TOWARDS THE IDENTIFICATION OF ORDINARY DIFFERENTIAL EQUATIONS FROM MEASUREMENTS

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Summary: The identification problem of estimating certain functions in a system of linear ordinary differential equations from measured data of its state is considered. The approach consists in an imbedding of the problem into a family of parameter-dependent problems which can be solved at least numerically. The corresponding solutions are proved to converge to the unknown functions as the parameters tend to infinity. Stability results with respect to disturbances in the measurements and the initial data are developed as well. The method is applied to determine mass exchange rates in a compartmental system of pharmaco-kinetic models.

1. INTRODUCTION

In this paper we deal with the inverse problem of determining an unknown matrix $A^*(x)$ in the ordinary differential equation

$$\frac{d}{dx}u^*(x) + A^*(x)u^*(x) = f^*(x), \qquad x \in [0,1],$$

where the solution $u^*(x)$ is known (sometimes, only certain of its components). Problems of this type are treated in the description of the kinematic behaviour of a compartmental system considered in pharmaco-kinetics, biology and medicine.

A compartment is understood as a quantity of material which kinematically behaves in a characteristic and homogeneous way. It may not coincide with a physiologically realizable region of space. A compartmental system consists of interconnected compartments which exchange material either by physical transport or by chemical reaction. A compartmental system is therefore characterized by compartments and intercompartmental relations which can be described using graphs; e.g.:

$$C_i \underset{k_{ji}}{\overset{k_{ij}}{\rightleftarrows}} C_j \overset{k_{j\ell}}{\rightleftarrows} C_\ell$$

Let q_i , m_i , v_i , f_{oi} , f_{ji} be the amount of material present in compartment i , the production of material in compartment i , the rate of material entering compartment i from outside, the excretion flow from the i–th compartment, the transfer flow from compartment j to compartment i , respectively. A simple mass balance yields the following differential equation:

$$(1.1) \qquad \dot{q}_i = \sum_{\substack{j \neq i \\ j \neq 0}} f_{ji} + m_i + v_i - \sum_{\substack{j \neq i \\ j \neq 0}} f_{ij} - f_{oi} ,$$

where all the quantities may depend on times and state. Dot denotes the time derivative. The classical compartment theory assumes linearity and time invariance of the system equations (1.1). Hence

$$f_{ij} = k_{ij}\, q_i$$

holds for some (unknown) constants k_{ij} . If the mass production is negligible and the change in compartment i caused by v_i is small, then the following system of differential equation holds:

$$(1.2) \qquad \dot{u}_i = \sum_{\substack{j \neq i \\ j \neq 0}} k_{ji}\, u_j - \sum_{\substack{j \neq i \\ j \neq 0}} k_{ij}\, u_i - k_{oi}\, u_i + v_i .$$

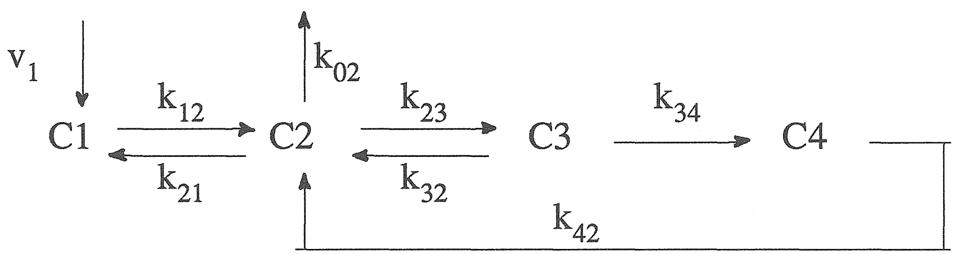
where $u_i := q_i - q_i^s$ with q_i^s denoting the material in compartment i in the stationary state. In matrix notation, (1.2) can be rewritten as

$$(1.3) \qquad \dot{u} = Au + Bv ,$$

where A , B are matrices, u is the state vector and v is an input vector. The matrix A contains the unknown coefficients k_{ij} , which are to be determined by a suitably chosen input–output experiment. More precisely, the problem becomes: *determine* A *from the output data* w *of system* (1.3), *which corresponds to a given input function* v ; namely, (1.4) $$w = Cu ,$$

where the input matrix B , as well as the output matrix C , is known.

As an example consider the ferrokinematics in a human body. The compartmental system consists of four compartments which represent extravascular space (compartment 1), plasma (compartment 2), bone marrow (compartment 3) and red blood cells (compartment 4). The following graph describes the dynamical behaviour of the system:



How to plan the input–output experiment in order to get full information on A is a question of system theory which is not the object of our paper. We take the standpoint of a numerical analyst who knows measured data w from a fixed experiment and wants to compute the matrix A using suitable adapted schemes. The common numerical algorithms for computing A are based on optimizing, with respect to the elements of A , the (nonlinear) L_2–fits between the measurements and the data generated by a specific solution of (1.3). It is well known that nonlinear L_2–fits can encounter certain numerical difficulties such as instability, poor convergence, and inadequate step–length control. Therefore, we propose a numerical scheme which avoids the use of minimization procedures and make it possible to compute A directly by solving a system of partial differential equations, and by considering the asymptotically stable steady state of its solution.

