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UNIQUENESS FOR A NONLINEAR ABSTRACT CAUCHY PROBLEM

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Let H be a complex Hilbert space, and let A be a linear, unbounded operator defined on a domain D in H. We show that the Cauchy problem for differential equations and inequalities involving the operator $d^n u/dt^n - Au$ as the principal part have at most one solution. No symmetry conditions are placed on the operator A.

1. Introduction. Let H be a complex Hilbert space and let A be a linear (in general, unbounded) operator defined on a domain D in H. We consider differential inequalities in which the principal part is given by

$$Lu = d^n u/dt^n - Au \tag{1.1}$$

where n is a fixed positive integer and neither symmetry nor semiboundedness conditions are placed on the operator A although there will be restrictions placed on the symmetric and antisymmetric parts of A. Our purpose, in short, is to extend the uniqueness results of Hile and Protter [**5**], where $n = 1, 2$ in (1.1) and A depends on t, to operators L in which n is arbitrary and A is independent of t. Furthermore, we obtain the uniqueness results of [**10**] as a special case. The method employed, developed originally in the study of elliptic equations (see e.g., [**12**]) and later extended to parabolic equations [**8**], is essentially the same as that used by Hile and Protter [**5**]. This same weighted L_2 argument has been employed in other similar contexts where A has been a specific partial differential operator. (See e.g., [**6**, **7**, **8**].)

Levine [**10**], generalizing previous results of Murray [**11**], proved that the only solution of $Lu = 0$ with $u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0$ is the zero function, provided the operator A is either symmetric or antisymmetric. The only other results for the operator L in which $n > 2$ and A is unbounded seem to be those of Fattorini [**2**, **3**] and Fattorini and Radnitz [**4**] who study the equation $Lu = 0$ under complete and incomplete Cauchy data. As Levine [**10**] points out, equations involving L in which A is bounded, or $n \leq 2$ and

A is a semibounded (especially from above) symmetric operator have been and continue to be studied extensively. As for uniqueness results when $n \leq 2$, the most general are those of Hile and Protter **[5]** who extended results of Agmon and Nirenberg **[1]** and some of those of Levine (see e.g., **[9]**).

In this article we consider differential inequalities of the form $(c \geq 0)$

$$\|Lu(t)\|^2 \leq c\left[\omega(t) + \int_0^t \omega(s)\, ds\right] \tag{1.2}$$

where

$$\omega(t) = \sum_{j=0}^{n-1} \left\|u^{(j)}(t)\right\|^2 \qquad \left(u^{(j)}(t) = d^j u(t)/dt^j,\ u^{(0)}(t) = u(t)\right)$$

and

$$\|Lu(t)\|^2 \leq c\left[\mu(t) + \int_0^t \mu(s)\, ds\right] \tag{1.3}$$

where

$$\mu(t) = |(Mu(t), u(t))| + \sum_{j=0}^{n-1} \left\|u^{(j)}(t)\right\|^2$$

and the operator M is the symmetric part of A. For $n = 2$ conditions (1.2) and (1.3) correspond precisely to those of **[5]**. Indeed, the two principal results of this article (Theorems 1 and 2) are those uniqueness results of **[5]**, when $n = 2$, restated for arbitrary n. However, unlike **[5]**, we require the operator A to be independent of t. Although our results are valid for arbitrary n, stronger results are known for $n = 1$. (See Theorem 1 of **[5]**.) It is unknown whether such a strong result can be extended to $n > 1$.

2. Main results. Let H be a complex Hilbert space with inner product $(\ ,\ )$ and norm $\|\ \|$. Let $C_m^*([0, T]; D)$ be the set of $u \in C^{m-1}([0, T]; H) \cap C^m((0, T]; H)$ such that $u^{(m)} \in L_2((0, T); H)$ and $u^{(j)}(t) \in D$ for all $t \in (0, T]$ and for all $j = 0, 1, \ldots, \langle m \rangle$ where $\langle m \rangle = [(m-1)/2] =$ greatest integer in $(m-1)/2$. We note that if $v, w \in C([0, T]; H) \cap C^1((0, T]; H)$ with $v', w' \in L_1((0, T))$ and $(w(0), v(0)) = (w(T), v(T)) = 0$, we have the integration by parts formula

$$\int_0^T (w(t), v'(t))\, dt = -\int_0^T (w'(t), v(t))\, dt. \tag{2.1}$$

We now give the requirements on the operator A which will be needed later.

Condition (*). Let A be a linear, in general unbounded, operator with domain D contained in H satisfying the following: $(D^n \equiv d^n/dt^n)$

(I) $A = M+N$ where M is a symmetric operator (i.e., $(Mx, y) = (x, My)$ for $x, y \in D$) and N is an antisymmetric operator (i.e., (Nx, y)
$= -(x, Ny)$ for $x, y \in D$);

(II) There exist nonnegative constants c_0 and c_1 such that

$$\operatorname{Re}(Mw, Nw) \geq -c_0 |(Mw, w)| - c_1 \|w\|^2, \qquad w \in D;$$

(III) For each $u \in C_n^*([0, T]; D)$, the functions Mu and Nu are in $C^{\langle n\rangle}((0, T]; H)$; and for each $t \in (0, T]$ and $0 \leq j \leq \langle n\rangle$, we have $D^j Mu(t) = M(D^j u(t))$, $D^j Nu(t) = N(D^j u(t))$. Furthermore, the functions $D^{\langle n\rangle} Mu$ and $D^{\langle n\rangle} Nu$ are bounded on $(0, T]$;

(IV) For every $v, w \in C_2^*([0, T]; D)$ for which the functions Mv, Mw, Nv are bounded and continuous on $(0, T]$, the functions $\operatorname{Re}(v'(t), Nv(t))$ and $\operatorname{Re}[(w'(t), Mv(t)) - (v'(t), Mw(t))]$ are differentiable on $(0, T]$ and satisfy

(a) $d/dt \operatorname{Re}(v'(t), Nv(t)) = \operatorname{Re}(v''(t), Nv(t))$,

(b)
$$d/dt \operatorname{Re}[(w'(t), Mv(t)) - (v'(t), Mw(t))] = \operatorname{Re}(w''(t), Mv(t)) - \operatorname{Re}(v''(t), Mw(t))$$
for all $t \in (0, T]$.

In condition (*), we note that (I) and (II) come from **[5]** while (III), for $n = 1, 2$, agrees with the results of **[5]** since in those cases $\langle n\rangle = 0$. In addition, (IV) allows for integration by parts in a manner comparable to inequalities (A) and (B) of **[5**, p. 70].

We now state our two main results. Theorem 1 is a generalization of results from both **[5]** and **[10]**. In particular, for $n = 2$ it coincides with Theorem 3 of **[5]** (when their operator A is independent of t); and for $c = c_0 = c_1 = 0$ (c, c_0 and c_1 come from (1.2) and (II)) it gives the uniqueness result of **[10]**. (See Theorem 3.1 of **[10]**.)

