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A NOTE ON DERIVATIONS WITH POWER CENTRAL VALUES ON A LIE IDEAL

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Let R be a prime ring of characteristic $\neq 2$ with a derivation $d \neq 0$ and a non-central Lie ideal U such that $d(u)^n$ is central, for all $u \in U$. We prove that R must satisfy s_4, the standard identity in 4 variables; hence R is either commutative or an order in a 4-dimensional simple algebra. This result extends a theorem of Herstein to Lie ideals.

In [**2**] Herstein shows that if R is a prime ring with center Z and if $d \neq 0$ is a derivation of R such that $d(x)^n \in Z$, for all $x \in R$, then R satisfies s_4, the standard identity in 4 variables. This theorem indicates that the global structure of a ring is often tightly connected to the behaviour of one of its derivations. The purpose of this note is to show that the same conclusion holds for prime rings of characteristic $\neq 2$ even if we assume only that $d(u)^n$ is central for those u in some non-central Lie ideal.

We will proceed by first proving the result when d is inner. We will then use Kharchenko's theorem on differential identities [**5**] to reduce to the case where d is inner on the Martindale quotient ring of R. By using Kharchenko's theorem, our proof will actually be somewhat simpler than the proof in [**2**].

In all that follows, unless stated otherwise, R will be a prime ring of characteristic $\neq 2$, U a non-central Lie ideal of R, $d \neq 0$ a derivation of R, and $n \geq 1$ a fixed integer such that $d(u)^n$ is central, for all $u \in U$. For any ring S, $Z = Z(S)$ will denote its center. For subsets $A, B \subset R$, $[A, B]$ will be the additive subgroup generated by all $[a, b] = ab - ba$; $a \in A$, $b \in B$. In addition, s_4 will denote the standard identity in 4 variables.

By a result of Herstein [**3**], $U \supset [I, R]$ for some $I \neq 0$, an ideal of R. Therefore, we will assume throughout that $U \supset [I, R]$.

We will also make frequent and important use of the following three results. We do not necessarily state them in their fullest generality.

1. (Carini-Giambruno [**1**].) If $U \not\subset Z(R)$ is a Lie ideal of a prime ring R of characteristic $\neq 2$, and if d is a derivation of R such that $d(u)^n = 0$, for all $u \in U$, then $d = 0$.

2. (Kharchenko [5].) If d is a derivation of a prime ring R and if there exist a_i, b_i, c_i, $e_i \in R$ such that $\sum a_i d(x) b_i = \sum c_i x e_i$, for all x in a non-zero ideal, then either d is inner in the Martindale quotient ring of R or $\sum a_i x b_i = \sum c_i x e_i = 0$ for all x in the ideal.

3. (Herstein-Procesi-Schacher [4].) If R is a prime ring satisfying a polynomial identity and if for all $x, y \in R$ there exists an $n = n(x, y) \geq 1$ such that $[x, y]^n \in Z(R)$, then R satisfies s_4.

We now begin the work necessary to prove our theorem with

LEMMA 1. *If $J \neq 0$ is an ideal of R then $J \cap Z(R) \neq 0$. Furthermore, R_Z, the localization of R at $Z(R)$, is simple with* 1.

Proof. Let $V = [I, J^2]$; it is easy to check that V is a non-central Lie ideal of R and $V \subset U$. Since $d(J^2) \subset Jd(J) + d(J)J \subset J$, we have

$$d(V) = d([I, J^2]) \subset [I, d(J^2)] + [d(I), J^2] \subset J.$$

By the result of Carini and Giambruno [1], there is some $v \in V$, such that $d(v)^n \neq 0$.

Hence $0 \neq d(v)^n \in J \cap Z(R)$.

If $K \neq 0$ is an ideal of R_Z, then $K \cap R$ is a non-zero ideal of R; hence $(K \cap R) \cap Z(R) \neq 0$. Therefore K contains invertible elements of R_Z and so, R_Z is simple with 1.

We can now prove the special case of our result when d is an inner derivation.

