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ON THE LATTICE OF VARIETIES OF BANDS OF GROUPS

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In this paper we prove that the lattice of varieties of bands of groups is modular and apply this to direct decompositions of various sublattices. The join of the varieties of bands and of completely simple semigroup is shown to be the variety of "pseudo-orthodox" bands of groups.

1. Introduction.

When considered as semigroups with an additional unary operation $x \to x^{-1}$, where x^{-1} denotes the (unique) inverse of x in the subgroup to which it belongs, the class $\boldsymbol{CR}$ of *completely regular* semigroups (often called *unions of groups*) forms a variety of universal algebras, containing as a subvariety thevariety $\boldsymbol{BG}$ of *bands of groups* (those completely regular semigroups on which $\mathscr{H}$ is a congruence) ([12]). In this paper results of Spitznagel [14] on the lattice of congruences on a band of groups are applied to show that $\mathscr{V}(\boldsymbol{BG})$, the lattice of subvarieties of $\boldsymbol{BG}$, is modular (Theorem 3.1). Petrich [12, 13] considered various subvarieties of $\boldsymbol{BG}$ but left open the problem [13, p.1196] of finding the join of the subvarieties $\boldsymbol{B}$ and $\boldsymbol{CS}$ (of bands and of completely simple semigroups respectively). We show that $\boldsymbol{B} \vee \boldsymbol{CS} = \boldsymbol{POBG}$, the variety of *pseudo-orthodox* bands of groups, and is thus strictly contained in $\boldsymbol{BG}$. (If V is a variety of completely regular semigroups and $S \in \boldsymbol{CR}$ we shall call S *pseudo-V* if $eSe \in V$ for every idempotent e of S.) This result is actually an immediate corollary to our characterization of the join $\boldsymbol{O} \vee \boldsymbol{NBG}$ of the varieties of orthodox completely regular semigroups and of normal bands of groups. Theorem 3.1 is also applied to directly decompose various sublattices of $\mathscr{V}(\boldsymbol{BG})$.

2. Preliminaries.

For background to this paper the reader is referred to [13] where defining identities are presented for most of the varieties encountered here. Various subvarieties of $\boldsymbol{CR}$ are shown on the diagram on p. 1172 of [13]. For easy reference we will give a list of our abbreviations for these:

$\boldsymbol{CR}$: completely regular semigroups
$\boldsymbol{BG}$: bands of groups
$\boldsymbol{NBG}$: normal bands of groups
$\boldsymbol{OBG}$: orthodox bands of groups
$\boldsymbol{B}$: bands
$\boldsymbol{CS}$: completely simple semigroups
$\boldsymbol{NB}$: normal bands

$\boldsymbol{SLG}$: semilattices of groups
$\boldsymbol{RG}$: rectangular groups
$\boldsymbol{SL}$: semilattices
$\boldsymbol{RB}$: rectangular bands
$\boldsymbol{G}$: groups
$\boldsymbol{T}$: the trivial variety

Further $\boldsymbol{O}$ will denote the variety of *orthodox* completely regular semigroups (those whose set of idempotents forms a subsemigroup).

We will otherwise use the terminology and notation of [7] (for semigroup theory) and [6] (for lattice theory). Throughout E_S denotes the set of idempotents of the semigroup S and $\langle E_S \rangle$ the subsemigroup of S generated by E_S (completely regular, by [3], when $S \in \boldsymbol{CR}$).

If ρ is a relation on S then ρ^* will denote the congruence on S generated by ρ and if $A \subseteq S$, $\rho | A$ will denote the restriction $\rho \cap (A \times A)$ of ρ to A. The symbol $\Lambda(S)$ represents the lattice of congruences on S, with ι and ω the smallest and largest elements respectively. A point which will be of importance in §3, in particular, is that $\Lambda(S)$ is a sublattice of the lattice $\Sigma(S)$ of all equivalences on S ([7], §I. 5).

3. Bands of groups.

We will prove the following theorem.

THEOREM 3.1. *The lattice $\mathscr{V}(\boldsymbol{BG})$ is modular.*

Before beginning the proof we need some lattice-theoretic concepts from [6], §III. 2. If L is a lattice and $a \in L$ then a is said to be *neutral* if

(i) a "separates" L:
if $x, y \in L$ then $a \wedge x = a \wedge y$ and $a \vee x = a \vee y$ together imply $x = y$,

(ii) the map $x \to a \vee x$ is a morphism

and

(iii) the map $x \to a \wedge x$ is a morphism.

