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# NOTES ON GENERALIZED BOUNDARY VALUE PROBLEMS IN BANACH SPACES, II INFINITE DIMENSIONAL EXTENSION THEORY

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**Let $X$, $Y$ be Banach spaces and $\mathscr{A}: X \to Y$, $\mathscr{B}: Y^* \to X^*$ be linear relations. Suppose $\mathscr{A}$ is a restriction of the adjoint (or preadjoint) $\mathscr{B}^*$ of $\mathscr{B}$ and the codimension of $G(\mathscr{A})$ in $G(\mathscr{B}^*)$ is not necessarily finite. Under certain hypotheses we can describe in computationally useful ways extensions $\mathscr{C}$ of $\mathscr{A}$ which are restrictions of $\mathscr{B}^*$ and their adjoints. The theory is applied to a number of examples and is a direct extension of a previous paper which mainly treated the finite dimensional case.**

**1. Introduction.** This paper is a continuation of [3] which outlined an adjoint and extension theory for generalized boundary value problems (g.b.v.p) in a Banach space setting.

Specifically [3] was concerned with two fundamental problems:

*The Adjoint Problem.* Let $A$ be a densely defined closed operator in a Banach space $X$ with range in a Banach space $Y$. Suppose further that $H$ is an operator on $D(A)$ with range in a locally convex topological vector space (l.c.t.v.s.) $F$ such that $D(A) \cap N(H) \neq 0$. Form the operators $A_H: X \to Y$ and $A^H: X \to Y \times F$ given by $A_H(x) = Ax$ on $N(H)$ and

$$A^H x = \begin{pmatrix} Ax \\ Hx \end{pmatrix} .$$

Then a question of interest is the determination of $A_H^*$ and $A^{H*}$ and the investigation of their mutual relationship.

*The Extension Problem.* Suppose $\mathscr{A}$ and $\mathscr{B}$ are two closed relations in $X \times Y$ such that $\mathscr{A} \subset \mathscr{B}$. We now wish to characterize $\mathscr{C}$ and $\mathscr{C}^*$ where $\mathscr{C}$ is an arbitrary closed relation such that $\mathscr{A} \subset \mathscr{C} \subset \mathscr{B}$. An interesting version of this problem is the case where $\mathscr{A} = A_H$ and $\mathscr{B} = B_K^*$ where $B: Y^* \to X^*$, $K$ is an operator on $D(B)$ with range in an l.c.t.v.s. $G$ such that $D(B) \cap N(K) \neq 0$, and $A$ is an operator restriction of the adjoint (or preadjoint) of $B^*$ (also an operator) so that $A_H \subset \mathscr{C} \subset B_K^*$ and $B_K \subset \mathscr{C}^* \subset A_H^*$. Here we wish to relate $\mathscr{C}$ and $\mathscr{C}^*$ to the structure—presumably already known—of $B_K^*$ and $A_H^*$.

In [3] solutions to both of the above problems were sought in terms of a theory which should be both applicable to particular b.v.p. (especially those determined by differential operators and multipoint, Stieltjes, or interface boundary conditions) and capable of extension to more difficult problems involving functional differential operators, PDE, evolution operators, etc.

The focus of the present paper will be on the extension problem. Our goal is to refine and generalize two distinct theories reflecting different points of view sketched in [3]. We will also mostly deal here with the most difficult case—when $H$ is an operator with infinite dimensional range.

The first approach which seeks to characterize $\mathscr{C}$ and $\mathscr{C}^*$ in terms of mutually adjoint boundary conditions is developed in § 3. The second which uses the Fredholm alternative rather than boundary operators is presented in § 4, § 5 contains a nonexhaustive list of examples. The last section consists of some historical remarks.

**2. Notation and preliminaries.** At this point to motivate the results of the next two sections and to make the paper notationally self-contained we will review some of the main ideas and conventions in [3].

We follow Arens [1] and call a closed linear possibly multivalued mapping from a Banach space $X$ to a Banach space $Y$ a linear relation.[1] It notationally convenient to distinguish between $\mathscr{A}$ and its graph $G(\mathscr{A})$ (a closed subspace of $X \times Y$) although logically the distinction is an artificial one. We often refer to $\mathscr{A}$ by the notation $\mathscr{A}: X \rightarrow Y$. But this will not imply either that $D(\mathscr{A}) = X$ or $R(\mathscr{A}) = Y$. (We will also often denote an operator with graph in $X \times Y$ with standard type rather than script; e.g., "$A$" instead of "$\mathscr{A}$".) Given a relation $\mathscr{A}$ we denote the image (a set) of $\alpha \in X$ under $\mathscr{A}$ by $\mathscr{A}(\alpha)$ and an arbitrary member of this set by $\mathscr{A}\alpha$. Clearly $\beta, \beta' \in \mathscr{A}(\alpha)$ if and only if $\beta \equiv \beta' \bmod \mathscr{A}(0)$. $\mathscr{A}^*$ means either the preadjoint or adjoint of $\mathscr{A}$ depending on whether or not $X$ and $Y$ are dual spaces. If $S \subset X$, $\bar{S}$ or "closure" of $S$ means weak* or topological closure (equivalently weak closure if $S$ is convex) according to whether or not $X$ is a dual space. We follow a similar policy with regard to terms like "complemented", "continuous", "annihilator", etc. Thus, for example, a complemented set $S$ in a dual space $X$ is weak* complemented in the sense that there exists a weak* continuous projection $P$ onto the (weak* closed) set $S$.

DEFINITION 2.1. *If $\mathscr{A}: X \rightarrow Y$ and $F$ is a* l.c.t.v.s. *then a boundary operator $H: X \times Y \rightarrow F$ is a linear operator such that $D(H) \supset$*

[1] One can also simply identify $\mathscr{A}$ with its graph $X \oplus Y$ and call $\mathscr{A}$ a "subspace" as is done for example in [**6**] or [**15**]-[**17**].

*$G(\mathscr{A})$, $N(H) \cap G(\mathscr{A}) \neq 0$, and $D(H^*)$ is total over $F$. The condition $H(y, \mathscr{A}y) = 0$ is called a boundary condition for $\mathscr{A}$. $\mathscr{A}_H$ will denote the restriction of $\mathscr{A}$ defined by $N(H) \cap G(\mathscr{A})$.*

It is easy to show from this definition that $H: G(\mathscr{A}) \to F$ is continuous if $G(\mathscr{A})$ is given the (relative) product topology as a subspace of $X \times Y$ and $F$ has the weak topology induced by $D(H^*)$. Thus $\mathscr{A}_H$ is a closed linear relation. Moreover, one can also show easily that every closed restriction of $\mathscr{A}$ is an "$\mathscr{A}_H$" with respect to a space $F$ which can be viewed either as a Banach space or as a l.c.t.v.s. under its weak topology. Surprisingly, the second choice fits some applications better (more will be said about this issue later).

In the case of an operator $A$ the structure of $A_H^*$ was completely determined in [3] when $R(H)$ is finite dimensional. Several characterizations of $A_H^*$ were also given in the more difficult infinite dimensional case under several sets of hypotheses. We restate two of the most useful ones:

THEOREM 2.2. *Suppose $N(A^*)$ is complemented in $Y^*$ and $H = M \circ A$. Assume that $\overline{D(M)}^{\perp}$ is complemented in $Y^*$ and that $D(M^*)$ is total over $F$. Then*

$$G(A_H^*) = \{(z, A^*\bar{z}): z + \psi \in D(A^*) \text{ and } \psi \in \overline{R(M^*)}\} .$$

*In other words $D(A_H^*) = D(A^*) - \overline{R(M^*)}$, and $G(A_H^*) = G(A^*) - (\overline{R(M^*)} \times \{0\})$.*

COROLLARY 2.3. *Suppose $A$ is a closed densely defined 1-1 operator, such that $N(A^*)$ is complemented. Let $A^+$ be a (not necessarily bounded) partial inverse satisfying $A^+A = I$. Assume further that $\overline{D(HA^+)}$ is complemented and that $D(HA^+)^*$ is total over $F$ then $A_H^*$ is the relation with graph*

$$\{(z, A^*(z + \psi)): z + \psi \in D(A^*) \text{ and } \psi \in \overline{R(HA^+)^*}\} .$$

We ought to remark here that the assumptions behind these two results are reasonable ones. For instance, if $D(M) = R(A)$ then $D(M)^{\perp} = N(A^*)$ and so it is complemented if and only if $N(A^*)$ is. Note also if $A$ is 1-1, $H$ can be a general boundary operator and not merely defined on $R(A)$. (The case where $A$ is not 1-1 can be handled by the extension theory.) Furthermore if $A$ is determined by a regular differential operator, $A^+ \equiv$ the Greens function. But even if $A$ is singular (with essential spectrum) $A^+$ can be identified with an "algebraic resolvent." For example, if $A$ is determined by

$y''$ viewed as an operator on $L^p[0, \infty) \to L^p[0, \infty)$ with boundary conditions $y(0) = y'(0) = 0$, $A^+$ is $\int_0^t (t - s)(\cdot)ds$.

At this point we indicate why $F$ is sometimes better viewed as a l.c.t.v.s. rather than a Banach space. Suppose for example $Hy = \langle c_i y(t_i) + d_i y'(t_{i+1})\rangle$ where $\{t_i\}$ is an infinite set of points with limit point $\infty$ on $[0, \infty)$. $R(H)$ may or may not be a Banach space[2] but $R(H) \subset R^{+,\omega}$ (my notation for the countable direct sum of $[0, \infty)$). We can make this into a l.c.t.v.s. by giving $R^{+,\omega}$ the weak topology vis a vis $R_{oo}^{+,\omega}$ (sequences in $R^{+,\omega}$ with finitely many nonzero terms). If we can find $\overline{R(HA^+)^*}$ or at least the main features of its structure we have a good idea of $A_H^*$. This generally is not difficult to do. This approach can be generalized to more complicated singular Stieltjes b.v.p. The results are similar to the parametric adjoints found for such problems by less general methods in [2]. (We will illustrate further this approach in § 5.)

Two approaches to the extension problem were outlined in [3]. The first one which completely solved the problem in the case $\dim F < \infty$ was a straightforward exercise in matrix theory and depended on the following generalized Green's identity.

THEOREM 2.4. *Suppose that $\mathscr{A}: X \to Y$, $\mathscr{B}: Y^* \to X^*$ are relations such that $\mathscr{A} \subset \mathscr{B}^*$ and $\dim G(\mathscr{A}^*)/G(\mathscr{B}) = G(\mathscr{B}^*)/G(\mathscr{A}) = n < \infty$. Then there exists an $n \times n$ nonsingular matrix $\hat{\mathscr{B}}$ and continuous operators $J: G(\mathscr{B}^*) \to \mathbb{C}^n$, $\tilde{J}: G(\mathscr{A}^*) \to \mathbb{C}^n$ with linearly independent coordinate functionals such that*

$$[\mathscr{B}^*y, z] - [y, \mathscr{A}^*z] = \tilde{J}(z, \mathscr{A}^*z)^*\hat{\mathscr{B}}J(y, \mathscr{B}^*y) \quad on \quad G(\mathscr{A}^*) \times G(\mathscr{B}^*) .$$

*Moreover, in a Hilbert space setting where $\mathscr{A}$ is symmetric and $\mathscr{A}^* = \mathscr{B}^*$ then $\hat{\mathscr{B}}$ is skew-hermitian.*

This result is easily proved using the linear dependence principle. An easy consequence is the fact that $\mathscr{A} \subset \mathscr{C} \subset \mathscr{B}^*$ if and only if $\mathscr{C}$ is determined by the boundary condition $DJ(G(\mathscr{B}^*)) = 0$ where $D$ is a $k \times n$, $k \leqq n$ matrix of full rank and that $\mathscr{C}^*$ is given by the adjoint boundary condition

$$[ND]^*\hat{\mathscr{B}}^*\tilde{J}(y, \mathscr{A}^*y) = 0 \tag{2.1}$$

where $[ND]$ is a matrix whose columns form a basis for the null space of $D$.

