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POWER-ASSOCIATIVE ALGEBRAS AND RIEMANNIAN CONNECTIONS

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Let G/H be a reductive homogeneous space with the corresponding Lie algebra decomposition $g = m + h$ where the complementary subspace m satisfies the condition $(\operatorname{ad} H)m \subset m$. It has been shown that the G-invariant connections on G/H correspond to certain nonassociative algebras (m, α) and that these algebras, in turn, correspond to certain local analytic multiplications on G/H. These correspondences generalize many of the results of Lie theory; it has been shown, for example, that there is a change of coordinates at $\bar{e} = eH$ which makes the algebras associated with a local multiplication anti-commutative. However, if G/H has pseudo-Riemannian structures and we require that the change of coordinate maps be local isometries, then the existence of a change of coordinates which gives an anti-commutative algebra is no longer guaranteed. Thus it is natural to ask when an algebra (m, α) inducing a pseudo-Riemannian connection is anti-commutative and it is shown in this paper that a necessary and sufficient condition is basically that (m, α) be power-associative.

1. Basics. Let G be a connected Lie group with Lie algebra g and let H be a closed (Lie) subgroup with Lie algebra h. Then the pair (G, H) or (g, h) is called a *reductive pair* if there exists a subspace m of g such that $g = m + h$ (subspace direct sum) and $(\operatorname{ad} H)m \subset m$. The corresponding analytic manifold $M = G/H$ is called a *reductive homogeneous space* and m is identified with the tangent space $M_{\bar{e}}$. For a reductive space with a fixed Lie algebra decomposition $g = m + h$ it is shown in [2], [6] that there is a 1-1 correspondence between G-invariant connections ∇ and nonassociative algebras (m, α) with $\operatorname{ad} H \subset \operatorname{Aut}(m, \alpha)$. ($\alpha$ is the bilinear algebra multiplication on m and $\operatorname{Aut}(m, \alpha)$ is the automorphism group of the algebra (m, α).)

A G-invariant pseudo-Riemannian connection on a reductive homogeneous space G/H corresponds to an algebra (m, α) with a nondegenerate symmetric bilinear form C such that for all $X, Y, Z \in m$ and $U \in h$

$$C((\operatorname{ad} U)X, Y) + C(X, (\operatorname{ad} U)Y) = 0 \quad \text{and} \tag{1}$$

$$C(\alpha(Z, X), Y) + C(X, \alpha(Z, Y)) = 0\,. \tag{2}$$

We denote such algebras by (m, α, C) and they are discussed in

[4], [6], [7]. In particular since the torsion tensor is zero we have from [2] that for $X, Y \in m$

$$\alpha(X, Y) - \alpha(Y, X) = XY \tag{3}$$

where we use the notation $XY = [X, Y]_m$(resp. $h(X, Y)$) for the projection of $[X, Y]$ in g onto m (resp. h). Thus the algebra (m, α, C) is reductive Lie admissible [5] and in particular for $h = \{0\}$ the algebra (g, α, C) is Lie admissible [1].

As an example let $\pi: G \to G/H$ be the canonical projection of G onto the reductive space G/H. For any $X \in m$ the curves $\gamma(t) = \pi \exp tX$ are geodesics relative to the G-invariant pseudo-Riemannian connection ∇ given by (m, α, C) if and only if $\alpha(X, Y) = (1/2)XY$. This connection is called the *pseudo-Riemannian connection of the first kind* [2], [4] and we use the notation $(m, (1/2)XY, B)$ for the corresponding algebra where B now denotes the nondegenerate form. In particular, let g and h be semi-simple and let Kill denote the Killing form of g. Since $\mathrm{Kill} \mid h \times h$ is nondegenerate we can write $g = m + h$ with $m = h^{\perp}$ relative to the Killing form. Thus (g, h) is a reductive pair. The form $B = \mathrm{Kill} \mid m \times m$ and the multiplication $\alpha(X, Y) = (1/2)XY$ give an algebra $(m, (1/2)XY, B)$ which satisfies conditions (1) and (2) and therefore induces a pseudo-Riemannian connection of the first kind. (One, of course, considers $B = -\mathrm{Kill} \mid m \times m$ in case $\mathrm{Kill} \mid m \times m$ is negative definite as is the case for $G = SO(n)$ and $H = SO(k)$.)

Now let the reductive space G/H have a pseudo-Riemannian connection of the first kind given by the algebra $(m, (1/2)XY, B)$ and suppose G/H has another pseudo-Riemannian connection given by the algebra (m, α, C). Then the nondegeneracy of B and C implies the existence of an $S \in GL(m)$ such that

$$C(X, Y) = B(SX, Y)$$

for all $X, Y \in m$. Also by the symmetry and equation (1) we obtain

$$S^b = S \text{ and } [\mathrm{ad}\, U, S] = 0 \tag{*}$$

for all $U \in h$, where b denotes the adjoint relative to B. In [3], [4], [6] it is noted that the set, J, of endomorphisms of m satisfying (*) forms a Jordan algebra relative to the usual multiplication $S_1 \cdot S_2 = (1/2)(S_1S_2 + S_2S_1)$. Also the formula for α is given by

$$2\alpha(X, Y) = XY + S^{-1}[X(SY) - (SX)Y]$$

where $XY = [X, Y]_m$ is the multiplication in the algebra $(m, (1/2)XY, B)$. Many examples of the algebras (m, α, C) determined by the Jordan algebra J are given in [4]. In the next section we discuss some of the algebraic identities which these algebras may satisfy. These identities for the algebras (m, α, C) are related to isometric coordinate changes and H-spaces $(G/H, \mu)$ as discussed in [7].

