Nihonkai Math. J.
Vol.25(2014), 119–125

DISCONTINUOUS MAPS WHOSE ITERATIONS ARE CONTINUOUS

KOUKI TANIYAMA

Abstract. Let X be a topological space and $f : X \to X$ a bijection. Let $\mathcal{C}(X, f)$ be a set of integers such that an integer n is an element of $\mathcal{C}(X, f)$ if and only if the bijection $f^n : X \to X$ is continuous. A subset S of the set of integers $\mathbb{Z}$ is said to be realizable if there is a topological space X and a bijection $f : X \to X$ such that $S = \mathcal{C}(X, f)$. A subset S of $\mathbb{Z}$ containing 0 is called a submonoid of $\mathbb{Z}$ if the sum of any two elements of S is also an element of S. We show that a subset S of $\mathbb{Z}$ is realizable if and only if S is a submonoid of $\mathbb{Z}$. Then we generalize this result to any submonoid in any group.

1. Introduction

Let X be a topological space and $f : X \to X$ a bijection. By $f^{-1} : X \to X$ we denoted the inverse mapping of f. For each integer n we define a bijection $f^n : X \to X$ by

$$f^n = \begin{cases} \underbrace{f \circ f \circ \cdots \circ f}_{n} & (n > 0) \\ \mathrm{id}_X & (n = 0) \\ \underbrace{f^{-1} \circ f^{-1} \circ \cdots \circ f^{-1}}_{-n} & (n < 0). \end{cases}$$

We note that $f^n \circ f^m = f^{m+n}$ for any integers m and n. Let $\mathbb{Z}$ be the set of all integers. We define a subset $\mathcal{C}(X, f)$ of $\mathbb{Z}$ by

$$\mathcal{C}(X, f) = \{n \in \mathbb{Z} | f^n : X \to X \text{ is continuous.}\}.$$

2010 *Mathematics Subject Classification.* 20M99, 54C05.

Key words and phrases. Continuous map, discontinuous map, bijection, group, monoid, subgroup, submonoid.

The author was partially supported by Grant-in-Aid for Scientific Research (C) (No. 24540100), Japan Society for the Promotion of Science.

A subset S of $\mathbb{Z}$ is said to be *realizable* if there is a topological space X and a bijection $f : X \to X$ such that $S = \mathcal{C}(X, f)$. A subset S of $\mathbb{Z}$ is called a *submonoid* of $\mathbb{Z}$ if S satisfies the following two conditions.

(1) S contains 0,
(2) if S contains a and b then S contains $a + b$.

Note that it is not necessary that S contains $a - b$.

Example 1.1. The following subsets of $\mathbb{Z}$ are submonoids of $\mathbb{Z}$.
$\mathbb{Z}$, $\{n \in \mathbb{Z} | n \geq 0\}$, $\{0\} \cup \{n \in \mathbb{Z} | n \leq -3\}$, $\{2n | n \in \mathbb{Z}\}$, $\{0\} \cup \{3n | n \in \mathbb{Z}, n \geq 2\}$, $\{3a + 5b | a, b \in \mathbb{Z}, a, b \geq 0\} = \{0, 3, 5, 6, 8\} \cup \{n \in \mathbb{Z} | n \geq 9\}$, $\{0\}$.

Theorem 1.1. *A subset S of the set of all integers $\mathbb{Z}$ is realizable if and only if S is a submonoid of $\mathbb{Z}$.*

We generalize Theorem 1.1 to any submonoid in any group in the third section.

2. Proof of Theorem 1.1

Proposition 2.1. *Let X be a topological space and $f : X \to X$ a bijection. Then the subset $\mathcal{C}(X, f)$ of $\mathbb{Z}$ is a submonoid of $\mathbb{Z}$.*

Proof. Since $f^0 = \mathrm{id}_X$ is continuous the set $\mathcal{C}(X, f)$ contains 0. Suppose that $\mathcal{C}(X, f)$ contains a and b. Then f^a and f^b are continuous. Then the composition $f^b \circ f^a = f^{a+b}$ is also continuous. Therefore $\mathcal{C}(X, f)$ contains $a + b$. □

Proof of Theorem 1.1. It follows from Proposition 2.1 that if S is realizable then S is a submonoid of $\mathbb{Z}$. We will show that if S is a submonoid of $\mathbb{Z}$ then S is realizable. Let S be a submonoid of $\mathbb{Z}$. For each integer n we define a subset X_n of the 2-dimensional Euclidean space $\mathbb{R}^2$ as follows.

$$X_n = \begin{cases} \{n\} \times [0, 2) & (n \in S) \\ \{n\} \times ([0, 1) \cup [2, 3)) & (n \in (\mathbb{Z} \setminus S)). \end{cases}$$

Let $X = \bigcup_{n \in \mathbb{Z}} X_n$. Then X is a topological subspace of $\mathbb{R}^2$. Let $f : X \to X$ be a bijection defined by the followings.