The details of our paper are organized as follows. In Section 2, we present the system of partial differential equations which will be shown to be the starting point of our numerical algorithm. An energy estimate for the solution is derived. In Section 3, convergence properties are analysed for the full system as well as for its finite Galerkin approximations, including the limit case when the dimensions of the Galerkin spaces tend to infinity. Section 4 contains a stability analysis with respect to disturbances in the measurements and input, as well as the initial conditions for the partial differential equations. The paper is based on the ideas contained in [1], but the results are more comprehensive.

2. THE PROBLEM AND ITS IMBEDDING

In this section we assume that the solution u^* and the right hand side f^* of the system equation are known exactly.

Let $u^* \in H^1(0,1)$ and $f^* \in L^2(0,1)$ be given. The problem is to find a positive definite $m \times m$ matrix A^* with $L^2(0,1)$ elements (written $A^* \in L^2(0,1)$) which satisfies the differential equation

$$\text{(2.1)} \qquad \frac{d}{dx} u^*(x) + A^*(x)u^*(x) = f^* \quad \text{a.e. in } (0,1) .$$

In this connection a matrix $A^* \in L^2(0,1)$ is called positive definite if there is a global constant $\gamma > 0$ such that

$$\text{(2.2)} \qquad w^T A^*(x)\, w \geq \gamma |w|^2$$

holds for all $w \in \mathbb{R}^m$ and a.e. in $(0,1)$. Equation (2.1) is a generalization of the differential equation governing the compartmental system where the matrix elements are assumed to be constant. Since we are not interested in the existence of matrices A^* satisfying (2.1), we will always assume that the following condition (A1) is satisfied:

$$\text{(A1)} \qquad S := \left\{ A^* \in L^2(0,1) \;\middle|\; \frac{d}{dx} u^* + A^*u^* = f^* \right\}$$

contains at least one element of A^* .

For the construction of the partial differential equation mentioned above, $A^*(x)$ is assumed to be embedded into a family of parameter-dependent matrices $A(x,t)$ which satisfy the following system of initial boundary value problems of parabolic type:

(2.3) $$\frac{\partial}{\partial t} u(x,t) + \frac{\partial}{\partial x} u(x,t) + A(x,t)u(x,t) = f^*(x)\,, \quad (x,t) \subset (0,1) \times \mathbb{R}_+$$

$$u(x,0) = u^o(x) \ \text{ a.e. in } (0,1)\,,$$

$$\left.\begin{aligned} u(0,t) &= u^*(0) \\ u(1,t) &= u^*(1) \end{aligned}\right\} \text{ for } t \geq 0\,.$$

(2.4) $$\frac{\partial}{\partial t} A(x,t) = (u - u^*)(x,t)\, u(x,t)^T\,, \quad (x,t) \in (0,1) \times \mathbb{R}_+\,,$$

$$A(x,0) = A^o(x) \ \text{ a.e. in } (0,1)\,.$$

The initial values $u^o \in H^1(0,1)$ and $A^o \in L^2(0,1)$ can be chosen arbitrarily. The aim is to study the limits of $u(x,t)$ and $A(x,t)$ as t tends to infinity. Initially, we derive an *a priori* estimate for (u,A) satisfying (2.3), (2.4).

Let us define $w: = u - u^*$ and $R: = A - A^*$. From (2.1) and (2.3) one obtains

(2.5) $$\frac{\partial}{\partial t} w(x,t) + \frac{\partial}{\partial x} w(x,t) + A^*(x,t)w(x,t) + R(x,t)u(x,t) = \theta\,,$$

$$w(x,0) = w^o(x) \ \text{ in } (0,1)\,,$$

$$w(0,t) = w(1,t) = \theta \ \text{ for } t \geq 0\,.$$

Multiplying (2.5) by $w(x,t) \in \mathbb{R}^m$ leads to

(2.6) $$\frac{1}{2}\left[\frac{\partial}{\partial t}\,|w(x,t)|^2 + \frac{\partial}{\partial x}\,|w(x,t)|^2\right] + w(x,t)^T A^*(x)w(x,t) + + \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(x,t)\, R_{ij}(x,t)u_j(x,t) = 0$$

a.e. in $(0,1)$ and $t \geq 0$, where $R(x,t) = (R_{ij}(x,t))_{i,j=1,\dots,m}$. The matrix relation multiplied by $R_{ij}(x,t)$ elementwise and summed up over i and j gives

(2.7) $$\sum_{i=1}^{m}\sum_{j=1}^{m} \frac{1}{2}\frac{\partial}{\partial t}(R_{ij}(x,t))^2 - \sum_{i=1}^{m}\sum_{j=1}^{m} w_i(x,t)R_{ij}(x,t)u_j(x,t) = 0\,, \quad (x,t) \in (0,1) \times \mathbb{R}_+$$

where $\frac{\partial}{\partial t} A(x,t) = \frac{\partial}{\partial t} R(x,t)$ was used. Introducing the Erhard–Schmid matrixnorm $\|\cdot\|_{ES}$ and adding the formulas (2.6) and (2.7) one obtains

$$(2.8) \qquad \frac{1}{2}\left[\frac{\partial}{\partial t} |w(x,t)|^2 + \frac{\partial}{\partial x} |w(x,t)|^2 + \frac{\partial}{\partial t} \|R(x,t)\|_{ES}^2\right] =$$

$$= -w(x,t)^T A^*(x) w(x,t), \quad \text{a.e. in } (0,1), \quad t \geq 0 .$$

Hence

$$(2.9) \qquad \frac{1}{2}\frac{\partial}{\partial t} \int_0^1 \left[|w(x,t)|^2 + \|R(x,t)\|_{ES}^2\right] dx +$$

$$+ \frac{1}{2}(|w(1,t)|^2 - |w(0,t)|^2 + \int_0^1 w(x,t)^T A^*(x) w(x,t)\, dx = 0 .$$