Unfortunately if $\dim G(\mathscr{B}^*)/G(\mathscr{A}) = \infty$, the foregoing analysis

[2] See §6 for further discussion of this point.

breaks down because the linear dependence principle no longer holds. To cover this case an alternative approach was developed in [3]. The underlying idea is simple. When $A_H \subset \mathscr{C} \subset B_K^*$, $B_K \subset \mathscr{C}^* \subset A_H^*$. If it is assumed that $R(\mathscr{C})$ is closed, one can describe $\mathscr{C}$ as that restriction of $B_K^*$ whose range is orthogonal to a certain subspace of $N(A_H^*)$ and whose null space is a subspace of $B_K^*$. Conversely starting with known subspaces of $N(A_H^*)$ and $N(B_K^*)$ one can describe an extension $\mathscr{C}$. Interchanging these subspaces describes $\mathscr{C}^*$. Unfortunately the assumption of closed range is too strong because it rules out b.v.p. for singular differential operators with essential spectrum.

This survey of [3] brings us to the point of departure for the present paper. The next section will show that—suitably reinterpreted—the simple formulas of the finite dimensional theory (e.g., Theorem 2.4) can be extended to the infinite dimensional case. This is the most significant accomplishment of the paper. In § 4 we investigate the simple Fredholm alternative approach. This section is mostly a refinement of previous work. But, it is shown that a large portion of the theory may be salvaged if we abandon the closed range hypothesis. Fortunately, the portion that remains is just what we need to calculate extensions in concrete problems.

**3. A theory of boundary operators for the infinite dimensional extension problem.** As mensioned above, the purpose of this section is to show how to preserve the results of [3]—e.g., Theorem 2.4 or (2.1) above—describing finite dimensional extensions in the infinite dimensional setting. However, we state at the outset that this will be done at a price. $\hat{\mathscr{B}}$, $D$, etc., will no longer be matrices but rather weakly continuous operators. Also, certain restrictions must be placed on the boundary operators $J$, $\tilde{J}$ as well as on the class of extensions considered.

The finite dimensional theory depended on linear algebra arguments and especially the "linear dependence principle": If $\psi_i$, $i = 1, \cdots, n$ and $\phi$ are linear functionals on a space $X$ such that $N(\phi) \supset \cap N(\psi_i)$, then $\phi$ is a linear combination of the $\psi_i$. Our first step is to replace this principle by an obvious generalization.

LEMMA 3.1. *Suppose $X$, $F$, $G$, are linear spaces and $\pi: X \to F$, $\phi: X \to G$ are linear operators such that $N(\pi) \supset N(\phi)$ then there exists $\lambda: G \to F$ such that $\lambda \circ \phi = \pi$.*

Next we add a refinement to the idea of a boundary operator.

DEFINITION 3.2. *Suppose $\mathscr{A} \subset \mathscr{C} \subset \mathscr{B}^*$. We say that $\mathscr{C}$ is* *regular if there is a boundary operator $H$ for $\mathscr{B}^*$* (cf. *Definition* 2.1) *defining $\mathscr{C}$*, i.e., $\mathscr{C} = \mathscr{B}^*_H$ *such that $H^*$ has closed range in $Y^* \times X^*$. In these circumstances $H$ is said to be a regular boundary operator.*

REMARK. Recall from Definition 2.1 that $H$ is considered a mapping from $X \times Y$ to $F$. So that $H^*$ has range in $Y^* \times X^*$. But although $D(H) \supset G(\mathscr{B}^*)$ $H$ need not be densely defined so that $H^*$ may be multivalued.

LEMMA 3.3. *Suppose $D(H) = G(B^*)$ then $\mathscr{C}$ is regular if and only if*

$$R(H^*) = G(-\mathscr{C}^*)^{-1} . \tag{3.1}$$

*Proof.* If (3.1) holds $\mathscr{C}$ is regular since $\mathscr{C}^*$ is closed. Conversely, if $\mathscr{C}$ is regular the fact that $N(H) \equiv G(\mathscr{C})$ implies

$$(N(H))^{\perp} = G(-\mathscr{C}^*)^{-1} = R(H^*) . \qquad \square$$

We now make the following assumptions: (i) $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$ are regular; (ii) the boundary operators $J$ for $\mathscr{A}$ and $\tilde{J}$ for $\mathscr{B}$ (i.e., $\mathscr{A} = \mathscr{B}_J{}^*$, $\mathscr{B} = \mathscr{A}_{\tilde{J}}{}^*$) are *onto* certain fixed l.c.t.v.s. $F$ and $G$; (iii) $D(J) = G(B^*)$, $D(\tilde{J}) = G(A^*)$; $\mathscr{C}$—unless stated otherwise—is a regular relation which is an extension of $\mathscr{A}$ and contained in $\mathscr{B}^*$.

Throughout the paper we also adopt the following conventions concerning the topologies of $F$ and $G$. Unless otherwise mentioned $F^*$ denotes $D(J^*)$ endowed with the weak* topology induced by $F$ and $F$ will have the weak topology induced by $F^*$. Thus $F^*$ is the dual of $F$ and $F^{**} = F$. A similar policy will be followed with regard to $G$. It follows (since $D(J^*)$ and $D(\tilde{J}^*)$ are total) that $J$, $\tilde{J}$ or any other boundary operator may be viewed as continuous on $G(\mathscr{B}^*)$ or $G(\mathscr{A}^*)$.

LEMMA 3.4. *If $G(\mathscr{C})$ is a subspace (but perhaps neither closed or regular) of $G(\mathscr{B}^*)$ then $\mathscr{C}$ is closed if and only if $J(G(\mathscr{C}))$ is closed.*

*Proof.* Since $J(\mathscr{C})$ is a subspace it is sufficient to show that $J(G(\mathscr{C}))$ is weakly closed. Let $\gamma$ be a weak limit point of $J(G(\mathscr{C}))$. Then there exists a net $\langle y_l, \mathscr{B}^* y_l\rangle$ in $G(\mathscr{C})$ such that

$$[J(y_l, \mathscr{B}^* y_l), \phi] \longrightarrow [\gamma, \phi] \tag{3.2}$$

for all $\phi$ in $F^*$. Since $J$ is onto $\gamma = J(y, \mathscr{B}^* y)$ for some $y$. Taking adjoints we have,

$$[(\mathscr{B}^*y_l, y_l), J^*\phi] \longrightarrow [(\mathscr{B}^*y, y), J^*\phi]$$

since $\mathscr{A} \subset \mathscr{C} \subset \mathscr{B}^*$, $\mathscr{B} \subset \mathscr{C}^* \subset \mathscr{A}^*$. Since $\mathscr{C} \supset N(J) = \mathscr{A}$,

$$-\mathscr{C}^* = \mathscr{C}^\perp \subset N(J)^\perp = \mathscr{A}^\perp = -\mathscr{A}^* .$$

Since $N(J)^\perp = R(J^*)$, (because $J$ is regular!), it follows that $-\mathscr{C}^* \subset R(J^*)$. Hence we can select $\phi$ so that $J^*\phi$ is an arbitrary member $(z, \mathscr{A}^*z)$ in $G(-\mathscr{C}^*)$. Then (3.2) can be written

$$[\mathscr{B}^*y_l, z] - [y_l, \mathscr{A}^*z] \longrightarrow [\mathscr{B}^*y, z] - [y, \mathscr{A}^*z] .$$

Since all the terms on the left vanish it follows that $(y, \mathscr{B}^*y) \in G(\mathscr{C})$ so that $\gamma \in J(G(\mathscr{C}))$ and $J(G(\mathscr{C}))$ is weakly closed.

Now suppose $J(G(\mathscr{C}))$ is closed. Let $(y, \mathscr{B}^*y) \in \overline{G(\mathscr{C})}$. Then there is a sequence $\langle y_l, \mathscr{B}^*y_l\rangle$ in $G(\mathscr{C})$ converging to $(y, \mathscr{B}^*y)$. By the (weak) continuity of $J$,

$$[J(y_l, \mathscr{B}^*y_l), \phi] \longrightarrow [J(y, \mathscr{B}^*y), \phi]$$

for all $\phi$ in $F^*$. This implies that $J(y, \mathscr{B}^*y)$ is a limit point of $J(G(\mathscr{C}))$ and hence belongs to $J(G(\mathscr{C}))$. Since $J^{-1}J(y, \mathscr{B}^*y) \equiv G(\mathscr{C})$ mod $G(\mathscr{A})$, $(y, \mathscr{B}^*y) \in G(\mathscr{C})$. □

The next two results generalize the Greens formula—Theorem 1.2, derived in [3] for finite dimensional $F$. Together they will serve as the foundations of our extension theory.

THEOREM 3.5. *There exists a* 1-1 *continuous operator* $\hat{\mathscr{B}}: G \to F^*$ *such that*

$$[\mathscr{B}^*y, z] - [y, \mathscr{A}^*z] = [J(y, \mathscr{B}^*y), \hat{\mathscr{B}}\tilde{J}(z, \mathscr{A}^*z)] . \tag{3.3}$$

*Proof.* Fix an element $(\alpha, \beta)$ in $G(\mathscr{A}^*)$. Then $[\mathscr{B}^*y, \alpha] - [y, \beta]$ is a functional $\psi_{\alpha\beta}$ continuous on $G(\mathscr{B}^*)$ whose null space contains $N(J)$. Consequently by Lemma 3.1 $\kappa(\alpha, \beta) := \psi_{\alpha\beta}J^{-1}$ is well-defined and

$$[\mathscr{B}^*y, \alpha] - [y, \beta] = \kappa(\alpha, \beta) \circ J(y, \mathscr{B}^*y) . \tag{3.4}$$

We now show that $\kappa(\alpha, \beta) \in F^*$. Suppose $\langle \gamma_l\rangle$ is a weakly convergent net with limit $\gamma$ in $F$. Since $J$ is onto, there exists a net $\langle y_l, \mathscr{B}^*y_l\rangle$ and a pair $(y, \mathscr{B}^*y)$ such that $J(y_l, \mathscr{B}^*y_l) = \gamma_l$ and $J(y, \mathscr{B}^*y) = \gamma$. Hence by the weak convergence of $\langle \gamma_l\rangle$

$$[J(y_l, \mathscr{B}^*y_l), \phi] \longrightarrow [J(y, \mathscr{B}^*y), \phi]$$

for all $\phi$ in $F^* := D(J^*)$. Taking adjoints yields

$$[(y_l, \mathscr{B}^* y_l), J^*\phi] \longrightarrow [(y, \mathscr{B}^* y), J^*\phi] .$$

By Lemma 3.3 $R(J^*) = G(-\mathscr{A}^*)^{-1}$. Hence, choosing $\phi$ so that $J^*\phi = (-\beta, \alpha)$,

$$[\mathscr{B}^* y_l, \alpha] - [y_l, \beta] \longrightarrow [\mathscr{B}^* y, \alpha] - [y, \beta] . \tag{3.5}$$

Putting (3.4) and (3.5) together we conclude that $\kappa(\alpha, \beta) \circ \gamma_n \rightarrow \kappa(\alpha, \beta) \circ \gamma$. Thus $\kappa(\alpha, \beta) \in F^*$. (3.4) also shows that the mapping $(\alpha, \beta) \xrightarrow{\hat{\kappa}} \kappa(\alpha, \beta)$ is linear and continuous with respect to the graph topology on $G(\mathscr{A}^*)$ and the weak* topology of $F^*$. Further $(\alpha, \beta) \in G(\mathscr{B})$ if and only if $\tilde{J}(\alpha, \beta) = 0$. In this case $[J(y, \mathscr{B}^* y), \kappa(\alpha, \beta)] = 0$ on $G(\mathscr{B}^*)$. Since $J$ is onto, $(\alpha, \beta) \in N(\hat{\kappa})$. By Lemma 3.1 there exists an operator $\hat{\mathscr{B}}: G \rightarrow F^*$ such that $\hat{\mathscr{B}} \circ \tilde{J} = \hat{\kappa}$ so that from (3.4)

$$[\mathscr{B}^* y, \alpha] - [y, \beta] = (\hat{\mathscr{B}} \circ \tilde{J}(\alpha, \beta)) \circ J(y, \hat{\mathscr{B}}^* y) . \tag{3.6}$$

To derive the Greens relation (3.3) it remains to show that $\hat{\mathscr{B}}$ has the stated continuity properties. Let $\langle g_l \rangle$ be a net in $G$ converging to $g$. We write $g_l = \tilde{J}(\alpha_l, \mathscr{A}^* \alpha_l)$ and $g = \tilde{J}(\alpha, \mathscr{A}^* \alpha)$. Then

$$[\tilde{J}(\alpha_l, \mathscr{A}^* \alpha_l), \phi] \longrightarrow \tilde{J}(\alpha, \mathscr{A}^* \alpha), \phi]$$

for all $\phi$ in $G^*$. Taking adjoints yields

$$[(\alpha_l, \mathscr{A}^* \alpha_l), \tilde{J}^*\phi] \longrightarrow [(\alpha, \mathscr{A}^* \alpha), \tilde{J}^*\phi] . \tag{3.7}$$

Since $G(-\mathscr{B}^*)^{-1} = R(\tilde{J}^*)$ (Lemma 3.3) (3.7) can be written

$$[\mathscr{B}^* y, \alpha_l] - [y, \mathscr{A}^* \alpha_l] \longrightarrow [\mathscr{B}^* y, \alpha] - [y, \mathscr{A}^* \alpha]$$

on $G(\mathscr{B}^*)$. Thus from (3.6)

$$(\hat{\mathscr{B}} \circ g_l) \circ J(y, \mathscr{B}^* y) \longrightarrow (\hat{\mathscr{B}} \circ g) \circ J(y, \mathscr{B}^* y) .$$

Since $J(y, \mathscr{B}^* y)$ is an arbitrary element of $F$, $\hat{\mathscr{B}}$ is continuous. Finally $\hat{\mathscr{B}}$ is 1-1 for if $\hat{\mathscr{B}}\tilde{J}(z, \mathscr{A}^* z) = 0$ and (3.3) is true, then $(z, \mathscr{A}^* z) \in G(\mathscr{B})$, implying $\tilde{J}(z, \mathscr{A}^* z) = 0$.