THEOREM 1. *Suppose $u \in C_n^*([0, T]; D)$ satisfies (1.2). In addition, suppose the operator A satisfies condition (*) with $c_0 = 0$. If $u(0) = u'(0) = \cdots = u^{(n-1)}(0) = 0$, then $u \equiv 0$ on $[0, T]$.*

In addition to Theorem 1, we shall also show that solutions of inequality (1.3) having homogeneous Cauchy data must be identically zero provided, that in addition to the operators M and N satisfying (II)–(IV), the symmetric part M satisfies an additional constraint. In particular, we require the operator M to satisfy one of the following semiboundedness conditions: There exists a nonnegative constant c_2 such that

$$(2.2)\qquad (Mv\,,\,v) \le c_2\,\|v\|^2\,,\qquad v\in D,$$

or

$$(2.3)\qquad (Mv\,,\,v) \ge -c_2\,\|v\|^2\,,\qquad v\in D.$$

We now have the following theorem which, when $n=2$, coincides with Theorem 4 of **[5]** (when their operator A is independent of t). Thus the uniqueness results of Hile and Protter **[5]** for $n=2$ generalize nicely to arbitrary n provided the linear operator A is independent
of t.

THEOREM 2. *Suppose* $u\in C_n^*([0\,,T]\,;D)$ *and satisfies* (1.3). *In addition, suppose the operator* A *satisfies condition* (*) *and either* (2.2) *or* (2.3). *If* $u(0)=u'(0)=\cdots=u^{(n-1)}(0)=0$, *then* $u\equiv 0$ *on* $[0\,,T]$.

As mentioned earlier, our method of proof is quite similar to that of **[5]** but must be modified in important ways. Of particular note is the unwieldiness of the weight functions used in **[5]** in the case of arbitrary n. For this reason, we have instead opted for a weight function used previously (see e.g., **[7]**) which is simply a variation on the one introduced by Lees and Protter **[8]** for backward-in-time parabolic inequalities.

Prior to proving the above stated theorems, we need to establish a series of important propositions and lemmas. Indeed, the proof of the theorems themselves are rather anticlimactic once the preliminary lemmas have been established.

DEFINITION. Let $\lambda(t)=t+\eta$ where $\eta\ge 1$ and define the operators B and B^* on $C^n((0\,,T]\,;H)$ as follows:

$$Bu(t)=\lambda^{-k}(t)\,D^n\left[\lambda^k(t)\,u(t)\right],$$
$$B^*u(t)=(-1)^n\,\lambda^k(t)\,D^n\left[\lambda^{-k}(t)\,u(t)\right]$$

where k is an arbitrary positive integer. (Thus B^* is the formal adjoint of B.) Furthermore, we define the operators B_+ and B_- as follows:

$$B_+=(B+B^*)/2\,,\quad B_-=(B-B^*)/2.$$

Thus B_+ and B_- are the symmetric and antisymmetric parts, respectively, of the operator B. In the sequel (as in this definition) the dependence of all these operators on the positive integer k is suppressed for ease of notation.

PROPOSITION 1. *Suppose* $v, w \in C([0, T]; D) \cap C^1((0, T]; H)$ *such that* $v', w' \in L_1((0, T); H)$ *and the functions* Mv, Mw, Nv *are bounded and continuous on* $(0, T]$. *Then the functions* $\mathrm{Re}(Mv(t), w(t))$ *and* $\mathrm{Re}(Nv(t), w(t))$ *are differentiable on* $(0, T]$ *and*

$$d/dt \,\mathrm{Re}\,(Mv(t), w(t)) = \mathrm{Re}\,(Mv(t), w'(t)) + \mathrm{Re}\,(Mw(t), v'(t)), \quad t \in (0, T]; \tag{2.4}$$

and

$$d/dt \,\mathrm{Re}\,(Nv(t), w(t)) = \mathrm{Re}\,(Nv(t), w'(t)) - \mathrm{Re}\,(Nw(t), v'(t)), \quad t \in (0, T]. \tag{2.5}$$

Furthermore, if $v(0) = v(T) = 0$ *or* $w(0) = w(T) = 0$, *then*

$$\int_0^T \mathrm{Re}\,(Mv(t), w'(t))\, dt = -\int_0^T \mathrm{Re}\,(Mw(t), v'(t))\, dt, \tag{2.6}$$

$$\int_0^T \mathrm{Re}\,(Nv(t), w'(t))\, dt = \int_0^T \mathrm{Re}\,(Nw(t), v'(t))\, dt. \tag{2.7}$$

Proof. We prove (2.4) and (2.6) and omit the proof of (2.5) and (2.7) since their proofs are similar. We let $r(t) = \mathrm{Re}(Mv(t), w(t))$ and show that r is differentiable on $(0, T]$. Notice that for h small (if $t = T$, we take $h < 0$)

$$\begin{aligned}
[r(t+h) - r(t)]/h &= \mathrm{Re}\,(M[v(t+h) - v(t)]/h, w(t+h)) \\
&\quad + \mathrm{Re}\,(Mv(t), [w(t+h) - w(t)]/h) \\
&= \mathrm{Re}\,([v(t+h) - v(t)]/h, Mw(t+h)) \\
&\quad + \mathrm{Re}\,(Mv(t), [w(t+h) - w(t)]/h) \\
&= \mathrm{Re}\,(Mw(t+h), [v(t+h) - v(t)]/h) \\
&\quad + \mathrm{Re}\,(Mv(t), [w(t+h) - w(t)]/h).
\end{aligned}$$

Letting $h \to 0$, we obtain (2.4) since v and w are (strongly) differentiable and the function Mw is continuous on $(0, T]$. To prove

(2.6), note that the left side of (2.4) is continuous on $(0, T]$ since the right side is also continuous on $(0, T]$ by virtue of the continuity of v', w', Mv and Mw on $(0, T]$. Thus, for $\varepsilon > 0$, we integrate (2.4) to get

$$\int_\varepsilon^T d/dt \operatorname{Re}(Mv(t), w(t))\, dt = \int_\varepsilon^T \operatorname{Re}\big(Mv(t), w'(t)\big)\, dt + \int_\varepsilon^T \operatorname{Re}\big(Mw(t), v'(t)\big)\, dt$$

which yields

$$\text{(2.8)} \quad \operatorname{Re}(Mv(T), w(T)) - \operatorname{Re}(Mv(\varepsilon), w(\varepsilon)) = \int_\varepsilon^T \operatorname{Re}\big(Mv(t), w'(t)\big)\, dt + \int_\varepsilon^T \operatorname{Re}\big(Mw(t), v'(t)\big)\, dt.$$

Since Mv is bounded in $(0, T]$ and w is continuous on $[0, T]$ with $w(0) = w(T) = 0$ (or $v(0) = v(T) = 0$), the left side (2.8) goes to zero as $\varepsilon \downarrow 0$. Also since Mv and Mw are bounded on $(0, T]$ while v', $w' \in L_1((0, T); H)$, we know the two terms on the right side of (2.4) are integrable on $(0, T)$. Thus, letting $\varepsilon \downarrow 0$ in (2.8) gives

$$0 = \operatorname{Re} \int_0^T \big(Mv(t), w'(t)\big)\, dt + \operatorname{Re} \int_0^T \big(Mw(t), v'(t)\big)\, dt.$$

This completes the proof.