(1) if $n, n + 1 \in S$, then $f((n, x)) = (n + 1, x)$ for each $x \in [0, 2)$,
(2) if $n, n + 1 \in (\mathbb{Z} \setminus S)$, then $f((n, x)) = (n + 1, x)$ for each $x \in ([0, 1) \cup [2, 3))$,
(3) if $n \in S$ and $n + 1 \in (\mathbb{Z} \setminus S)$, then $f((n, x)) = (n + 1, x)$ for each $x \in [0, 1)$ and $f((n, x)) = (n + 1, x + 1)$ for each $x \in [1, 2)$,
(4) if $n \in (\mathbb{Z} \setminus S)$ and $n + 1 \in S$, then $f((n, x)) = (n + 1, x)$ for each $x \in [0, 1)$ and $f((n, x)) = (n + 1, x - 1)$ for each $x \in [2, 3)$.

By definition we have $f^n(X_m) = X_{m+n}$ for any integers m and n. Suppose that $n \in (\mathbb{Z} \setminus S)$. Since $X_0 = \{0\} \times [0, 2)$ is connected and $f^n(X_0) = X_n = \{n\} \times ([0, 1) \cup [2, 3))$ is not connected, we see that f^n is discontinuous. Therefore n is not an element of $\mathcal{C}(X, f)$. Suppose that $n \in S$. For each $m \in (\mathbb{Z} \setminus S)$ we see that f^n maps $X_m = \{m\} \times ([0, 1) \cup [2, 3))$ onto X_{m+n}. If $m + n \in S$ then $X_{m+n} = \{m + n\} \times [0, 2)$ and $f^n((m, x)) = (m + n, x)$ for each $x \in [0, 1)$ and $f^n((m, x)) = (m + n, x - 1)$ for each $x \in [2, 3)$. Therefore f^n maps X_m continuously onto X_{m+n}. If $m + n \in (\mathbb{Z} \setminus S)$ then $X_{m+n} = \{m + n\} \times ([0, 1) \cup [2, 3))$ and $f^n((m, x)) = (m + n, x)$ for each $x \in ([0, 1) \cup [2, 3))$. Therefore f^n maps X_m homeomorphically onto X_{m+n}. Thus we see that $f^n|_{X_m}$ is continuous for each $m \in (\mathbb{Z} \setminus S)$. Suppose that m is an element of S. Then $X_m = \{m\} \times [0, 2)$. Since S is a submonoid of $\mathbb{Z}$ we see that $m+n$ is also an element of S. Therefore $X_{m+n} = \{m+n\} \times [0, 2)$. We see that $f^n((m, x)) = (m+n, x)$ for each $x \in [0, 2)$. Therefore f^n maps X_m homeomorphically onto X_{m+n}. Thus we see that $f^n|_{X_m}$ is continuous for each $m \in S$. Therefore f^n is continuous. Therefore n is an element of $\mathcal{C}(X, f)$. Thus we have $S = \mathcal{C}(X, f)$ as desired. $\square$

Example 2.1. Figure 1 illustrates X and $f : X \to X$ in the proof of Theorem 1.1 where $S = \mathcal{C}(X, f) = \{0\} \cup \{n \in \mathbb{Z} | n \geq 3\}$.