COROLLARY 3.6. *Suppose $X = Y$ is a Hilbert space ("$\mathscr{H}$") and $\mathscr{B}^* = \mathscr{A}^*$ so that $\mathscr{A} \subset \mathscr{A}^*$. Then $\hat{\mathscr{B}}$ is skew-hermetian. Since the argument is no different from the proof in* [3] (*Lemma* 4.2), *we do not repeat it.*

The previous theorem has shown that given regular $\mathscr{A}$, $\mathscr{B}$ and certain conditions on $J$, $\tilde{J}$ then a Green's relation may be constructed.

However, in many applications the Green's formula is already known. The following theorem therefore is sort of a converse of Theorem 3.5 and shows that in certain circumstances given a Green's formula, $\mathscr{A}$ and $\mathscr{B}$ are regular.

THEOREM 3.7. *Let $W$, $Z$ be linear spaces and suppose $J\colon X \times Y \to W$, $\tilde{J}\colon Y^* \times X^* \to Z$ are linear operators with domains $G(\mathscr{B}^*)$ and $G(\mathscr{A}^*)$ respectively. Suppose further that there is a 1-1 operator $\hat{\mathscr{B}}$ from $R(\tilde{J})$ onto a total space of functionals over $R(J)$ such that*

$$[\mathscr{B}^*y, z] - [y, \mathscr{A}^*z] = [J(y, \mathscr{B}^*y), \hat{\mathscr{B}}\tilde{J}(z, \mathscr{A}^*z)] . \tag{3.8}$$

*Then $J$ and $\tilde{J}$ are regular boundary operators $\mathscr{A}$ and $\mathscr{B}$.*

*Proof.* We regard $R(J)$ as a l.c.t.v.s. $F$ under the weak topology determined by $R(\hat{\mathscr{B}}\tilde{J}) := F^*$ and $F^*$ as a l.c.t.v.s. endowed with the weak* topology determined by $F$. (3.8) implies that $J\colon G(\mathscr{B}^*) \to F$ and $\hat{\mathscr{B}}\tilde{J}\colon G(\mathscr{A}^*) \to F^*$ are continuous. Hence $\mathscr{B}_J{}^*$ and $\mathscr{A}^*_{\hat{\mathscr{B}}\tilde{J}} = \mathscr{A}_{\tilde{J}}{}^*$ are closed. (3.8) also shows that $\mathscr{B}_J{}^* \subset \mathscr{A} = \mathscr{A}^{**}$ and $\mathscr{A}^* \subset \mathscr{B} = \mathscr{B}^{**}$. On the other hand if $(y, \mathscr{B}^*y) \in G(\mathscr{A})$

$$[J(y, \mathscr{B}^*y), \hat{\mathscr{B}}\tilde{J}(z, \mathscr{A}^*z)] = 0 \tag{3.9}$$

for all elements of $G(\mathscr{A}^*)$. Our assumption of totalness implies that $(y, \mathscr{B}^*y) \in G(\mathscr{B}_J{}^*)$. Hence $\mathscr{A} \subset \mathscr{B}_J{}^*$ and the two relations are equal. Similarly if $(z, \mathscr{A}^*z) \in G(\mathscr{B})$ (3.9) holds for all elements of $G(\mathscr{B}^*)$—i.e., on $R(J)$. This means (since $F$ is automatically total over $F^*$) that $\hat{\mathscr{B}}\tilde{J}(z, \mathscr{A}^*z) = 0$. Since $\hat{\mathscr{B}}$ is 1-1, $J(z, \mathscr{A}^*z) = 0$ and $\mathscr{B} = \mathscr{A}_{\tilde{J}}{}^*$. Since $J$ is continuous, standard theory implies that $J^*$ exists and $D(J^*) = F^*$. We now show that $R(J^*) = G((-\mathscr{A}^*)^{-1})$ (showing in particular that $R(J^*)$ is closed). To see this we rewrite (3.8) in the form

$$[G(\mathscr{B}^*), (-\mathscr{A}_\alpha{}^*, \alpha) - J^*\hat{\mathscr{B}}\tilde{J}(\alpha, \mathscr{A}_\alpha{}^*)] = 0$$

for fixed $(\alpha, \mathscr{A}_\alpha{}^*)$ in $G(\mathscr{A}^*)$. It follows that

$$(-\mathscr{A}_\alpha{}^*, \alpha) - J^*\hat{\mathscr{B}}\tilde{J}(\alpha, \mathscr{A}_\alpha{}^*) \subset G((-\mathscr{B})^{-1}) , \tag{3.10}$$

so that

$$J^*\hat{\mathscr{B}}\tilde{J}(\alpha, \mathscr{A}_\alpha{}^*) \subset G(-\mathscr{A}^{*-1})$$

showing that $R(J^*) \subset G(-\mathscr{A}^{*-1})$. However from what was proven above $(z, \mathscr{A}^*z) \subset G(\mathscr{B})$ if and only if $\tilde{J}(z, \mathscr{A}^*z) = 0$. This and (3.8) shows that $J^*(0) = G(-\mathscr{B}^{-1})$. Therefore from (3.10)

$$(\mathscr{A}_\alpha{}^*, \alpha) \in J^*\hat{\mathscr{B}}(\tilde{J}(\alpha, \mathscr{A}_\alpha{}^*))$$

for any $\alpha$. Thus $G(-\mathscr{A}^{*-1}) \subset R(J^*)$ and we conclude that $R(J^*) = G(-\mathscr{A}^{*-1})$. We now consider the operator $\tilde{J}$. As noted previously $F$ is automatically a total family of functionals over $F^* := R(\hat{\mathscr{B}}\tilde{J})$. Since $\hat{\mathscr{B}}$ is 1-1 the pairing

$$[f, \hat{\mathscr{B}}\tilde{J}(z, \mathscr{A}^*z)]\,, \quad f \in F \tag{3.11}$$

induces a total family of functionals in $G := R(J)$. Thus the usual way $G$ and $F$ can be regarded as mutually dual l.c.t.v.s. with respect to the pairing (3.11). The operator $\hat{\mathscr{B}}: G \to F^*$ is automatically continuous. Its adjoint $\hat{\mathscr{B}}^*: F \to F^*$ exists and is continuous. $\hat{\mathscr{B}}^*$ is defined by the equation

$$f \xrightarrow{\hat{\mathscr{B}}^*} [f, \hat{\mathscr{B}}(\cdot)]\,.$$

Also, $\hat{\mathscr{B}}^*$ is trivially onto and 1-1. In terms of $\hat{\mathscr{B}}^*$ we can write the Green's relation (3.8) as

$$[\mathscr{B}^*y, z] - [y, \mathscr{A}^*z] = [\hat{\mathscr{B}}^*J(y, \mathscr{B}^*y), \tilde{J}(z, \mathscr{A}z)]\,.$$

Here $R(\hat{\mathscr{B}}^*J)$ defines a total family of functionals over $\tilde{J}$. By reasoning paralleling that for $J$ in the first part of the proof, we can prove that $\tilde{J}$ is a regular boundary operator. $\square$

REMARK 3.8. In most practical cases $\hat{\mathscr{B}}^*$ is known and we need not formally define it as has been done in the previous theorem.

To develop an extension theory paralleling the finite dimensional case, it is necessary to put a further restriction on the intermediate relation $\mathscr{C}$. At this point we assume that in addition to being regular $\mathscr{C} = \mathscr{B}_{DJ}^*$ where $D: F \to F$ is a weakly continuous operator. Let us call such $\mathscr{C}$ (or the boundary operator defining it) "admissible".

The connection between this assumption and the assumption of regularity is made clear by the following lemma.

LEMMA 3.9. *Suppose $D: F \to F$ is a weakly continuous operator. Then $DJ$ is regular if and only if $D^*$ has closed range.*

*Proof.* Since $D$ is weakly continuous $D^*$ exists and the domain of $D^* = F^*$. This implies that $(DJ)^* = J^*D^*$. For if $y$ is in the domain of $(DJ)^*$

$$[Jx, D^*y] = [x, J^*D^*y] = [x, (DJ)^*y]$$

so that $J^*D^*y = (DJ)^*y$ and thus $J^*D^* \supset (DJ)^*$. On the other hand the reverse inclusion is trivial. Now suppose that $D^*$ has closed range. Let $\lambda_l$ be a net in $R(DJ)^*$ converging to $\lambda$. Then $\lambda_l =$

$(DJ)^*\phi_l$. Further

$$[\theta, (DJ)^*\phi_l] = [\theta, J^*D^*\phi_l] \longrightarrow [\theta, \lambda]$$

for all $\theta$ in $F$. Since $\lambda_l \in R(J^*)$ and $J^*$ has closed range, (recall $J$ is assumed regular) $\lambda = J^*\eta$. Hence

$$[J\theta, D^*\phi_l] \longrightarrow [J\theta, \eta] .$$

Since $J$ is onto and $D^*$ has closed range, $\eta = D^*\phi$. It follows that

$$\lambda = J^*D^*\phi = (DJ)^*\phi .$$

So $(DJ)^*$ has closed range. Conversely if $(DJ)^*$ has closed range and $\lambda_l = D^*\phi_l \to \lambda$ is a net in $R(D^*)$, then

$$[J\theta, D^*\phi_l] \longrightarrow [J\theta, \lambda] .$$

So that

$$[\theta, (DJ)^*\phi_l] \longrightarrow [\theta, J^*\lambda] .$$

Hence

$$J^*\lambda = (DJ)^*\phi = J^*D^*\phi$$

implying since $J^*$ is 1-1 that $\lambda = D^*\phi$. □

We can now prove our main result concerning admissible relations.

THEOREM 3.10. *Suppose $\mathscr{C}$ is admissible. Then*

$$G(\mathscr{C}^*) = \{(z, \mathscr{A}^*z) : \hat{\mathscr{B}}\tilde{J}(z, \mathscr{A}^*z) \in R(\mathscr{D}^*)\} .$$

*Proof.* By Green's relation $(z, \mathscr{A}^*z) \in \mathscr{C}^*$ if and only if

$$[J(\mathscr{C}), \hat{\mathscr{B}}\tilde{J}(z, \mathscr{A}^*z)] = 0 , \tag{3.12}$$

that is,

$$\mathscr{C}^* = \{(z, \mathscr{A}^*z) : \hat{\mathscr{B}}\tilde{J}(z, \mathscr{A}^*z) \in J(\mathscr{C})^\perp\} .$$

By hypothesis $N(DJ) = \mathscr{C}$ so that $N(\mathscr{D}) = J(\mathscr{C})$ because $J$ is surjective. Thus

$$J(\mathscr{C})^\perp = N(D)^\perp = R(D^*) \tag{3.13}$$

because $\mathscr{C}$ is admissible and $R(D^*)$ is closed by Lemma 3.9. □

If $\mathscr{C}$ is regular but not admissible we have the following result.