Elements satisfying (i) and (ii) are called *standard*. It is almost immediate that a is neutral if and only if the map $x \to (a \wedge x, a \vee x)$ is an isomorphism of L upon a subdirect product of the sublattices $\{x \in L : x \leqq a\}$ and $\{x \in L : x \geqq a\}$. Clearly neutrality is a self-dual notion. (For equivalent formulations of neutrality see [6], Theorem III. 2.4.)

We now quote some results of Spitznagel, rephrased in the above terminology.

RESULT 3.2 [14, Theorem 3.9]. On any band of groups S, the congruence $\mathscr{H}$ is a standard element of $\Lambda(S)$. Further it is a neutral element of the sublattice $[\iota, \mathscr{D}]$ of $\Lambda(S)$.

REMARK 3.3. Using similar methods to those of Spitznagel it may be verified that an analogous result is true in *any* completely regular semigroup S. For example $\mathscr{H}$ separates $\Lambda(S)$ (in the above sense), the map $\rho \to \rho \vee \mathscr{H}$ is a complete morphism of the sublattice $\Lambda(S)$ of $\Sigma(S)$ (see §2) into $\Sigma(S)$ and the map $\rho \to \rho \cap \mathscr{H}$ is a complete morphism of the sublattice $[\iota, \mathscr{D}]$ of $\Lambda(S)$ into $\Sigma(S)$ (the join being interpreted as join of equivalences where necessary). A consequence of the last statement will be used in the proof of Theorem 5.3: for any collection $\{\tau_k\}_{k \in K}$ of congruences on S, each contained in $\mathscr{D}$, $\vee\{\tau_k: k \in K\} \cap \mathscr{H} = V\{\tau_k \cap \mathscr{H}: k \in K\}$, so that in particular, if $\tau_k \cap \mathscr{H} = \iota$ for each $k \in K$ then $V\{\tau_k: k \in K\} \cap \mathscr{H} = \iota$.

Now in any variety of algebras the lattice of subvarieties is dually isomorphic with the lattice of fully invariant congruences on the free algebra F on a countable set of generators in that variety, and moreover the lattice of fully invariant congruences on any algebra A forms a sublattice of the lattice of all congruences on A. (See, for example [10]. A congruence ρ on an algebra A is *fully invariant* if $a\rho b$ implies $a\theta \;\rho\; b\theta$ for every endomorphism θ of A.)

Noticing that both $\mathscr{H}$ and $\mathscr{D}$ are fully invariant congruences on any band of groups, we see from Result 3.2 that in this case $\mathscr{H}$ is a neutral element in the lattice of fully invariant congruences contained in $\mathscr{D}$. Since $\mathscr{H}$ defines the variety $\boldsymbol{B}$ (within $\boldsymbol{BG}$) and $\mathscr{D}$ defines $\boldsymbol{SL}$, we see that $\boldsymbol{B}$ is a neutral element in the sublattice $[\boldsymbol{SL}, \boldsymbol{BG}]$ of $\mathscr{V}(\boldsymbol{BG})$. We then have

PROPOSITION 3.4. *The sublattice $[\boldsymbol{SL}, \boldsymbol{BG}]$ is a subdirect product of the sublattices $[\boldsymbol{SL}, \boldsymbol{B}]$ and $[\boldsymbol{B}, \boldsymbol{BG}]$ and is therefore modular.*

Proof. The first statement follows from the remarks on the definition of neutrality. For the second, we quote the result of [1, 2, 4] that $\mathscr{V}(\boldsymbol{B})$ is distributive (whence modular) and then note again that since $\boldsymbol{B}$ corresponds to the fully invariant congruence $\mathscr{H}$ on F (the free band of groups on a countable set of generators), the sublattice $[\boldsymbol{B}, \boldsymbol{BG}]$ is dually isomorphic with the lattice of all fully invariant congruences on F contained in $\mathscr{H}$, which in turn is a sublattice of the lattice of *all* congruences on F contained in $\mathscr{H}$. Since this last lattice is modular [9], so is $[\boldsymbol{B}, \boldsymbol{BG}]$ and therefore $[\boldsymbol{SL}, \boldsymbol{BG}]$ also.

The proof of Theorem 3.1 will be completed by the following proposition, since $\mathscr{V}(\boldsymbol{SL})$ consists of just $\boldsymbol{T}$ and $\boldsymbol{SL}$ and is therefore distributive. Note that this result is true for *arbitrary* varieties containing $\boldsymbol{SL}$ (and contained in $\boldsymbol{CR}$).

