c\rangle |^2 \leqq \| g(\cdot)c \|^2 \| c \|^2 = \langle f(\cdot)c, c\rangle \| c \|^2$ for all c in $\mathscr{C}$, so $\langle g(\cdot)c, c\rangle \in L^2(R)$ for all c in $\mathscr{C}$. (21) now follows from Theorem 2.6 and the fact that $\langle g(\cdot)c, c\rangle \in \mathscr{M}(1, u, R)$.

As an example suppose $F(\cdot)$ is an entire function taking values in $B(\mathscr{C})$ such that $\langle F(x)c, c\rangle \geqq 0$ whenever $c \in \mathscr{C}$ and $x \in R$, and there exists $\tau \geqq 0$ and $K > 0$ with

$$\| F(z) \|^2 \leqq Ky^{-1} (1 + |z|^2) \sinh 2\tau y, \ z = x + iy \in \Omega \, .$$

Then F is factorable, $F(z) = \tilde{G}(z)G(z)$, where $G(\cdot)$ is an entire function taking values in $B(\mathscr{C})$. This follows from Theorems 3.1 and 3.2 (i) with $U(z) = e^{i\tau z}$. $G(\cdot)$ is entire by the last statement in Theorem 3.1. It also is deducible from Theorem 3.6 of [5].

If in addition to above $F(\cdot)$ satisfies $\int_{-\infty}^{\infty} \langle F(x)c, c\rangle \, dx < \infty$, then by (21) G satisfies

$$| \langle G(z)c, c\rangle |^2 \leqq K_c y^{-1} (1 + |z|^2) (1 - e^{-\tau y}) \, ,$$

for all $z = x + iy$ with $y \neq 0$ and $c \in \mathscr{C}$. K_c is a constant depending on c.

4. A Fourier type transform and the Paley-Wiener representation. As before let U and V be inner functions in Ω and denote the space $\mathscr{M}(u, v, R) \cap L^2(R)$ by $\mathscr{M}^2(u, v, R)$. This space is easily seen to be a Hilbert subspace of $L^2(R)$. As noted in the introduction $\mathscr{M}^2(e^{ix\tau}, e^{ix\tau}, R)$ is the restriction to the real axis of a classical Paley-Wiener space of entire functions. That

$$\mathscr{M}^2 (e^{ix\tau}, e^{ix\tau}, R) = \mathscr{F}^* L^2 (-\tau, \tau) \, ,$$

(where $\mathscr{F}$ is the Fourier-Plancherel operator on $L^2(R)$), is the content of a well known theorem of Paley and Wiener.

In [4] one of the present authors generalized this theorem to give an integral representation for any of the spaces $\mathscr{M}^2(u, v, R)$. In this section we combine this result with Theorem 2.6. First we shall set down some basic facts from [4]. For simplicity we assume that U and V have no zeros and are normalized so that $U(i)$ and $V(i)$ are positive. U then has a factorization $U(z) = S(z)e^{i\alpha z}$ where S is a singular inner function in Ω and $\alpha \geqq 0$. Using the usual representation for singular inner functions we can combine the two factors in the following convenient form:

$$U(z) = \exp\left(i \int_{R^*} \frac{1 + tz}{t - z} \mu(dt)\right) \tag{22}$$

where μ is a finite positive measure on the extended real numbers $R^* = R \cup \{\infty\}$ whose restriction to R is singular and with $\mu(\{\infty\}) = \alpha$. In the integrand, and elsewhere below, we always take $(z\,\infty)/\infty = z$ for any complex z. V has a similar representation with corresponding measure γ.

Let τ be the total variation of μ and suppose that a is an R^*-valued measurable function defined on $[0, \tau]$ such that $m(a^{-1}(E)) = \mu(E)$ for every subinterval E of R^*. For example, we could take $a(t) = \inf\{x \in R^*: \mu((-\infty, x]) \geqq t\}$. Extend the definition of a to $[0, \infty)$ by setting $a(t) = \infty$ if $t > \tau$. For each $t \geqq 0$ let

$$U_t(z) = \exp\left(i \int_0^t \frac{1 + za(x)}{a(x) - z} dx \right).$$

It is clear from (22) and a change of variables that $U_\tau = U$. Moreover, U_t is an inner function for each t and U_s divides U_t if $0 \leqq s < t$.

In a like manner one can associate $\sigma, b: [0, \sigma] \rightarrow R^*$ and V_t (analogous to τ, a and U_t) with the inner function V. Note that $V_\sigma = V$. U_t and V_t have pseudo-continuations to Ω_- given by (12). For any z in $\Omega \cup \Omega_-$ let

$$H_z^+(t) = V_t(z) \frac{b(t) - i}{b(t) - z}$$

and

$$H_z^-(t) = U_t(z)^{-1} \frac{a(t) + i}{a(t) - z}, \quad t \geqq 0.$$

Now let $H^2(\Omega)$ and $H^2(\Omega_-)$ denote the usual Hardy spaces of functions analytic in Ω and Ω_- respectively, which can also be considered as orthogonal complements of each other in $L^2(R)$. It was shown in [4] that the mappings W_1 and W_2 given by

$$(W_1 g)(z) = (2\pi)^{-1/2} \int_0^\infty H_z^+(t) g(t)\, dt, \ \operatorname{Im} z > 0$$

and

$$(W_2 g)(z) = (2\pi)^{-1/2} \int_0^\infty H_z^-(t) g(t) dt, \ \operatorname{Im} z < 0 ,$$

are isometries from $L^2(0, \infty)$ onto $H^2(\Omega)$ and $H^2(\Omega_-)$ respectively.

Let E: $L^2(-\infty, 0) \to L^2(0, \infty)$ be the operator $(Eg)(t) = g(-t)$. The $W_2 E \oplus W_1$ can be considered as a unitary operator from

$$L^2(-\infty, 0) \oplus L^2(0, \infty) = L^2(R)$$

onto $H^2(\Omega_-) \oplus H^2(\Omega) = L^2(R)$. This operator takes $L^2(-s, t)$ onto $\mathscr{M}^2(u_s, v_t, R)$ for all $s, t \geqq 0$. If μ and γ are supported on the singleton $\{\infty\}$ or, equivalently, if $a(t) = b(t) = \infty$ a.e., then $W_2 E \oplus W_1$ is the adjoint of the Fourier-Plancherel operator. Combining this with Theorem 2.6 yields the following result.

THEOREM 4.1. *Let U and V be inner functions of type $\mathfrak{C}$. Let F be analytic in $\Omega \cup \Omega_-$ and suppose that the two sided boundary function $f(x) = \lim_{|y| \to 0} F(x + iy)$ exists* a.e. *and lies in $L^2(R)$. Let $s, t \geqq 0$. Then the following are equivalent.*

(i)

$$| F(z) |^2 \leqq K (\operatorname{Im} z)^{-1} (1 + | z |^2) (| U_s(z) |^{-2} - | V_t(z) |^2), \ z \in \Omega \cup \Omega_- .$$

(ii) *There exist* a.e. *unique functions g_1 in $L^2(0, t)$ and g_2 in $L^2(0, s)$ such that*

$$F(z) = (2\pi)^{-1/2} \int_0^t H_z^+(z) g_1(x)\, dx + (2\pi)^{-1/2} \int_0^s H_z^-(x) g_2(x) dx, \ \operatorname{Im} z \neq 0 .$$

Moreover, $\| f \|_2^2 = \| g_1 \|_2^2 + \| g_2 \|_2^2$.

Added in proof. We refer the reader to the papers.

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and,

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for more detailed information on meromorphic continuation and $(uH^2)^{\perp}$.

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