PROPOSITION 2. *Suppose* $z \in C_n^*([0, T]; D)$ *and* $z^{(j)}(0) = z^{(j)}(T) = 0$ *for* $0 \le j \le n-1$. *If the operator* A *satisfies condition* (*) *except possibly* (II), *then*

(2.9)
$$\operatorname{Re} \int_0^T (B_+ z(t), Nz(t))\, dt = \operatorname{Re} \int_0^T (B_- z(t), Mz(t))\, dt = 0.$$

Proof. We prove $\operatorname{Re} \int_0^T (B_+ z(t), Nz(t)) dt = 0$ and omit the proof of the rest of (2.9) because of its similarity to this one. Note firstly that the function $(B_+ z(t), Nz(t))$ is integrable on $(0, T)$ since $|(B_+ z(t), Nz(t))| \le \|B_+ z(t)\| \|Nz(t)\|$ and $B_+ z(t)$ is a linear combination (coefficients being C^∞ functions of t) of z, z', $\dots$, $z^{(n)}$ all of which are integrable (by definition of $C_n^*([0, T]; D)$) on $(0, T)$ while $\|Nz(t)\|$ is bounded. Furthermore, using (III) and applying the integration by parts formula (2.1) $\langle n \rangle$ times, we obtain ($\nu \equiv n - \langle n \rangle$ and all integrals are taken over $[0, T]$)

$$\text{(2.10)}\quad \begin{aligned} \operatorname{Re}\int (Bz, Nz)\,dt &= \operatorname{Re}\int \lambda^{-k}(D^n[\lambda^k z], Nz)\,dt \\ &= \operatorname{Re}\int (D^n[\lambda^k z], N[\lambda^{-k}z])\,dt \\ &= (-1)^{\langle n\rangle}\operatorname{Re}\int (D^\nu[\lambda^k z], D^{\langle n\rangle}N[\lambda^{-k}z])\,dt. \end{aligned}$$

If n is odd, then $\nu = \langle n\rangle + 1$. Using this and (2.7) we get

$$\text{(2.11)}\quad \begin{aligned} &(-1)^{\langle n\rangle}\operatorname{Re}\int \left(D^\nu[\lambda^k z], D^{\langle n\rangle}N[\lambda^{-k}z]\right) dt \\ &= (-1)^{\langle n\rangle}\operatorname{Re}\int \left(\left[D^{\langle n\rangle}[\lambda^k z]\right]', N\left[D^{\langle n\rangle}[\lambda^{-k}z]\right]\right) dt \\ &= (-1)^{\langle n\rangle}\operatorname{Re}\int \left(\left[D^{\langle n\rangle}[\lambda^{-k} z]\right]', N\left[D^{\langle n\rangle}[\lambda^{k}z]\right]\right) dt. \end{aligned}$$

In the last integral in (2.11) we now integrate by parts ($\langle n\rangle$ times) to obtain

$$\text{(2.12)}\quad \begin{aligned} &(-1)^{\langle n\rangle}\operatorname{Re}\int \left(\left[D^{\langle n\rangle}[\lambda^{-k} z]\right]', N\left[D^{\langle n\rangle}[\lambda^{k}z]\right]\right) dt \\ &= \operatorname{Re}\int \left(D^{\langle n\rangle+1}\left[D^{\langle n\rangle}[\lambda^{-k}z]\right], N[\lambda^k z]\right) dt \\ &= \operatorname{Re}\int \lambda^k\left(D^n[\lambda^{-k}z], Nz\right) dt = -\operatorname{Re}\int (B^*z, Nz)\,dt \end{aligned}$$

where the last equality is valid for n odd. Combining (2.10), (2.11) and (2.12), we get

$$\operatorname{Re}\int (Bz, Nz)\,dt = -\operatorname{Re}\int (B^*z, Nz)\,dt$$

from which (2.9) follows for n odd. If n is even, then $\nu = \langle n\rangle + 2$, and we use analysis similar to that of (2.10) and (2.11) to get

$$\text{(2.13)}\quad \begin{aligned} &\operatorname{Re}\int (Bz, Nz)\,dt \\ &= (-1)^{\langle n\rangle}\operatorname{Re}\int \left(\left[D^{\langle n\rangle}[\lambda^k z]\right]'', N\left[D^{\langle n\rangle}[\lambda^{-k}z]\right]\right) dt. \end{aligned}$$

To handle (2.13), we note that by applying (IV), part (a), to $v + w$, we obtain

$$\begin{aligned} &d/dt\operatorname{Re}\left[(w'(t), Nv(t)) + (v'(t), Nw(t))\right] \\ &\quad = \operatorname{Re}(w''(t), Nv(t)) + \operatorname{Re}(v''(t), Nw(t)). \end{aligned}$$

Furthermore, if $w'(0) = w'(T) = v'(0) = v'(T) = 0$, we may integrate to get

$$\operatorname{Re}\int (w''(t)\,,\, Nv(t))\,dt = -\operatorname{Re}\int (v''(t)\,,\, Nw(t))\,dt.$$

Application of this to the right side of (2.13) using

$$w(t) = D^{\langle n\rangle}[\lambda^k(t)z(t)] \quad \text{and} \quad v(t) = D^{\langle n\rangle}[\lambda^{-k}(t)z(t)]$$

yields

$$\begin{aligned}\operatorname{Re}\int &(Bz\,,\, Nz)\,dt \\ &= -(-1)^{\langle n\rangle}\operatorname{Re}\int \left(\left[D^{\langle n\rangle}[\lambda^{-k}z]\right]''\,,\, N\left[D^{\langle n\rangle}[\lambda^k z]\right]\right)\,dt.\end{aligned}$$

Now integrate the last expression by parts $\langle n\rangle$ times to get

$$\begin{aligned}\operatorname{Re}\int (Bz\,,\, Nz)\,dt &= -\operatorname{Re}\int \lambda^k\left(D^{\nu}\left[D^{\langle n\rangle}[\lambda^{-k}z]\right]\,,\, Nz\right)\,dt \\ &= -\operatorname{Re}\int \lambda^k\left(D^n[\lambda^{-k}z],\, Nz\right)\,dt \\ &= -\operatorname{Re}\int (B^*z\,,\, Nz)\,dt\end{aligned}$$

and hence (2.9) follows for n even. This completes the proof.

LEMMA 1. *Let j, n be positive integers such that $1 \le j \le n$. Then*

$$\sum_{i=0}^{2j}(-1)^i\, i\,(i-1)\binom{n}{i}\binom{n}{2j-i} = (-1)^{j+1}\, j\,(n-2j+1)\binom{n}{j}, \tag{2.14}$$

$$\sum_{i=1}^{j}(-1)^{i+1}(2i-1)\binom{n}{j+i-1}\binom{n}{j-i} = j\binom{n}{j} \tag{2.15}$$

where ( ) denotes the binomial coefficient. ($\binom{n}{\alpha}$ is understood to be zero if $\alpha < 0$ or $\alpha > n$. Thus the upper (lower) limit on the summations may be larger (smaller) without altering the result of the lemma.)

Proof. To prove (2.14), we let $a_i = (-1)^i i(i-1)\binom{n}{i}$, $b_i = \binom{n}{i}$ and notice that the left side of (2.14) is precisely $\sum_{i=0}^{2j} a_i b_{2j-i}$ which, by the Cauchy product formula, is the coefficient on x^{2j} in the product $[\sum_{i=0}^{n} a_i x^i][\sum_{i=0}^{n} b_i x^i]$. It is easy to show

$$\sum_{i=1}^{n} a_i x^i = n(n-1)x^2(1-x)^{n-2} \quad \text{and} \quad \sum_{i=1}^{n} b_i x^i = (1+x)^n .$$

Hence

$$\begin{aligned}\sum_{i=0}^{2j}(-1)^i i(i-1)\binom{n}{i}\binom{n}{2j-i} &= \text{coefficient of } x^{2j} \text{ in the expression} \\ &\quad\; n(n-1)x^2(1-x)^{n-2}(1+x)^n .\end{aligned}$$

Since $x^2(1-x)^{n-2}(1+x)^n = (x^2+2x^3+x^4)(1-x^2)^{n-2}$, an elementary calculation produces (2.13).