Integrating (2.9) over t and using $w(1,t) = w(0,t) = \theta$ leads to

$$(2.10) \quad \frac{1}{2}(\|w(t)\|^2 - \|w(0)\|^2 + \|R(t)\|^2 - \|R(0)\|^2) + \int_0^t \int_0^1 w(x,\tau)^T A^*(x) w(x,\tau)\, dx\, d\tau = 0 ,$$

where

$$\|w(t)\|^2 := \int_0^1 |w(x,t)|^2 dx \quad \text{and} \quad \|R(t)\|^2 := \int_0^1 \|R(x,t)\|_{ES}^2 dx$$

was introduced. Since $A^*(x)$ is positive definite uniformly in x one obtains finally the following *a priori* estimate:

$$(2.11) \qquad \sup_{t \geq 0} \{\|w(t)\|^2 + \|R(t)\|^2\} + \int_0^\infty \|w(\tau)\|^2 d\tau \leq C$$

for some constant $C < \infty$. This estimate will appear as a key in the convergence proof presented in the next section.

3. CONVERGENCE RESULTS

In this paper we shall ignore existence and uniqueness questions in solving (2.3), (2.4) and assume that the following condition (A2) always holds:

(A2) Equations (2.3), (2.4) have a unique global solution

$$u \in H^1(0,T; H^1(0,1)), \quad A \in H^1(0,T; L^2(0,1)) \text{ for all } T > 0 .$$

THEOREM 1: *Let* (A1) *and* (A2) *be satisfied and* $\{t_n\}$ *be any sequence in* $\mathbb{R}_+$ *with* $t_n \to \infty$.

Then, there exists a subsequence $\{t_{n_k}\}$ *of* $\{t_n\}$ *such that* $A\{t_{n_k}\}$ *converges weakly in* $L^2(0,1)$ *to a matrix* $A_\infty \in S$.

PROOF. Let $\{t_n\}$, $t_n \to \infty$, be any sequence. By (A1) there exists a positive definite matrix A^* whch solves (2.1). For $w := u - u^*$ and $R := A - A^*$ the *a priori* estimate (2.11) holds. This implies that $\{A(t_n)\}$ is bounded in $L^2(0,1)$. Hence, a subsequence of $\{t_n\}$ exists which we denote by $\{t_n\}$ again, with $A(t_n) \to A_\infty \in L^2(0,1)$ weakly in $L^2(0,1)$. It remains to show that $A_\infty \in S$, that means that

$$\frac{\partial}{\partial x} u^*(x) + A_\infty(x)u^* = f^*(x)\,, \quad \text{a.e. in } (0,1)\,.$$

From equation (2.4) using (2.11) one concludes

$$\left\| \frac{\partial}{\partial t} R(t) \right\|_{L^1(0,1)} \le C\|w(t)\| \tag{3.1}$$

for $t \ge 0$. Hence,

$$\begin{aligned} \sup_{|s|\le 1} \|A_{ij}(t_n+s) - A_{ij}(t_n)\|_{L^1(0,1)} &= \sup_{|s|\ge 1} \int_0^1 \left| \int_{t_n}^{t_n+s} \dot{A}_{ij}(x,t)\,dt \right| dx \le \\ &\le \int_0^1 \int_{t_n-1}^{t_n+1} |\dot{A}_{ij}(x,t)|\,dt\,dx \le C \left| \int_{t_n-1}^{t_n+1} \|w(t)\|^2\,dt \right|^{1/2} \end{aligned} \tag{3.2}$$

is valid for all elements A_{ij} of the matrix A. But the right hand side of (3.2) tends to zero as $n \to \infty$. Therefore

$$\sup_{|s|\le 1} \|A(t_n+s) - A(t_n)\|_{L^1(0,1)} \to 0 \quad \text{for } n \to \infty\,. \tag{3.3}$$

Now let $\phi \in H^1(0,1)$ be any fixed test function. Then

$$<\dot{w}(t),\phi> + \left\langle \frac{\partial}{\partial x} u(t),\phi \right\rangle + <A(t)u(t),\phi> = <f^*,\phi> \tag{3.4}$$

holds, where $<\cdot,\cdot>$ is the inner product in $L^2(0,1)$. Integrating equation (3.4) over t from t_n+s-1 to t_n+s gives

$$(3.5)\quad <w(t_n+s) - w(t_n+s-1),\phi> + \int_{t_n+s-1}^{t_n+s} \left\langle \frac{\partial}{\partial x} u(t),\phi \right\rangle dt + \int_{t_n+s-1}^{t_n+s} <A(t)u(t),\phi> = <f^*,\phi> .$$

The second term on the left hand side converges to $\left\langle \frac{\partial u^*}{\partial x},\phi \right\rangle$ uniformly in s as $n \to \infty$:

$$\int_{t_n+s-1}^{t_n+s} \left\langle \frac{\partial}{\partial x} u(t) - \frac{\partial}{\partial x} u^*,\phi \right\rangle dt = -\int_{t_n+s-1}^{t_n+s} \left\langle w(t), \frac{\partial \phi}{\partial x} \right\rangle dt \le$$

$$\le \int_{t_n+s-1}^{t_n+s} \|w(t)\| \left\|\frac{\partial \phi}{\partial x}\right\| dt \le C \left| \int_{t_n+s-1}^{t_n+s} \|w(t)\|^2 dt \right|^{1/2} .$$

Using (2.11) it follows that

$$(3.6)\qquad \lim_{n\to\infty} \int_{t_n+s-1}^{t_n+s} \left\langle \frac{\partial}{\partial x} u(t),\phi \right\rangle dt = \left\langle \frac{\partial}{\partial x} u^*,\phi \right\rangle$$

uniformly in s.