PROPOSITION 3.5. *The variety $\boldsymbol{SL}$ is a neutral element in $\mathscr{V}(\boldsymbol{CR})$ and therefore in any sublattice containing $\boldsymbol{SL}$. In particular, therefore, $\mathscr{V}(\boldsymbol{BG})$ is a subdirect product of $\mathscr{V}(\boldsymbol{SL})$ and $[\boldsymbol{SL}, \boldsymbol{BG}]$.*

Proof. It will prove convenient here to show directly that the map $\chi: \boldsymbol{K} \to (\boldsymbol{K} \cap \boldsymbol{SL}, \boldsymbol{K} \vee \boldsymbol{SL})$ is an isomorphism of $\mathscr{V}(\boldsymbol{CR})$ into $\mathscr{V}(\boldsymbol{SL}) \times [\boldsymbol{SL}, \boldsymbol{CR}]$. Clearly χ is order-preserving.

Suppose, on the other hand, that $\boldsymbol{K}, \boldsymbol{L} \subseteq \boldsymbol{CR}$ and $\boldsymbol{K} \cap \boldsymbol{SL} \subseteq \boldsymbol{L} \cap \boldsymbol{SL}$, $\boldsymbol{K} \vee \boldsymbol{SL} \subseteq \boldsymbol{L} \vee \boldsymbol{SL}$. Now since $\mathscr{V}(\boldsymbol{SL}) = \{\boldsymbol{T}, \boldsymbol{SL}\}$, either $\boldsymbol{SL} \subseteq \boldsymbol{L}$ or $\boldsymbol{L} \cap \boldsymbol{SL} = \boldsymbol{T}$. In the former case $\boldsymbol{K} \subseteq \boldsymbol{K} \vee \boldsymbol{SL} \subseteq \boldsymbol{L} \vee \boldsymbol{SL} = \boldsymbol{L}$. Otherwise $\boldsymbol{L} \cap \boldsymbol{SL} = \boldsymbol{T}$, whence $\boldsymbol{K} \cap \boldsymbol{SL} = \boldsymbol{T}$ also. But on any completely regular semigroup S, $\mathscr{D}$ is an SL-congruence and if $S/\mathscr{D} \in \boldsymbol{T}$ then $S \in \boldsymbol{CS}$. Hence $\boldsymbol{K}, \boldsymbol{L} \subseteq \boldsymbol{CS}$, and so $\boldsymbol{K} \vee \boldsymbol{SL}$, $\boldsymbol{L} \vee \boldsymbol{SL} \subseteq \boldsymbol{CS} \vee \boldsymbol{SL} = \boldsymbol{NBG}$.

From [13, Theorem 4.7], the map $\boldsymbol{V} \to \boldsymbol{V} \cap \boldsymbol{CS}$ is a lattice morphism of $\mathscr{V}(\boldsymbol{NBG})$ upon $\mathscr{V}(\boldsymbol{CS})$. Thus $\boldsymbol{K} = (\boldsymbol{K} \cap \boldsymbol{CS}) \vee (\boldsymbol{SL} \cap \boldsymbol{CS}) = (\boldsymbol{K} \vee \boldsymbol{SL}) \cap \boldsymbol{CS}$ and similarly $\boldsymbol{L} = (\boldsymbol{L} \vee \boldsymbol{SL}) \cap \boldsymbol{CS}$, whence $\boldsymbol{K} \vee \boldsymbol{SL} \subseteq \boldsymbol{L} \vee \boldsymbol{SL}$ implies $\boldsymbol{K} \subseteq \boldsymbol{L}$.

Hence χ is an order isomorphism and $\boldsymbol{SL}$ is neutral.

COROLLARY 3.6. *The lattice $\mathscr{V}(\boldsymbol{CS})$ is modular.*

From Theorem 3.1, direct decompositions of various sublattices of $\mathscr{V}(\boldsymbol{BG})$ may be obtained by applying the following result.

RESULT 3.7 ([6], Theorem IV. 1.14). If a, b are elements of a modular lattice L then the sublattice of L generated by $[a \wedge b, a]$ and $[a \wedge b, b]$ is isomorphic to $[a \wedge b, a] \times [a \wedge b, b]$.

For modular lattices in general, the sublattice generated by $[a \wedge b, a]$ and $[a \wedge b, b]$ need not be $[a \wedge b, a \vee b]$. However in our situation this does occur.

COROLLARY 3.8 ([12], *Theorem* 3.3). $\mathscr{V}(\boldsymbol{OBG}) \cong \mathscr{V}(\boldsymbol{B}) \times \mathscr{V}(\boldsymbol{G})$.

Proof. Using, for example, the results of [7, Chapter VI], we have that any orthodox band of groups is a subdirect product of its maximum $\boldsymbol{B}$-image and its maximum $\boldsymbol{SLG}$-image (since the inverse completely regular semigroups are just the semilattices of groups).