COROLLARY 3.11. $\hat{\mathscr{B}}\tilde{J}(\mathscr{C}^*) = \overline{R(D^*)}$.

If $\mathscr{C}$ is not regular we can still say something about $\mathscr{C}^*$.

COROLLARY 3.12. *Two relations $\mathscr{C}$ and $\mathscr{C}^+$ are mutually adjoint if and only if*

$$[J(G(\mathscr{C})), \hat{\mathscr{B}}\tilde{J}(G(\mathscr{C}^+))] = 0 .$$

*Proof.* This follows from (3.12) above. □

COROLLARY 3.13. *Suppose $\overline{R(D^*)}$ is complemented and $\mathscr{C}$ is regular. Then $G(\mathscr{C}^*)$ is characterized by the adjoint boundary condition*

$$(I - P)\hat{\mathscr{B}}\tilde{J}(G(\mathscr{C}^*)) = 0 \tag{3.14}$$

*where $P$ is the projection on $\overline{R(\mathscr{D}^*)}$.*

COROLLARY 3.14. *If $\mathscr{C}$ is admissible and $R(D^*)$ is complemented, then $\mathscr{C}^*$ is admissible.*

*Proof.* Looking at the boundary condition (3.14) characterizing $\mathscr{C}^*$ we must prove that $T := ((I - P)\hat{\mathscr{B}}\tilde{J})^*$ has closed range. Now

$$((I - P)\hat{\mathscr{B}}J)^* = \tilde{J}^*\hat{\mathscr{B}}^*(I - P)^* .$$

It is also easily shown that $I - P^*$ is a (weakly continuous) projection on $N(D)$.

Let $\lambda_l := T\phi_l$ be a net $\to \lambda$ in $R(T)$ since $\lambda_l \in R(\tilde{J}^*)$ and $R(\tilde{J}^*)$ is closed ($J$ being admissible) $\lambda = \tilde{J}^*_\eta$. Since $\hat{\mathscr{B}}^*$ is onto, $\eta = \hat{\mathscr{B}}^*\psi$. Thus

$$[T\phi_l, \theta] \longrightarrow [\tilde{J}^*\hat{\mathscr{B}}^*\psi, \theta]$$

or

$$(-(I - P^*)\phi_l, \hat{\mathscr{B}}\tilde{J}\theta] \longrightarrow [\psi, \hat{\mathscr{B}}\tilde{J}\theta]$$

for all $\theta$ in $F^*$. Since $\hat{\mathscr{B}}\tilde{J}$ runs over $F^*$ and $N(D)$ is (weakly) closed, $\psi = (I - P^*)\phi$. Going backwards these fact imply that $\lambda = T\psi$ so that $T$ has closed range. □

*Self-adjoint and symmetric extensions.* Suppose $X = Y = \mathscr{H}$, $F = G = H$ where $\mathscr{H}$ and $H$ are hilbert spaces and $\mathscr{A} = \mathscr{B}$. In this case $\mathscr{A}$ is symmetric and $\hat{\mathscr{B}}$ is skew-Hermetian. We wish to characterize self-adjoint and symmetric extensions $\mathscr{C}$ of $\mathscr{A}$.

Suppose $\mathscr{C}$ is admissible and regular then:

THEOREM 3.15. *Let $P$ be the orthogonal projection onto $N(\mathscr{D})$. Then $\mathscr{C}$ is antisymmetric,* i.e., *$\mathscr{C} \supset \mathscr{C}^*$ if and only if $N(P\hat{\mathscr{B}}) \subset N(D)$ and $\mathscr{C}$ is self-adjoint if and only if $N(P\hat{\mathscr{B}}) = N(D)$.*

*Proof.* We have $\mathscr{C} = (\mathscr{A}^*)_{DJ}$ and, by Corollary 3.13, $\mathscr{C}^* = (\mathscr{A}^*)_{P\hat{\mathscr{B}}J}$, so that $\mathscr{C} = N(DJ)$ and $\mathscr{C}^* = N(P\hat{\mathscr{B}}J)$. Since $J$ is surjective it follows that $J(\mathscr{C}) = N(D)$ and $J(\mathscr{C}^*) = N(P\hat{\mathscr{B}})$. Let $\mathscr{C} \supset \mathscr{C}^*$. Then $J(\mathscr{C}) \supset J(\mathscr{C}^*)$ and so $N(D) \supset N(P\hat{\mathscr{B}})$. Conversely, let $N(D) \supset N(P\hat{\mathscr{B}})$. Then $N(DJ) \supset N(P\hat{\mathscr{B}}\mathscr{J})$ so that $\mathscr{C} \supset \mathscr{C}^*$. The second assertion follows from the foregoing lines after replacing the inclusion signs by equality signs. □

COROLLARY 3.16. *Suppose $P$ is an orthogonal projection with the property that $P\hat{\mathscr{B}}P = 0$. Then the intermediate admissible relation $\mathscr{C}$ detemined by $DJ$ where $D = P\hat{\mathscr{B}}$ is antisymmetric and is self-adjoint if and only if $N(P\hat{\mathscr{B}}) = R(P)$.*

*Proof.* Let $Q$ be the orthogonal projection onto $N(D)$. Then $P\hat{\mathscr{B}}P = 0$ implies that $N(Q\hat{\mathscr{B}}) \subset N(D)$ so that by Theorem 3.15 $\mathscr{C}$ is antisymmetric. To see that $N(Q\hat{\mathscr{B}}) \subset N(D)$, let $x \in N(Q\hat{\mathscr{B}})$. Then $Q\hat{\mathscr{B}}x = 0$ so that $\hat{\mathscr{B}}x \in N(D)^\perp = R(D^*)$ since $\mathscr{C}$ is admissible (see Lemma 3.9). Hence

$$\hat{\mathscr{B}}^*x = -\hat{\mathscr{B}}x = -D^*y = -\hat{\mathscr{B}}^*Py \quad \text{for some} \quad y .$$

Here we used the fact that $\hat{\mathscr{B}}$ is skew-hermetian. Since $\hat{\mathscr{B}}^*$ is 1-1 we have that $x = -Py$ so that $P\hat{\mathscr{B}}x = -D\hat{\mathscr{B}}Py = 0$ showing that $N(Q\hat{\mathscr{B}}) \subset N(D)$. The second assertion of the corollary follows from the second assertion of Theorem 3.15 once we have proved that $N(Q\hat{\mathscr{B}}) = N(D)$ if and only if $N(P\hat{\mathscr{B}}) = R(P)$. Assume the first equality. If $x \in N(P)$ then $x \in N(Q)$ and hence

$$\hat{\mathscr{B}}x \in N(D)^\perp = R(D^*) = R(\hat{\mathscr{B}}D) .$$

It follows that $\hat{\mathscr{B}}x = \hat{\mathscr{B}}Py$ for some $y$. Since $\hat{\mathscr{B}}$ is 1-1 we have $x \in Py \in R(D)$. Hence $N(P\hat{\mathscr{B}}) \subset R(P)$. Clearly $P\hat{\mathscr{B}}P = 0$ implies $R(P) \supset N(P\hat{\mathscr{B}})$ so that $N(P\hat{\mathscr{B}}) = R(P)$. Now assume $N(P\hat{\mathscr{B}}) = R(P)$. If $x \in N(Q\hat{\mathscr{B}}) = N(D)$, then

$$\hat{\mathscr{B}}x \in N(D)^\perp = N(\hat{\mathscr{B}})^\perp = R(P)^\perp = N(P)$$

so that $x \in N(P\hat{\mathscr{B}}) = N(D)$. Hence $N(Q\hat{\mathscr{B}}) \subset N(D)$. The converse inclusion can be proved in a similar way.

COROLLARY 3.17. *Let $S$ be a closed subspace of $H$. Then if*

$\mathscr{C} \subset \mathscr{B}^*$ *is determined by the condition* $[J(y, \mathscr{B}^*y), s] = 0$ $s \in S$. $\mathscr{C}$ *is self-adjoint if and only if* $\hat{\mathscr{B}}^*$ *maps* $S^\perp$ *onto* $S$.

*Proof.* Let $D$ be the orthogonal projection onto $S$. Then $\mathscr{C} = (\mathscr{A}^*)_{D\mathscr{J}}$. Since $D$ is continuous and $R(D^*) = R(D) = S$ is closed, $\mathscr{C}$ is admissible. Now by Theorem 3.15 $\mathscr{C}$ is self-adjoint if and only if $N((I-D)\hat{\mathscr{B}}) = N(D)$. Taking orthogonal complements we see that $\mathscr{C}$ is self-adjoint if and only if $R(\hat{\mathscr{B}}^*(I-D)) = R(D)$. Here we use the fact that $R(\hat{\mathscr{B}}^*(I-D))$ is closed because of Corollaries (3.14), (3.13) and Lemma 3.9. The last equality is equivalent to $\hat{\mathscr{B}}^*S^\perp$.

Note that since $\hat{\mathscr{B}}^*$ is 1-1 and onto a necessary condition that $\mathscr{C}$ be self-adjoint is that $\dim S^\perp = \dim S$.

COROLLARY 3.18. *Let* $S$ *be a* (*weak**) *complemented subspace of* $F^*$. *Then if* $\mathscr{C}$ *is the restriction of* $\mathscr{B}^*$ *determined by the condition* $[J(y, \mathscr{B}^*y), s] = 0$ $s \in S$. $\mathscr{C}^*$ *is the restriction of* $\mathscr{A}^*$ *determined by the condition* $[s', \hat{\mathscr{B}}J(z, \mathscr{A}^*z)]$, $s' \in S^\perp$.

*Proof.* The condition determining $\mathscr{C}$ can be written $P^*J(y, \mathscr{B}^*y) = 0$ where $P$ is the projection on $S$. By Theorem 3.10 $(z, \mathscr{A}^*z) \in G(\mathscr{C}^*)$ if and only if $\hat{\mathscr{B}}J(z, \mathscr{A}^*z) \in R(P)$. Equivalently

$$(I-P)\hat{\mathscr{B}}\tilde{J}(z, \mathscr{A}^*z) = 0 ,$$

or

$$[v(I-P^*), \hat{\mathscr{B}}\tilde{J}(z, \mathscr{A}^*z)] , \quad v \in F ,$$

or

$$[s', \hat{\mathscr{B}}\tilde{J}(z, \mathscr{A}^*z)] , \quad s' \in S^\perp .$$

**4. Extension theory via Fredholm alternatives.** In the previous section we have attempted to characterize extensions $\mathscr{C}$ and $\mathscr{C}^*$ in terms of their boundary operators. In this section we pursue a different strategy by describing $\mathscr{C}$ and $\mathscr{C}^*$ "parametrically"—by relating $\mathscr{C}$ to a "known" extension $C$ and then describing $\mathscr{C}$ in terms of certain subspaces $S_c$, $S_c^*$.

We assume throughout the setting of Theorem 2.2. Recall that $A: X \to Y$ and $B: Y^* \to X^*$ are closed operators such that $A \subset B^*$ and $N(A^*)$, $N(B^*)$ are complemented subspaces of $X$. $H$ and $K$ are boundary operators (cf. Definition 2.1) for $A$ and $B$ respectively such that $H = M \circ A$ and $K = N \circ B$. The operators $M$ and $N$ have ranges in the t.v.s.s. $F$ and $G$. $\overline{D(M)^\perp}$ and $\overline{D(N)^\perp}$ are complemented in $Y^*$ and $D(M^*)$, $D(N^*)$ are total over $F$ and $G$. Under these hypotheses

$$\begin{aligned} G(A_H^*) &= G(A^*) - (\overline{R(M^*)} \times \{0\}) \\ G(B_K^*) &= G(B^*) - (R(N^*) \times \{0\}) . \end{aligned} \tag{4.1}$$

We will often use the notation $(z, A^*\bar{z})$ or $(z, C\bar{z})$ if $C \subset A^*$ where $\bar{z}$ means $z + \psi$ for an appropriately chosen element in $\overline{R(M^*)}$ to refer to an element in $G(A_H^*)$ or in the graph of a certain restriction of $A_H^*$. Similarly $(y, B^*\bar{y})$ will denote an element of $G(B_K^*)$.