The third term in (3.5) is estimated as follows:

$$\int_{t_n+s-1}^{t_n+s} <A(t)u(t),\phi> dt = \int_{t_n+s-1}^{t_n+s} <A(t)w(t),\phi> dt +$$

$$+ \int_{t_n+s-1}^{t_n+s} <(A(t) - A(t_n))u^*,\phi> dt + \int_{t_n+s-1}^{t_n+s} <A(t_n)u^*,\phi> dt .$$

Each expression in this formula is estimated separately. Since $u^* \in C[0,1]$ and $A(t_n) \rightharpoonup A_\infty$ we obtain

$$<A(t_n)u^*,\phi> \to <A_\infty u^*,\phi> .$$

Estimating the second term leads to

$$\left| \int_{t_n+s-1}^{t_n+s} <(A(t) - A(t_n))u^*,\phi> dt \right| \le C \int_{t_n+s-1}^{t_n+s} \sum_{i,j=1}^{m} \int_0^1 |A_{ij}(x,t) - A_{ij}(x,t_n)| \, dx \, dt$$

$$\leq C \int_{t_n+s-1}^{t_n+s} \|A(t) - A(t_n)\|_{L^1(0,1)}\, dt \to 0$$

uniformly in s as $n \to \infty$, because of (3.3).

For the first term we obtain

$$\left| \int_{t_n+s-1}^{t_n+s} \langle A(t)w(t), \phi \rangle \, dt \right| \leq \sum_{i=1}^{m} \sum_{j=1}^{m} \int_{t_n+s-1}^{t_n+s} \left| \int_0^1 |A_{ij}(x,t)|\, |w_j(x,t)|\, |\phi_i(x)|\, dx \right| dt$$

$$\leq C \sum_{i,j=1}^{m} \int_{t_n+s-1}^{t_n+s} \|A_{ij}(t)\|\, \|w_j(t)\|\, dt \leq C \sum_{i=1}^{m} \left| \sum_{j=1}^{m} \left[\int_{t_n+s-1}^{t_n+s} \|A_{ij}(t)\|^2 dt \right]^{1/2} \right| \cdot$$

$$\cdot \left[\int_{t_n+s-1}^{t_n+s} \|w_j(t)\|^2 dt \right]^{1/2} \leq$$

$$\leq C \sum_{i=1}^{m} \left[\left[\sum_{j=1}^{m} \int_{t_n+s-1}^{t_n+s} \|A_{ij}(t)\|^2 dt \right]^{1/2} \cdot \left[\sum_{j=1}^{m} \int_{t_n+s-1}^{t_n+s} \|w_j(t)\|^2 dt \right]^{1/2} \right] \leq$$

$$\leq C \left[\int_{t_n-1}^{t_n+1} \|w_j(t)\|^2 dt \right]^{1/2} \cdot \sqrt{m} \left[\sum_{i,j=1}^{m} \int_{t_n+s-1}^{t_n+s} \|A_{ij}(t)\|^2 dt \right]^{1/2} \leq$$

$$\leq C \cdot \sqrt{m} \left[\int_{t_n-1}^{t_n+1} \|w(t)\|^2 dt \right]^{1/2} \cdot \left[\int_{t_n+s-1}^{t_n+s} \|A(t)\|^2 dt \right]^{1/2}$$

$$\leq C \left[\int_{t_n-1}^{t_n+1} \|w(t)\|^2 dt \right]^{1/2},$$

since $\|A(t)\|^2$ is bounded on $t \geq 0$.

But, because of (2.11), the last integral tends to zero as $n \to \infty$. Thus we have proved

$$\int_{t_n+s-1}^{t_n+s} \left\langle \frac{\partial}{\partial x} u(t), \phi \right\rangle dt + \int_{t_n+s-1}^{t_n+s} \langle A(t)u(t), \phi \rangle \, dt \to \left\langle \frac{\partial}{\partial x} u^* + A_\infty u^*, \phi \right\rangle$$

as $n \to \infty$ uniformly in s.

Integrating (3.5) over $[-1,1]$ and passing to the limit gives:

$$\lim_{n\to\infty} \int_{-1}^{+1} \langle w(t_n+s) - w(t_n+s-1), \phi \rangle \, ds = \left\langle f^* - \frac{\partial}{\partial x} u^* - A_\infty u^*, \phi \right\rangle .$$

The left hand side converges to zero. Hence we have proved that

$$\left\langle f^* - \frac{\partial}{\partial x} u^* - A_\infty u^*, \phi \right\rangle = 0$$

for all $\phi \in H^1(0,1)$. Since $H^1(0,1)$ is dense in $L^2(0,1)$ one finally concludes that $A_\infty \in S$. #

For practical applications it is sufficient to study the finite–dimensional Galerkin approximations of the equations (2.3), (2.4). In this case the convergence results of Theorem 1 can be improved.