Thus if $S \in \boldsymbol{K}$ and $\boldsymbol{K} \subseteq \boldsymbol{OBG}$, then $S \in (\boldsymbol{K} \cap \boldsymbol{B}) \vee (\boldsymbol{K} \cap \boldsymbol{SLG})$. But by, for example, (6) of [13],

$$\begin{aligned} \boldsymbol{K} \cap \boldsymbol{SLG} &= ((\boldsymbol{K} \cap \boldsymbol{SLG}) \cap \boldsymbol{SL}) \vee ((\boldsymbol{K} \cap \boldsymbol{SLG}) \cap \boldsymbol{CS}) \\ &= (\boldsymbol{K} \cap \boldsymbol{SL}) \vee (\boldsymbol{K} \cap \boldsymbol{G}) \,, \end{aligned}$$

so $S \in (\boldsymbol{K} \cap \boldsymbol{B}) \vee (\boldsymbol{K} \cap \boldsymbol{G})$ (since $\boldsymbol{SL} \subseteq \boldsymbol{B}$).

Therefore $\boldsymbol{K} = (\boldsymbol{K} \cap \boldsymbol{B}) \vee (\boldsymbol{K} \cap \boldsymbol{SLG})$ and so $\boldsymbol{K}$ belongs to the sublattice generated by $\mathscr{V}(\boldsymbol{B})$ and $\mathscr{V}(\boldsymbol{G})$. Hence $\mathscr{V}(\boldsymbol{B})$ and $\mathscr{V}(\boldsymbol{G})$ generate $\mathscr{V}(\boldsymbol{BG})$ and since $\boldsymbol{B} \cap \boldsymbol{G} = \boldsymbol{T}$ an application of Result 3.7 yields the result.

REMARK 3.9. Since the description of $\mathscr{V}(\boldsymbol{NBG})$ as a direct product of $\mathscr{V}(\boldsymbol{SL})$ and $\mathscr{V}(\boldsymbol{CS})$ ([13], §4) was essentially used in the proof of Theorem 3.1, we cannot of course apply the theorem to that situation.

4. A closure operator. For each subvariety $\boldsymbol{U}$ of $\boldsymbol{CR}$ we let $\boldsymbol{P}(\boldsymbol{U})$, or just $\boldsymbol{PU}$, be the class consisting of those completely regular semigroups S whose (completely regular) subsemigroups eSe belong to $\boldsymbol{U}$ for every $e \in E_S$.

PROPOSITION 4.1. *For any variety $\boldsymbol{U} \subseteq \boldsymbol{CR}$, $\boldsymbol{PU}$ is a variety containing $\boldsymbol{U}$. In fact the operator $\boldsymbol{U} \to \boldsymbol{PU}$ is a closure operator on $\mathscr{V}(\boldsymbol{CR})$.*

Proof. If $S \in \boldsymbol{PU}(\boldsymbol{U} \subseteq \boldsymbol{CR})$, T is a (completely regular) subsemigroup of S and $e \in E_T$ then $e \in E_S$ and eTe is a (completely regular) subsemigroup of eSe and so belongs to U. If T is a morphic image of S, under ϕ, say, and $e \in E_T$ then by a lemma of Lallement [8], $e = f\phi$ for some $f \in E_S$, whence $eTe = (fSf)\phi \in \boldsymbol{U}$. That $\boldsymbol{PU}$ is closed under direct products is immediate upon noting that an element of a direct product of semigroups is idempotent if and only if each of its components is idempotent. Hence $\boldsymbol{PU}$ is a variety.

Clearly $\boldsymbol{U} \subseteq \boldsymbol{PU}$, and $\boldsymbol{U} \subseteq \boldsymbol{V}$ implies $\boldsymbol{PU} \subseteq \boldsymbol{PV}$. If $S \in \boldsymbol{P}(\boldsymbol{PU})$ and $e \in E_S$ then $eSe \in \boldsymbol{PU}$. But $e \in E_{eSe}$ and so $eSe \in \boldsymbol{U}$. Therefore $eSe \in \boldsymbol{U}$, whence $\boldsymbol{P}(\boldsymbol{PU}) = \boldsymbol{PU}$.

We call $\boldsymbol{PU}$ the variety of *pseudo-$\boldsymbol{U}$ semigroups*.

LEMMA 4.2. $\boldsymbol{P}(\boldsymbol{BG}) = \boldsymbol{BG}$.