To prove (2.15), we make the change of variable $(i' = j-i)$ to get (after replacing i' with i)

$$\begin{aligned}(2.16)\quad S &\equiv 2\sum_{i=1}^{j}(-1)^{i+1}(2i-1)\binom{n}{j+i-1}\binom{n}{j-i} \\ &= 2(-1)^{j+1}\sum_{i=0}^{j-1}(-1)^i(2j-2i-1)\binom{n}{2j-i-1}\binom{n}{i}.\end{aligned}$$

Symmetry of the binomial coefficients involved allows us to get

$$S = (-1)^{j+1}\sum_{i=0}^{2j-1}(-1)^i(2j-2i-1)\binom{n}{2j-i-1}\binom{n}{i}.$$

Letting $a_i = (-1)^i d_i$, $b_i = id_i$, $c_i = (-1)^i id_i$ and $d_i = \binom{n}{i}$, we note that

$$(2.17)\qquad S = (-1)^{j+1}(S_1 - S_2)$$

where

$$S_1 = \sum_{i=0}^{2j-1} a_i b_{2j-i-1} = \text{coefficient of } x^{2j-1} \text{ in the product } \left[\sum_{i=0}^{n} a_i x^i\right]\left[\sum_{i=0}^{n} b_i x^i\right]$$

and

$$S_2 = \sum_{i=0}^{2j-1} c_i d_{2j-i-1} = \text{ coefficient of } x^{2j-1} \text{ in}$$
$$\text{the product } \left[\sum_{i=0}^{n} c_i x^i\right]\left[\sum_{i=0}^{n} d_i x^i\right].$$

Since

$$\sum_{i=0}^{n} a_i x^i = (1-x)^n\,, \quad \sum_{i=0}^{n} b_i x^i = nx\,(1+x)^{n-1}\,,$$
$$\sum_{i=0}^{n} c_i x^i = -nx\,(1-x)^{n-1}\,, \quad \sum_{i=0}^{n} d_i x^i = (1+x)^n\,,$$

we get

$$S_1 - S_2 = \text{ coefficient of } x^{2j-1} \text{ in the expression } 2nx(1-x^2)^{n-1}.$$

However $2nx(1-x^2)^{n-1} = -d/dx[(1-x^2)^n]$ and hence

$$S_1 - S_2 = (-2j)\left[\text{ coeff of } x^{2j} \text{ in } (1-x^2)^n\right] = -2j\,(-1)^j \binom{n}{j}.$$

Combining this with (2.16) and (2.17) yields (2.15) and the proof is complete.

LEMMA 2. *Suppose* $v \in C^{m+p}((0,T];H) \cap C^{m+p-1}([0,T];H)$, $v^{(m+p)} \in L_1((0,T);H)$ *and* $v^{(j)}(0) = v^{(j)}(T) = 0$ *for* $0 \le j \le m+p-1$. *Let* s *be a positive integer. Then there exist real numbers* $K_j(p)$ *depending only on* p *and* j *such that* ([ ] = *greatest integer function*)

$$\text{(2.18)} \qquad \operatorname{Re} \int_0^T \lambda^{-s}(t)\left(v^{(m)}(t)\,,\, v^{(m+p)}(t)\right) dt$$
$$= \sum_{j=0}^{[p/2]} K_j(p)\,(s+p-2j-1)!/(s-1)!$$
$$\times \int_0^T \lambda^{-s-p+2j}(t)\left\|v^{(m+j)}(t)\right\|^2 dt$$

in which the constants $K_j(p)$ *satisfy the following conditions*:

(i)

$$K_0(p) = \begin{cases} 1 & \textit{if } p = 0\,, \\ 1/2 & \textit{if } p \ge 1\,, \end{cases}$$

(ii) $K_j(2j) = (-1)^j$ *if* $j \geq 1$,
(iii) $K_j(2j+1) = (-1)^j(2j+1)/2$ *if* $j \geq 0$,
(iv) $K_j(p) = 0$ *for* $p \leq 2j-1$ *if* $j \geq 1$,
(v) $K_j(p) = K_j(p-1) - K_{j-1}(p-2)$ *if* $1 \leq j \leq [(p-1)/2]$.

(*Note: When using this lemma later in this article, only* (i)–(iii) *will be needed. However, it is convenient to state and use* (iv) *and* (v) *in the proofs of* (i)–(iii). *Also note that* (iv) *is a redundancy since it is a statement that the right side of* (2.18), *if written as a sum with upper limit greater than* $[p/2]$, *has zero coefficients for* j *greater than* $[p/2]$.)

Proof (*by induction on* p). For $p = 0$, identity (2.18) is trivially true with $K_0(0) = 1$ and likewise for $p = 1$ it is true with $K_0(1) = 1/2$. Thus suppose the conclusion of the lemma is true for $0, 1, \dots, p$ and for all $s \geq 1$ and $m \geq 0$. For $p+1$ the left side of (2.18) may be integrated by parts to get (all integrals are taken over $[0, T]$)

$$\begin{aligned}\operatorname{Re} &\int \lambda^{-s}(v^{(m)}, v^{(m+p+1)})\,dt \\ &= \operatorname{Re}\int s\lambda^{-s-1}(v^{(m)}, v^{(m+p)})\,dt - \operatorname{Re}\int \lambda^{-s}(v^{(m+1)}, v^{(m+p)})\,dt.\end{aligned}$$

We now apply the induction hypothesis to both integrals on the right side, letting $F(s, p, j) = K_j(p)(s+p-2j-1)!/(s-1)!$, to get

$$\begin{aligned}(2.19)\quad \operatorname{Re} &\int \lambda^{-s}(v^{(m)}, v^{(m+p+1)})\,dt \\ &= \sum_{j=0}^{[p/2]} sF(s+1, p, j)\int \lambda^{-s-p+2j-1}\|v^{(m+j)}\|^2\,dt \\ &\quad - \sum_{j=0}^{[(p-1)/2]} F(s, p-1, j)\int \lambda^{-s-p+2j+1}\|v^{(m+1+j)}\|^2\,dt \\ &= \sum_{j=0}^{[p/2]} sF(s+1, p, j)\int \lambda^{-s-p+2j-1}\|v^{(m+j)}\|^2\,dt \\ &\quad - \sum_{j=1}^{[(p+1)/2]} F(s, p-1, j-1)\int \lambda^{-s-p+2j-1}\|v^{(m+j)}\|^2\,dt.\end{aligned}$$

Note that we have changed the summation index in the last expression. Observe that the right side of (2.19) is exactly

$$\sum_{j=0}^{[(p+1)/2]} F(s, p+1, j)\int \lambda^{-s-p+2j-1}\|v^{(m+j)}\|^2\,dt$$

provided

$$F(s,p+1,j) = \begin{cases} sF(s+1,p,0) & \text{if } j=0, \\ sF(s+1,p,j) - F(s,p-1,j-1) & \text{if } 1 \le j \le [p/2], \\ -F(s,p-1,[(p-1)/2]) & \text{if } j = [(p+1)/2] \\ & \quad > [p/2], \\ 0 & \text{if } j > [(p+1)/2] \ge 1. \end{cases}$$

Hence $F(s,p+1,j) = K_j(p+1)(s+p-2j)!/(s-1)!$ where

$$K_j(p+1) = \begin{cases} 1/2 & \text{if } j=0,\ p=0, \\ K_j(p) & \text{if } j=0,\ p \ge 1, \\ K_j(p) - K_{j-1}(p-1) & \text{if } 1 \le j \le [p/2], \\ -K_{j-1}(p-1) & \text{if } j=[(p+1)/2] > [p/2], \\ 0 & \text{if } j > [(p+1)/2] \ge 1. \end{cases} \tag{2.20}$$

The proofs of (i), (iv) and (v) now follow directly from (2.20). Likewise (ii), using (iv) and (v), and (iii), using (v) and (ii), are easily done using induction. This completes the proof.