Let $V \subset H^1(0,1)$ and $W \subset L^2(0,1)$ be finite–dimensional subspaces with $u^o, u^* \in V$ and $A^o \in W$.

We define by

$$S_V := \left\{ \hat{A} \in L^2(0,1) \;\middle|\; \left\langle \frac{\partial}{\partial x} u^* + \hat{A} u^* - f^*, \phi \right\rangle = 0 \quad \forall \phi \in V \right\}$$

the set of Galerkin solutions of the original problem (2.1) and consider the Galerkin equations for (2.3), (2.4):

(3.7)
$$\left\langle \frac{\partial}{\partial x} u(t) + \frac{\partial}{\partial x} u(t) + A(t)u(t) - f^*, \phi \right\rangle = 0 , \quad \phi \in V ,$$
$$u(0) = u^o$$
$$u(0,t) = u^*(0) , \qquad u(1,t) = u^*(1) ;$$

(3.8)
$$\left\langle \frac{\partial}{\partial x} A(t) - (u(t) - u^*) u(t)^T, \Phi \right\rangle , \quad \Phi \in W ,$$
$$A(0) = A^o .$$

This system of ordinary differential equations has a unique solution $u \in C^1(0,T;V)$, $A \in C^1(0,T;W)$ for small $T > 0$. Its convergence properties are similar to those presented in [1].

THEOREM 2: *Let* (A1) *be satisfied and assume that the compatibility condition*

(A3) $$\{(u - u^*)u^T \mid u \in V\} \subset W$$

holds.

Then, the system (3.7), (3.8) *has a unique global solution* $u(t), A(t)$ *which satisfies*

1. *There exists* $A_\infty \in S_V : \lim_{t\to\infty} \|A(t) - A_\infty\| = 0$,
2. $\lim_{t\to\infty} \|u(t) - u^*\| = 0$.

PROOF: Obviously $S \subset S_V$ holds.

Initially, we derive an *a priori* estimate (2.11) for the present situation. This proves also that (3.7), (3.8) have a unique global solution $u \in C^1(0,\infty;V)$, $A \in C^1(0,\infty;W)$. Let $w: = u - u^*$ and $R: = A - A^*$.

We recall that A^* is in S_V and is positive definite. From (3.7) one obtains

$$\left\langle \frac{\partial}{\partial x} w(t) + \frac{\partial}{\partial x} w(t) + A^*w(t) + R(t)u(t),\phi \right\rangle = 0$$

for all $\phi \in V$, where $w(0) = w^o \in V$ and $w(0,t) = w(1,t) = 0$. Choose $\phi: = w$. Then

(3.9) $$\frac{1}{2}\frac{\partial}{\partial t} \|w(t)\|^2 + \langle A^*w(t), w(t)\rangle + \langle R(t)u(t), w(t)\rangle = 0 .$$

In (3.8) we choose as a test function $\Phi: = A - PA^*$, where P is the orthogonal projection in $L^2(0,1)$ onto W . This implies that

$$0 = \left\langle \frac{\partial}{\partial t} A(t) - w(t)u^T, A(t) - PA^* \right\rangle =$$
$$= \left\langle \frac{\partial}{\partial t} A(t) - w(t)u^T, R(t) \right\rangle + \left\langle \frac{\partial}{\partial t} A(t) - w(t)u^T, A^* - PA^* \right\rangle .$$

Since $A^* - PA^* \in W^\perp$, we obtain using (A3)

$$\left\langle \frac{\partial}{\partial t} R(t), R(t) \right\rangle = \langle R\, u(t), w(t)\rangle$$

and hence with (3.9):

(3.10) $$\frac{1}{2}\frac{\partial}{\partial t}(\|w(t)\|^2 + \|R(t)\|^2) + \langle A^*w(t), w(t)\rangle = 0\ .$$

This yields the required *a priori* estimate

(3.11) $$\sup_{t\geq 0}(\|w(t)\|^2 + \|R(t)\|^2) + \int_0^1 (\|w(t)\|^2\, dt \leq C < \infty\ .$$

It should be noted that the estimate (3.11) holds for each positive definite matrix $A^* \in S_V$ and that the constant C depends on A^* but not on the subspaces V and W. In particular, (3.11) yields that (3.7), (3.8) has a global solution $u(t)$, $A(t)$, and that $\{\|A(t)\|\}$ is bounded. We conclude that $\{A(t_n)\}$ has a cluster point $A_\infty \in W$ for each sequence $t_n \to \infty$. Proceeding in the same way as we did in the proof of Theorem 1 it follows that each such cluster point A_∞ is in S_V.

Finally we prove that $\{A(t)\}$ has only one cluster point for $t \to \infty$ (compare [1]).

Fix a positive definite matrix $\hat{A} \in S_V$ and consider a cluster point A_∞ of some sequence $\{A(t_n)\}$, $t_n \to \infty$.

From (3.10) it follows:

$$\frac{1}{2}\frac{\partial}{\partial t}(\|w(t)\|^2 + \|R(t)\|^2) = -\langle \hat{A}\, w(t), w(t)\rangle \leq 0$$

and hence

(3.12) $$\ell := \lim_{t\to\infty}(\|w(t)\|^2 + \|R(t)\|^2)$$

exists and ℓ does not depend on A_∞ but on $\hat{A}$.