Next suppose $C$ is closed extension $A$ contained in $B^*$. Let $S_c$ and $S_c^*$ be closed subspaces of $\overline{R(N^*)}$ and $\overline{R(M^*)}$. Define $C'$ as the restriction of $C$ whose range is orthogonal to $S_c^*$ and let $\mathscr{C}$ be the linear relation in $X \times Y$ with graph $G(\mathscr{C}) = G(C') - (S_c \times \{0\})$. We shall say that $\mathscr{C}$ is determined by $S_c$, $S_c^*$, and $\mathscr{C}$. From (4.1) $C \subset B_K^*$. Clearly $A_H \subset A \subset C$. Also $R(A_H)$ is orthogonal to $N(A_H^*)$ and therefore also to $S_c^*$. Hence $A_H \subset C'$. Since $C' \subset C$ we conclude that $A_H \subset C \subset B_K^*$.

At this point we introduce a new assumption: that every closed subspace of $N(A^*)$ or $N(B^*)$ is complemented.[3] This will certainly be true if $N(A^*)$ and $N(B^*)$ is finite dimensional or the setting is in a Hilbert space.

$C'$ is easily shown to be closed (see Lemma 4.13 [3]). $N(C') = N(C)$ and is a closed subspace of $N(B^*)$. Since it is complemented

$$S_c = S_c \cap N(C') \oplus S_c \cap N(C)^c .$$

Further

$$\begin{aligned} G(C) &= G(C') + ((S_c \cap N(C') \oplus S_c \cap N(C')^c) \times \{0\}) \\ &= G(C') + ((S_c \cap N(C')^c) \times \{0\}) . \end{aligned}$$

The intersection of $G(C')$ and $N((C')^c) \times \{0\}$ is clearly trivial. If $Q$ is a projection of $X$ onto $N(C')^c$ we define a projection $\tilde{Q}$ from $X \times Y$ to $N(C')^c \times \{0\}$ by $\tilde{Q}(x, y) := (Qx, 0)$. It follows that $(S_c \cap N(C'))\ \{0\} \subset R(\tilde{Q})$ and $G(C') \subset N(\tilde{Q})$. Since both $(S_c \cap N(C')) \times \{0\}$ and $G(C')$ are closed and contained in complementary subspaces $G(C)$ must be closed.

These remarks prove the first part of the following result.

THEOREM 4.1. *Suppose every closed subspace of $N(A^*)$ or $N(B^*)$ is complemented and let $C$ be an extension of $A$ contained in $B^*$. Then for every pair of closed subspaces $S_c \subset \overline{R(N^*)}$ and $S_c^* \subset \overline{R(M^*)}$, there exists a closed extension $\mathscr{C}$ of $A_H$ contained in $B_K^*$ determined by $C$, $S_c$ and $S_c^*$. Morerover, its adjoint $\mathscr{C}^*$ is determined by $C^*$, $S_c^*$ and $S_c$;* i.e., *$S_{c^*} = S_c^*$ and $S_{c^*}^* = S_c$.*

*Proof.* It is sufficient to show that $\mathscr{C}^*$ is determined by $C^*$, $S_c^*$ and $S_c$. By repeating the argument for the first part of the theorem we can show that there is an extension $\mathscr{C}^+$ of $B_K$ contained in $A_H^*$ determined by $C^*$, $S_c^*$ and $S_c$. Suppose $(\alpha, \beta) \in \mathscr{C}^*$. Since

[3] This assumption needs to be introduced to correct an error in the discussion of Theorem 4.13 in [**3**]. The argument we give here works with minor changes there.

$\mathscr{C}^* \subset A_H^*$ $\beta \in A^*\bar{\alpha}$. Further

$$(4.2)\quad [\mathscr{C}y, \alpha] - [y, \mathscr{C}^*\alpha] = 0 = [C\bar{y}, \bar{\alpha}] - [\bar{y}, A^*\bar{\alpha}] - [C\bar{y}, \psi_\alpha] - [\eta_y, A^*\bar{\alpha}]$$

for all $y \in D(\mathscr{C})$, where $\psi_\alpha \in \overline{R(M^*)}$ and $\eta_y \in \overline{R(N^*)}$. Now

$$\psi_\alpha \in N(\mathscr{C}^*) = R(\mathscr{C})^\perp = R(C)^\perp$$

and

$$\eta_y \in N(\mathscr{C}) = R(\mathscr{C}^*)^\perp = R(A^*\bar{z})^\perp\ , \quad z \in D(\mathscr{C}^*)$$

showing that $\psi_\alpha \in S_c^*$, $\eta_y \in S_c$. The last two terms of (4.2) vanish, so that $(\bar{\alpha}, A^*\bar{\alpha}) \in G(C^*)$. We conclude that $\mathscr{C}^* \subset \mathscr{C}^+$. The reverse inclusion follows, since

$$\begin{aligned}[\mathscr{C}y, z] - [y, \mathscr{C}^+z] &= [C\bar{y}, \bar{z}] - [\bar{y}, C^*\bar{z}] - [C\bar{y}, \psi_z] + [\eta_y, C^*\bar{z}] \\ &= 0\ .\end{aligned}$$

It is natural to attempt to determine the class of extensions between $A_H$ and $B_K^*$ determined by a $C$ between $A$ and $B^*$ and a pair of subspaces $S_c$ and $S_c^*$. In particular when does this class equal all extensions between $A_H$ and $B_K^*$? We have not settled these questions in general but there are interesting partial results (already sketched in [3]) for extensions with closed range or in a Hilbert space setting when $A$ is symmetric.

LEMMA 4.2. *Suppose $C$ is an extension of $A_H$ contained in $B^*$ with closed range. Then there is a closed subspace $S_c^*$ of $\overline{R(M^*)}$ and an extension $\tilde{C}$ between $A$ and $B^*$ such that $C$ is determined by $\tilde{C}$, $\{0\}$, and $S_c^*$.*

*Proof.* Define $\tilde{C}$ such that

$$G(\tilde{C}) = \{(y, B^*y) : y \in D(B_c^*) \oplus N(C),\ B^*y \perp (R(C)^\perp \cap N(A^*))\}$$

where $B_c^*$ is the restriction of $B^*$ to $D(B^*) \cap N(B^*)^c$. Now $\tilde{C}$ is closed (cf. Lemma 4.12 [3]). Also $C \subset \tilde{C}$. Set $S_c^* := R(C)^\perp \cap N(A^*)^c$. Let

$$G(C^+) = \{(y, \tilde{C}y) : \tilde{C}y \perp S_c^*\}\ .$$

Suppose $(\alpha, \beta) \in G(C)$. Since $R(C) \subset S_c^{*\perp}$ and $C \subset \tilde{C}$, $(\alpha, \beta) \in G(C^+)$ and $C \subset C^+$. On the other hand if $(\alpha, \beta) \in G(C^+)$

$$\beta \perp (S_c^* + (R(C^\perp) \cap N(A^*)) = R(C^\perp))\ ,$$

so that $\beta \in R(C)$ (since $C$ has closed range). If $(\alpha', \beta) \in G(C)$ we find by subtraction that $\alpha - \alpha' \in N(\tilde{C}) = N(C)$ so that $\alpha \in D(C)$ proving that $C^+ \subset C$. □

LEMMA 4.3. *Suppose that $\mathscr{C}$ is an extension of $A_H$ contained in $B_K^*$ with closed range. Then there is an extension $C$ of $A_H$ contained in $B^*$ and a closed space $S_c \subset \overline{R(N^*)}$ such that $\mathscr{C}$ is determined by $C$, $S_c$, and $\{0\}$.*

*Proof.* The argument is similar to that of Lemma 4.2. Define $C$ by

$$G(C) := \{(\bar{y}, B^*\bar{y}) : y \in D(\mathscr{C})\} .$$

That is, $G(C) = (y + \psi, B^*(y + \psi)$ when $y$ runs over $D(\mathscr{C})$ and $\psi$ is an element in $\overline{R(N^*)}$ such that $y + \psi \in D(B^*)$. One checks that $C$ is an operator restriction of $B^*$ but perhaps not closed. Set $S_c := N(\mathscr{C}) \cap \overline{R(N^*)}$. Define $C^+$ by

$$G(C^+) := \{(z, B^*z) : z \in D(B_c^*) \oplus N(C),\ B^*z \perp N(\mathscr{C}^*)\} . \tag{4.3}$$

Since $N(C) \subset N(B^*)$ is assumed to be closed, it is easy to show (Lemma 4.12, [3]) that $C^+$ is closed. Also

$$D(C) = D(C) \cap N(B^*)^c \oplus D(C) \cap N(B^*) \subset D(B_c^*) + N(C) .$$

And $R(C) \perp N(C^*)$ since $R(C) = R(\mathscr{C})$. It follows that $C \subset C^+$. Since $R(\mathscr{C})$ is closed by the Fredholm alternative, we have

$$R(C) = R(C^+) = N(C^*)^\perp .$$

So $R(C^+) = R(C)$. Suppose $(\alpha, \beta) \in G(C^+)$ then $\beta = C\alpha'$ for some $\alpha' \in D(C)$. Hence

$$(\alpha' - \alpha, 0) \in N(C^+) .$$

By (4.3) $N(C^+) = N(C)$. Hence $(\alpha, \beta) \in G(C)$, so that $C^+ \subset C$ and the two operators are equal. (This also shows the incidental fact not obvious from the definition that $C$ is closed.) Since $D(A_H) \subset D(B_c^*)$, $N(\mathscr{C}^*) \subset N(A_H^*)$ and $R(A_H) \perp N(\mathscr{C}^*)$ so that $A_H \subset C^+$. By definition $\mathscr{C}$ is determined by $C$, $S_c$, $\{0\}$. □

THEOREM 4.4. *Suppose $\mathscr{C}$ is an extension of $A_H$ contained in $B_K^*$ with closed range. Then there is an extension $A \subset C \subset B^*$ and closed subspaces $S_c \subset \overline{R(N^*)}$, $S_c^* \subset \overline{R(M^*)}$ such that $\mathscr{C}$ is determined by $C$, $S_c$ and $S_c^*$.*

*Proof.* By Lemma 4.2 there exists $A_H \subset C' \subset B^*$ with closed range and closed $S_c \subset \overline{R(N^*)}$ determining $\mathscr{C}$. By Lemma 4.4 there exists $A \subset C \subset B^*$ and closed $S_c^* \subset \overline{R(M^*)}$ determining $\mathscr{C}'$. □

COROLLARY 4.5. *Suppose $R(B^*)$ is closed. Then every extension*

*$A_H \subset \mathscr{C} \subset B_K^*$ with complemented nullspace is characterized according to Theorem* 4.4.

*Proof.* If $R(\mathscr{C}) = R(B^*)$ we can apply Theorem 4.4. If $R(\mathscr{C}) \subset R(B^*)$ we observe that $\mathscr{C}_c$ (the restriction of $\mathscr{C}$ to $D(\mathscr{C}) \cap N(\mathscr{C})^c$) is 1-1, closed, and has a bounded inverse. It is well-known that these conditions imply that $\mathscr{C}_c$ has closed range (cf. Goldberg [10], Lemma IV. 1.1). □

In the Hilbert space setting these special hypotheses are no longer needed. We close this section by quoting some results concerning self-adjoint extensions of a symmetric $A$ proved in [3].

THEOREM 4.6. *Let $A$ be a symmetric operator defined on a Hilbert space $\mathscr{H}$. Then if $A$ has a self-adjoint extension $C$, for each closed subspace $S$ of $\overline{R(M^*)}$ there exists a self-adjoint extension $\mathscr{C}_s$ of $A_H$ such that*

$$D(\mathscr{C}_s) = \{y \in D(C) - S \colon A^*\bar{y} \perp S\}$$
$$G(\mathscr{C}_s) = \{(y, A^*\bar{y}) \colon y \in D(\mathscr{C}_s)\}\ .$$

THEOREM 4.7. *Let $A$ be a symmetric operator defined on a Hilbert space $\mathscr{H}$. Suppose $\mathscr{C}$ is a self-adjoint extension of $A_H$. Then $A$ has a self-adjoint extension $C$. Moreover if $S := R(\mathscr{C})$, $\mathscr{C}$ is the self-adjoint extension $\mathscr{C}_s$ determined by $C$ and $S$ given by Theorem* 4.4.