THEOREM 2. *Let R be a prime ring of characteristic $\neq 2$ and let $a \in R$, $a \notin Z(R)$ be such that $(au - ua)^n \in Z(R)$, for all $u \in U$, where $U \not\subset Z(R)$ is a Lie ideal of R. Then R satisfies s_4.*

Proof. By Lemma 1 we may localize R at $Z(R)$, and it follows that $(au - ua)^n \in Z(R_Z)$, for all $u \in [R_Z, R_Z]$. Therefore in order to prove that R satisfies s_4, we may assume that R is simple with 1 and $U \supset [R, R]$. Since $a \notin Z$, $[(a(xy - yx) - (xy - yx)a)^n, z]$ is a non-trivial generalized polynomial identity for R. Thus, by a result of Martindale [6], R is primitive with minimal right ideal, whose commuting ring D is a division ring finite dimensional over $Z(R)$. However, since R is simple with 1, R must be artinian; thus $R = D_k$ for some $k \geq 1$.

If $Z(R)$ is infinite, let F be a maximal subfield of D and consider $\overline{R} = R \otimes_{Z(R)} F = F_m$, for some $m \geq 1$.

A Vandermonde determinant argument shows that in $\overline{R}$, $[(a(xy - yx) - (xy - yx)a)^n, z]$ is still an identity. If $Z(R)$ is finite then D is a finite division ring; hence $D = Z(R)$. In either case, to prove that R satisfies s_4 it is enough to consider the case where $R = Z(R)_m$, for some $m \geq 1$, and $(a(xy - yx) - (xy - yx)a)^n \in Z(R)$ for all $x, y \in R$.

R is now the ring of linear transformations over $Z(R)$ of an m-dimensional vector space V. Since $a \notin Z(R)$, we can find some $v \in V$ such that v and va are linearly independent.

It suffices to show that $m \leq 2$, so we suppose not and let $t_1, t_2, \ldots, t_{m-2} \in V$ be such that $T = \{v, va, t_1, \ldots, t_{m-2}\}$ is a basis for V over $Z(R)$.

Let $x \in R$ such that $vax = t_1$ and $tx = 0$, for all other $t \in T$. In addition, let $y \in R$ be such that $t_1 y = v$ and $ty = 0$, for all other $t \in T$. Therefore

$$v(a(xy - yx) - (xy - yx)a) = v \quad \text{and so,}$$
$$v(a(xy - yx) - (xy - yx)a)^n = v.$$

Hence $(a(xy - yx) - (xy - yx)a)^n$ is a non-zero element of $Z(R)$. Thus m equals the rank of $(a(xy - yx) - (xy - yx)a)^n$ as a linear transformation. However, $xy - yx$ has rank 1; therefore $a(xy - yx) - (xy - yx)a$ has rank ≤ 2 and so, $(a(xy - yx) - (xy - yx)a)^n$ also has rank ≤ 2.

Therefore $m \leq 2$ and we have proved that R must satisfy s_4.

At this point we would like to reduce the general case down to the case where d is inner. To do this, in addition to using Kharchenko's theorem [**5**], we need the following lemma which is similar to Lemma 4 of [**2**].

LEMMA 3. *If* $\operatorname{char} R = p > 0$ *and if* $d(Z) \neq 0$, *then* R *satisfies* s_4.

Proof. Let $\gamma \in Z$ be such that $d(\gamma) \neq 0$ and let $K = \{\alpha \in Z \mid d(\alpha) = 0\}$. Since $\operatorname{char} R = p > 0$, for every $\beta \in Z$ we have $\beta^p \in K$. Now, if K were finite, then K would be a field and the quotient field of Z would be algebraic over K. Therefore there would exist an integer $m \geq 1$ such that $\gamma^{p^m} = \gamma$, resulting in the contradiction $0 = d(\gamma^{p^m}) = d(\gamma)$. Hence K is infinite.

In R_Z, let $T = \{d(\gamma)/(\alpha + \gamma) \mid \alpha \in K\}$; by the preceding argument, T is an infinite subset of $Z(R_Z)$. If $u \in [I, R]$ we note that $(\alpha + \gamma)u \in [I, R]$, for all $\alpha \in K$. Hence

$$d((\alpha + \gamma)u)^n = (d(\gamma)u + (\alpha + \gamma)d(u))^n \in Z(R)$$

for all $u \in [I, R]$ and $\alpha \in K$. Dividing by $(\alpha + \gamma)^n$ results in $(\lambda u + d(u))^n \in Z(R_Z)$, for all $\lambda \in T$ and $u \in [I, R]$. We again use a Vandermonde determinant argument, since T is infinite, to see that $u^n \in Z(R_Z) \cap R = Z(R)$.