Proof. Let $S \in P(BG)$, $x, y \in S$, $x\mathscr{H}y$ and $s \in S$. Now $x\mathscr{R}y$ implies $sx\mathscr{R}sy$. Let $e = xx^{-1} = yy^{-1}$. Then since $H_e \subseteq eSe$, a band of groups, we have

$$esx = (ese)x\mathscr{H}(ese)y = esy \,.$$

But $esx\mathscr{J}sx\mathscr{J}sy\mathscr{J}esy$ so (since $\mathscr{J}$-classes of S are completely simple)

$sx\mathscr{L}esx\mathscr{H}esy\mathscr{L}sy$. Therefore $sx\mathscr{H}sy$. Similarly $xs\mathscr{H}ys$ and thus $\mathscr{H}$ is a congruence, that is $S \in \boldsymbol{BG}$.

As noted above $\boldsymbol{BG} \subseteq \boldsymbol{P}(\boldsymbol{BG})$ and the result follows.

As a result of this lemma, if $\boldsymbol{U} \subseteq \boldsymbol{BG}$, then $\boldsymbol{PU} \subseteq \boldsymbol{BG}$ also. Observe that $\boldsymbol{P}(\boldsymbol{T}) = \boldsymbol{RB}$ and $\boldsymbol{P}(\boldsymbol{G}) = \boldsymbol{CS}$. By Theorem 4.1 of [11], a completely regular semigroup S is a normal band of groups if and only if it satisfies "$\mathscr{D}$-majorization": if $e, f, g \in E_S$, $e \geqq f$, $e \geqq g$, $f\mathscr{D}g$ then $f = g$. Clearly this is equivalent to saying each $\mathscr{D}$-class of eSe contains precisely one idempotent for every $e \in E_S$, that is each $eSe \in \boldsymbol{SLG}$. Hence

LEMMA 4.3. *$\boldsymbol{P}(\boldsymbol{SLG}) = \boldsymbol{NBG}$ and $\boldsymbol{P}(\boldsymbol{SL}) = \boldsymbol{NB}$.*

If $\boldsymbol{U} \in \mathscr{V}(\boldsymbol{CR})$, a congruence on a semigroup S will be called a *U-congruence* if $S/\rho \in \boldsymbol{U}$. If S is completely regular then S has a least $\boldsymbol{U}$-congruence. We now show how to derive the least $\boldsymbol{PU}$-congruence on S in terms of $\boldsymbol{U}$-congruences on the subsemigroups eSe, $e \in E_S$.

PROPOSITION 4.4. *If $\boldsymbol{U} \subseteq \boldsymbol{CR}$ and $S \in \boldsymbol{CR}$ then the least $\boldsymbol{PU}$-congruence on S is the congruence generated by the union of the least $\boldsymbol{U}$-congruences on the subsemigroups eSe, $e \in E_S$.*

Proof. Denote by ρ_e the least $\boldsymbol{U}$-congruence on eSe, $e \in E_S$, and put $\rho = (\cup\{\rho_e : e \in E_S\})^\#$, $T = S/\rho$.

If $f \in E_T$ then, as above, $f = e\rho$ for some $e \in E_S$ and so $fTf \cong (eSe)/(\rho | eSe)$. Since $\rho_e \subseteq \rho | eSe$, there is a morphism of eSe/ρ_e upon $eSe/(\rho | eSe)$ and thus since $eSe/\rho_e \in \boldsymbol{U}$, $fTf \in \boldsymbol{U}$. Hence $T \in \boldsymbol{PU}$ and ρ is a $\boldsymbol{PU}$-congruence.

Now let τ be an arbitrary $\boldsymbol{PU}$-congruence. For each $e \in E_s$, $\tau | eSe$ is a congruence on eSe and in fact a $\boldsymbol{U}$-congruence, since $eSe/(\tau | eSe) \cong (e\tau)(S/\tau)(e\tau) \in \boldsymbol{U}$ (since $e\tau \in E_{S/\tau}$ and $S/\tau \in \boldsymbol{PU}$). Therefore $\rho_e \subseteq \tau | eSe \subseteq \tau$ for each $e \in E_S$ and $\rho \subseteq \tau$, by definition.

Hence ρ is the least $\boldsymbol{PU}$-congruence on S.

A result which is clearly relevant to this proposition and which is required in the next lemma is the following.