From the monotonicity we obtain:

$$\lim_{n\to\infty}\left|\int_{t_n}^{t_n+1}\|A(t) - \hat{A}\|^2\, dt + \int_{t_n}^{t_n+1}\|w(t)\|^2\, dt\right| = \ell\ .$$

Using (3.11) leads to

(3.13) $$\|A_\infty - \hat{A}\|^2 = \ell\ .$$

Now taking any two cluster points A^1_∞, A^2_∞ and using (3.13) yields:

$$\|A_\infty^1 - \hat{A}\|^2 = \|A_\infty^2 - \hat{A}\|^2 \tag{3.14}$$

which holds for all positive definite matrices $\hat{A} \in S_V$.

Define

$$\hat{A} := A^* + \varepsilon(A_\infty^1 - A_\infty^2) \in S_V ,$$

which is positive definite for sufficiently small $|\varepsilon|$. It follows from (3.14) that

$$\|A_\infty^1 - A^* - \varepsilon(A_\infty^1 - A_\infty^2)\|^2 = \|A_\infty^2 - A^* - \varepsilon(A_\infty^1 - A_\infty^2)\|^2$$

and hence

$$\langle A_\infty^1 - A^*, A_\infty^1 - A_\infty^2\rangle = \langle A_\infty^2 - A^*, A_\infty^1 - A_\infty^2\rangle$$

which proves $A_\infty^1 = A_\infty^2$.

It remains to show that $\lim_{t\to\infty} u(t) = u^*$.

This follows easily from (3.12) on using (3.13). #

Now we investigate convergence properties of the Galerkin solutions as the dimension of the subspaces tend to infinity.

Let V_k and $W_{p(k)}$ be subspaces of $H^1(0,1)$ and $L^2(0,1)$ respectively with $\dim V_k = k$, $\dim W_{p(k)} = p(k) < \infty$. Furthermore we assume that $u^o, u^* \in V_k$ and $A^o \in W_{p(k)}$ for all $k \geq 2$ holds. Additionally, let the compatibility condition (A3), $(u - u^*)u^T \in W_{p(k)}$, be satisfied for $u \in V_k$ and all $k \geq 2$.

THEOREM 3. *Let* V_k *and* $W_{p(k)}$ *be the Galerkin subspaces of* $H^1(0,1)$ *and* $L^2(0,1)$ *respectively with* $V_k \subset V_{k+1}$ *and* $W_{p(k)} \subset W_{p(k+1)}$ *for all* $k \geq 1$ *having the property that*

$$\overline{\bigcup_{k=1}^{\infty} V_k}^{\|\cdot\|_{L_2}} = L^2(0,1) . \tag{A4}$$

Furthermore, let (A1) *and* (A2) *be satisfied for all* $k \geq 1$ *and let* $A_{\infty,k} \in S_V$ *be the limit* *point of* $A_k(t)$ *as* t *tends to infinity where* $A_k(t)$ *is constructed by the Galerkin process with respect to* V_k, $W_{p(k)}$ *according to Theorem* 2.

Then each subsequence $\{A_{\infty,k_1}\}$ of $\{A_{\infty,k}\}$ *has a weak cluster point in* $L^2(0,1)$ *which belongs to* S.

PROOF. Let $A^* \in S$. Then $A^* \in S_{V_k}$ since $S \subset S_{V_k}$ for all $k \in \mathbb{N}$.

Define $R_k := A_k - A^*$ and $w_k := u_k - u^*$.

Then, from (2.10), it follows that

$$\frac{1}{2}(\|w_k(t)\|^2 + \|R_k(t)\|^2) \le \frac{1}{2}\|w^o\|^2 + \|R^o\|^2 \le C,$$

where C does not depend on k.

Hence passing to the limit for $t \to \infty$ implies

$$\|A_{\infty,k} - A^*\| \le C,$$

and C is independent on k. Then each subsequence $\{A_{\infty,k_\ell}\}$ of $\{A_{\infty,k}\}$ has a weak cluster point $\hat{A} \in L^2(0,1)$. We have to show that $\hat{A} \in S$.

Let $\phi \in L^2(0,1)$ be any test function.

For all $\phi_{k_\ell} \in V_{k_\ell}$ we have:

$$\left|\left\langle \frac{\partial}{\partial x} u^* + \hat{A}\, u^* - f^*, \phi \right\rangle\right| \le \left|\left\langle \frac{\partial}{\partial x} u^* + A_{\infty,k_\ell} u^* - f^*, \phi - \phi_{k_\ell} \right\rangle\right| + |\langle \hat{A} - A_{\infty,k_\ell}) u^*, \phi \rangle| \le$$

$$\le \left\|\frac{\partial}{\partial x} u^* + A_{\infty,k_\ell} u^* - f^*\right\| \|\phi - \phi_{k_\ell}\| + |\langle \hat{A} - A_{\infty,k_\ell}) u^*, \phi \rangle| .$$

The first term is bounded since $\{A_{\infty,k_\ell}\}$ is bounded. The expression $\|\phi - \phi_{k_\ell}\|$ can be made arbitrarily small by a special choice of ϕ_{k_ℓ} for ℓ sufficiently large because of condition (A4), and the last term converges to zero with $\ell \to \infty$ since $A_{\infty,k_\ell} \rightharpoonup \hat{A}$ in $L^2(0,1)$.