THEOREM 4.8. *Suppose $A$ is a symmetric operator on a Hilbert space $\mathscr{H}$ with equal deficiency indices. Let $R(H)$ be a Hilbert space and let the hypotheses of Corollary* 2.3 *be satisfied. Further let $S$ be a closed subspace of $R(H)$. Then $A_H$ has a self-adjoint extension determined by the boundary conditions*

$$[HA^+(A^*\bar{z}), \phi] = 0\ , \quad \phi \in S\ , \quad \bar{z} \in C$$

*where $C$ is a self-adjoint extension of $A$.*

**5. Examples.** In this section we present a few examples to illustrate some of the main idea in the previous sections.

1. *Differential operators with Stieltjes boundary conditions on compact intervals.* Let $ly = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_n y$ be an $n$th order regular differential operator on $[a, b]$ ($a_i = C^{n-1}[a, b]$, all $i$). We let $ly$ generate a 1-1 closed operator in $L^p[a, b]$ by setting

$$D := \{y \in L^p[a, b] \cap AC^{n-1}[a, b] \colon y^{(j)}(a) = y^{(j)}(b) = 0,$$
$$j = 0, \cdots, n-1,\ ly \in L^p[a, b]\}$$

where $AC^{n-1}[a, b]$ is the space of functions on $[a, b]$ having a $n - 1$st absolutely continuous derivative, and defining $A$ by $ly$ on $D$. We restrict $A$ by the system of boundary conditions

$$\sum_{i=1}^{n} \int_a^b dw_{ij} y^{(n-i)} = 0 , \quad j = 1, \cdots, m ,$$

where the $\{dw_{ij}\}$ are a family of Stieltjes measures each of finite variation on $(a, b)$. It will be understood that the boundary conditions are independent so that $F = R^m = F^*$. Also $dw_{ij}$, $j = 1, \cdots, n - 1$, and for all $i$ are singular with respect to Lebesgue measure. This assumption is made to keep the computations simple; in most cases it is not unduly restrictive. Since $dw_{ij}$ has an absolutely continuous part $dw_{ij}^c$ it satisfies

$$\int_a^b dw_{ij}^c y^{(n-j)} = \int_a^b \frac{dw_{ij}^c}{dt} y^{(n-j-1)} dt .$$

By assuming $dw_{ij}^c/dt$ is also of bounded variation, etc., repeated integration by parts results in singular measures. Finally, we supose that $dw_{nj}/dt \in L^q[a, b]$. Using the standard Euclidian pairing, we set $d\bar{w}_i = (dw_{ii} \cdots dw_{im})^t$ and write

$$[Hy, \phi] = \phi^t \sum_{i=1}^{n} \int_a^b d\bar{w}_i y^{(n-i)} .$$

By the method of variation of parameters we can produce a Green's kernel $g(t, s)$ determing $A^+$, i.e.,

$$A^+ Ay = \int_a^t g(t, s) ly ds$$

where $g(t, s)$ has the following properties (cf. [6], Ch. 7):

(i) $D_t^{(j)} g(t, s)$ is continuous on $[a, b] \times [a, b]$, $j = 0, \cdots, n - 2$;

(ii) $g(t, s) = 0$, $t < s$;

(iii) $D_t^j g(t, t^-) = 0$, $j = 0, \cdots, n - 2$;

(iv) $D_t^{(n-1)} g(t, t^-) = 1$.

Then

$$\begin{aligned}[Hy, \phi] &= [HA^+ Ay, \phi] \\ &= \sum_{i,j} \int_a^b dw_{ij} D_t^{(n-i)} \int_a^t g(t, s) ly ds \phi_j \\ &= \phi^t \sum_i \int_a^b d\bar{w}_i D_t^{(n-i)} \int_a^t g(t, s) ly ds .\end{aligned}$$

Integration by parts yields

$$\sum_i \phi^t \Big\{ \bar{w}_i[a, b] D_t^{(n-i)} \int_a^t g(t, s) ly ds \Big|_{t=b} - \int_a^b \Big( \bar{w}_i[a, t] D_t^{(n-i+1)} \int_a^t g(t, s) ly ds \Big) dt \Big\} \tag{5.1}$$

or

$$\sum_i \phi^t \Big\{ \bar{w}_i[a, b] \int_a^b D_t^{(n-i)} g(t, s) \Big|_{t=b} lyds$$
$$+ \int_a^b \Big( \bar{w}_i[a, t] \int_a^t D_t^{(n-i+1)} g(t, s) ly\, ds \Big) + w_1[a, t] lydt \Big\} .$$

Finally, interchanging the order of integration and rearrangement gives

$$\int_a^b ly \Big\{ \phi^t \big( \sum_i (\bar{w}_i[a, b] D_t^{(n-i)} g(t, s)|_{t=b})$$
$$+ \int_a^b (\bar{w}_i[a, t] D_t^{(n-i+1)} g(t, s) dt) + w_1[a, s) \Big\} dt .$$

The expression in brackets is $(HA^+)^*\phi$. Then

$$\begin{aligned} D(A_H^+) &= D(A^*) - R(HA^+)^* \\ G(A_H^+) &= (z, l_n^+ \bar{z}) \end{aligned} \tag{5.2}$$

where $\bar{z} = z + (HA^+)^*\phi$ and Corollary 2.2 says that $A_H^* = A_H^+$. The following description of $A_H^*$ has been given in several previous papers (e.g., [1], [2]). Introduce the "partial adjoint" expressions:

$$\begin{aligned} \tilde{l}_0^+ z &:= z \\ \tilde{l}_1^+ z &:= -(l_0^+ z + \phi^t \bar{w}_1[0, t))' + a_1 z \\ &\vdots \\ \tilde{l}_{j+1}^+ z &:= -(l_i^+ z + \phi^t \bar{w}_{j+1}[a, t))' + a_{j+1} z \\ &\vdots \\ \tilde{l}_n^+ z &:= -(l_{n-1}^t z + \phi^t \bar{w}_n[a, t))' + a_n z . \end{aligned}$$

Let $D^+$ be the subspace of functions $z \subset L^q[a, b]$ such that $\tilde{l}_j^+ z$ exists in $L^q[a, b]$ for appropriate $\phi$ in $R^m$ and that $\tilde{l}_j^+ z + \phi^t \bar{w}_{j+1}[a, t]$, $j = 0, \cdots, n-1$. Note that $D^+ \supset D(A^*) = D(l_n^+)$ and that $\tilde{l}_n^+ z = -l_n^+ z + \phi^t(d\bar{w}_n/dt)$. Now set

$$G(L^+) := \{(z, \tilde{l}_n^+ z) : z \in D^+\} .$$

THEOREM. $A_H^* = L^+$.

We now show that $L^+ = A_H^+$ so that these two characterizations of $A_H^*$ are the same. We require a lemma.

LEMMA 5.1. $(HA^+)^*\phi - \sum_{j=0}^{n-1} (-1)^j I^j \phi^t w_{j+1}[a, s) \in D(A^*)$ *where* $I^j$ *stands for the* $j$ *fold integration operator*

$$\int_a^{t_j} \int_a^{t_{j-1}} \cdots \int_a^{t_0} (\cdot) dt_0, \cdots, dt_{j-1} .$$

*Proof.* From (5.1) and (5.2) we see that

$$\begin{aligned} D^{(j)}(HA^{+})^{*} &\equiv (-1)^{j}\phi^{t}w_{j+1}[a, s]D_t^{(n-j+1+j-1)}g(t, s)|_{t=s^+} \\ &\equiv (-1)^{j}\phi^{t}w_{j+1}[a, s] \mod AC[a, b]\ , \\ &\qquad\qquad j = 0, \cdots, n-1\ . \end{aligned}$$

By assumption $dw_n/ds \in L^q[a, b]$ and thus $D^{(n)}(HA^{+})^{+}\phi \in L^q[a, b]$. The smoothness assumptions of the coefficients of $l_n^+$ imply that $D(A^*) = \{z \in AC^{n-1}[a, b]: z^{(n)} \in L^q[a, b]\}$. The lemma follows.

Now suppose $z \in D(A_H^+)$. A calculation (similar to the above) shows that

$$z - \sum_{j=0}^{n-1}(-1)^{j}I^{j}\phi^{t}w_{j+1}[a, s) \in D(A^*)\ .$$

By Lemma 5.1 it follows that $z \in D^+$ so that $D(A_H^+) \subset D^+$. Further, it is easy to verify using the singularity of the measures $dw_1, \cdots, dw_{n-1}$ that

$$l_n^+\bar{z} = l_n^+ z + \phi^t\frac{d\bar{w}_n}{dt} = \tilde{l}_n^+ z$$

so that $A_H^+ \subset L^+$. On the other hand the following Green's formula involving $L^+$ can be proven by repeated integration by parts (cf. [2]).

$$\int_a^b (zly - \tilde{l}_n^+ zy)dt = \phi^t Hy\ .$$

2. *An operator with multipoint conditions on* $[0, \infty)$. Let $A: L^p[0, \infty) \to L^p[0, \infty)$ be given by $y^{(n)}$ on

$$\begin{aligned} D := \{y \in L^p[0, \infty) \cap AC^{n-1}[0, \infty)&: y^{(j)}(0) = 0, \\ & j = 0, 1, \cdots, n-1,\ y^{(n)} \in L^p[0, \infty)\}\ . \end{aligned}$$

It is known that $A$ is a 1-1 closed operator. Let $T = \{t_0 < t_1 < \cdots < t_j < \cdots\}$ be an ordered set of points in $(0, \infty)$ and restrict $A$ by the boundary conditions

$$\sum_{j=0}^{\eta_i} \alpha_{ij}y(t_{i+j}) = 0\ .$$

Let $R^\omega$ be the countable Cartesian product of $R$. We view elements of $R^\omega$ as infinite column vectors. Let $R_{oo}^\omega$ be the subspace of elements of $R^\omega$ with finitely many nonzero components. We set $F = \boldsymbol{R}^\omega$ and $F^* = R_{oo}^\omega$, giving $F$ and $F^*$ respectively the weak and weak* topologies induced by the pairing

$$[f, g] = g^t f = \sum f_i g_i f \in F,\ g \in F^*\ .$$

Then

$$Hy = \left\langle \sum_{j=0}^{\eta_i} \alpha_{ij} y(t_{i+j}) \right\rangle \in R^{\omega} .$$

Also

$$A^+ z = \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} z ds$$

for $z \in R(A)$. Note that $A^+$ is not a bounded operator. Hence

$$\begin{aligned}
[HA^+ z, \phi] &= \phi^t \sum_{i,j} \alpha_{ij} \int_0^{t_{i+j}} \frac{(t_{i+j}-s)^{n-1}}{(n-1)!} z ds \\
&= \int_0^\infty z \left\{ \sum_{i,j} \alpha_{ij} \frac{(t_{i+j}-s)^{n-1}}{(n-1)!} + \phi_i \right\} ds , \\
& \qquad\qquad i = 0, 1, \cdots ; \quad 0 \leqq j \leqq \eta_i . \\
x_+ &:= \begin{cases} x & \text{if} \quad x > 0 \\ 0 & \text{otherwise} \end{cases} \\
&= [z, (HA^+)^* \phi] .
\end{aligned}$$

Suppose $\eta \in \overline{R(HA^+)^*}$. Since $R(HA^+)^*$ restricted to some compact interval $\Delta$ is a finite dimensional space and $\eta|_\Delta \in \overline{R(HA^+)^*|_\Delta}$,

$$\eta|_\Delta = \lambda(s)_\Delta \sum_{i,j} \alpha_{ij} \frac{(t_{i+j}-s)}{(n-1)!} + \tilde{\phi}_i .$$

Hence $\eta$ has the same structure as an element in $R(HA^+)^*$ except that $\langle \tilde{\phi}_i \rangle \in R^\omega$ instead of $R^\omega_{oo}$, i.e., the admissible parameters corresponding to $\overline{R(HA^+)^*}$ comprise some subspace of $R^\omega$. What subspace this is an interesting question. We are able to answer it in some circumstances as will be shown in Example 3 below.