PROPOSITION 4.5. *If S is any semigroup, $e \in E_S$ and ρ is a congruence on eSe, then $\rho^\# | eSe = \rho$.*

Proof. Clearly $\rho \subseteq \rho^\# | eSe$. Conversely, suppose $(x, y) \in \rho^\# | eSe$. Then there exists a sequence

$$x = x_0 \longrightarrow x_1 \longrightarrow x_2 \longrightarrow \cdots \longrightarrow x_n = y$$

of elements of S such that for each $i (1 \leqq i \leqq n)$, $x_{i-1} = s_i a_i t_i$ and $x_i = s_i b_i t_i$ for some $s_i, t_i \in S^1$, $(a_i, b_i) \in \rho$. Since $exe = x$ and $eye = y$, the sequence may be replaced by the sequence

$$x = ex_0 e \longrightarrow ex_1 e \longrightarrow ex_2 e \longrightarrow \cdots \longrightarrow ex_n e = y ,$$

where for each i, $ex_{i-1}e = es_i a_i t_i e = (es_i e)a_i(et_i e)$, since $a_i \in eSe$, and $ex_i e = (es_i e)b_i(et_i e)$ similarly. Since $(a_i, b_i) \in \rho$ and $es_i e$, $et_i e \in eSe$, $(ex_{i-1}e, ex_i e) \in \rho$, $1 \leqq i \leqq n$, hence $(x, y) \in \rho$, as required.

The next lemma is required in §5.

LEMMA 4.6. *If $S \in \boldsymbol{CR}$, $e \in E_S$ and ρ is a congruence on eSe such that $\rho \cap \mathscr{H} = \iota$ (on eSe) then $\rho^\sharp \cap \mathscr{H} = \iota$ on S.*

Proof. Suppose $(x, y) \in \rho^\sharp \cap \mathscr{H}$, $x, y \in S$, $x \neq y$. Then as above, $x = s_1 a_1 t_1$ for some $s_1, t_1 \in S^1$, $a_1 \in eSe$. Therefore $J_x \leqq J_{a_1} \leqq J_e$ and since S is a semilattice of completely simple semigroups, $exe \mathscr{D} x$. Let $u \in R_{exe} \cap L_x$ and let u' be the inverse of u in $R_x \cap L_{exe}$. Then uxu', $uyu' \in H_{exe}$ and we have (noting that eSe consists of complete $\mathscr{H}$-classes of S) $(uxu', uyu') \in \rho^\sharp | eSe$. By the previous proposition, $\rho^\sharp | eSe = \rho$, so $uxu' = uyu'$, whence $x = u'(uxu')u = u'(uyu')u = y$, a contradiction.

5. Lattice joins.

In this section we find the join $\boldsymbol{O} \vee \boldsymbol{NBG}$, and as a corollary $\boldsymbol{OBG} \vee \boldsymbol{NBG}$. From the results of [13], it may be observed that $\boldsymbol{OBG} \vee \boldsymbol{NBG} = \boldsymbol{B} \vee \boldsymbol{CS}$.

Since, as was seen in §4, $\boldsymbol{NBG} = \boldsymbol{P}(\boldsymbol{SLG})$ and $\boldsymbol{SLG} \subseteqq \boldsymbol{O}$ we have $\boldsymbol{NBG} \subseteqq \boldsymbol{PO}$ and so $\boldsymbol{O} \vee \boldsymbol{NBG} \subseteqq \boldsymbol{PO}$. In particular $\boldsymbol{OBG} \vee \boldsymbol{NBG} \subseteqq \boldsymbol{POBG} (= \boldsymbol{PO} \cap \boldsymbol{BG})$. In view of the next example, $\boldsymbol{OBG} \vee \boldsymbol{NBG}$ is properly contained in $\boldsymbol{BG}$.

EXAMPLE 5.1. Let C be a nonorthodox completely simple semigroup and $S = C^1$. Since $\mathscr{H}$ is a congruence on C, it is also a congruence on S, that is $S \in \boldsymbol{BG}$. But $S \notin \boldsymbol{POBG}$ since $S = 1.S.1$ is not orthodox.

Consider now the property:

$$(*) \qquad apb \mathscr{H} app^{-1}b \quad \text{for all} \quad p \in \langle E_S \rangle,\ a, b \in S .$$

Any band of groups satisfies (*) since of course $p \mathscr{H} pp^{-1}$ and $\mathscr{H}$ is a congruence on a band of groups. Moreover any orthodox completely regular semigroup also satisfies (*), for then $\langle E_S \rangle = E_S$ and $p = pp^{-1}$.

We now show (*) may be expressed in terms of identities and so the class $\boldsymbol{I}$ of all completely regular semigroups satisfying (*) forms a subvariety of $\boldsymbol{CR}$ containing, as we have just seen $\boldsymbol{O} \vee \boldsymbol{BG}$.