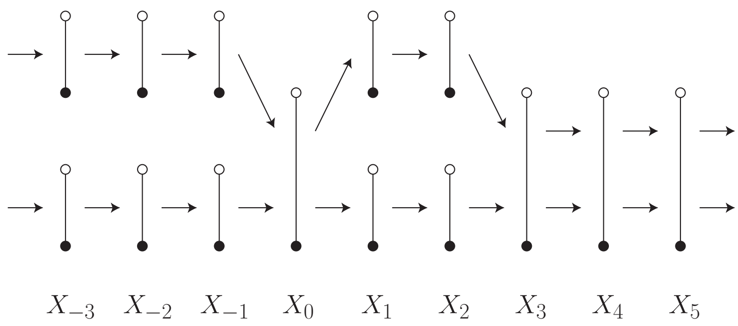

FIGURE 1

We note that the topological type of the topological space X in the proof of Theorem 1.1 is independent of the choice of the subset S of $\mathbb{Z}$. Actually X is a disjoint union of countably many semi-open intervals. Thus we have shown the following proposition.

Proposition 2.2. *Let X be a disjoint union of countably many semi-open intervals. Then for any submonoid S of $\mathbb{Z}$ there is a bijection $f : X \to X$ such that $S = \mathcal{C}(X, f)$.*

We note that not all topological spaces have such a property as X in Proposition 2.2. For example, let X be a compact Hausdorff space. Then a continuous bijection from X to X is a homeomorphism. Therefore, for any bijection $f : X \to X$ the set $\mathcal{C}(X, f)$ is invariant under the map $r : \mathbb{Z} \to \mathbb{Z}$ defined by $r(x) = -x$, $x \in \mathbb{Z}$.

3. Generalization

In this section we reformulate and generalize Theorem 1.1 as follows. Let G be a group and e the unit element of G. A subset S of G is called a *submonoid* of G if S satisfies the following two conditions.

(1) S contains e,
(2) if S contains a and b then S contains ab.

Let X be a topological space. By $\mathcal{B}(X)$ we denote the set of all bijections from X to X. Then $\mathcal{B}(X)$ forms a group under the composition of maps. Let $A(X)$ be a subgroup of $\mathcal{B}(X)$. By $\mathcal{C}(A(X))$ we denote the set of all continuous bijections in $A(X)$. Since $\mathrm{id}_X : X \to X$ is continuous and the composition of two continuous maps is continuous, we see that $\mathcal{C}(A(X))$ is a submonoid of $A(X)$. Let G and H be groups and S and T submonoids of G and H respectively. We say that the pair (G, S) is *isomorphic* to the pair (H, T) if there is a group isomorphism $h : G \to H$ such that $h(S) = T$.

Theorem 3.1. *Let G be a group and S a submonoid of G. Then there is a topological space X and a subgroup $A(X)$ of $\mathcal{B}(X)$ such that the pair (G, S) is isomorphic to the pair $(A(X), \mathcal{C}(A(X)))$.*

Proof. Let G be a group and S a submonoid of G. We give a discrete topology to G. Let $\mathbb{R}$ be the 1-dimensional Euclidean space and $G \times \mathbb{R}$ the product topological space. For each element n in G we define a subspace X_n of $G \times \mathbb{R}$ as follows.

$$X_n = \begin{cases} \{n\} \times [0, 2) & (n \in S) \\ \{n\} \times ([0, 1) \cup [2, 3)) & (n \in (G \setminus S)). \end{cases}$$

Let $X = \bigcup_{n \in G} X_n$. Then X is a topological subspace of $G \times \mathbb{R}$. For each element n in G we define a bijection $f_n : X \to X$ by the followings.

(1) if $m, mn \in S$, then $f_n((m, x)) = (mn, x)$ for each $x \in [0, 2)$,
(2) if $m, mn \in (G \setminus S)$, then $f_n((m, x)) = (mn, x)$ for each $x \in ([0, 1) \cup [2, 3))$,
(3) if $m \in S$ and $mn \in (G \setminus S)$, then $f_n((m, x)) = (mn, x)$ for each $x \in [0, 1)$ and $f_n((m, x)) = (mn, x + 1)$ for each $x \in [1, 2)$,
(4) if $m \in (G \setminus S)$ and $mn \in S$, then $f_n((m, x)) = (mn, x)$ for each $x \in [0, 1)$ and $f_n((m, x)) = (mn, x - 1)$ for each $x \in [2, 3)$.