Hence $\hat{A} \in S$. #

REMARK: It should be noted that equations (2.3), (2.4) can be solved at least numerically and thus define a numerical algorithm to identify A^* in (2.1).

4. STABILITY

All considerations of Section 3 were based on the assumption that u^* is known exactly. The question of stability will now be studied.

As before we consider the differential equation

$$\frac{\partial}{\partial x} u^* + A^* u^* = f^* \qquad \text{a.e. in } (0,1)$$

with $u^* \in H^1(0,1)$, $A^* \in L^2(0,1)$ and positive definite, and $f^* \in L^2(0,1)$. Let the data u^* corresponding to f^* be disturbed by a small variation $w \in H^1(0,1)$ corresponding to $g \in L^2(0,1)$, and define

$$u := u^* + w , \qquad f := f^* + g .$$

The disturbed identification problem consists in finding a matrix $B \in L^2(0,1)$ such that

$$B\, u = g - A^* w - \frac{\partial}{\partial x} w \tag{4.1}$$

holds. For convenience we define $h := g - A^* w - \frac{\partial}{\partial x} w$ and note that $h \in L^2(0,1)$. Equation (4.1) has a solution if there is a constant $\mu > 0$ such that

$$|u_{j^*}(x)| \geq \mu > 0 \tag{A4}$$

holds for some index j^* uniformly in $x \in (0,1)$; but in general the solution will not be unique.

Therefore we consider the following optimization problem (P) :

(P) Minimize $\int_0^1 \|B(x)\|_{ES}^2\,dx$, $B \in L^2(0,1)$, subject to $B\,u = h$.

This problem has a unique solution $B^* \in L^2(0,1)$ and $\int_0^1 \|B(x)\|_{ES}^2\,dx$ is estimated in the following theorem.

THEOREM 4: *Let* $B^* \in L^2(0,1)$ *be a solution of problem* (P) *and let condition* (A4) *be satisfied. Then*

$$\|B^*\| \le C\left[\|g\| + \|A^*w\| + \left\|\frac{\partial}{\partial x} w\right\|\right] \tag{4.2}$$

holds for some constant C .

PROOF. For each $B \in L^2(0,1)$ satisfying $B\,u = h$ the inequality

$$\int_0^1 \|B^*(x)\|_{ES}^2\,dx \le \int_0^1 \|B(x)\|_{ES}^2\,dx$$

holds.

Define

$$\tilde{b}_{ij^*}(x) := \frac{1}{u_j^*(x)}\,h_i(x) \quad \text{for } i = 1,2,\dots,m$$

and

$$\tilde{b}_{ij}(x) = 0 \quad \text{for } i = 1,2,\dots,m \text{ and } j = 1,2,\dots,m,\ j \ne j^* .$$

Then $\tilde{B} = (\tilde{b}_{ij}) \in L^2(0,1)$ and satisfies the constraints.

We obtain

$$\|B^*\|^2 = \int_0^1 \|B^*(x)\|_{ES} \le \int_0^1 \sum_{i=1}^m \sum_{j=1}^m \tilde{b}_{ij}(x)^2\,dx =$$

$$= \int_0^1 \sum_{i=1}^m \tilde{b}_{ij}^*(x)^2\,dx = \int_0^1 \frac{1}{u_j^*(x)} \sum_{i=1}^m h_i(x)^2\,dx \le$$

$$\le \frac{1}{\mu^2}\int_0^1 |h(x)|\,dx \le \frac{1}{\mu^2}\|h\|^2 ,$$

whence (4.2) follows. #

REMARK. This result shows that the error B^* depends on the variations g and w as well as on the matrix A^* itself.

Next we show that the Galerkin approximations of our identification procedure described in Section 3 are stable against disturbances in the initial data. We recall that u and A are solutions of the equations:

$$\left\langle \dot{u}(t) + \frac{\partial}{\partial x} u(t) + A(t)u(t) - f^*, \phi \right\rangle = 0 , \quad \phi \in V , \tag{4.3}$$

$$u(0) = u^o$$

$$u(0,t) = u^*(0) , \quad u(1,t) = u^*(1) ,$$

$$<\dot{A}(t) - (u(t) - u^*)\, u(t)^T, \Phi> = 0 , \quad \Phi \in W , \tag{4.4}$$

$$A(0) = A^o ,$$

where V and W are finite dimensional subspaces of $H^1(0,1)$ and $L^2(0,1)$, respectively.

Obviously u and A depend smoothly on A^*, u^*, A^o and u^o in each fixed finite time interval $[0,T]$. Furthermore we know from the convergence result in Section 3 that

$$u^* = \lim_{t \to \infty} u(t) ,$$

$$A_V \ni A^\infty = \lim_{t \to \infty} A(t) ,$$

$$\int_0^\infty \|w(t)\|^2 \, dt < \infty . \tag{4.5}$$

THEOREM 5. *Let* (A1) *and* (A3) *be fulfilled and let* A^∞ *be positive definite. Then* $A_\varepsilon^\infty \to A^\infty$ *as* $\varepsilon \to 0$ *provided that* $u_\varepsilon^o \to u^o$ *and* $A_\varepsilon^o \to A^o$ *as* $\varepsilon \to 0$.