Lemma 3. *Suppose $v \in C^m((0,T];H) \cap C^{m-1}([0,T];H)$, $v^{(m)} \in L_2((0,T);H)$ and $v^{(j)}(0) = v^{(j)}(T) = 0$ for $0 \le j \le m-1$. Let $z(t) = \lambda^{-k}(t)v(t)$ where $\lambda(t) = t+\eta$, $\eta \ge 1$. Then there exist constants $p_j(k)$, having polynomial dependence on k with the polynomial coefficients dependent only on j and m, such that*

$$\begin{aligned} k\int_0^T \lambda^{-2}(t)\left\|z^{(m)}(t)\right\|^2 dt &\ge k\int_0^T \lambda^{-2k-2}(t)\left\|v^{(m)}(t)\right\|^2 dt \\ &\quad + \sum_{j=1}^m p_j(k)\int_0^T \lambda^{-2k-2j-2}(t)\left\|v^{(m-j)}(t)\right\|^2 dt. \end{aligned} \tag{2.21}$$

Furthermore, the degree of $p_j(k)$ (in k) is no larger than $2j+1$.

Proof. It is easy to see that (all integrals are taken over $[0, T]$)

$$\begin{aligned}
k \int \lambda^{-2} \|z^{(m)}\|^2 \, dt &= k \int \lambda^{-2} \left\| (\lambda^{-k} v)^{(m)} \right\|^2 dt \\
&= k \int \lambda^{-2} \left\| \lambda^{-k} v^{(m)} + \sum_{j=1}^{m} \binom{m}{j} (\lambda^{-k})^{(j)} v^{(m-j)} \right\|^2 dt \\
&\geq k \int \lambda^{-2k-2} \|v^{(m)}\|^2 \, dt \\
&\quad + 2k \operatorname{Re} \sum_{j=1}^{m} \binom{m}{j} \int \lambda^{-k-2} (\lambda^{-k})^{(j)} (v^{(m)}, v^{(m-j)}) \, dt.
\end{aligned}$$

Denoting the last summation by W, we now do the differentiation of λ^{-k} indicated in the integrand and apply (2.18) to get

$$\begin{aligned}
W &= 2k \operatorname{Re} \sum_{j=1}^{m} \binom{m}{j} (-1)^j \frac{(k+j-1)!}{(k-1)!} \\
&\quad \times \int \lambda^{-2k-j-2} (v^{(m)}, v^{(m-j)}) \, dt \\
&= 2k \sum_{j=1}^{m} \sum_{\alpha=0}^{[j/2]} D(j, k, \alpha) \int \lambda^{-2k-2j+2\alpha-2} \|v^{(m-j+\alpha)}\|^2 \, dt
\end{aligned} \tag{2.22}$$

where

$$\begin{aligned}
D(j, k, \alpha) = \binom{m}{j} (-1)^j \frac{(k+j-1)!}{(k-1)!} K_\alpha(j) \\
\times (2k + 2j - 2\alpha + 1)! / (2k + j + 1)!.
\end{aligned}$$

Letting $\alpha = j - s$ and changing the summation in (2.22) so that we sum over s and j instead of α and j, we get

$$W = 2k \sum_{s=1}^{m} \sum_{j=s}^{2s} D(j, k, j-s) \int \lambda^{-2k-2s-2} \|v^{(m-s)}\|^2 \, dt.$$

Thus (2.21) holds with $p_s(k) = 2k \sum_{j=s}^{2s} D(j, k, j-s)$. It is also clear that $p_j(k)$ is a polynomial in the integer k and in fact

$$\begin{aligned}
&D(j, k, j-s) \\
&\quad = \binom{m}{j} (-1)^j K_{j-s}(j) \frac{(k+j-1)!}{(k-1)!} \frac{(2k+2j-2(j-s)+1)!}{(2k+j+1)!} \\
&\quad = \binom{m}{j} (-1)^j K_{j-s}(j) (k^j + \text{l.d.t.})(k^{2s-j} + \text{l.d.t.}) \propto k^{2s} + \text{l.d.t.}
\end{aligned}$$

Hence $p_s(k)$ has degree not exceeding $2s+1$. This completes the proof.

LEMMA 4. *Suppose* $v \in C_n^*([0, T]; D)$, $v^{(j)}(0) = v^{(j)}(T) = 0$ *for* $0 \le j \le n-1$ *and the operator* A *satisfies condition* (*). *Then, for* $\lambda(t) = t + \eta$, $\eta \ge 1$, *there exists a positive number* $\varepsilon(n, \eta, T)$, *independent of* k, *such that for all* k *sufficiently large*

$$(2.23)\quad 2\int_0^T \lambda^{-2k}(t)\,\|Lv(t)\|^2\,dt \ge \varepsilon(n, \eta, T)\sum_{j=1}^{n} k^{2j-1}\int_0^T \lambda^{-2k-2j}(t)\left\|v^{(n-j)}(t)\right\|^2 dt + 4\,\mathrm{Re}\int_0^T \lambda^{-2k}(t)\,(Mv(t), Nv(t))\,dt.$$

Proof. Let $z(t) = \lambda^{-k}(t)v(t)$ where $\lambda(t) = t + \eta$, $\eta \ge 1$, and note $Lv \in L_2((0, T); H)$ by (II) and the definition of $C_n^*([0, T]; D)$. Then elementary calculations along with (2.9) yields (all integrals are taken over the interval $[0, T]$)

$$\begin{aligned} 2\int \lambda^{-2k}\|Lv\|^2\,dt &= 2\int \|Bz - Az\|^2\,dt \\ &= \int \|(B_+z - Mz) + (B_-z - Nz)\|^2\,dt \\ &\ge 4\,\mathrm{Re}\int ((B_+z - Mz), (B_-z - Nz))\,dt \\ &= 4\,\mathrm{Re}\int (B_+z, B_-z)\,dt + 4\,\mathrm{Re}\int (Mz, Nz)\,dt \\ &= \int \left[\|Bz\|^2 - \|B^*z\|^2\right] dt + 4\,\mathrm{Re}\int \lambda^{-2k}(t)\,(Mv(t), Nv(t))\,dt. \end{aligned}$$

Thus it suffices to show, for k sufficiently large,

$$(2.24)\qquad \int \left[\|Bz\|^2 - \|B^*z\|^2\right] dt \ge \sum_{j=1}^{n} k^{2j-1}\int \lambda^{-2j}\|z^{(n-j)}\|^2\,dt$$

and the existence of a positive number $\varepsilon(n, \eta, T)$, independent of k, such that

$$(2.25)\qquad \sum_{j=1}^{n} k^{2j-1}\int \lambda^{-2j}\|z^{(n-j)}\|^2\,dt \ge \varepsilon(n, \eta, T)\sum_{j=1}^{n} k^{2j-1}\int \lambda^{-2k-2j}\|v^{(n-j)}\|^2\,dt.$$

To prove (2.24), notice that straightforward calculation produces

$$Q \equiv \int \left[\|Bz\|^2 - \|B^*z\|^2\right] dt = \int \left\| \sum_{j=0}^{n} \binom{n}{j} \frac{k!}{(k-j)!} \lambda^{-j} z^{(n-j)} \right\|^2 dt$$
$$- \int \left\| \sum_{j=0}^{n} \binom{n}{j} \frac{(k+j-1)!}{(k-1)!} (-1)^j \lambda^{-j} z^{(n-j)} \right\|^2 dt.$$