At any rate, Corollary 2.2 implies that

$$G(A_H^*) = \{(u, A^* \bar{u})\}$$

where $A^*$ is given by $(-1)^n z^{(n)}$ on

$$\begin{aligned}
D^* = \{ & z \in L^q[0, \infty] \cap AC^{n-1}[0, \infty) : \lim_{t \to \infty} \sum_i (-1)^i z^{(n-i)} y^{(i)}(t) = 0, \\
& y \in D(A); z^{(n)} \in L^q[0, \infty) \} .
\end{aligned}$$

Note that $u^{(j)} \in D(A_H^*)$, $i = 0, \cdots, n-2$, is continuous, but that

$$z^{(n-1)}(t_i^+) - z^{(n-1)}(t_i^-) = -\sum_{j=0}^{\eta_i} \alpha_{ij} \phi_i .$$

3. *Operators on $R$ with interface conditions.* Let $T = \{t_{i-1} < t_i < t_{i+1}\}$ be a biinfinite set of points in $R$. Set $\Delta_i = [t_i, t_{i+1}]$, $|\Delta_i| = t_{i+1} - t_i$, and

$$D_p^* := \{y \in L^p(R) \cap AC^{n-1}[\Delta_i] \text{ all } i\colon y^{(n)} \in L^p(R)\} .$$

Define $B^*\colon L^p(R) \to L^p(R)$ by $y^{(n)}$ on $D_p^*$ and $A^*\colon L^q(R) \to L^q(R)$ by $(-1)^n z^{(n)}$ on $D_q$, $1/p + 1/q = 1$. It can be shown that $A$ is given by $y^{(n)}$ on

$$D_p \subset D_P^* := \{y \in L^p(R) \cap AC^{n-1}(R)\colon y^{(j)}(t_i) = j = 0, \cdots, n-1 ,\\ y^{(n)} \in L^p(R)\} .$$

Similarly $B$ is given by $(-1)z^{(n)}$ on $D_q$.

Let $\mathscr{S}$ be the space of biinfinite sequences $\langle \cdots \phi_{i-1} < \phi_i < \phi_{i+1} \cdots \rangle$ and $E\colon \mathscr{S} \to \mathscr{S}$ be the shift operator defined by $E\langle \phi_i \rangle = \langle \phi_{i+1} \rangle$. Introduce the notation for elements in $\mathscr{S}$:

$$y_-^j = \langle y^{(j)}(t_i^-) \rangle$$
$$y_+^j = \langle y^{(j)}(t_i^+) \rangle$$

and define $J\colon D_p^* \to \mathscr{S}$ and $\tilde{J}\colon D_q^* \to \mathscr{S}$ by

$$Jy = \begin{pmatrix} y_+ \\ Ey_- \\ \vdots \\ y_+^{n-1} \\ Ey_-^{n-1} \end{pmatrix} \qquad \tilde{J}z = \begin{pmatrix} z_+ \\ Ez_- \\ \vdots \\ z_+^{n-1} \\ Ez_-^{n-1} \end{pmatrix} .$$

We also construct a pairing in $\sum_{\oplus}^{2n} \mathscr{S} \times \sum_{\oplus}^{2n} \mathscr{S}$ by

$$[\alpha, \beta] := \sum_{i,j} \alpha_{ji}\beta_{ji} , \quad 1 \leqq j \leqq 2n , \quad -\infty \leqq i \leqq \infty$$

where

$$\alpha = \begin{pmatrix} \langle \alpha_{1i} \rangle \\ \vdots \\ \langle \alpha_{2ni} \rangle \end{pmatrix} , \qquad \beta = \begin{pmatrix} \langle \beta_{1i} \rangle \\ \vdots \\ \langle \beta_{2ni} \rangle \end{pmatrix}$$

for those $\alpha$, $\beta$ such that the pairing exists. Then since

$$[y^{(n)}, z] - [y, (-1)^n z^{(n)}] = \sum_i (-1)^{n-1}(y^{(l-1)} z^{(n-1)}|_{t_{i+1}^-} - y^{(l-1)} z^{(n-1}|_{t_{i+1}^+}) .$$

We arrive at the Green's formula

$$[B^* y, z] - [y, A^* z] = [Jy, \hat{\mathscr{B}}\tilde{J}z] \tag{5.3}$$

where $\hat{\mathscr{B}}$ is the $2n \times 2n$ matrix

$$\begin{pmatrix} & & & (-1)^{n-1}H \\ \bigcirc & & \cdot^{\cdot^{\cdot}} & \\ (-1)^{n-j}H & & & \\ \cdot^{\cdot^{\cdot}} & & & \bigcirc \\ H & & & \end{pmatrix}$$

where

$$H = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

and the pairing on the right in (5.3) is interpreted according to (5.2).

Since $H^* = -H$, $\hat{\mathscr{B}}$ is skew symmetric. Further if $n$ is even, $\hat{\mathscr{B}}$ is unitary since

$$\begin{aligned}
\hat{\mathscr{B}}\hat{\mathscr{B}}^* &= \begin{pmatrix} \bigcirc & & (-1)^{n-1}H \\ & \cdot^{\cdot^{\cdot}} & \\ (-1)^{n-j}H & & \\ & \cdot^{\cdot^{\cdot}} & \bigcirc \\ H & & \end{pmatrix}\begin{pmatrix} \bigcirc & & (-1)^{n-1}H^* \\ & \cdot^{\cdot^{\cdot}} & \\ (-1)^{n-j}H^* & & \\ & \cdot^{\cdot^{\cdot}} & \bigcirc \\ H^* & & \end{pmatrix} \\
&= \begin{pmatrix} (-1)^{n-1}HH^* & & \bigcirc \\ & \ddots & \\ \bigcirc & & (-1)^{n-1}HH^* \end{pmatrix} \\
&= I_{2n} .
\end{aligned}$$

We are interested in using the theory of § 5 to describe intermediate extensions $C$ between $A$ and $B^*$ and their adjoints. (This problem has also been considered by Lee [15], in the case $n = p = q = 2$.) We begin by proving that $J$ and $\tilde{J}$ are regular boundary operators. This will be done by showing that—provided $|\Delta_i|$ is uniformly bounded—$J$ and $\tilde{J}$ are onto certain Banach spaces. The closed range theorem then implies that $R(J^*)$ and $R(\tilde{J}^*)$ are closed.

First we require a technical lemma.

LEMMA 5.2. *Let $P_{ij}(t)$, $Q_{ij}(t)$ be polynomials of degree $2n - 1$ on $\Delta_i$ interpolating the data.*

$$\begin{aligned} P_{ij}^{(k)}(t_i) &= \delta_{kj}\phi_{ij} , & P_{ij}^{(k)}(t_{i+1}) &= 0 , \\ Q_{ij}^{(k)}(t_i) &= 0 , & Q_{ij}^{(k)}(t_{i+1}) &= \delta_{kj}\phi_{ij} , \end{aligned}$$

$k = 0, \cdots, n-1$.

Then on $\Delta_i$

$$|Q_{ij}(t)|, |P_{ij}(t)| \leqq |\phi_{ij}||\Delta_i|^j C_1 \tag{5.4}$$

$$|Q_{ij}^{(n)}(t)|, |P_{ij}^{(n)}(t)| \leqq |\phi_{ij}||\Delta_i|^{j-n} C_2 \tag{5.5}$$

where the constants $C_1$, $C_2$ depend only on $n$ and $j$.

*Proof.* We will prove (5.4) and (5.5) for $Q_{ij}(t)$ and $Q_{ij}^{(n)}(t)$ only, since

$$P_{ij}(s) := (-1)^j Q_{ij}(t_{i+1} + t_i - s)$$

is the unique polynomial interpolating the data for $P_{ij}$ and on $\Delta_i|P_{ij}| = |Q_{ij}|$.

By the Newton formula for osculatory interpolation (cf. [6] p. 233).

$$Q_{ij}(t) = \sum_{l=0}^{2n-1} Q_{ij}[\tilde{t}_0, \cdots, \tilde{t}_l] \prod_{\rho=0}^{l-1} (t - \tilde{t}_\rho)$$

where

$$\tilde{t}_l = \begin{cases} t_i & \text{if } l < n \\ t_{i+1} & \text{if } l \geqq n . \end{cases}$$

The boundary conditions on $Q_{ij}$ imply that $Q_{ij}[\tilde{t}_0, \cdots, \tilde{t}_l] = 0$, $l = 0, \cdots, n-1$, so that

$$\begin{aligned} Q_{ij}(t) &= \sum_{l=n}^{2n-1} Q_{ij}[\tilde{t}_0, \cdots, \tilde{t}_l] \prod_{\rho=0}^{l-1} (t - \tilde{t}_\rho) \\ &= \sum_{k=1}^{n} Q_{ij}[\tilde{t}_0, \cdots, \tilde{t}_{n-1}, \tilde{t}_n, \cdots, \tilde{t}_{n-1+k}] \prod_{\rho=0}^{n+k-2} (t - \tilde{t}_\rho) . \end{aligned} \tag{5.6}$$

Define

$$g_{ij}(x) := Q_{ij}[\tilde{t}_0, \cdots, \tilde{t}_{n-1}, x] .$$

Then

$$\begin{aligned} \frac{g_{ij}^{(k-1)}(\tilde{t}_n)}{(k-1)!} &= Q_{ij}[\tilde{t}_0, \cdots, \tilde{t}_{n-1}, \underbrace{t_{i+1}, \cdots, t_{i+1}}_{|\leftarrow k \text{ times} \rightarrow|}] \\ &= Q_{ij}[\tilde{t}_0, \cdots, \tilde{t}_{n-1}, \tilde{t}_n, \cdots, \tilde{t}_{n-1+k}] \\ &= \frac{g^{(k-1)}(t_{i+1})}{(k-1)!} \end{aligned}$$

(see [6] p. 230 Ex. 4.6-8).

The boundary conditions on $Q_{ij}$ and the definition of the divided difference imply:

$$\begin{aligned} g_{ij}(x) &= \frac{Q_{ij}[\tilde{t}_1, \cdots, \tilde{t}_{n-1}, x] - Q_{ij}[\tilde{t}_0, \cdots, \tilde{t}_{n-1}]}{x - t_i} \\ &= \frac{Q_{ij}[t_1, \cdots, t_{n-1}, x]}{x - t_i} \\ &\cdots \\ &= \frac{Q_{ij}(x)}{(x-t)^n} . \end{aligned}$$

Hence

$$\begin{aligned}\frac{g_{ij}^{(k-1)}(\tilde{t}_n)}{(k-1)!} &= \sum_{r=0}^{k-1}\binom{k-1}{r}Q_{ij}^{k-1-r}(x)D^{(r)}(x-t_i)^{-n}|_{x=t_{i+1}} \\ &= \binom{k-1}{k-1-j}Q_{ij}D^{(k-1-j)}(x-t_i)^{-1}|_{x=t_{i+1}} \\ &= \binom{k-1}{k-1-j} - n(-n-1)\cdots \\ &\qquad (-n-(k-1-j)+1)|\Delta_i|^{-n-k+1+j} .\end{aligned}$$

So that from (5.7)

$$|Q_{ij}(t)| \leqq |\phi_{ij}||\Delta_i|^j \sum_{k=1}^{n}\binom{k-1}{k-1-j}(n)(n+1)\cdots(n+(k-1-j)-1)$$

proving (5.4).

Also

$$\begin{aligned}|Q_{ij}^{(n)}(t)| &\leqq |\phi_{ij}|\sum_{k=1}^{n}\binom{k-1}{k-1-j}(n)(n+1)\cdots(n+(k-1-j)-1) \\ &\quad\times |\Delta|^{-n-k+1+j}\max_t D_t^{(n)}\prod_{\rho=0}^{n+k-2}(t-\tilde{t}_\rho) \\ &\leqq |\phi_{ij}||\Delta_i|^{j-n}C_2 ,\end{aligned}$$

where

$$C_2 = \sum_{k=1}^{n}\binom{k-1}{k-1-j}(n)(n+1)\cdots(n+(k-1-j)-1)2^n n!\, k! .$$

Here we are using the estimate

$$\begin{aligned}D_t^{(n)}\prod_{\rho=0}^{n+k-2}(t-t_\rho) &= D_t^{(n)}(t-t_i)^n(t-t_{i+1})^{k-1} \\ &= \sum_{r=0}^{n}\binom{n}{r}D_t^{(n-r)}(t-t_i)^n D_t^r(t-t_{i+1})^{k-1} \\ &\leqq 2^n n!\, k!\, |\Delta_i|^{k-1} .\end{aligned}$$

PROPOSITION 5.3. *Suppose* $0 < N \leqq |\Delta_i| \leqq M < \infty$ *all* $i$. *Then* $J(D(B^*))$ *and* $\tilde{J}(D(A^*))$ *are continuous operators onto* $\sum_{\oplus}^{2n} l_p$, $\sum_{\oplus}^{2n} l_q$ *for* $1 \leqq p \leqq \infty$, $1/q + 1/p = 1$.