For any two elements m and n in G we see by definition that $f_n \circ f_m = f_{mn}$. Let $A(X)$ be the subgroup of $\mathcal{B}(X)$ defined by $A(X) = \{f_n | n \in G\}$. Then we see that the group $A(X)$ is isomorphic to the group G. Then by an entirely analogous argument as in the proof of Theorem 1.1 we see that $\mathcal{C}(A(X)) = \{f_n | n \in S\}$. Thus we see that the pair $(A(X), \mathcal{C}(A(X)))$ is isomorphic to the pair (G, S) as desired. $\square$

Remark 3.1. (1) In general the group $\mathcal{B}(X)$ is so big that we should take a subgroup $A(X)$ of $\mathcal{B}(X)$ as in the statement of Theorem 3.1. In fact there is a group G that is not isomorphic to $\mathcal{B}(X)$ for any set X. For example, it is easy to check that $\mathcal{B}(X)$ is not isomorphic to a cyclic group of order 3 for any set X.
(2) Even in the case that a group G is isomorphic to $\mathcal{B}(X)$ for some set X, not all pair (G, S) is realized by the pair $(\mathcal{B}(X), \mathcal{C}(\mathcal{B}(X)))$ under any topology on X. Let $G = S_3$ be a symmetric group of degree 3. Note that every submonoid of a finite group G is a subgroup of G. We will see that the pair (S_3, C_3) is not realized where C_3 is a cyclic group of order 3. It is clear that $\mathcal{B}(X)$ is isomorphic to S_3 if and only if X contains exactly 3 points. Therefore we may suppose without loss of generality that $X = \{a, b, c\}$. Then, up to self-homeomorphism, there are 9 topologies on X. They are $\mathcal{D}_1 = \{\emptyset, X\}$, $\mathcal{D}_2 = \{\emptyset, \{a\}, X\}$, $\mathcal{D}_3 = \{\emptyset, \{a, b\}, X\}$, $\mathcal{D}_4 = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\mathcal{D}_5 = \{\emptyset, \{a\}, \{b, c\}, X\}$, $\mathcal{D}_6 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$, $\mathcal{D}_7 = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, X\}$, $\mathcal{D}_8 = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$ and $\mathcal{D}_9 = 2^X$. Then we see that the subgroup $\mathcal{C}(\mathcal{B}(X, \mathcal{D}_i))$ of $\mathcal{B}(X, \mathcal{D}_i)$ is the trivial group for $i = 4, 8$, a cyclic group of order 2 for $i = 2, 3, 5, 6, 7$ and the symmetric group of degree 3 $\mathcal{B}(X, \mathcal{D}_i)$ for $i = 1, 9$. Thus $\mathcal{C}(\mathcal{B}(X, \mathcal{D}_i))$ is not a cyclic group of order 3 for any i.

Next we give a variation of Theorem 3.1 as follows. A *monoid* M is a semigroup with the unit element e. Namely M has an associative binary operation such that $xe = ex = x$ for any element $x \in M$. A subset S of a monoid M is said to be a *submonoid* of M if e is an element of S and for any elements a and b of S the element ab is an element of S. Let X be a topological space. By $\mathcal{M}(X)$ we denote the set of all maps from X to X. Then $\mathcal{M}(X)$ forms a monoid under the composition of maps. Let $A(X)$ be a submonoid of $\mathcal{M}(X)$. By $\mathcal{C}(A(X))$ we denote the set of all continuous maps in $A(X)$. Then we see as before that $\mathcal{C}(A(X))$ is a submonoid of $A(X)$. Let M and N be monoids and S and T submonoids of M and N respectively. We say that the pair (M, S) is *isomorphic* to the pair (N, T) if there is a monoid isomorphism $h : M \to N$ such that $h(S) = T$.

Theorem 3.2. *Let M be a monoid and S a submonoid of M. Then there is a topological space X and a submonoid $A(X)$ of $\mathcal{M}(X)$ such that the pair (M, S) is isomorphic to the pair $(A(X), \mathcal{C}(A(X)))$.*

Proof. We define a topological space X to be a subspace of $M \times \mathbb{R}$ as in the proof of Theorem 3.1. The map $f_n : X \to X$ is also defined in the same way for each element

n of M. The only difference is that the map f_n is not a bijection in general. Note that in the proof of Theorem 3.1 the assumption that n has an inverse element in the group G assured the fact that $f_n : X \to X$ is a bijection. Then the rest of the proof is the same as the proof of Theorem 3.1. $\square$

Finally we give another variation of Theorem 3.1 as follows. As we have already remarked, if X is compact Hausdorff and $f : X \to X$ is a continuous bijection, then $f^{-1} : X \to X$ is also continuous. Therefore for any subgroup $A(X)$ of $\mathcal{B}(X)$ the submonoid $\mathcal{C}(A(X))$ of $A(X)$ is a subgroup of $A(X)$. Then we have the following theorem.