Doing the indicated multiplication in the integrands and noticing that the square of the $j = 0$ term adds out, we get

$$\begin{aligned}(2.26) \qquad Q = &\sum_{j=1}^{n} \binom{n}{j}^2 \left\{ \frac{k!}{(k-j)!} \right\}^2 \int \lambda^{-2j} \|z^{(n-j)}\|^2 \, dt \\ &- \sum_{j=1}^{n} \binom{n}{j}^2 \left\{ \frac{(k+j-1)!}{(k-1)!} \right\}^2 \int \lambda^{-2j} \|z^{(n-j)}\|^2 \, dt \\ &+ 2 \sum_{j=1}^{n} \sum_{i=0}^{j-1} \binom{n}{i} \binom{n}{j} \left\{ \frac{k!}{(k-i)!} \right\} \left\{ \frac{k!}{(k-j)!} \right\} \\ &\times \operatorname{Re} \int \lambda^{-i-j} (z^{(n-i)}, z^{(n-j)}) \, dt \\ &- 2 \sum_{j=1}^{n} \sum_{i=0}^{j-1} \binom{n}{i} \binom{n}{j} \frac{(k+i-1)! \, (k+j-1)!}{\{(k-1)!\}^2} (-1)^{i+j} \\ &\times \operatorname{Re} \int \lambda^{-i-j} (z^{(n-i)}, z^{(n-j)}) \, dt.\end{aligned}$$

Since we are interested in what happens for large k, we use the identities

$k!/(k-i)! = k^i - \{i(i-1)/2\} k^{i-1} +$ l.d.t. (lower degree terms in k)

and

$$(k+i-1)!/(k-1)! = k^i + \{i(i-1)/2\} k^{i-1} + \text{ l.d.t.}$$

in (2.26). Letting

$$a_{ij} = \binom{n}{i} \binom{n}{j} \left\{1 - (-1)^{i+j}\right\} \quad \text{and}$$
$$b_{ij} = \left\{1 + (-1)^{i+j}\right\} \binom{n}{i} \binom{n}{j} \{[i(i-1) + j(j-1)]/2\}$$

we get

$$(2.27)\qquad \begin{aligned} Q = & -2\sum_{j=1}^{n}\binom{n}{j}^2\left\{j(j-1)k^{2j-1} + \text{ l.d.t.}\right\} \\ & \times\int \lambda^{-2j}\|z^{(n-j)}\|^2\,dt \\ & + 2\sum_{j=1}^{n}\sum_{i=0}^{j-1}(a_{ij}k^{i+j} - b_{ij}k^{i+j-1} + \text{ l.d.t.}) \\ & \times \operatorname{Re}\int \lambda^{-i-j}(z^{(n-i)},\, z^{(n-j)})\,dt \\ \equiv\ & Q_1 + Q_2 \end{aligned}$$

Now apply (2.18) to Q_2 and let

$$C(i\,,\,j\,,\,\alpha) = [(2j-2\alpha-1)!/(i+j-1)!]K_\alpha(j-i)$$

to get $(J \equiv [(j-i)/2])$

$$\begin{aligned} Q_2 = & 2\sum_{j=1}^{n}\sum_{i=0}^{j-1}\sum_{\alpha=0}^{J}(a_{ij}k^{i+j} - b_{ij}k^{i+j-1} + \text{ l.d.t.}) \\ & \times C(i\,,\,j\,,\,\alpha)\int \lambda^{-2j+2\alpha}\|z^{(n-j+\alpha)}\|^2\,dt. \end{aligned}$$

Interchanging the second and third summations yields

$$(2.28)\qquad \begin{aligned} Q_2 = & 2\sum_{j=1}^{n}\sum_{i=0}^{j-1}(a_{ij}k^{i+j} - b_{ij}k^{i+j-1} + \text{ l.d.t.}) \\ & \times C(i\,,\,j\,,\,0)\int \lambda^{-2j}\|z^{(n-j)}\|^2\,dt \\ & + 2\sum_{j=2}^{n}\sum_{\alpha=1}^{[j/2]}\sum_{i=0}^{j-2\alpha}(a_{ij}k^{i+j} - b_{ij}k^{i+j-1} + \text{ l.d.t.}) \\ & \times C(i\,,\,j\,,\,\alpha)\int \lambda^{-2j+2\alpha}\|z^{(n-j+\alpha)}\|^2\,dt \\ \equiv\ & Q_3 + Q_4. \end{aligned}$$

Making the summation variable change $\alpha = j-s$ in (2.28) and changing the summations over j and α into summations over s and j

gives $(N \equiv \min\{n, 2s\})$

$$Q_4 = 2\sum_{s=1}^{n}\sum_{j=s+1}^{N}\sum_{i=0}^{2s-j}(a_{ij}k^{i+j} - b_{ij}k^{i+j-1} + \text{ l.d.t.}) \times C(i, j, j-s)\int \lambda^{-2s}\|z^{(n-s)}\|^2\,dt. \tag{2.29}$$

Using (2.29) in (2.28) and in turn using (2.28) in (2.27), we get

$$\int\left[\|Bz\|^2 - \|B^*z\|^2\right]dt = \sum_{s=1}^{n} C_s(k)\int \lambda^{-2s}\|z^{(n-s)}\|^2\,dt \tag{2.30}$$

where

$$\begin{aligned} C_s(k) = & -2\binom{n}{s}^2\left\{s(s-1)k^{2s-1} + \text{ l.d.t.}\right\} \\ & + 2\sum_{i=0}^{s-1}(a_{is}k^{i+s} - b_{is}k^{i+s-1} + \text{ l.d.t.})C(i, s, 0) \\ & + 2\sum_{j=s+1}^{N}\sum_{i=0}^{2s-j}(a_{ij}k^{i+j} - b_{ij} + k^{i+j-1}\text{ l.d.t.})C(i, j, j-s). \end{aligned}$$

Rearranging these terms we get

$$C_s(k) = \left[\sum_{j=s+1}^{N} a_{2s-j,j}\right]k^{2s} + \widetilde{C}(s)k^{2s-1} + \text{ l.d.t.} \tag{2.31}$$

where

$$\begin{aligned} \widetilde{C}(s) = & -2\binom{n}{s}^2 s(s-1) + 2a_{s-1,s}C(s-1, s, 0) \\ & + 2\sum_{j=s+1}^{N} a_{2s-j-1,j}C(2s-j-1, j, j-s) \\ & - 2\sum_{j=s+1}^{N} b_{2s-j,j}C(2s-j, j, j-s). \end{aligned}$$

Since

$$\sum_{j=s+1}^{N} a_{2s-j,j} = \sum_{j=s+1}^{N}\binom{n}{2s-j}\binom{n}{j}\left\{1-(-1)^{2s}\right\} = 0, \tag{2.32}$$

it is necessary for us to evaluate $\widetilde{C}(s)$. Using the definitions of a_{ij}, b_{ij} and $C(i, j, \alpha)$ and the properties of $K_j(p)$ given in Lemma 2, it is easy to see that

$$\begin{aligned}\widetilde{C}(s) = & -2s(s-1)\binom{n}{s}^2 \\ & + 2(2s-1)\sum_{j=s}^{N}(-1)^{j-s}(2j-2s+1)\binom{n}{j}\binom{n}{2s-j-1} \\ & - 2\sum_{j=s+1}^{N}(-1)^{j-s}\binom{n}{j}\binom{n}{2s-j} \\ & \times\{(2s-j)(2s-j-1)+j(j-1)\} \\ \equiv & R_1 + R_2 + R_3.\end{aligned}$$