2. Power-associative algebras.

An algebra A over a field F is power-associative if every element $X \in A$ generates an associative subalgebra $F[X]$; see [9]. We now assume the algebra (m, α, C) discussed in § 1 is power-associative and use the notation $X^n = \alpha(X, \cdots, \alpha(X, X)\cdots)$ where X occurs n times; this notation is used only for the algebra (m, α, C) and is not to be confused with the product XY in $(m, (1/2)XY, B)$. The following result indicates that an algebra (m, α, C) which defines an invariant Riemannian connection on a reductive space G/H does not satisfy the "usual" identities unless the algebra is anti-commutative; that is, unless the connection is of the first kind.

THEOREM 1. *Let (G, H) be a reductive pair with a corresponding Lie algebra decomposition $g = m + h$.*

(a) *If the algebra (m, α, C) defines an invariant Riemannian connection on G/H, then $\alpha(X^2, X) = \alpha(X, X^2)$ if and only if $\alpha(X, Y) = (1/2)XY$ for all $X, Y \in m$.*

(b) *Let G/H have an invariant Riemannian connection of the first kind which is determined by the algebra $(m, (1/2)XY, B)$. If the algebra (m, α, C) defines an invariant pseudo-Riemannian connection on G/H, then the algebra (m, α, C) is power associative if and only if $\alpha(X, Y) = (1/2)XY$ for all $X, Y \in m$.*

Proof. Since an anti-commutative algebra is power-associative, we need only prove the converses of the above statements.

(a) From formula (2) the positive definite form C must satisfy $C(V, \alpha(U, V)) = 0$ for all $U, V \in m$. Now using this and formula (2) we see that for any $X \in m$

$$\begin{aligned} C(\alpha(X, X), \alpha(X, X)) &= -C(X, \alpha(X, \alpha(X, X))) \\ &= -C(X, \alpha(\alpha(X, X), X)) \\ &= 0 . \end{aligned}$$

where the identity $\alpha(X, X^2) = \alpha(X^2, X)$ is used for the second equality. Thus $\alpha(X, X) = 0$. Using (3), we obtain $\alpha(X, Y) = (1/2)XY$.

(b) If we are given an algebra $(m, (1/2)XY, B)$ which induces a Riemannian connection of the first kind and a second algebra (m, α, C) which induces another pseudo-Riemannian connection, then, as remarked in § 1, we can write $C(X, Y) = B(SX, Y)$ and $2\alpha(X, Y) = XY + S^{-1}[X(SY) - (SX)Y]$ for some $S \in GL(m)$. Using the fact that the positive definite form B satisfies $B(ZX, Y) + B(X, ZY) = 0$, we now show that the algebra (m, α, C) has no nonzero idempotent elements. For suppose $E = \alpha(E, E)$; then from the above formula $E = S^{-1}[E(SE)]$ so that $SE = E(SE)$. From this $SE = E(E(SE))$ and therefore

$$\begin{aligned} B(SE, SE) &= B(SE, E(E(SE))) \\ &= -B(E(SE), E(SE)) \\ &= -B(SE, SE) \end{aligned}$$

so that $B(SE, SE) = 0$ and $SE = 0$. As S is nonsingular, $E = 0$.

Since the power-associative algebra (m, α, C) contains no idempotents, the associative subalgebra $F[X]$ generated by any $X \in m$ is nil [9; Prop. 3.3]; that is, for each $X \in m$, there exists a positive integer p such that $X^p = 0$ in the algebra (m, α, C). By power-associativity if $X^{r+t} = 0$ for positive integers r and t, then

$$0 = X^{r+t} = \alpha(X^r, X^t) = \frac{1}{2} X^r X^t + \frac{S^{-1}}{2} [X^r(SX^t) - (SX^r)X^t] .$$

Thus using $\alpha(X, Y) - \alpha(Y, X) = XY$ we also see $X^r X^t = \alpha(X^r, X^t) - \alpha(X^t, X^r) = X^{r+t} - X^{r+t} = 0$ which implies

$$X^r(SX^t) = (SX^r)X^t \tag{4}$$

whenever $X^{r+t} = 0$.

We now show $X^3 = 0$ implies $X^2 = 0$. For suppose $X^3 = 0$; then from formula (4) we obtain

$$X(SX^2) = (SX)X^2 .$$

Using the formula for $\alpha(X, Y)$ we note $SX^2 = X(SX)$ and have

$$\begin{aligned} B(SX^2, SX^2) &= B(X(SX), SX^2) \\ &= -B(SX, X(SX^2)) \\ &= -B(SX, (SX)X^2) \\ &= -B((SX)(SX), X^2) \\ &= 0 \end{aligned}$$

using the anti-commutativity $ZZ = 0$ in $(m, (1/2)XY, B)$. Thus $SX^2 = 0$ which implies $X^2 = 0$.