For each $n \geqq 1$ let I_n be the identity

$$(ap_nb)(ap_nb)^{-1} = (ap_np_n^{-1}b)(ap_np_n^{-1}b)^{-1}$$

where $p_n = (x_1x_1^{-1})(x_2x_2^{-1}) \cdots (x_nx_n^{-1})$, $a, b, x_i \in S$, $i \geqq 1$. Since any product $p = e_1 \cdots e_n$ of idempotents of S can be expressed in the form $p = (e_1e_1^{-1}) \cdots (e_ne_n^{-1})$ it is evident that a completely regular semigroup has property (*) if and only if it satisfies the set $\{I_n\}_{n\geqq 1}$ of identities. The main property of such semigroups which we exploit is the following.

LEMMA 5.2. *If $S \in \boldsymbol{I}$ then the congruence on S generated by $\mathscr{H}_{\langle E_S\rangle}$ is contained in $\mathscr{H}$. Hence the least $\boldsymbol{O}$-congruence on S is contained in $\mathscr{H}$.*

Proof. Since $\mathscr{H}$ is an equivalence on S it is sufficient to show that if $(p, q) \in \mathscr{H}_{\langle E_S\rangle}$ then $apb\mathscr{H}aqb$ for all $a, b \in S^1$. But $p\mathscr{H}pp^{-1} = qq^{-1}\mathscr{H}q$, so $apb\mathscr{H}app^{-1}b = aqq^{-1}b\mathscr{H}aqb$, using (*).

The least $\boldsymbol{O}$-congruence is clearly generated by all pairs $((ef)^2, ef)$, $e, f \in E_S$. Since $((ef)^2, ef) \in \mathscr{H}_{\langle E_S\rangle}$ the second statement follows immediately.

Our main theorem for this section is now

THEOREM 5.3. *A completely regular semigroup $S \in \boldsymbol{O} \vee \boldsymbol{NBG}$ if and only if S is pseudo-orthodox and satisfies* (*). *That is, $\boldsymbol{O} \vee \boldsymbol{NBG} = \boldsymbol{POI} (= \boldsymbol{PO} \cap \boldsymbol{I})$.*

Proof. That any semigroup in $\boldsymbol{O} \vee \boldsymbol{NBG}$ is pseudo-orthodox and satisfies (*) has been established. The converse will be proved by showing that any such completely regular semigroup S is a subdirect product of an orthodox semigroup and a normal band of groups; this will follow from our proof that, on such a semigroup, the least $\boldsymbol{O}$-congruence and the least $\boldsymbol{NBG}$-congruence have trivial intersection (see, for example [5], Theorem 20.2).

By the previous lemma the least $\boldsymbol{O}$-congruence on S, α, say, is contained in $\mathscr{H}$. Let η be the least $\boldsymbol{NBG}$-congruence on S. Since $\boldsymbol{NBG} = \boldsymbol{P}(\boldsymbol{SLG})$, η is the congruence generated by the union of the least $\boldsymbol{SLG}$-congruences on the subsemigroups eSe, $e \in E_S$ (using Proposition 4.4). Since $\boldsymbol{SLG}$ is precisely the class of completely regular semigroups which are also inverse semigroups, the least $\boldsymbol{SLG}$-congruence on eSe is just the least inverse congruence $\mathscr{Y}_{eSe}$. Moreover since $S \in \boldsymbol{PO}$, each eSe is orthodox and therefore $\mathscr{Y}_{eSe} \cap \mathscr{H} = \iota$ on eSe (see, for example [7, p. 191]). By Lemma 4.6, $\mathscr{Y}_{eSe}{}^{\#} \cap \mathscr{H} = \iota$ on S.

At this stage we apply the comment made in Remark 3.3: *if $\{\tau_k\}_{k\in K}$ is a collection of congruences contained in $\mathscr{D}$ on a completely* *regular semigroup S, and if $\tau_k \cap \mathscr{H} = \iota$ for all $k \in K$, then* $\vee\{\tau_k : k \in K\} \cap \mathscr{H} = \iota$ (the join denoting the join in the lattice of congruences on S).

Therefore $\eta \cap \mathscr{H} = (\vee \mathscr{Y}_{eSe}{}^{\#}) \cap \mathscr{H} = \iota$ and so for any $S \in \boldsymbol{POI}$ we have $\eta \cap \alpha \subseteq \eta \cap \mathscr{H} = \iota$, as required.

COROLLARY 5.4. $\boldsymbol{B} \vee \boldsymbol{CS} = \boldsymbol{OBG} \vee \boldsymbol{NBG} = \boldsymbol{POBG}$.

Proof. As noted earlier, $\boldsymbol{BG} \subseteq \boldsymbol{I}$.

COROLLARY 5.5 (to the proof).