The expressions R_1 and R_3 can be combined to get (remembering $\binom{n}{j} = 0$ if $j > n$)

$$R_1 + R_3 = -2\sum_{j=0}^{2s}(-1)^{j-s} j(j-1)\binom{n}{j}\binom{n}{2s-j}$$

and hence (2.14) produces $R_1 + R_3 = 2s(n-2s+1)\binom{n}{s}$. Changing the summation variable in R_2 $(j' = j - s + 1)$ yields

$$R_2 = 2(2s-1)\sum_{j=1}^{s}(-1)^{j+1}(2j-1)\binom{n}{s-j}\binom{n}{s+j-1}$$

and thus (2.15) gives $R_2 = 2s(2s-1)\binom{n}{s}$. Hence

$$\begin{aligned}\widetilde{C}(s) &= 2s(n-2s+1)\binom{n}{s} + 2s(2s-1)\binom{n}{s} \\ &= 2ns\binom{n}{s} \geq 2 \quad \text{for } 1 \leq s \leq n\end{aligned}$$

which, along with (2.30), (2.31), and (2.32), yields (2.24) for k sufficiently large.

To prove (2.25), we use induction. If $n = 1$, then $\varepsilon(1, \eta, T) = 1$ and equality holds in (2.25). Thus suppose (2.25) holds for $n = m$. For $n = m + 1$, the left side of inequality (2.25) may be rewritten using a change in the summation index as

$$\begin{aligned}
&\sum_{j=1}^{m+1} k^{2j-1} \int \lambda^{-2j} \|z^{(m+1-j)}\|^2 \, dt \\
&\quad = k \int \lambda^{-2} \|z^{(m)}\|^2 \, dt + \sum_{j=2}^{m+1} k^{2j-1} \int \lambda^{-2j} \|z^{(m+1-j)}\|^2 \, dt \\
&\quad = k \int \lambda^{-2} \|z^{(m)}\|^2 \, dt + \sum_{j=1}^{m} k^{2j+1} \int \lambda^{-2j-2} \|z^{(m-j)}\|^2 \, dt \\
&\quad \equiv Y_1 + Y_2.
\end{aligned}$$

Using the induction hypothesis along with the estimates $\lambda^{-2} \geq (T+\eta)^{-2}$ and $\lambda^2 \geq \eta^2$, we get

$$\begin{aligned}
Y_2 &\geq k^2 (T+\eta)^{-2} \varepsilon(m) \sum_{j=1}^{m} k^{2j-1} \int \lambda^{-2k-2j} \|v^{(m-j)}\|^2 \, dt \\
&\geq \eta^2 (T+\eta)^{-2} \varepsilon(m) \sum_{j=1}^{m} k^{2j+1} \int \lambda^{-2k-2j-2} \|v^{(m-j)}\|^2 \, dt.
\end{aligned} \tag{2.33}$$

Choose $\delta > 0$ (independent of k) so that, for k sufficiently large and $1 \leq j \leq m$,

$$\eta^2 (T+\eta)^{-2} \varepsilon(m) k^{2j+1} + \delta p_j(k) \geq \delta k^{2j+1} \tag{2.34}$$

where $p_j(k)$ comes from Lemma 3. Now using (2.34) in (2.33) and applying (2.21) to Y_1, we get

$$\begin{aligned}
\delta Y_1 + Y_2 &\geq \delta k \int \lambda^{-2k-2} \|v^{(m)}\|^2 \, dt \\
&\quad + \sum_{j=1}^{m} \left\{ \eta^2 (T+\eta)^{-2} \varepsilon(m) k^{2j+1} + \delta p_j(k) \right\} \\
&\quad \times \int \lambda^{-2k-2j-2} \|v^{(m-j)}\|^2 \, dt \\
&\geq \delta \sum_{j=0}^{m} k^{2j+1} \int \lambda^{-2k-2j-2} \|v^{(m-j)}\|^2 \, dt.
\end{aligned}$$

Changing the last expression so that the summation limits are 1 and $m+1$ produces (2.25) for $n = m+1$. This completes the proof.

Proof of Theorem 1. Choose $t_1 \in (0, T)$ and we shall show that $u = 0$ on $[0, t_1]$ which proves the theorem. Thus choose $0 < t_1 < t_2 < t_3 < T$ and let ζ be a real-valued infinitely differentiable function such that $\zeta(t) = 1$ for $t \in [0, t_2]$, $\zeta(t) = 0$ for $t \in [t_3, T]$ and $0 \le \zeta(t) \le 1$ for $t \in [t_2, t_3]$. Let $v(t) = \zeta(t)u(t)$ and notice that v satisfies the hypothesis of Lemma 4. It is easy to see that

$$\begin{aligned} 2(t_2+\eta)^{-2k} \int_{t_2}^{t_3} \|Lv\|^2 \, dt &\ge 2 \int_{t_2}^{t_3} \lambda^{-2k} \|Lv\|^2 \, dt \\ &= 2 \int_0^T \lambda^{-2k} \|Lv\|^2 \, dt - 2 \int_0^{t_2} \lambda^{-2k} \|Lu\|^2 \, dt. \end{aligned}$$

Application of (2.23) and (1.2) yields, for k sufficiently large,

$$\begin{aligned} (2.35) \qquad & 2(t_2+\eta)^{-2k} \int_{t_2}^{t_3} \|Lv\|^2 \, dt \\ & \ge \varepsilon(n, \eta, T) \sum_{j=1}^{n} k^{2j-1} \int_0^T \lambda^{-2k-2j} \|v^{(n-j)}\|^2 \, dt \\ & + 4 \operatorname{Re} \int_0^T \lambda^{-2k} (Mv, Nv) \, dt \\ & - 2c \int_0^{t_2} \lambda^{-2k} \left[\omega(t) + \int_0^t \omega(s) \, ds \right] dt. \end{aligned}$$

Since $T + \eta > 1$ and thus $(T+\eta)^{-2j} \ge (T+\eta)^{-2n}$, the first term on the right side of (2.35) admits the estimate

$$\begin{aligned} & \sum_{j=1}^{n} k^{2j-1} \int_0^T \lambda^{-2k-2j} \|v^{(n-j)}\|^2 \, dt \\ & \qquad \ge k(T+\eta)^{-2n} \int_0^T \lambda^{-2k} \sum_{j=1}^{n} \|v^{(n-j)}\|^2 \, dt. \end{aligned}$$

Using this and (II) (with $c_0 = 0$) in (2.35) gives, for k sufficiently large,

$$(2.36)\quad \begin{aligned} &2(t_2+\eta)^{-2k}\int_{t_2}^{t_3}\|Lv\|^2\,dt \\ &\geq \left\{\varepsilon(n,\eta,T)\,k\,(T+\eta)^{-2n}-4c_1\right\} \\ &\quad\times\int_0^T \lambda^{-2k}\sum_{j=1}^{n}\|v^{(n-j)}\|^2\,dt \\ &\quad-2c\int_0^{t_2}\lambda^{-2k}\left[\omega(t)+\int_0^t\omega(s)\,ds\right]dt. \\ &\geq \left\{\varepsilon(n,\eta,T)\,k\,(T+\eta)^{-2n}-4c_1\right\}\int_0^{t_2}\lambda^{-2k}\omega(t)\,dt \\ &\quad-2c\int_0^{t_2}\lambda^{-2k}\left[\omega(t)+\int_0^t\omega(s)\,ds\right]dt. \end{aligned}$$