Theorem 3.3. *Let G be a group and H a subgroup of G. Then there is a compact Hausdorff space X and a subgroup $A(X)$ of $\mathcal{B}(X)$ such that the pair (G, H) is isomorphic to the pair $(A(X), \mathcal{C}(A(X)))$.*

Proof. We give a discrete topology to G. Let $G \times [0, 1]$ be the product topological space and $X = (G \times [0, 1]) \cup \{\infty\}$ the one-point compactification of $G \times [0, 1]$. Then X is a compact Hausdorff space. For each element n in G we define a bijection $f_n : X \to X$ by the followings.

(1) For each m in G and x in $(0, 1)$, $f_n((m, x)) = (mn, x)$.
(2) If $m, mn \in H$ or $m, mn \in (G \backslash H)$, then $f_n((m, 0)) = (mn, 0)$ and $f_n((m, 1)) = (mn, 1)$.
(3) If $m \in H$ and $mn \in (G \setminus H)$, or $m \in (G \setminus H)$ and $mn \in H$, then $f_n((m, 0)) = (mn, 1)$ and $f_n((m, 1)) = (mn, 0)$.
(4) $f_n(\infty) = \infty$.

We see by the definition that the composition $f_n \circ f_m$ is equal to f_{mn} for any elements m and n in G. Let $A(X)$ be the subgroup of $\mathcal{B}(X)$ defined by $A(X) = \{f_n | n \in G\}$. Then we see that the group $A(X)$ is isomorphic to the group G. We will show that $\mathcal{C}(A(X)) = \{f_n | n \in H\}$. First we will show that f_n is continuous at ∞ for any n in G. Let U be an open neighbourhood of ∞. Then $X \setminus U$ is a compact subset of $G \times [0, 1]$. Therefore there is a finite subset F of G such that $X \setminus U$ is contained in $F \times [0, 1]$. Let $V = X \setminus ((Fn^{-1}) \times [0, 1])$. Then V is an open neighbourhood of ∞ such that $f_n(V) = X \setminus (F \times [0, 1])$ is contained in U as desired. Suppose that $n \in (G \setminus H)$. Then f_n maps $\{e\} \times [0, 1]$ to $\{n\} \times [0, 1]$. Since the unit element e is in H, $f_n((e, 0)) = (n, 1)$ and $f_n((e, 1)) = (n, 0)$. Therefore f_n is not continuous and f_n is not in $\mathcal{C}(A(X))$. Suppose that $n \in H$. Let m be an element of G. Then we see that mn is an element of H if and only if m is an element of H. Therefore f_n maps $\{m\} \times [0, 1]$ to $\{mn\} \times [0, 1]$ by the formula $f_n((m, x)) = (mn, x)$ for each x in $[0, 1]$. Therefore the restriction map $f_n|_{\{m\} \times [0,1]}$ is continuous for each m in G.

Therefore f_n is an element of $\mathcal{C}(A(X))$. Thus the pair (G, H) is isomorphic to the pair $(A(X), \mathcal{C}(A(X)))$. $\square$

Remark 3.2. Theorem 3.1, Theorem 3.2 and Theorem 3.3 concern the pairs (G, S), (M, S) and (G, H) respectively. There are some known results not on a pair but on a single group or a single monoid. It is shown in [1] that for any group H there exists a topological space X such that the group of all self-homeomorphisms of X is isomorphic to H. It is shown in [2] that for any monoid S there exists a topological space X such that the monoid of all nonconstant continuous maps from X to X is isomorphic to S.

Acknowledgements. The author is grateful to Professor Kazuhiro Kawamura for his helpful comments.

References

[1] J. de Groot, *Groups represented by homeomorphism groups I*, Math. Ann. **138** (1959), 80–102.

[2] V. Trnková, *Non-constant continuous mappings of metric or compact Hausdorff spaces*, Comment. Math. Univ. Carolinae **13** (1972), 283–295.

(Kouki Taniyama) Department of Mathematics, School of Education, Waseda University, Nishi-Waseda 1-6-1, Shinjuku-ku, Tokyo, 169-8050, Japan

E-mail address: taniyama@waseda.jp

Received July 10, 2014