(i) *Any semigroup in* $\boldsymbol{POI}$ *is a subdirect product of an orthodox semigroup and a normal band of groups (in fact of the maximum orthodox morphic image and the maximum* $\boldsymbol{NBG}$ *morphic image).*

(ii) *Any pseudo-orthodox band of groups is a subdirect product of a band and a normal band of groups.*

Proof. (i) is immediate. From the proof of the theorem we see that if $S \in \boldsymbol{POBG}$ then S is a subdirect of an orthodox band of groups and a normal band of groups. However more strongly, $\eta \cap \mathscr{H} = \iota$, where η is the least $\boldsymbol{NBG}$-congruence and, in a band of groups, $\mathscr{H}$ is the *least B-congruence*. Thus (ii) follows.

Before completing this section we show that pseudo-orthodoxy and property (*) are independent. As noted earlier every band of groups satisfies (*) but need not be pseudo-orthodox (Example 5.1). We now give an example to show that pseudo-orthodoxy need not imply (*).

EXAMPLE 5.6. Let C be a nonorthodox completely simple semigroup and let $c \to \bar{c}$ be a bijection of C upon a disjoint set R. Define a product on $S = C \cup R$ by extending that on C, putting right zero product on R and

$$\bar{a}b = (\overline{ab}) ,$$
$$a\bar{b} = \bar{b} , \qquad \text{for all } \ a, b \in C .$$

It is routine to verify that S is a pseudo-orthodox completely regular semigroup.

Now let e, f be two idempotents of C whose product is nonidempotent and let h be the idempotent in H_{ef}. Then

$$\bar{e}(ef) = \overline{ef} \quad \text{and} \quad \bar{e}h = \overline{eh} = \bar{h}$$

since C is completely simple. Hence S does not satisfy (*), for $(ef, h) \in \mathscr{H}_{\langle E_S \rangle}$ but $(\bar{e}(ef)\bar{e}h) \notin \mathscr{H}$ (since R has trivial subgroups).

Finally a direct decomposition similar to that of $\mathscr{V}(\boldsymbol{OBG})$ (Corollary 3.8) may be found for $[\boldsymbol{RB}, \boldsymbol{POBG}]$, using Corollary 5.5.

COROLLARY 5.7. $[\boldsymbol{RB}, \boldsymbol{POBG}] \cong [\boldsymbol{RB}, \boldsymbol{B}] \times [\boldsymbol{RB}, \boldsymbol{CS}]$.

Proof. Since $\boldsymbol{B} \cap \boldsymbol{CS} = \boldsymbol{RB}$ it will be sufficient, by Result 3.7, to show that $[\boldsymbol{RB}, \boldsymbol{B}]$ and $[\boldsymbol{RB}, \boldsymbol{CS}]$ generate $[\boldsymbol{RB}, \boldsymbol{POBG}]$.

Let $\boldsymbol{K} \subseteq \boldsymbol{POBG}$ and $S \in \boldsymbol{K}$. By Corollary 3.8, S is a subdirect product of the maximum $\boldsymbol{B}$-image and the maximum $\boldsymbol{NBG}$-image, so that $S \in (\boldsymbol{K} \cap \boldsymbol{B}) \vee (\boldsymbol{K} \cap \boldsymbol{NBG})$ and $\boldsymbol{K} = (\boldsymbol{K} \cap \boldsymbol{B}) \vee (\boldsymbol{K} \cap \boldsymbol{NBG})$.

Applying (6) of [13],

$$\begin{aligned} \boldsymbol{K} \cap \boldsymbol{NBG} &= ((\boldsymbol{K} \cap \boldsymbol{NBG}) \cap \boldsymbol{SL}) \vee ((\boldsymbol{K} \cap \boldsymbol{NBG}) \cap \boldsymbol{CS}) \\ &= (\boldsymbol{K} \cap \boldsymbol{SL}) \vee (\boldsymbol{K} \cap \boldsymbol{CS}) , \end{aligned}$$

so

$$\boldsymbol{K} = (\boldsymbol{K} \cap \boldsymbol{B}) \vee (\boldsymbol{K} \cap \boldsymbol{CS}) .$$

Thus if $\boldsymbol{K} \in [\boldsymbol{RB}, \boldsymbol{POBG}]$, $\boldsymbol{K}$ belongs to the lattice generated by $[\boldsymbol{RB}, \boldsymbol{B}]$ and $[\boldsymbol{RB}, \boldsymbol{CS}]$, as required.

REMARK 5.8. By generalizing the methods of [12] this corollary may be proved directly (with rather more difficulty).

Added in Proof. Since this paper was accepted, the authors have learned that our Corollary 5.4 has also been obtained by V. V. Rasin, "On varieties of bands of groups", in XV All-Union Algebra Conference, 1979.

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