Since

$$\begin{aligned} \int_0^{t_2}\lambda^{-2k}(t)\int_0^t\omega(s)\,ds\,dt &\leq \int_0^{t_2}\int_0^t\lambda^{-2k}(s)\,\omega(s)\,ds\,dt \\ &\leq t_2\int_0^{t_2}\lambda^{-2k}(s)\,\omega(s)\,ds, \end{aligned}$$

for k sufficiently large, inequality (2.36) can be simplified to get ($\beta_k \equiv \varepsilon(n,\eta,T)k(T+\eta)^{-2n}-4c_1-2c-2ct_2$)

$$(2.37)\quad \begin{aligned} 2(t_2+\eta)^{-2k}\int_{t_2}^{t_3}\|Lv\|^2\,dt &\geq \beta_k\int_0^{t_2}\lambda^{-2k}\omega(t)\,dt \\ &\geq \beta_k\int_0^{t_1}\lambda^{-2k}\omega(t)\,dt. \end{aligned}$$

Since $\lambda^{-2k}(t) \geq (t_1+\eta)^{-2k}$ for $t \in [0, t_1]$, it is easy to see that inequality (2.37) may be manipulated to produce

$$(2.38)\quad 2(\beta_k)^{-1}\{(t_1+\eta)/(t_2+\eta)\}^{2k}\int_{t_2}^{t_3}\|Lv\|^2\,dt \geq \int_0^{t_1}\omega(t)\,dt.$$

Letting $k \to \infty$ in (2.38), it is clear that the left side of (2.38) approaches zero and thus the right side, which is independent of k, must be identically zero. This completes the proof.

LEMMA 5. *Suppose the hypothesis of Lemma 4 holds and the operator M satisfies either (2.2) or (2.3). Then there exists a positive number $\hat{\varepsilon}(n, \eta, T)$, independent of k, such that*

$$(2.39)\; 3\int_0^T \lambda^{-2k}(t)\,\|Lv(t)\|^2\,dt \geq \hat{\varepsilon}(n, \eta, T)\sum_{j=1}^{n} k^{2j-1}\int_0^T \lambda^{-2k-2j}(t)\left\|v^{(n-j)}(t)\right\|^2 dt + k^{1/2}\int_0^T \lambda^{-2k}(t)\,|(Mv(t), v(t))|\,dt + 4\,\mathrm{Re}\int_0^T \lambda^{-2k}(t)\,(Mv(t), Nv(t))\,dt$$

for all k sufficiently large.

Proof. For $n = 1$ a stronger result holds and is easily proven as in [5]. (See Lemma 1 of [5].) Thus we omit that part of the proof and assume $n \geq 2$. As in the proof of [5; Theorem 4], we note that if either (2.2) or (2.3) holds, then (all integrals are taken over $[0, T]$)

$$(2.40)\qquad \int \lambda^{-2k}\,|(Mv, v)|\,dt \leq \left|\int \lambda^{-2k}(Mv, v)\,dt\right| + 2c_2\int \lambda^{-2k}\,\|v\|^2\,dt.$$

Also note that

$$(2.41)\quad \begin{aligned} k^{1/2}\int \lambda^{-2k}(Mv, v)\,dt &= k^{1/2}\,\mathrm{Re}\int \lambda^{-2k}(-Lv + v^{(n)} - Nv, v)\,dt \\ &= -k^{1/2}\,\mathrm{Re}\int \lambda^{-2k}(Lv, v)\,dt + k^{1/2}\,\mathrm{Re}\int \lambda^{-2k}(v^{(n)}, v)\,dt \\ &\equiv J_1 + J_2. \end{aligned}$$

An elementary estimate gives

$$(2.42)\qquad |J_1| \leq \int \lambda^{-2k}\,\|Lv\|^2\,dt + (k/4)\int \lambda^{-2k}\,\|v\|^2\,dt.$$

We use (2.18) and the identity $(2k+n-2j-1)!/(2k-1)! = (2k)^{n-2j} + \text{l.d.t.}$ to get

$$\text{(2.43)} \qquad |J_2| = \left| k^{1/2} \sum_{j=0}^{[n/2]} K_j(n) \frac{(2k+n-2j-1)!}{(2k-1)!} \times \int \lambda^{-2k-n+2j} \|v^{(j)}\|^2 \, dt \right|$$
$$\leq \sigma \sum_{j=0}^{[n/2]} k^{n-2j+1/2} \int \lambda^{-2k-n+2j} \|v^{(j)}\|^2 \, dt$$
$$\leq \sigma \sum_{j=0}^{n-1} k^{n-2j+1/2} \int \lambda^{-2k-n+2j} \|v^{(j)}\|^2 \, dt$$

where the positive constant σ depends only on n. Now substitute (2.42) and (2.43) into (2.41) to obtain,

$$\text{(2.44)} \qquad k^{1/2} \left| \int \lambda^{-2k} (Mv, v) \, dt \right| \leq \int \lambda^{-2k} \|Lv\|^2 \, dt + \sigma \sum_{j=0}^{n-1} k^{n-2j+1/2} \int \lambda^{-2k-n+2j} \|v^{(j)}\|^2 \, dt$$

for k sufficiently large. (We have absorbed the last term in (2.42) into the last term in (2.44).) Substitution of (2.44) into (2.40) gives

$$\text{(2.45)} \qquad k^{1/2} \int \lambda^{-2k} |(Mv, v)| \, dt \leq \int \lambda^{-2k} \|Lv\|^2 \, dt + 2c_2 k^{1/2} \int \lambda^{-2k} \|v\|^2 \, dt + \sigma \sum_{j=0}^{n-1} k^{n-2j+1/2} \int \lambda^{-2k-n+2j} \|v^{(j)}\|^2 \, dt.$$

Changing the summation index in the summation in (2.45) and then adding (2.45) to (2.23) produces

$$
\begin{aligned}
(2.46)\quad & 3\int \lambda^{-2k}\,\|Lv\|^2\,dt \\
&\geq \varepsilon(n,\eta,T)\sum_{j=1}^{n} k^{2j-1}\int \lambda^{-2k-2j}\|v^{(n-j)}\|^2\,dt \\
&\quad - 2c_2k^{1/2}\int \lambda^{-2k}\,\|v\|^2\,dt \\
&\quad - \sigma\sum_{j=1}^{n} k^{2j-n+1/2}\int \lambda^{-2k+n-2j}\|v^{(n-j)}\|^2\,dt \\
&\quad + k^{1/2}\int \lambda^{-2k}\,|(Mv,v)|\,dt + 4\,\mathrm{Re}\int \lambda^{-2k}\,(Mv,Nv)\,dt.
\end{aligned}
$$

Since $2j-n+1/2 < 2j-1$ for $n \geq 2$ and $\lambda^n(t) \leq (T+\eta)^n$ for $t \in [0,T]$, it is clear that the summations (2.46) may be combined to get the summation on the right side of (2.39) for k sufficiently large with $\hat{\varepsilon}$ depending only on n, T, η, c_2 and σ; in particular, $\hat{\varepsilon}$ is independent of k. This completes the proof.

Proof of Theorem 2. (Since the proof of Theorem 2 is virtually identical to that of Theorem 1 (with the only difference being that Lemma 5 is used instead of Lemma 4), we omit its proof.)

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