Thus $[i, r]^n \in Z(R)$, for all $i \in I$, $r \in R$; which implies that R satisfies a polynomial identity and, after localizing at $Z(R)$, we have $[x, y]^n \in Z(R_Z)$, for all $x, y \in R_Z$.

By the result of Herstein-Procesi-Schacher **[4]**, R_Z satisfies s_4, hence R satisfies s_4.

We can now prove the main result of this note.

THEOREM 4. *Let R be a prime ring of characteristic $\neq 2$ with a derivation $d \neq 0$ and a Lie ideal $U \not\subset Z(R)$ such that $d(u)^n \in Z(R)$, for all $u \in U$. Then R satisfies s_4; hence R is commutative or an order in a* 4 *dimensional simple algebra.*

Proof. Suppose d is inner in the Martindale quotient ring of R and is induced by some element q. However, by Lemma 1, the Martindale quotient ring of R is R_Z; hence $q = a/\alpha$, for some $a \in R$, $\alpha \in Z(R)$. Therefore the inner derivation of R induced by a satisfies all the hypotheses of Theorem 2, thus in this case, R satisfies s_4.

As a result, it now suffices to show that d is inner in the Martindale quotient ring of R. Let $x_1, \ldots, x_n \in I$ and $y_1, \ldots, y_n \in R$ and let $z_i = [x_i, y_i]$.

By hypothesis $d(z_1 + \cdots + z_n)^n \in Z(R)$ and, by linearization, we obtain

$$\sum_{\pi \in S_n} d(z_{\pi(1)}) \cdots d(z_{\pi(n)}) \in Z(R),$$

where S_n denotes the symmetric group in n letters.

Expanding each $d(z_i)$ we obtain

$$(*) \qquad \sum_{\pi \in S_n} \left(\left[d(x_{\pi(1)}), y_{\pi(1)} \right] + \left[x_{\pi(1)}, d(y_{\pi(1)}) \right] \right) \cdots \left(\left[d(x_{\pi(n)}), y_{\pi(n)} \right] + \left[x_{\pi(n)}, d(y_{\pi(n)}) \right] \right) \in Z(R).$$

If we multiply out the terms of $(*)$ we will obtain a sum of terms all of whom mention y_1 or $d(y_1)$ exactly once. Therefore, after commuting $(*)$ with any $r \in R$, we obtain an expression of the form

$$\sum a_i y_1 b_i + \sum c_i d(y_1) e_i = 0$$

where a_i, b_i, c_i, e_i are products obtained from r and the x_i, $d(x_i)$, y_i, and $d(y_i)$, but not y_1 or $d(y_1)$. By Kharchenko's result **[5]**, if d is not inner in the Martindale quotient ring then

$$\sum a_i y_1 b_i + \sum c_i y_1 e_i = 0, \quad \text{for all } y_1 \in R.$$

Thus in $(*)$ we may replace all occurrences of $d(y_1)$ by y_1. Similarly, we may sequentially replace each $d(y_i)$ by y_i and then each $d(x_i)$ by x_i to finally obtain

$$(**) \qquad 2^n \sum_{\pi \in S_n} \left[x_{\pi(1)}, y_{\pi(1)}\right] \cdots \left[x_{\pi(n)}, y_{\pi(n)}\right] \in Z(R).$$

Since $\operatorname{char} R \neq 2$, R satisfies a polynomial identity. However, by Lemma 3, if $\operatorname{char} R = p > 0$ and if R fails to satisfy s_4, then $d(Z) = 0$.

But now, by the Skolem-Noether theorem, d is inner on R_Z since $d(Z(R_Z)) = 0$. Thus in this case the proof is complete.

Finally, if $\operatorname{char} R = 0$ then let $x_1 = x_2 = \cdots = x_n$ and $y_1 = y_2 = \cdots = y_n$; therefore $(**)$ becomes $2^n n! [x_1, y_1]^n \in Z(R)$ and so, $[x_1, y_1]^n \in Z(R)$. As in the proof of Lemma 3, by **[4]**, R satisfies s_4, thereby completing the proof.

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