Next we show $X^{n+1} = 0$ implies $X^n = 0$ for $n \geqq 3$ and consequently by induction $X^{n+1} = 0$ implies $X^2 = 0$. For suppose $X^{n+1} = 0$; then $X^{2n-1} = 0$ and from formula (4) we obtain

$$X(SX^n) = (SX)X^n \text{ and } X^{n-1}(SX^n) = (SX^{n-1})X^n .$$

Using these we see

$$\begin{aligned} B(X(SX^{n-1}), SX^n) &= -B(SX^{n-1}, X(SX^n)) \\ &= -B(SX^{n-1}, (SX)X^n) \\ &= B((SX^{n-1})X^n, SX) \end{aligned}$$

and

$$\begin{aligned} B((SX)X^{n-1}, SX^n) &= B(SX, X^{n-1}SX^n)) \\ &= B(SX, (SX^{n-1})X^n) . \end{aligned}$$

Thus using $X^{n-1}X = \alpha(X^{n-1}, X) - \alpha(X, X^{n-1}) = X^n - X^n = 0$, we obtain $2SX^n = X(SX^{n-1}) - (SX)X^{n-1}$ and

$$\begin{aligned} 2B(SX^n, SX^n) &= B(X(SX^{n-1}) - (SX)X^{n-1}, SX^n) \\ &= B(X(SX^{n-1}), SX^n) - B((SX)X^{n-1}, SX^n) \\ &= 0 \end{aligned}$$

and therefore $X^n = 0$. Since the algebra (m, α, C) is nil, we have for every $X \in m$ that $X^p = 0$ for some integer p. Thus by the above $0 = X^2 = \alpha(X, X)$. Using (3), we obtain $\alpha(X, Y) = (1/2)XY$.

REMARKS. The conclusion of Theorem 1 that $\alpha(X, Y) = (1/2)XY$ need not imply the forms B and C are equal. However, let us consider the algebra $(m, (1/2)XY, B)$ as given where we can assume B is just nondegenerate. Then the endomorphism S which determines C for another algebra (m, α, C) with $\alpha(X, Y) = (1/2)XY$ is in the multiplication centralizer of $(m, (1/2)XY, B)$. To see this first recall that the multiplication centralizer, Γ, of the algebra $(m, (1/2)XY, B)$ consists of those endomorphisms T of m satisfying $L(X)T = TL(X)$ for all $X \in m$, where $L(X)\colon m \to m\colon Y \to XY$. In [9; p. 15] the multiplication centralizer is discussed in general. It is proven that Γ is a subalgebra of the algebra of all endomorphisms of m and if the algebra $(m, (1/2)XY, B)$ is simple, Γ is a field. Now, to see that S is in Γ we use formula (2) and $\alpha(X, Y) = (1/2)XY$ and note that

$$\begin{aligned} B(S(XY), Z) &= C(XY, Z) \\ &= 2C(\alpha(X, Y), Z) \\ &= -2C(Y, \alpha(X, Z)) \\ &= -C(Y, XZ) \\ &= -B(SY, XZ) \\ &= B(X(SY), Z) . \end{aligned}$$

Since B is nondegenerate, $S(XY) = X(SY)$; that is, $SL(X) = L(X)S$ which implies $S \in \Gamma$. Conversely, a nonsingular endomorphism S in $\Gamma \cap J$ determines an algebra (m, α, C) with $\alpha(X, Y) = (1/2)XY$. In particular, if S is chosen so that C is positive definite, then the corresponding connection is Riemannian.

As an example, let the pseudo-Riemannian connection determined by the nonzero algebra $(m, (1/2)XY, B)$ be holonomy irreducible. Then as discussed in [3], [4], [6], the algebra $(m, (1/2)XY, B)$ is simple. If we require that the algebra $(m, (1/2)XY, C)$ be such that C is positive definite, then the following computations prove S is symmetric relative to C. For $X, Y \in m$,

$$\begin{aligned} C(X, SY) &= B(SX, SY) \\ &= B(SY, SX) \\ &= C(Y, SX) \\ &= C(SX, Y) \end{aligned}$$

so that $S^c = S$, where c denotes the adjoint relative to C. Therefore, S has a nonzero real characteristic root λ and the characteristic root space $n = \{X \in m: SY = \lambda Y\}$ is a nonzero ideal of $(m, (1/2)XY, B)$; this uses $L(X)S = SL(X)$ for all $X \in m$. Since $(m, (1/2)XY, B)$ is simple, we see $n = m$ and consequently $S = \lambda I$; thus the original form B must be definite in this case. More generally, if $(m, (1/2)XY, B)$ is semi-simple (that is, a direct sum of simple ideals), then the corresponding S is diagonalizable. These semi-simple algebras often occur when g and h are semi-simple Lie algebras as discussed in [4], [8].

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