PROOF: Let A_ε, u_ε be the solution of (4.3), (4.4) with initial data A_ε^o, u_ε^o. Then

$$\frac{\partial}{\partial t}(u_\varepsilon - u) + \frac{\partial}{\partial x}(u_\varepsilon - u) + (A_\varepsilon - u) + (A_\varepsilon - A)\, u_\varepsilon = 0 \quad \text{in } V ,$$

$$\frac{\partial}{\partial t}(A_\varepsilon - A) = (u_\varepsilon - u)\, u_\varepsilon^T + (u - u^*)(u_\varepsilon - u)^T \quad \text{in } W .$$

Multiplying these equations by $(u_\varepsilon - u)$ and $(A_\varepsilon - A)$, respectively, integrating over $(0,1)$ and adding leads to

$$\frac{1}{2}\frac{\partial}{\partial t}(\|(u_\varepsilon - u)(t)\|^2 + \|A_\varepsilon(t) - A(t)\|^2) +$$
$$+ \int_0^1 (u_\varepsilon(x,t) - u(x,t))^T A(x,t)(u_\varepsilon(x,t) - u(x,t))\, dx -$$
$$- \int_0^1 (u(x,t) - u^*(x))^T (A_\varepsilon(x,t) - A(x,t))(u_\varepsilon(x,t) - u(x,t))\, dx = 0 .$$

Set

$$\phi_\varepsilon(t) := \frac{1}{2}(\|(u_\varepsilon - u)(t)\|^2 + |A_\varepsilon(t) - A(t)\|^2) ,$$

then

$$0 = \dot{\phi}_\varepsilon(t) + \langle(u_\varepsilon(t) - u(t)), A(t)(u_\varepsilon(t) - u(t))\rangle -$$
$$- \langle(u(t) - u^*, (A_\varepsilon(t) - A(t))(u_\varepsilon(t) - u(t))\rangle$$

and hence

$$\dot{\phi}_\varepsilon(t) + \langle(u_\varepsilon(t) - u(t), A(t)(u_\varepsilon(t) - u(t))\rangle =$$

$$= \sum_{i,j=1}^{m} \int_0^1 (u(x,t) - u^*(x,t))_i\, (A_\varepsilon(x,t) - A(x,t))_{ij}\, (u_\varepsilon(x,t) - u(x,t))_j\, dx \leq$$

$$= \sum_{i,j=1}^{m} \int_0^1 |(u(x,t) - u^*(x))_i|\; |(A_\varepsilon(x,t) - A(x,t))_{ij}|\; |(u_\varepsilon(x,t) - u(x,t))_j|\, dx \leq$$

$$\leq \sum_{i,j=1}^{m} \sigma \int_0^1 |(u_\varepsilon(x,t) - u(x,t))_j|^2\, dx + \frac{1}{4\sigma}\int_0^1 |(A_\varepsilon(x,t) - A(x,t))_{ij}|^2\, |(u(x,t) - u^*(x))_i|^2\, dx$$

$$\leq m\sigma\, \|u_\varepsilon(t) - u(t)\|^2 + \frac{1}{4\sigma} \max_{1\leq i\leq m} \|(u(t) - u^*)_i\|^2_{L^\infty(0,1)}\; \|A_\varepsilon(t) - A(t)\|^2$$

holds for all $\sigma > 0$.

Since $A(t) \to A^\infty$ and A^∞ is positive definite, there exists a time τ and a constant δ such that

$$u^T A(x,t)u \geq \delta |u|^2$$

holds for all $u \in \mathbb{R}^m$ and $t \geq \tau$, a.e. in $(0,1)$.

Choose $\sigma := \frac{\delta}{m}$, then

$$\dot{\phi}_\varepsilon(t) + \delta\|u_\varepsilon(t) - u(t)\|^2 \leq \delta\|u_\varepsilon(t) - u(t)\|^2$$

$$+ \frac{m}{4\delta} \max_{1 \leq i \leq m} \|(u(t) - u^*)_i\|^2_{L^\infty(0,1)} \|A_\varepsilon(t) - A(t)\|^2$$

for all $t \geq \tau$.

Hence $\dot{\phi}_\varepsilon(t) \leq C\|u(t) - u^*\|^2 \phi_\varepsilon(t)$ is obtained for all $t \geq \tau$ and some constant C.

Gronwall's Lemma gives

$$\phi_\varepsilon(t) \leq \phi_\varepsilon(\tau) \exp(C \int_\tau^t \|u(s) - u^*(s)\|^2 ds)$$

and consequently for all $t \geq \tau$ and $\varepsilon \geq 0$

$$\phi_\varepsilon(t) \leq \phi_\varepsilon(\tau) \exp(C \int_0^\infty \|u(s) - u^*(s)\|^2 ds) .$$

The last expression is finite because of (4.5).

Hence

$$\frac{1}{2} \|A_\varepsilon^\infty - A^\infty\|^2 = \lim_{t \to \infty} \phi_\varepsilon(t) \leq \phi_\varepsilon(\tau) \cdot C .$$

But $\lim_{\varepsilon \to 0} \phi_\varepsilon(\tau) = 0$, since on a finite interval the solutions of (4.3), (4.4) depend continuously on the data.

This completes the proof. #

REMARK: The numerical test computations have not been fully satisfactory so far. In [2], our equation (2.4) was modified by averaging the error expression on the right hand side. Instead of (2.4), they use the equation

$$\frac{\partial}{\partial t} A(x,t) = \int_0^t (u - u^*)(x,t)\, u(x,t)^T \, dx$$

to get very nice numerical results for test problems.

This modification seems to suppress the oscillation we have observed in our calculation.

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