*Proof.* Let $z$ be a function in $L^q(\Delta_i)$ such that $z^{(n-1)}$ is absolutely continuous $z^{(n)} \in L^q(\Delta_i)$ and satisfying the boundary conditions

$$z^{(k)}(t_i^+) = \delta_{kj}z^{(k)}(t_{i+1}^-) = 0 , \quad k = 0, \cdots, n-1 . \tag{5.8}$$

Then if $y \in D(B^*)$

$$[y^{(n)}, z]_{\Delta_i} = [y, (-1)^n z^{(n)}]_{\Delta_i} = (-1)^{j-1}y^{(n-j)}(t_i^+) .$$

So that

$$|y^{(n-j)}(t_i^+)| \leqq (\|y\|_{p,\Delta_i} + \|y^{(n)}\|_{p,\Delta_i})C_i$$

where

$$C_i = \|z\|_{q,\Delta_i} + \|z^{(n)}\|_{q,\Delta_i} .$$

If the $t_i$ are equally spaced, $C_i$ does not depend on $i$—the same function $z$ translated works for all intervals. Hence

$$\sum |y^{(n-j)}(t_i^+)|^p \leqq (\|y\|_p^p + \|y^{(n)}\|_p^p)C) \tag{5.9}$$

and so $\langle y^{(n-j)}(t_i^+)\rangle \in l^p$.

In any event we can choose for $z_i$ the polynomial $P_{ij}(t)$ of degree $2n-1$ interpolating the data (5.8). By Lemma 5.2

$$|P_{ij}(t)| \leqq |\Delta_i|^j C_1$$
$$|P_{ij}^{(n)}(t)| \leqq |\Delta_i|^{j-n} C_2 .$$

Hence

$$\begin{aligned} \|P_{ij}(t)\|_{q,\Delta_i} &\leqq C_1|\Delta_i|\frac{jq+1}{q} \\ \|P_{ij}^{(n)}(t)\|_{q,\Delta_i} &\leqq C_2|\Delta_i|\frac{(j-n)q+1}{q} . \end{aligned} \tag{5.10}$$

Our hypothesis that the $|\Delta_i|$ are uniformly bounded above and below implies that

$$\|P_{ij}(t)\|_{q,\Delta_i} + \|P_{ij}^{(n)}(t)\|_{q,\Delta_i} \equiv \|z\|_{q,\Delta_i} + \|z^{(n)}\|_{q,\Delta_i} \leqq C , \quad \text{all} \quad i$$

so that (5.9) is true. Varying $j$ shows that $J$ maps into $\sum_{\oplus}^{2n} l_p$. The argument for $\tilde{J}$ is the same changing $p$ to $q$. We now show that $\tilde{J}(D(B^*))$ and $\tilde{J}(D(A^*))$ are onto, again giving the argument only for $J$. Let

$$\begin{pmatrix} \langle\phi_{i0}\rangle \\ \vdots \\ \langle\phi_{i2n-1}\rangle \end{pmatrix} \in \sum_{\oplus}^{2n} l_p .$$

It is sufficient to find $2n$ functions $y_{ji}$ satisfying the boundary conditions

$$y_{ji}^{(k)}(t_i^+) = \delta_{kj}\phi_{ij}$$
$$y_{ji}^{(k)}(t_i^-) = 0 , \quad k = 0, \cdots, n-1 , \quad \text{for} \quad j \leqq n-1$$

or

$$y_{ji}^{(k)}(t_i^+) = 0$$
$$y_{ji}^{(k)}(t_i^-) = \delta_{kj}\phi_{ij} , \quad k = 0, \cdots, n-1 , \quad \text{for} \quad n \leqq j \leqq 2n-j ,$$

so that

$$y_j := \sum_i \lambda(t)_{\Delta_i} y_{ji} \in D(B^*) .$$

Then a function $y$ satisfying the boundary conditions

$$\begin{aligned} y^{(j)}(t_i^+) &= \phi_{ij} , \quad 0 \leqq j \leqq n-1 \\ y^{(j)}(t_i^-) &= \phi_{ij} , \quad n \leqq j \leqq 2n-1 \end{aligned}$$

in $D(B^*)$ can be obtained by superposition. We define the $y_{ij}$ as

$$y_{ij} = \begin{cases} P_{ij}(t) & \text{if} \quad 0 \leqq j \leqq n-1 \\ Q_{ij}(t) & \text{if} \quad n \leqq j \leqq 2n-1 . \end{cases}$$

Lemma 5.2 now implies that $y_j \in D(B^*)$. That $J$ and $\tilde{J}$ are continuous with respect to the norm topologies on $X \times Y$ or $Y^* \times X^*$ and the weak or weak* topologies on their ranges is obvious. For $1 \leqq p < \infty$, $1 \leqq q < \infty$ the boundary oprerators are continuous with respect to the norm topologies on their ranges (Ch. [10] V. 3.15).

We are now in a position to determine extensions and their adjoints between $A$ and $B^*$. For example suppose $C$ is given by $y''$ and the boundary conditions

$$\begin{aligned} y(t_i^+) &= 2y'(t_i^-) \\ 2y(t_i^-) &= y'(t_i^+) . \end{aligned}$$

Then

$$D = \begin{pmatrix} 1 & 0 & 0 & -2E^{-1} \\ 0 & 2E^{-1} & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} .$$

The adjoint boundary condition

$$\hat{\mathscr{B}} Jz = R(D^*)$$

may be written

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} z_+ \\ Ez_- \\ z'_+ \\ Ez'_- \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2E^{-1} & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -2E^{-1} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \langle \phi_{1i} \rangle \\ \langle \phi_{2i} \rangle \\ \langle \phi_{3i} \rangle \\ \langle \phi_{4i} \rangle \end{pmatrix}$$

which implies the boundary conditions

$$\begin{aligned} z(t_{i+2}^-) &= -2z'(t_i^+) \\ -z'(t_{i+2}^-) &= 2z(t_i^+) . \end{aligned}$$

4. *A family of self-adjoint extensions.* We use the same setting as Example 3 with $p = q = 2$ and $n$ an even integer so that $A^* = B^*$, $A = B$ and $J = \tilde{J}$. Let $S_1, \cdots, S_n$ be closed subspaces of $l^2$ and $P_1, \cdots, P_n$ be orthogonal projections onto these subspaces. Define a projection $P$ in $\sum_{\oplus}^{2n} l^2$ by

$$\begin{pmatrix} P_1 & & & & & \\ & \ddots & & & \bigcirc & \\ & & P_n & & & \\ & & & Q_1 & & \\ & \bigcirc & & & \ddots & \\ & & & & & Q_n \end{pmatrix},$$

where

$$Q_i = \begin{cases} I - P_{n-1} & \text{if } i \text{ is odd} \\ I - P_{n+2-i} & \text{if } i \text{ is even} . \end{cases}$$

Then $P\hat{\mathscr{B}}P = 0$ and

$$N(P\hat{\mathscr{B}}) = \begin{pmatrix} R(P_1) \\ \vdots \\ R(P_n) \\ R(Q_i) \\ \vdots \\ R(Q_n) \end{pmatrix}.$$

It follows from Corollary 3.16 that the operator $C$ determined by the boundary condition $P\hat{\mathscr{B}}Jy$ is self-adjoint. For example if $k = 4$ and $P_1 = I$, $P_2 = I$, $P_3 = 0$, $P_4 = 0$ then $Q_1 = I$, $Q_2 = I$, $Q_3 = 0$, $Q_4 = 0$ and the (self-adjoint) boundary conditions are

$$\begin{aligned} E\langle z_-^{(iii)}\rangle &= 0 \\ \langle z_+^{(iii)}\rangle &= 0 \\ E\langle z'_-\rangle &= 0 \\ \langle z'_+\rangle &= 0 \end{aligned}$$

with arbitrary jumps at the other points of $T$ in the other derivatives, i.e., the operator is determined by

$$\begin{aligned} & y^{(iv)} \\ & y'(t_i) = 0 \\ & y^{(iii)}(t_i) = 0 \\ & y(t_i^+) - y(t_i^-) = \phi_{1i} \\ & y''(t_i^+) - y''(t_i^-) = \phi_{2i} . \end{aligned}$$

It would be worthwhile to determine the structure of all projections $P$ inducing self-adjoint extensions for the minimal operator determined by $y^{(n)}$ with boundary conditions $(t_i) = 0$, $j = 0, \cdots, n-1$, all $i$.

**6. Conclusion.** We close the paper with a few historical remarks concerning generalized b.v.p. and brief mention of some unsolved problems.

We should point out first of all that concrete b.v.p. posed by differential equations and nonstandard boundary conditions (often arising from specific physical problems) have been of interest to the mathematical community for many years. Perennial problems have been the determination of adjoints, extensions, Green's functions and eigenfunction expansions for increasing general systems. Good examples of recent work on these questions include papers of Krall [12], [14], Kemp and Lee [11], this writer [1], [2], and the book of Schwabik, Tvrdý, and Vejvoda [18]. Additional historical information can be found in the surveys of Krall [13], and Whyburn [19].

An increasing tendency to abstraction has been evident in the last decade characterized by the introduction of functional analysis and spectral theory. This process culminated in the paper [4], of E. A. Coddington and A. Dijksma. Their achievement was to introduce an abstract setting divorced from particular problems. In [4] for example, $A$ is a closed subspace of $X \times Y$ and $A_H = A \cap {}^*B$ where ${}^*B$ is the preadjoint of a finite dimensional subspace $B$ in $Y^* \times X^*$. Such a representation is always possible if $H$ is continuous on $G(A)$ and $F$ is finite dimensional. This "subspace" intepretation of $A_H$ leads to a simple construction of $(A \cap {}^*B)^*$ and also to a more complicated solution of the extension problem) (including self-adjoint extensions of symmetric b.v.p.). The results (particularly with respect to the determination of adjoints) are equivalent to those of [3] in the case where $H$ has finite dimentional range. For some applications of this approach see [8]. An eigenfunction expansion theory associated with the problems in [4] can be found in [5]. The theory has also been extensively further developed by Lee [15], [16], and [17].

While the finite dimensional theory is fairly clear, the same cannot be said for the infinite dimensional case. In order to get usable results special hypotheses seem necessary. There is a certain freedom of choice here. Different assumptions can and do lead to different theories, and no theory yet seems able to characterize in a *computationally useful* fashion every extension $C$ between $A$ and $B^*$. For example the efforts of Lee to treat the infinite dimensional case assume that $F$ is $l^2$. This will be true for example when $B^*$ and $A$ are defined in a separable Hilbert space or when $G(B^*)/G(A) \cong l^2$. By means of the theory of Hilbert Besselian bases and a Green's formula similar to our own, Lee is able to solve both the adjoint and extension problem. Similarly both the theories presented in this paper, although useful computationally, hold for restricted classs of problems. In § 3, for example, we must first show that $R(J^*)$ is closed and that $H = DJ$. While the second condition seems natural the first is hard to verify unless $R(J)$ happens to be a Banach space so that the closed range theorem applies. But to check that $R(J)$ is a Banach space and to describe that space can be difficult for even simple $J$ (cf. § 5 Example 3—especially Proposition 5). When $J$ is more complicated—for example defined by a Stieltjes measures—the question is unsolved.

For these reasons the theory presented in § 4 may be more convenient to use. It can be applied directly to many natural extensions. However, it will not describe all extensions unless all have closed range.

At this writing relation between these two theories is unclear. In particular we call attention to the two apparently unrelated descriptions of self-adjoint extensions given by Theorem 3.15 and Theorem 4.8. Obviously further work needs to be done on these issues. It should also be interesting to apply these ideas to difficult examples such as b.v.p. involving partial differential of functional differential operators.

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*Added in proof.* We also refer the reader to the paper of C. Bennewitz, *Spectral theory for pairs of differential operators*, Ark. Mat., 15 (1977), which includes an extension of the classical deficiency index theory for symmetric operators to the setting of linear relations.

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