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FIXED POINT SETS OF PEANO CONTINUA

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For each positive integer $n = 1, 2, \cdots$, it is shown that there is an $(n+1)$-dimensional acyclic LC^{n-1} continuum X_n containing an n-dimensional sphere which is not the fixed point set of any self-map of X_n.

1. Introduction. A subset A of a topological space X is called a *fixed point set* of X if there is a (continuous) map $f: X \to X$ such that $f(x) = x$ iff $x \in A$. If X is Hausdorff, then A is closed, and, clearly, every retract of X is a fixed point set of X. It is possible that a space X may have the property that each of its nonempty closed subsets is a fixed point set of X. The problem of determining which spaces have this property, called the *complete invariance property* by L. E. Ward, Jr. in [5], has been investigated by H. Robbins, Helga Schirmer, and L. E. Ward, Jr. Some spaces known to have the complete invariance property include n-cells [1], dendrites [2], convex subsets of Banach spaces [5], compact manifolds without boundary [3], and all compact triangulable manifolds with or without boundary [4].

The general question as to what properties a space must satisfy to insure that it has the complete invariance property has not been resolved. In fact, in [5, p. 553] L. E. Ward, Jr. asks the following question.

Does every Peano continuum have the complete invariance property?

The purpose of this note is to show that even acyclic Peano continua which possess higher order local connectedness need not have the complete invariance property. Indeed, for each positive integer $n = 1, 2, \cdots$, we give an example of an $(n+1)$-dimensional acyclic LC^{n-1} continuum X_n which fails to have the complete invariance property. Moreover, X_n contains an n-dimensional sphere which is not a fixed point set of X_n.

2. Notation and the construction of X_n. We shall let E^n denote Euclidean n-dimensional space, and we shall consider E^m to be canonically imbedded in E^n if $m < n$. The closed unit ball in E^{n+1} shall be denoted by B^{n+1} and the boundary $\mathrm{Bd}\, B^{n+1}$ of B^{n+1} shall be denoted by S^n.

Consider the rectangle in $E^2 \subset E^{n+2}$ with vertices $(1, -1, 0, \cdots 0)$, $(0, -1, 0, \cdots, 0)$, $(0, 1, 0, \cdots, 0)$, $(1, 1, 0, \cdots, 0)$. Let D denote the closed disk consisting of this rectangle together with its interior in

E^2, and let A denote the segment in $\operatorname{Bd} D$ with endpoints $(0, -1, 0, \cdots, 0)$ and $(0, 1, 0, \cdots, 0)$. We shall let C denote the closure of the curve in E^2 whose equation is given by $y = \sin \pi/x$ for $0 < x \leqq 1$.

Let $\{B_t^{n+1} | 1 \leqq t < \infty\}$ be a disjoint collection of $(n+1)$-dimensional balls in E^{n+2} satisfying the following properties.

(1) $\bigcup_{1 \leqq t < \infty} B_t^{n+1}$ is homeomorphic to $B^{n+1} \times [1, \infty)$ under a homeomorphism which sends B_t^{n+1} onto $B^{n+1} \times \{t\}$ for each t in the half-open interval $[1, \infty)$.

(2) $\lim_{t \to \infty} \delta(B_t^{n+1}) = 0$, where $\delta(B_t^{n+1})$ denotes the diameter of B_t^{n+1}.

(3) For each t in $[1, \infty)$,

$$B_t^{n+1} \cap E^2 = \left\{\left(\frac{1}{t}, \sin \pi t, 0, \cdots, 0\right)\right\} .$$

Let $S_t^n = \operatorname{Bd} B_t^{n+1}$. If J is an interval in the real line, we define

$$B_J^{n+1} = \bigcup_{t \in J} B_t^{n+1} \quad \text{and} \quad S_J^n = \bigcup_{t \in J} S_t^n .$$

Definition. For $n = 1, 2, \cdots$, we define

$$X_n = D \cup S^n_{[1,\infty)} .$$

Essentially, $S^n_{[1,\infty)}$ is obtained by taking a cone over an n-sphere, removing the vertex, and then winding the resulting tube of n-spheres in a "$\sin 1/x$" fashion in E^{n+2} so as to converge to the limit interval A. This procedure is carried out so that the intersection of E^2 with the closure of $S^n_{[1,\infty)}$ is precisely C. The disk D is then attached to the closure of $S^n_{[1,\infty)}$ along C to obtain X_n.

3. The properties of X_n.

Property 1. *Each X_n is an $(n+1)$-dimensional acyclic continuum.*

Proof. By acyclic we mean that X_n has the Čech cohomology of a point. Since X_n is clearly an $(n+1)$-dimensional continuum, we need only show that it is acyclic. But

$$X_n = \bigcap_{t \geqq 1} (D \cup S^n_{[1,\infty)} \cup B^{n+1}_{[t,\infty)})$$

and, for each t, $D \cup B^{n+1}_{[t,\infty)}$ is a deformation retract of $D \cup S^n_{[1,\infty)} \cup B^{n+1}_{[t,\infty)}$. Since $D \cup B^{n+1}_{[t,\infty)}$ is contractible, each of the spaces $D \cup S^n_{[1,\infty)} \cup B^{n+1}_{[t,\infty)}$ is contractible. Therefore, by the continuity axiom for Čech cohomology, X_n is acyclic.

Property 2. *Each* X_n *is* LC^{n-1} *at points of* A, *and locally contractible elsewhere.*

Proof. Clearly, every point p in $X_n - A$ has a neighborhood which is finitely triangulable, and therefore X_n is locally contractible at p. If $p \in A$, then p has a compact neighborhood consisting of a closed disk K and infinitely many disjoint cylinders of the form $S^n_{[t,t+\lambda]}$, each of which intersects K in an arc. It then follows that X_n is LC^{n-1} at p.

Property 3. S^n_1 *is not a fixed point set of* X_n.

Proof. Since S^n_1 is not contractible in $S^n_{[1,\infty)} \cup A$ and $(S^n_{[1,\infty)} \cup A) \cap D = C$, it follows that S^n_1 is not contractible in X_n.

Now suppose that $f: X_n \to X_n$ is a map whose fixed point set is precisely S^n_1. If $f(A) \subset A$, then f has a fixed point in A. Since this is not possible, there is a point p in A such that $f(p) \in X_n - A$. Let V be a contractible neighborhood of $f(p)$ in X_n, and let $\beta: V \times I \to V$ denote a homotopy which deforms V to a point. Since f is continuous at p, there is a neighborhood U of p in X_n such that $f(U) \subset V$. Then, for some r in $[1, \infty)$, we have $S^n_r \subset U$. Let $\alpha: S^n_1 \times I \to S^n_{[1,r]}$ be a homotopy which deforms S^n_1 onto S^n_r.

Define a homotopy $H: S^n_1 \times I \to X_n$ by

$$H(x, t) = \begin{cases} f(\alpha(x, 2t)) , & (\text{if } 0 \leqq t \leqq \frac{1}{2}) , \\ \beta(f(\alpha(x, 1)), 2t - 1) , & (\text{if } \frac{1}{2} \leqq t \leqq 1) . \end{cases}$$

It is easy to check that H is a homotopy which deforms S^n_1 to a point in X_n. This contradiction shows that S^n_1 is not a fixed point set of X_n.

4. Problem. Let us restrict our discussion to the class of compacta. Then all the examples of spaces having the complete invariance property which are mentioned in the introduction of this paper are ANR-spaces. In fact, the first three examples are AR-spaces. In view of this, it seems appropriate to suggest the following problem.

Does every AR-*space* (ANR-*space*) *have the complete invariance property*?

References

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